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Research Article

Convergence of Variational Iteration Method for Fractional Delay Integrodifferential-Algebraic Equations

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Fractional order delay integrodifferential-algebraic equations are often used for many practical modeling problems in science and engineering, which have time lag, memory, constraint limit, and so forth. These yield some difficulties in numerical computation. The iterative methods are good choice. In the present paper, we construct variational iteration method for solving them by using the appropriate restricted variation. This overcomes the difficulties caused by limitations of large storage amount and algebraic constraint and extends the previous conclusions.

1. Introduction

Fractional delay integrodifferential-algebraic equations (FDIDAEs) are often used for modeling many science and engineering problems with memory and algebraic constraints, such as flexible multibody dynamics and integrated circuits. Recently, fractional integrodifferential equations (FIDEs) have received much attention; for instance, the stability and asymptotic stability of FIDEs are studied in [1–3]; the numerical methods for solving FIDEs can be found in [4–7]; for the approximate analytical methods for solving FIDEs, the readers can refer to [8–10]. The studies on differential-algebraic equations (DAEs) are mainly concentrated in qualitative analysis as well as convergence and stability of numerical methods. For instance, the structural characteristics and asymptotic stability of the (neutral) DAEs are presented in [11, 12]; the convergence results of one-leg methods, Runge-Kutta methods, BDF methods, and linear multistep methods for DAEs are obtained in [13–16]; the stability of Runge-Kutta methods and Rosenbrock methods for (neutral) DAEs are studied in [17, 18]. As for integrodifferential-algebraic equations, only a few studies have been undertaken, for instance, the convergence and stability of Runge-Kutta methods [19–21].

The variational iteration method (VIM) is one of the important methods used to obtain approximate analytical solutions [22–25] and possesses some good properties, such as flexibility, convenience, accuracy, and less storage. In particular, this method was used to solve pantograph equations [26, 27], differential (integral) equations [28–30], fractional differential (integral) equations [31–34], delay differential-algebraic equations [35] and fractional differential-algebraic equations [36], and so forth.

As far as we know, there are few works about numerical methods (including the VIM) for FDIDAEs. The aim of this paper is to use the VIM to solve FDIDAEs and obtain the corresponding convergence results.

2. Convergence

Consider the initial value problems of FDIDAEs.

$$\begin{aligned}
&D_*^{\alpha} x(t) = f\left(x(t), x(\omega(t)), \int_{t-\tau}^{t} h(x(s), y(s))\, ds, y(t), y(\phi(t))\right), \quad t \in [0,T], \\
&0 = g(x(t), y(t)), \quad t \in [0,T], \\
&x(t) = \varphi(t), \\
&y(t) = \psi(t), \\
&\qquad\qquad t \in [-\tau, 0].
\end{aligned} \tag{1}$$

$D_*^\alpha x(t)$ denotes the *Caputo* derivative of order $\alpha, m-1 < \alpha \le m$, $m \in N$, the delay functions $\omega(t)$ and $\phi(t)$ satisfy $\omega(t) \in [-\tau, t]$, $\phi(t) \in [-\tau, t]$, $f : R^{n_1} \times R^{n_1} \times R^{n_1} \times R^{n_2} \times R^{n_2} \to R^{n_1}$, and $g : R^{n_1} \times R^{n_2} \to R^{n_2}$ are smooth vector functions on the real Euclidean spaces, and $h : R^{n_1} \times R^{n_2} \to R^{n_1}$ satisfies the Lipschitz condition

$$\left\| h(\nu_1, \theta_1) - h(\nu_2, \theta_2) \right\| \le \gamma_1 \left\| \nu_1 - \nu_2 \right\| + \gamma_2 \left\| \theta_1 - \theta_2 \right\|, \tag{2}$$

where $\nu_1, \nu_2 \in R^{n_1}$, $\theta_1, \theta_2 \in R^{n_2}$, the $(m-1)$-order derivatives of initial value functions $\varphi : [-\tau, 0] \to R^{n_1}$ and $\psi : [-\tau, 0] \to R^{n_2}$ are continuous, the Jacobian matrix g_y is invertible, and f_i' is bounded (f_i, $i = 1, 2, \ldots, 5$, denotes the partial derivatives of the function f to ith variable) in a neighborhood of the exact solution. We assume that system (1) has smooth solutions $x(t)$, $y(t)$. Throughout this article, $\|\cdot\|$ denotes the standard Euclidean norm, and the matrix norm is subordinate to $\|\cdot\|$.

Applying the VIM to (1), we can construct the correction functional

$$\begin{aligned} x_{n+1}(t) &= x_n(t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\xi)^{\alpha-1} \lambda(t,\xi) \\ &\quad \cdot \left[D_*^\alpha x_n(\xi) - \widetilde{f}\left(x_n(\xi), x_n(\omega(\xi)), \right.\right. \\ &\quad \left.\left. \int_{\xi-\tau}^{\xi} h(x_n(s), y_n(s))\, ds, y_n(\xi), y_n(\phi(\xi)) \right) \right] d\xi, \end{aligned} \tag{3a}$$

$$0 = g(x_{n+1}(t), y_{n+1}(t)), \tag{3b}$$

where $\lambda(t,\xi)$ is a general Lagrange multiplier, which can be defined optimally by variational theory, and $\widetilde{f}$ denotes the restrictive variation; that is, $\delta\widetilde{f} = 0$. In order to obtain $\lambda(t,\xi)$, we select $\alpha = 1$ (see [36]) and have

$$\begin{aligned} \delta x_{n+1}(t) &= \delta x_n(t) + \int_0^t \lambda(t,\xi)\, \delta \left[x_n'(\xi) - \widetilde{f}\left(x_n(\xi), \right.\right. \\ &\quad x_n(\omega(\xi)), \int_{\xi-\tau}^{\xi} h(x_n(s), y_n(s))\, ds, y_n(\xi), \\ &\quad \left.\left. y_n(\phi(\xi)) \right) \right] d\xi. \end{aligned} \tag{4}$$

By using part integral to (4), the stationary conditions are obtained as

$$\begin{aligned} \left. \frac{\partial \lambda(t,\xi)}{\partial \xi} \right|_{\xi=t} &= 0, \\ 1 + \left. \lambda(t,\xi) \right|_{\xi=t} &= 0. \end{aligned} \tag{5}$$

Moreover, the general Lagrange multiplier can be readily identified by

$$\lambda(t,\xi) = -1. \tag{6}$$

Therefore, the variational iteration formula can be written as

$$\begin{aligned} x_{n+1}(t) &= x_n(t) - J^\alpha \left[D_*^\alpha x_n(t) - f\left(x_n(t), \right.\right. \\ &\quad x_n(\omega(t)), \int_{t-\tau}^{t} h(x_n(s), y_n(s))\, ds, y_n(t), \\ &\quad \left.\left. y_n(\phi(t)) \right) \right], \end{aligned} \tag{7a}$$

$$0 = g(x_{n+1}(t), y_{n+1}(t)). \tag{7b}$$

Theorem 1. *Let $x(t), x_i(t) \in (C^1[-\tau, T])^{n_1}$ and $y(t), y_i(t) \in (C^1[-\tau, T])^{n_2}$, $i = 1, 2, \ldots$. Then the sequences $\{x_n(t)\}_{n=1}^\infty$ and $\{y_n(t)\}_{n=1}^\infty$ defined by (7a) and (7b) with $x_0(t) = \varphi(t)$, $y_0(t) = \psi(t)$, and $t \in [-\tau, 0]$ converge to the solutions of (1).*

Proof. From system (1), we have

$$\begin{aligned} x(t) &= x(t) - J^\alpha \left[D_*^\alpha x(t) - f\left(x(t), x(\omega(t)), \right.\right. \\ &\quad \left.\left. \int_{t-\tau}^{t} h(x(s), y(s))\, ds, y(t), y(\phi(t)) \right) \right], \end{aligned} \tag{8a}$$

$$0 = g(x(t), y(t)). \tag{8b}$$

Let $E_n x(t) = x_n(t) - x(t)$, $E_n y(t) = y_n(t) - y(t)$, $n = 0, 1, \ldots$, and $E_j x(t) = E_j y(t) = 0$ when $t < 0$, $j = 0, 1, \ldots$.

From (7a)–(8b), we obtain

$$\begin{aligned} E_{n+1} x(t) &= J^\alpha \left[f\left(x_n(t), x_n(\omega(t)), \right.\right. \\ &\quad \int_{t-\tau}^{t} h(x_n(s), y_n(s))\, ds, y_n(t), y_n(\phi(t)) \\ &\quad - f\left(x(t), x(\omega(t)), \int_{t-\tau}^{t} h(x(s), y(s))\, ds \right), \\ &\quad \left.\left. y(t), y(\phi(t)) \right) \right], \\ 0 &= g(x_{n+1}(t), y_{n+1}(t)) - g(x(t), y(t)). \end{aligned} \tag{9}$$

Based on the fact that the functions f, g are smooth, the matrix g_y is invertible, and hence we have

$$\begin{aligned} E_{n+1} x(t) &= J^\alpha \left[f_1' E_n x(t) + f_2' E_n x(\omega(t)) \right. \\ &\quad + f_3' \int_{t-\tau}^{t} \left[h(x_n(s), y_n(s)) - h(x(s), y(s)) \right] ds \\ &\quad \left. + f_4' E_n y(t) + f_5' E_n y(\phi(t)) \right], \end{aligned} \tag{10a}$$

$$E_{n+1} y(t) = -g_y^{-1} g_x E_{n+1} x(t), \tag{10b}$$

where f_i' $(i = 1, 2, \ldots, 5)$ denotes the partial derivative of the function f to ith variable.

Let $l_i = \|f_i'\|$ and $K = \|-g_y^{-1}g_x\|$. From (2), (10a), and (10b), we have

$$\begin{aligned}\begin{pmatrix}\|E_{n+1}x(t)\|\\ \|E_{n+1}y(t)\|\end{pmatrix} &\le \begin{pmatrix} l_1 & l_4\\ Kl_1 & Kl_4\end{pmatrix}\begin{pmatrix}J^{\alpha}\|E_nx(t)\|\\ J^{\alpha}\|E_ny(t)\|\end{pmatrix} + \begin{pmatrix} l_2 & l_5\\ Kl_2 & Kl_5\end{pmatrix}\begin{pmatrix}J^{\alpha}\|E_nx(\omega(t))\|\\ J^{\alpha}\|E_ny(\phi(t))\|\end{pmatrix}\\ &+ l_3\tau\begin{pmatrix}\gamma_1 & \gamma_2\\ K\gamma_1 & K\gamma_2\end{pmatrix}\begin{pmatrix}J^{\alpha}\max_{-\tau\le\xi\le T}\|E_nx(\xi)\|\\ J^{\alpha}\max_{-\tau\le\xi\le T}\|E_ny(\xi)\|\end{pmatrix}.\end{aligned} \tag{11}$$

We can derive

$$\begin{pmatrix}\|E_{n+1}x(t)\|\\ \|E_{n+1}y(t)\|\end{pmatrix} \le \begin{pmatrix} l_1+l_2+l_3\tau\gamma_1 & l_4+l_5+l_3\tau\gamma_2\\ K(l_1+l_2+l_3\tau\gamma_1) & K(l_4+l_5+l_3\tau\gamma_2)\end{pmatrix}\begin{pmatrix}J^{\alpha}\max_{-\tau\le t\le T}\|E_nx(t)\|\\ J^{\alpha}\max_{-\tau\le t\le T}\|E_ny(t)\|\end{pmatrix}. \tag{12}$$

Now, we proceed as follows:

$$\begin{gathered}\begin{pmatrix}\|E_1x(t)\|\\ \|E_1y(t)\|\end{pmatrix} \le \begin{pmatrix} l_1+l_2+l_3\tau\gamma_1 & l_4+l_5+l_3\tau\gamma_2\\ K(l_1+l_2+l_3\tau\gamma_1) & K(l_4+l_5+l_3\tau\gamma_2)\end{pmatrix}\begin{pmatrix}\max_{-\tau\le t\le T}\frac{1}{\Gamma(\alpha+1)}\|E_0x(t)\|t^{\alpha}\\ \max_{-\tau\le t\le T}\frac{1}{\Gamma(\alpha+1)}\|E_0y(t)\|t^{\alpha}\end{pmatrix},\\
\begin{pmatrix}\|E_2x(t)\|\\ \|E_2y(t)\|\end{pmatrix} \le \begin{pmatrix} l_1+l_2+l_3\tau\gamma_1 & l_4+l_5+l_3\tau\gamma_2\\ K(l_1+l_2+l_3\tau\gamma_1) & K(l_4+l_5+l_3\tau\gamma_2)\end{pmatrix}^2\begin{pmatrix}\max_{-\tau\le t\le T}\frac{1}{\Gamma(2\alpha+1)}\|E_0x(t)\|t^{2\alpha}\\ \max_{-\tau\le t\le T}\frac{1}{\Gamma(2\alpha+1)}\|E_0y(t)\|t^{2\alpha}\end{pmatrix},\\
\begin{pmatrix}\|E_3x(t)\|\\ \|E_3y(t)\|\end{pmatrix} \le \begin{pmatrix} l_1+l_2+l_3\tau\gamma_1 & l_4+l_5+l_3\tau\gamma_2\\ K(l_1+l_2+l_3\tau\gamma_1) & K(l_4+l_5+l_3\tau\gamma_2)\end{pmatrix}^3\begin{pmatrix}\max_{-\tau\le t\le T}\frac{1}{\Gamma(3\alpha+1)}\|E_0x(t)\|t^{3\alpha}\\ \max_{-\tau\le t\le T}\frac{1}{\Gamma(3\alpha+1)}\|E_0y(t)\|t^{3\alpha}\end{pmatrix},\\
\vdots\\
\begin{pmatrix}\|E_nx(t)\|\\ \|E_ny(t)\|\end{pmatrix} \le \begin{pmatrix} l_1+l_2+l_3\tau\gamma_1 & l_4+l_5+l_3\tau\gamma_2\\ K(l_1+l_2+l_3\tau\gamma_1) & K(l_4+l_5+l_3\tau\gamma_2)\end{pmatrix}^n\begin{pmatrix}\max_{-\tau\le t\le T}\frac{1}{\Gamma(n\alpha+1)}\|E_0x(t)\|t^{n\alpha}\\ \max_{-\tau\le t\le T}\frac{1}{\Gamma(n\alpha+1)}\|E_0y(t)\|t^{n\alpha}\end{pmatrix}.\end{gathered} \tag{13}$$

We have

$$\begin{pmatrix}\|E_nx(t)\|\\ \|E_ny(t)\|\end{pmatrix} \le \frac{(\tau+T)^{n\alpha}\rho^n}{\Gamma(n\alpha+1)}\begin{pmatrix}\max_{-\tau\le t\le T}\|E_0x(t)\|\\ \max_{-\tau\le t\le T}\|E_0y(t)\|\end{pmatrix}, \tag{14}$$

where $K, T, \tau, \gamma_1, \gamma_2, l_i$ $(i = 1, 2, \ldots, 5)$, $\max_{-\tau\le t\le T}\|E_0x(t)\|$, and $\max_{-\tau\le t\le T}\|E_0y(t)\|$ are constants and ρ is the spectral radius of the iterative matrix in the above inequality.

We select $M = \lfloor n\alpha \rfloor$, and therefore $\Gamma(n\alpha+1) \ge \Gamma(M+1)$, $\Gamma(M+1) = M!$. Moreover, we have

$$\begin{pmatrix}\|E_nx(t)\|\\ \|E_ny(t)\|\end{pmatrix} \le \frac{(\tau+T)^{n}\rho^n}{M!}\begin{pmatrix}\max_{-\tau\le t\le T}\|E_0x(t)\|\\ \max_{-\tau\le t\le T}\|E_0y(t)\|\end{pmatrix}. \tag{15}$$

By using *Stirling's* formula, we have

$$\begin{pmatrix}\|E_nx(t)\|\\ \|E_ny(t)\|\end{pmatrix} \le \frac{[(\tau+T)\rho e/M]^n}{\sqrt{2M\pi}(1+O(1/M))}\begin{pmatrix}\max_{-\tau\le t\le T}\|E_0x(t)\|\\ \max_{-\tau\le t\le T}\|E_0y(t)\|\end{pmatrix}, \tag{16}$$

and thus $(\|E_nx(t)\|, \|E_ny(t)\|)^T \to 0$ as $n \to \infty$. □

If the right function $f(x(t), x(\omega(t)), \int_{t-\tau}^{t} h(x(s), y(s))ds, y(t), y(\phi(t))) = Ax(t) + F(x(t), x(\omega(t)), \int_{t-\tau}^{t} h(x(s), y(s))ds,$

$y(t), y(\phi(t)))$, we consider the initial value problems of fractional delay integrodifferential-algebraic equations

$$\begin{aligned}
& D_*^{\alpha} x(t) = Ax(t) + F\Big(x(t), x(\omega(t)), \\
& \quad \int_{t-\tau}^{t} h(x(s), y(s))\,ds, y(t), y(\phi(t))\Big), \quad t \in [0,T], \\
& 0 = g(x(t), y(t)), \quad t \in [0,T], \\
& x(t) = \varphi(t), \\
& y(t) = \psi(t), \\
& \qquad t \in [-\tau, 0],
\end{aligned} \tag{17}$$

where the matrix $A = \operatorname{diag}(a_{11}, a_{22}, \ldots, a_{n_1 n_1}) \in R^{n_1 \times n_1}$, $F : R^{n_1} \times R^{n_1} \times R^{n_1} \times R^{n_2} \times R^{n_2} \to R^{n_1}$, and $g : R^{n_1} \times R^{n_2} \to R^{n_2}$ are smooth vector functions on the real Euclidean spaces, and $h : R^{n_1} \times R^{n_2} \to R^{n_1}$ satisfies the Lipschitz condition

$$\|h(v_1, \theta_1) - h(v_2, \theta_2)\| \le \gamma_1 \|v_1 - v_2\| + \gamma_2 \|\theta_1 - \theta_2\|, \tag{18}$$

where $v_1, v_2, \theta_1, \theta_2, \varphi, \psi, g_y, f_i'$ are defined in the same way as those in system (1). We assume that system (17) has smooth solutions $x(t), y(t)$.

Applying the VIM to (17), we can construct the correction functionals

$$\begin{aligned}
x_{n+1}(t) &= x_n(t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\xi)^{\alpha-1} \Lambda(t,\xi) \\
&\cdot \Big[D_*^{\alpha} x_n(\xi) - Ax_n(\xi) - \widetilde{F}\Big(x_n(\xi), x_n(\omega(\xi)), \\
&\int_{\xi-\tau}^{\xi} h(x_n(s), y_n(s))\,ds, y_n(\xi), y_n(\phi(\xi))\Big)\Big]\, d\xi,
\end{aligned} \tag{19a}$$

$$0 = g(x_{n+1}(t), y_{n+1}(t)). \tag{19b}$$

We select $\alpha = 1$, and thus

$$\begin{aligned}
\delta x_{n+1}(t) &= \delta x_n(t) + \int_0^t \Lambda(t,\xi)\, \delta \Big[x_n'(\xi) - Ax_n(\xi) \\
&- \widetilde{F}\Big(x_n(\xi), x_n(\omega(\xi)), \int_{\xi-\tau}^{\xi} h(x_n(s), y_n(s))\,ds, \\
&y_n(\xi), y_n(\phi(\xi))\Big)\Big]\, d\xi.
\end{aligned} \tag{20}$$

By using part integral to (19a) and (19b), the stationary conditions are obtained as

$$\begin{aligned}
\left.\frac{\partial \Lambda(t,\xi)}{\partial \xi}\right|_{\xi=t} + A\Lambda(t,\xi) = 0, \\
E + \Lambda(t,\xi)\big|_{\xi=t} = 0,
\end{aligned} \tag{21}$$

and, moreover, the general Lagrange multiplier can be readily identified by

$$\Lambda(t,\xi) = \left(-e^{a_{11}(t-\xi)}, -e^{a_{22}(t-\xi)}, \ldots, -e^{a_{n_1 n_1}(t-\xi)}\right)^T. \tag{22}$$

Theorem 2. *Let $x(t), x_i(t) \in (C^1[-\tau, T])^{n_1}$ and $y(t), y_i(t) \in (C^1[-\tau, T])^{n_2}$, $i = 1, 2, \ldots$. Then the sequences $\{x_n(t)\}_{n=1}^{\infty}$ and $\{y_n(t)\}_{n=1}^{\infty}$ defined by (19a) and (19b) with $x_0(t) = \varphi(t)$, $y_0(t) = \psi(t)$, and $t \in [-\tau, 0]$ converge to the solutions of (17).*

Proof. The proof process is similar to that in system (1). □

In general, the Lagrange multiplier $\Lambda(t, \xi)$ obtained with exponential form can increase convergence speed of iterative sequences.

3. Special Cases

Remark 3. If the right function of system (1) has no integral item and no delay item, it becomes the fractional differential-algebraic equation. The results obtained are consistent with the ones discussed in [36]. Moreover, we present a new way to prove the convergence.

Remark 4. When $\alpha = 1$, system (1) can be written as the delay integrodifferential-algebraic equation. Moreover, if the right function of system (1) has no integral item, it becomes the delay differential-algebraic equation discussed in [35].

In conclusion, we get the more general result, which extends the conclusions of existing literature.

4. Illustrative Examples

In this section, some illustrative examples are given to show the efficiency of the VIM for solving fractional delay differential-algebraic equations.

Example 1. Consider the initial value problem of fractional delay differential-algebraic equation

$$\begin{aligned}
& D_*^{\alpha} x(t) = x\left(\frac{t}{2}\right) y(t) - \frac{t^2}{8} y(t) - \frac{t^5 - t^3}{8} + \frac{6t^{1.5}}{\Gamma(2.5)}, \\
& \qquad t \in [0,T], \\
& 0 = x(t) - ty(t) + t^2, \quad t \in [0,T], \\
& x(0) = 0, \\
& y(0) = 0.
\end{aligned} \tag{23}$$

When $\alpha = 1.5$, the exact solution of system (23) is

$$\begin{aligned}
x(t) &= t^3, \\
y(t) &= t^2 + t.
\end{aligned} \tag{24}$$

Applying the VIM to (23), we can construct the correction functional

$$\begin{aligned}
x_{n+1}(t) &= x_n(t) - J^{1.5}\Big[D_*^{1.5} x_n(t) - x_n\left(\frac{t}{2}\right) y_n(t) \\
&+ \frac{t^2}{8} y_n(t) + \frac{t^5 - t^3}{8} - \frac{6t^{1.5}}{\Gamma(2.5)}\Big],
\end{aligned} \tag{25a}$$

$$y_{n+1}(t) = \frac{x_{n+1}(t)}{t} + t. \tag{25b}$$

Moreover, the iteration sequence with the initial approximations $x_0(t) = 0$ and $y_0(t) = 0$ is obtained from (25a) and (25b) as follows:

$$\begin{aligned}
x_1(t) &= 0.9945231827t^3 + 0.171607664t^{3.5} \\
&\quad + 0.101083967t^{4.5}, \\
y_1(t) &= t + 0.9945231827t^2 + 0.171607664t^{2.5} \\
&\quad + 0.101083967t^{3.5}, \\
x_2(t) &= 0.9998023770t^3 + 0.0066196316t^{3.5} \\
&\quad - 0.000964871t^{4.5} + 0.0098023770t^5 \\
&\quad - 0.420964871t^6, \\
y_2(t) &= t + 0.9998023770t^2 + 0.0066196316t^{2.5} \\
&\quad - 0.000964871t^{3.5} + 0.0098023770t^4 \\
&\quad - 0.420964871t^5, \\
x_3(t) &= 0.9999891996t^3 + 0.000403828233t^{3.5} \\
&\quad - 0.0000291441t^{4.5} - 0.11787329t^5 \\
&\quad + 0.003079509167t^6 + 0.000009896323t^{6.5} \\
&\quad + 0.000110548961t^{7.5}, \\
y_3(t) &= t + 0.9999891996t^2 + 0.000403828233t^{2.5} \\
&\quad - 0.0000291441t^{3.5} - 0.11787329t^4 \\
&\quad + 0.003079509167t^5 + 0.000009896323t^{5.5} \\
&\quad + 0.000110548961t^{6.5}, \\
&\vdots
\end{aligned} \tag{26}$$

The approximate solution and exact solution are plotted in Figures 1 and 2. The imaginary line is the curve of the approximate solution, and the solid line is the curve of the exact solution, which shows that the method gives a very good approximation to the exact solution.

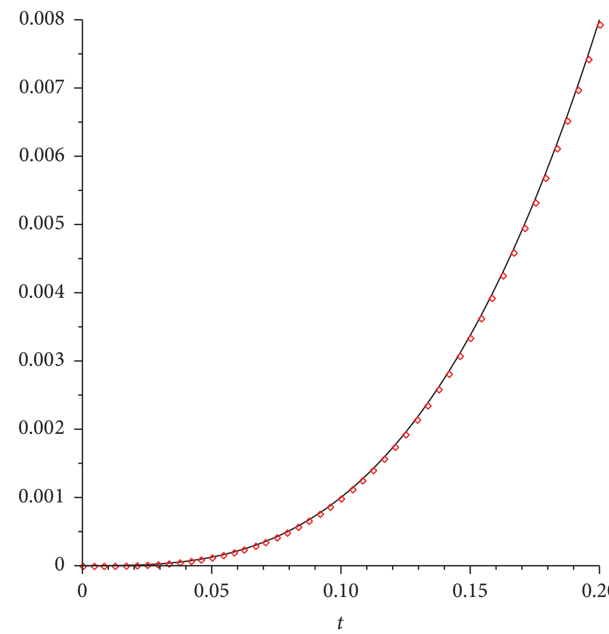

FIGURE 1: The comparison of the exact and approximate solutions of x.

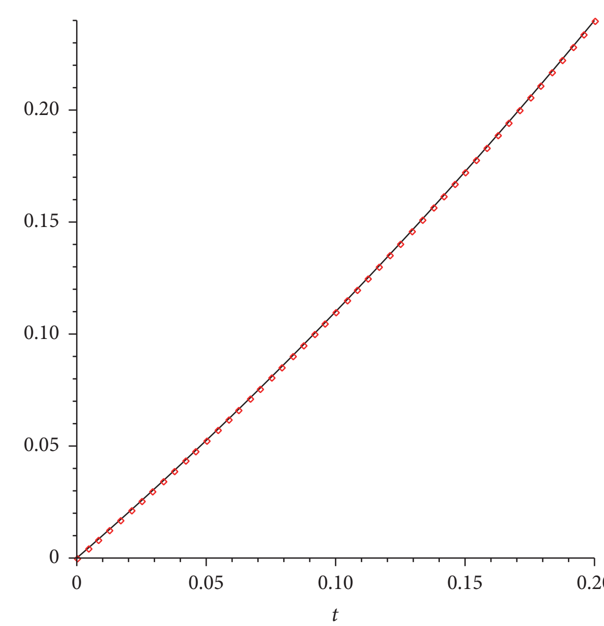

FIGURE 2: The comparison of the exact and approximate solutions of y.

Example 2. Consider the initial value problem of fractional delay differential-algebraic equation

$$\begin{aligned}
D_*^{\alpha} x(t) &= -x^2\left(\frac{t}{2}\right) + y(t) - \sin(t), \quad t \in [0, T], \\
0 &= x(t) - y(t) - e^{-t} + \sin(t), \quad t \in [0, T], \\
x(0) &= 1, \\
y(0) &= 0.
\end{aligned} \tag{27}$$

When $\alpha = 1$, the exact solution of system (27) is

$$\begin{aligned}
x(t) &= e^{-t}, \\
y(t) &= \sin(t).
\end{aligned} \tag{28}$$

Table 1: The errors of the iteration for $x(t)$.

t	$t = 0.02$	$t = 0.06$	$t = 0.1$	$t = 0.3$
$x_3(t)$	0.009999833333	0.9982000000	0.9950000000	1.4843655750
$x(t)$	0.009999833334	0.9982005399	0.9950041653	1.4843717880
Error	$6.7000E-09$	$5.3990E-07$	$4.1656E-06$	$6.2130E-06$

Table 2: The errors of the iteration for $y(t)$.

t	$t = 0.02$	$t = 0.06$	$t = 0.1$	$t = 0.3$
$y_3(t)$	0.01999866670	0.05996399737	0.09983329948	0.2955202500
$y(t)$	0.01999866666	0.05996400648	0.09983341666	0.2954917734
Error	$4.1000E-11$	$9.1100E-09$	$1.1718E-07$	$2.8476E-05$

Using the VIM in the previous section, we construct the following correction functional:

$$x_{n+1}(t) = x_n(t) - J^1\left[D_*^1 x_n(t) + x_n^2\left(\frac{t}{2}\right) - y_n(t) + \sin(t)\right], \tag{29a}$$

$$0 = x_{n+1}(t) - y_{n+1}(t) - e^{-t} + \sin(t). \tag{29b}$$

Moreover, the iteration sequence with the initial approximations $x_0(t) = 1$ and $y_0(t) = 0$ is obtained from (29a) and (29b) as follows:

$$\begin{aligned} x_1(t) &= 1 - t + \frac{1}{2}t^2 + \frac{1}{24}t^4, \\ y_1(t) &= t - \frac{1}{3}t^3 + \frac{1}{720}t^6, \\ x_2(t) &= 1 - t + \frac{1}{2}t^2 - \frac{1}{6}t^3 - \frac{1}{96}t^4 - \frac{1}{240}t^5 + o\left(t^5\right), \\ y_2(t) &= t - \frac{1}{3}t^3 + \frac{5}{96}t^4 + \frac{1}{240}t^5 + o\left(t^5\right), \\ x_3(t) &= 1 - t + \frac{1}{2}t^2 - \frac{1}{6}t^3 + \frac{1}{24}t^4 - \frac{13}{3840}t^5 + o\left(t^5\right), \\ y_3(t) &= t - \frac{1}{6}t^3 - \frac{13}{3840}t^5 + o\left(t^5\right), \\ &\vdots \end{aligned} \tag{30}$$

When the iteration number $n = 3$, the corresponding relative errors are shown in Tables 1 and 2.

Example 3. Consider the initial value problem of a fractional delay integrodifferential-algebraic equation

$$\begin{aligned} D_*^{\alpha} x(t) &= 4x\left(\frac{t}{2}\right) y(t) + y^2(t) - y^3(t) \\ &\quad + \frac{2}{\Gamma(3-\alpha)} t^{2-\alpha} \\ &\quad + \int_0^t \left(3x(s) - 2y(s)\right) ds, \quad t \in [0, T], \\ 0 &= y^2(t) - x(t), \quad t \in [0, T], \\ x(0) &= 0, \\ y(0) &= 0. \end{aligned} \tag{31}$$

When $\alpha = 1.8$, the exact solutions of system (31) are

$$\begin{aligned} x(t) &= t^2, \\ y(t) &= t. \end{aligned} \tag{32}$$

Using the VIM in the previous section, we construct the following correction functionals:

$$\begin{aligned} x_{n+1}(t) = x_n(t) - J^{1.8}\Big[& D_*^{1.8} x_n(t) - 4x_n\left(\frac{t}{2}\right) y_n(t) \\ & - y_n^2(t)\, y_n^3(t) - \frac{2}{\Gamma(1.2)} t^{0.2} \\ & - \int_0^t \left(3x_n(s) - 2y_n(s)\right) ds \Big], \end{aligned} \tag{33a}$$

$$0 = y_{n+1}^2(t) - x_{n+1}(t). \tag{33b}$$

Moreover, the iteration sequence with the initial approximations $x_0(t) = t$ and $y_0(t) = t$ is obtained from (33a) and (33b) as follows:

$$\begin{aligned} x_1(t) &= 1.040023124t^2 - 0.8231479768t^{4.8}, \\ y_1(t) &= 1.040023124t - 0.8231479768t^{3.8}, \\ x_2(t) &= 0.9991113151t^2 + 0.100545240t^{3.8} \\ &\quad - 0.0003301028t^{4.8} - 0.0008794265t^{6.6} + o\left(t^7\right), \\ y_2(t) &= 0.9991113151t + 0.100545240t^{2.8} \\ &\quad - 0.0003301028t^{3.8} - 0.0008794265t^{5.6} + o\left(t^6\right). \\ &\vdots \end{aligned} \tag{34}$$

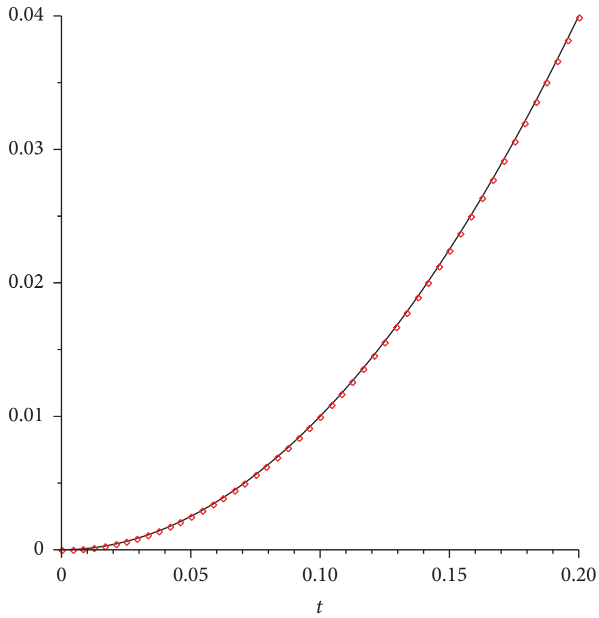

FIGURE 3: The comparison of the exact and approximate solutions of x.

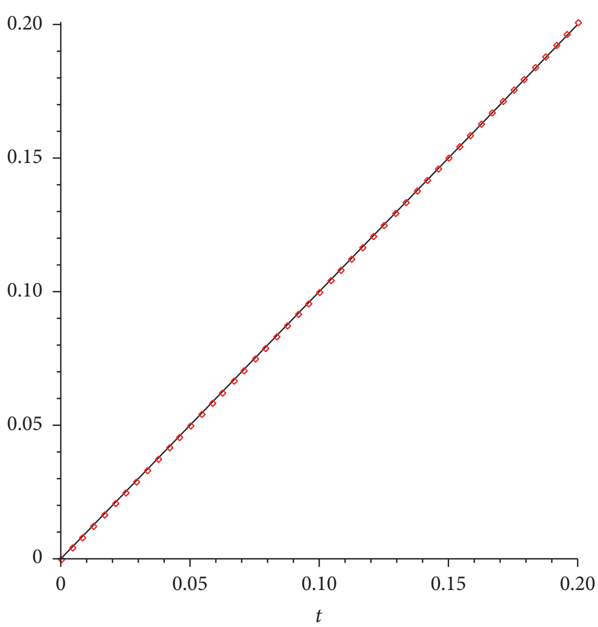

FIGURE 4: The comparison of the exact and approximate solutions of y.

The approximate solutions and exact solutions are plotted in Figures 3 and 4. The imaginary line is the curve of the approximate solution, and the solid line is the curve of the exact solution, which shows that the method gives a very good approximation to the exact solution.

Example 4. Consider the initial value problem of a fractional delay integrodifferential-algebraic equation

$$\begin{aligned} D_*^{\alpha} x(t) &= x(t) - 2x\left(\frac{t}{2}\right) y\left(\frac{t}{4}\right) \\ &\quad + \int_0^t \left[x(u) - \sqrt{y(u)}\right] du, \quad t \in [0,1], \\ 0 &= e^{-t} x(t) - y(t), \quad t \in [0,1], \\ x(0) &= 1, \\ y(0) &= 1. \end{aligned} \tag{35}$$

When $\alpha = 1$, the exact solution of system (35) is

$$\begin{aligned} x(t) &= e^{-t}, \\ y(t) &= e^{-2t}. \end{aligned} \tag{36}$$

We select $\Lambda(t,s) = -e^{(t-s)}$, and, using the VIM in the previous section, we construct the following correction functionals:

$$\begin{aligned} x_{n+1}(t) &= x_n(t) - J^1 \left[e^{t-s} \left(D_*^1 x_n(t) - x_n(t) \right. \right. \\ &\quad - 2x_n\left(\frac{t}{2}\right) y_n\left(\frac{t}{4}\right) \\ &\quad \left. \left. + \int_0^t \left[x_n(u) - \sqrt{y_n(u)} \right] du \right) \right], \end{aligned} \tag{37a}$$

$$0 = e^{-t} x_{n+1}(t) - y_{n+1}(t). \tag{37b}$$

Moreover, the iteration sequence with the initial approximations $x_0(t) = 1$ and $y_0(t) = 1$ is obtained from (37a) and (37b) as follows:

$$\begin{aligned} x_1(t) &= 1 - t, \\ y_1(t) &= 1 - 2t, \\ x_2(t) &= 1 - t + \frac{1}{2}t^2, \\ y_2(t) &= 1 - 2t + 2t^2, \\ x_3(t) &= 1 - t + \frac{1}{2}t^2 - \frac{7}{24}t^3, \\ y_3(t) &= 1 - 2t + 2t^2 - \frac{4}{3}t^3, \\ &\vdots \end{aligned} \tag{38}$$

When the iteration number $n = 3$, the corresponding relative errors are shown in Tables 3 and 4.

To compare the convergence speed of iterative sequences with different Lagrange multiplier $\Lambda(t,s)$, then, we

Table 3: The errors of the iteration for $x(t)$.

t	$t = 0.1$	$t = 0.15$	$t = 0.2$	$t = 0.3$
$x_3(t)$	0.9047083333	0.8602656250	0.8176666667	0.7371250000
$x(t)$	0.9048374180	0.8607079764	0.8187307531	0.7408182207
Error	$1.2908E-04$	$4.4235E-04$	$1.0640E-03$	$3.6932E-03$

Table 4: The errors of the iteration for $y(t)$.

t	$t = 0.1$	$t = 0.15$	$t = 0.2$	$t = 0.3$
$y_3(t)$	0.8186666667	0.7405000000	0.6693333333	0.5440000000
$y(t)$	0.8187307531	0.7408182207	0.6703200460	0.5488116361
Error	$6.4086E-05$	$3.1822E-04$	$9.8671E-04$	$4.8116E-03$

Table 5: The error comparison of $x(t)$ with different $\Lambda(t,s)$.

t	$t = 0.01$	$t = 0.03$	$t = 0.05$	$t = 0.07$	$t = 0.09$
Error x_1	$1.2540E-07$	$3.4085E-06$	$1.5882E-05$	$4.3861E-05$	$9.3810E-05$
Error x_2	$3.9850E-02$	$1.1866E-01$	$1.9627E-01$	$2.7266E-01$	$3.8518E-01$

Table 6: The error comparison of $y(t)$ with different $\lambda(t,s)$.

t	$t = 0.01$	$t = 0.03$	$t = 0.05$	$t = 0.07$	$t = 0.09$
Error y_1	$6.6000E-09$	$5.3360E-07$	$4.0847E-06$	$1.5568E-05$	$4.2211E-05$
Error y_2	$6.0052E-02$	$1.8051E-01$	$3.0149E-01$	$4.2307E-01$	$5.4544E-01$

select$\Lambda(t,s) = -1$ and construct the following correction functionals

$$x_{n+1}(t) = x_n(t) - J^1\left[D_*^1 x_n(t) - x_n(t) - 2x_n\left(\frac{t}{2}\right)y_n\left(\frac{t}{4}\right) + \int_0^t \left[x_n(u) - \sqrt{y_n(u)}\right]du\right], \tag{39a}$$

$$0 = e^{-t}x_{n+1}(t) - y_{n+1}(t). \tag{39b}$$

Moreover, the iteration sequence with the initial approximations $x_0(t) = 1$ and $y_0(t) = 1$ is obtained from (39a) and (39b), and when the iteration number $n = 3$, the error comparisons of different $\Lambda(t,s)$ are shown in Tables 5 and 6.

The Error x_1 denotes the error of $x(t)$ with $\Lambda(t,s) = -e^{t-s}$, and the Error x_2 denotes the error of $x(t)$ with $\Lambda(t,s) = -1$.

The Error y_1 denotes the error of $y(t)$ with $\Lambda(t,s) = -e^{t-s}$, and the Error y_2 denotes the error of $y(t)$ with $\Lambda(t,s) = -1$.

Tables 5 and 6 show that the Lagrange multiplier $\Lambda(t,\xi)$ with exponential form can increase convergence speed of iterative sequences in Example 4.

In conclusion, the convergence speed of iterative sequences can be increased by selecting the appropriate restrictive variation and constructing the corresponding Lagrange multiplier. The construction techniques of different Lagrange multipliers are discussed in [24].

5. Conclusion

In this paper, the convergence of VIM for FDIDAEs is established. Theoretical analysis and numerical experiments show that VIM can be used efficiently to solve FDIDAEs. This method offers significant advantages in terms of applicability and computational efficacy and accuracy.

Competing Interests

The authors declare that they have no competing interests.

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