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Research Article

A Mixture of Inverse Weibull and Inverse Burr Distributions: Properties, Estimation, and Fitting

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The new mixture model of the two components of the inverse Weibull and inverse Burr distributions (MIWIBD) is proposed. First, the properties of the investigated mixture model are introduced and the behaviors of the probability density functions and hazard rate functions are displayed. Then, the estimates of the five-dimensional vector of parameters by using the classical method such as the maximum likelihood estimation (MLEs) and the approximation method by using Lindley's approximation are obtained. Finally, a real data set for the proposed mixture model is applied to illustrate the proposed mixture model.

1. Introduction

The importance of mixture models comes from the fact that most available data can be considered as data coming from a mixture of two or more statistical models; see Sultan et al. [1]. For books that dealt with the models of the mixture, see Everitt and Hand [2] and McLachlan and Peel [3]. Because the mixing of statistical distributions gives a new distribution with the properties of its compounds, we in this paper propose the two-component mixture models of inverse Weibull and inverse Burr distributions (MIWIBD). For the importance of the inverse Weibull distribution (IWD) as a single component from its uses in physical phenomena, see Keller et al. [4]. Also, for the importance of the inverse Burr distribution (IBD) as one component from its uses in forestry applications, see Lindsay [5]. This importance for each distribution alone has made us merge the two distributions together to obtain new properties from the distributive compounds. It should be noted that the mixing of the IWIBD gives a mixture model with a unimodal and bimodal peak for the hazard rate functions and these forms are important in applications which will be displayed in Section 2. The probability density function (pdf) from the MIWIBD is as follows:

$$f(x;\Theta)=\sum_{i=1}^{2}p_i f_i(x,\Theta_i),\quad 0\le p_1\le 1,\ p_1+p_2=1, \tag{1}$$

where the (pdf) of the first component (inverse Weibull) is given by

$$f_1(x;\Theta_1)=\beta_1\alpha_1^{-\beta_1}x^{-(\beta_1+1)}e^{-(\alpha_1 x)^{-\beta_1}},\qquad x\ge 0,\ \alpha_1,\beta_1>0, \tag{2}$$

and the (pdf) of the second component (inverse Burr) is given by

$$f_2(x;\Theta_2)=\beta_2\alpha_2 x^{-(\beta_2+1)}\left(1+x^{-\beta_2}\right)^{-(\alpha_2+1)},\qquad x\ge 0,\ \alpha_2,\beta_2>0, \tag{3}$$

where $\Theta = (p_1, \alpha_1, \beta_1, \alpha_2, \beta_2)$, $\Theta_1 = (\alpha_1, \beta_1)$, and $\Theta_2 = (\alpha_2, \beta_2)$. Evidently, the cumulative density function (cdf) from the MIWIBD is as follows:

$$F(x;\Theta) = \sum_{i=1}^{2} p_i F_i(x, \Theta_i), \quad (4)$$

where the cdf for each distribution from the MIWIBD alone, respectively, is as follows:

$$F_1(x;\Theta_1) = e^{-(\alpha_1 x)^{-\beta_1}}, \quad x \geq 0,\ \alpha_1 > 0,\ \beta_1 > 0, \quad (5)$$

$$F_2(x;\Theta_2) = \left(1 + x^{-\beta_2}\right)^{-\alpha_2}, \quad x \geq 0,\ \beta_2 > 0,\ \alpha_2 > 0. \quad (6)$$

Some papers have dealt with the mixtures of two inverse Weibull distributions (MTIWD), for example, Sultan et al. [1] and Sultan and Al-Moisheer [6]. In addition, there are some researches that have discussed the mixtures of two inverse Burr distributions (MTIBD), for example, the works of Al-Moisheer [7]. Also, there is a mixture of one of its components which is IWD; see Sultan and Al-Moisheer [8].

In this paper, the order is as follows: in Section 2, we introduce few properties of the MIWIBD. In Section 3, through the method of maximum likelihood we find the five unknown parameters estimates of the MIWIBD. In Section 4, we use Lindley's approximation to estimate the unknown parameters of the MIWIBD. In Section 5, we apply the MIWIBD by fitting it to a real data collected from Jeddah city for measuring the carbon monoxide level in different locations. Finally, we draw expressions for Lindley's approximation matrix, and these are displayed in Appendix.

2. Some Properties for the MIWIBD

From (2) and (3), Keller et al. [4] and Abd-Elfattah and Alharbey [9] have discussed some properties of the IWD and IBD, respectively. In this section, we discuss some properties of the MIWIBD by merging the corresponding conclusions of the IWD and IBD.

2.1. Measures of Location and Dispersion (Mean and Variance). The measures of location and dispersion for the mean and variance of the MIWIBD in (1) are as follows:

$$\begin{aligned} E(X) &= \frac{p_1}{\alpha_1}\Gamma\left(1 - \frac{1}{\beta_1}\right) + p_2\alpha_2\Gamma\left(\alpha_2 - \frac{1}{\beta_2}\right)\Gamma\left(1 + \frac{1}{\beta_2}\right), \quad \beta_1 > 1,\ \beta_2, \alpha_2 > 1, \\ \operatorname{Var}(X) &= \frac{p_1}{\alpha_1^2}\left[\Gamma\left(1 - \frac{2}{\beta_1}\right) - \Gamma^2\left(1 - \frac{1}{\beta_1}\right)\right] + p_2\left[\alpha_2\Gamma\left(\alpha_2 - \frac{2}{\beta_2}\right)\Gamma\left(1 + \frac{2}{\beta_2}\right) - \alpha_2^2\Gamma^2\left(\alpha_2 - \frac{1}{\beta_2}\right)\Gamma^2\left(1 + \frac{1}{\beta_2}\right)\right], \\ &\qquad \beta_1 > 2,\ \beta_2, \alpha_2 > 2, \end{aligned} \quad (7)$$

where $\Gamma(\cdot)$ denotes the gamma function.

Table 1: Numerical results for the mode(s) and median of the MIWIBD.

$\Theta = (p_1, \alpha_1, \beta_1, \alpha_2, \beta_2)$	Mode(s)	Median
0.2, 1, 1.5, 2, 3	1.0517	1.8332
0.3, 1, 1.5, 2, 3	1.0695	1.7666
0.5, 1, 1.5, 2, 3	1.1257	1.6270
0.6, 1, 1.5, 2, 3	1.1717	1.5554
0.8, 1, 1.5, 2, 3	1.3660	1.4125
0.2, 1, 0.75, 4, 6	3.2311, 8.5172	1.6876
0.3, 1, 0.75, 4, 6	3.2311, 8.7123	1.6857
0.5, 1, 0.75, 4, 6	3.2311, 9.0552	1.6804
0.6, 1, 0.75, 4, 6	3.2311, 9.2414	1.6766
0.8, 1, 0.75, 4, 6	3.2311, 9.7475	1.6637

2.2. Measures of Location (Mode and Median). By solving the nonlinear equations with respect to x and from (4), the mode and median of the MIWIBD are obtained, respectively, by

$$\begin{aligned} &p_1\beta_1\alpha_1^{-\beta_1}x^{-(\beta_1+2)}e^{-(\alpha_1 x)^{-\beta_1}}\left[-(\beta_1+1) + \beta_1\alpha_1^{-\beta_1}x^{-(\beta_1)}\right] - p_2x^{-(\beta_2+2)}\left(1 + x^{-\beta_2}\right)^{-(\alpha_2+1)} \\ &\quad \cdot\left[\beta_2(\alpha_2+1)x^{-\beta_2}\left(1 + x^{-\beta_2}\right)^{-1} - (\beta_2+1)\right] = 0, \end{aligned} \quad (8)$$

$$p_1e^{-(\alpha_1 x)^{-\beta_1}} + p_2\left(1 + x^{-\beta_2}\right)^{-\alpha_2} = 0.5. \quad (9)$$

Table 1 shows the modes and median of the MIWIBD for some selections of the parameters.

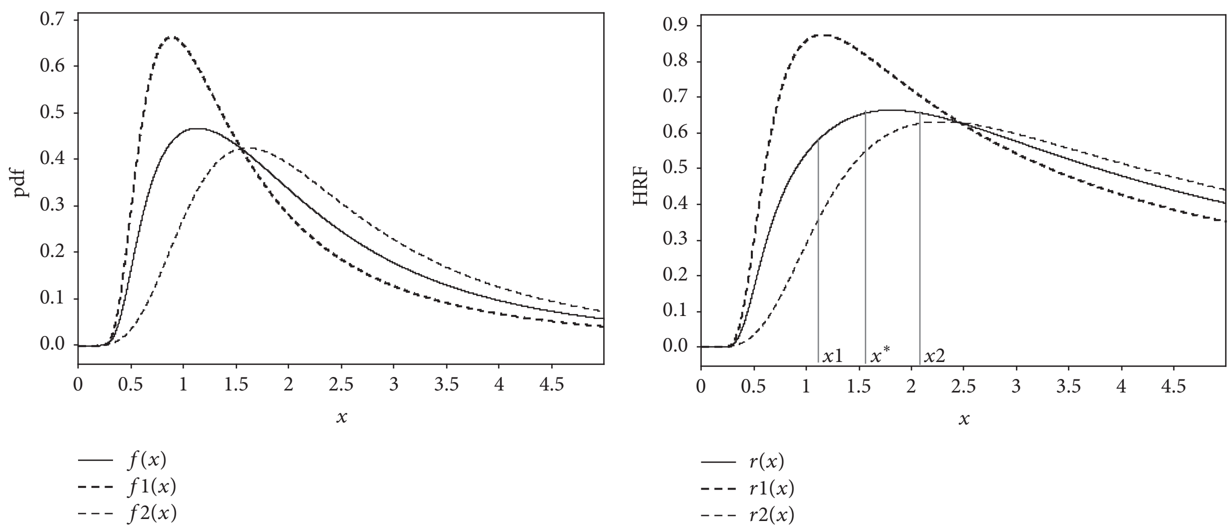
In Table 1, the five parameters $p_1, \alpha_1, \beta_1, \alpha_2$, and β_2 are selected to display the unimodal and bimodal shapes for the pdf of the proposed MIWIBD. Table 1 clearly shows that the modes are not much affected by changing the value of p_1, but the median was affected by changing the value of p_1. Figures 1(a) and 2(a) show the pdf between the components and their mixtures with parameters displaying the shapes for the peak of the unimodal and bimodal cases for the proposed MIWIBD.

2.3. Reliability and Failure Rate Functions. The following equation gives the reliability function of the MIWIBD

$$R(x) = p_1\left[1 - e^{-(\alpha_1 x)^{-\beta_1}}\right] + p_2\left[1 - \left(1 + x^{-\beta_2}\right)^{-\alpha_2}\right], \quad x \geq 0. \quad (10)$$

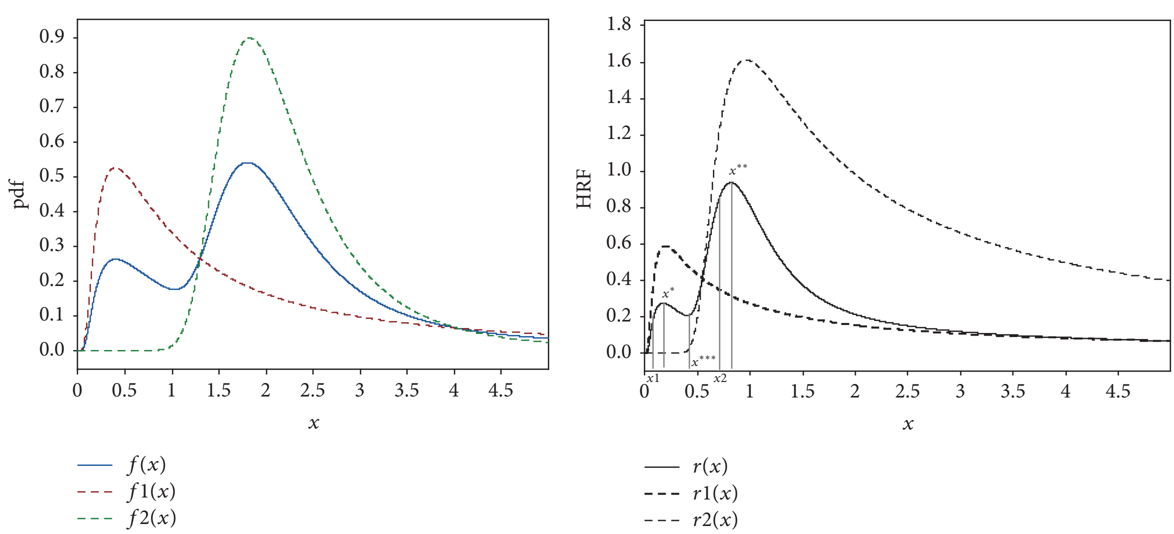
Equations (1) and (4) help us in finding the failure rate function (hazard rate function HRF) of the MIWIBD and are given as

$$r(x) = \frac{p_1\beta_1\alpha_1^{-\beta_1}x^{-(\beta_1+1)}e^{-(\alpha_1 x)^{-\beta_1}} + p_2\beta_2\alpha_2x^{-(\beta_2+1)}\left(1 + x^{-\beta_2}\right)^{-(\alpha_2+1)}}{p_1\left(1 - e^{-(\alpha_1 x)^{-\beta_1}}\right) + p_2\left(1 - \left(1 + x^{-\beta_2}\right)^{-\alpha_2}\right)}, \quad x \geq 0. \quad (11)$$



(a) (b)

FIGURE 1: (a) Density functions: components and their mixture with parameters (0.5, 1.0, 1.5, 2.0, and 3.0). (b) HR functions: components and their mixture with parameters (0.5, 1.0, 1.5, 2.0, and 3.0), unimodal case.



(a) (b)

FIGURE 2: (a) Density functions: components and their mixture with parameters (0.5, 1.0, 0.75, 4.0, and 6.0). (b) HR functions: components and their mixture with parameters (0.5, 1.0, 0.75, 4.0, and 6.0), bimodal case.

The above equation is expressed by viewing the result by Al-Hussaini and Sultan [10], as

$$r(x) = h(x) r_1(x) + (1 - h(x)) r_2(x), \qquad (12)$$

where

$$h(x) = \frac{1}{1 + p_2 R_2(x) / p_1 R_1(x)},$$

$$r_i(x) = \frac{f_i(x)}{R_i(x)}, \quad i = 1, 2,$$

$$R_1(x) = 1 - e^{-(\alpha x)^{-\beta}},$$

$$R_2(x) = 1 - \left(1 + x^{-\beta_2}\right)^{-\alpha_2}. \qquad (13)$$

By taking the derivative of the failure rate function, we get

$$r'(x) = h(x) r_1'(x) + (1 - h(x)) r_2'(x) - h(x)(1 - h(x))\left[r_1(x) - r_2(x)\right]^2. \qquad (14)$$

Observe that $h(x)$ and $1-h(x)$ assume values in interval $[0,1]\ \forall x$. Also, it follows from (14) that if the derivative of the failure rate function is less than zero ($r_i'(x)<0$, $\forall x$, $i=1,2$), then the derivative of the failure rate function is less than zero ($r'(x)<0$, $\forall x$). After few conversions, the derivative of the failure rate function can be reduced and is given in (14) where $h(x)$ is determined in (12) and the derivative of the failure rate function $r_i'(x)$, $i=1,2$, is as follows:

$$r_1'(x) = r_1(x)\, x^{-(\beta_1+1)}\left[\frac{\beta_1 \alpha_1^{-\beta_1}}{1-e^{-(\alpha_1 x)^{-\beta_1}}} - (\beta_1+1)\, x^{\beta_1}\right],$$
$$r_2'(x) = r_2(x) \cdot x^{-(\beta_2+1)}\left[\frac{\beta_2(\alpha_2+1)\left(1+x^{-\beta_2}\right)^{-1} - \alpha_1^{-1}(\beta_2+1)\, x^{\beta_2}}{1-\left(1+x^{-\beta_2}\right)^{-\alpha_2}}\right]. \quad (15)$$

Equation (11) that represents the failure function of the MIWIBD holds for the following limits.

Lemma 1. *We have*

$$\lim_{x\to 0} r(x) = 0, \quad (16)$$

if $\beta_2\alpha_2>1$, where β_2 and α_2 are the shape parameters in IBD. And

$$\lim_{x\to\infty} r(x) = 0. \quad (17)$$

Proof. First from (12) we see that $\lim_{x\to 0} h(x) = p_1$; then from the mixture components, IWD and IBD, respectively, we have $\lim_{x\to 0} r_1(x) = 0$ for the first component (IWD) from the mixture [see Sultan et al. [1]]. Now, from (13) it can be shown the failure rate function for the second component (IBD) from the mixture takes a form

$$r_2(x) = \frac{\beta_2\alpha_2 x^{-(\beta_2+1)}\left(1+x^{-\beta_2}\right)^{-(\alpha_2+1)}}{1-\left(1+x^{-\beta_2}\right)^{-\alpha_2}}, \quad (18)$$

and then in the IBD, we have three cases for the limit dependent on the multiple of two shape parameters β_2, α_2 and the limit for $r_2(x)$ can be written in the following form:

$$\lim_{x\to 0} r_2(x) = \begin{cases} \infty, & \beta_2\alpha_2 < 1, \\ 1, & \beta_2\alpha_2 = 1, \\ 0, & \beta_2\alpha_2 > 1, \end{cases} \quad (19)$$

so we put the condition on the two shape parameters $\beta_2\alpha_2>1$ in the (IBD) for $\lim_{x\to\infty} r_2(x) = 0$; thus (16) is proved. Second, from (12) it can be shown that $\lim_{x\to\infty} h(x) \le 1$, which means $0 \le h(x) \le 1$. Then for (IWD) we can see that $\lim_{x\to\infty} r_1(x) = 0$ [see Sultan et al. [1]]. Also for IBD the limit of failure rate function $r_2(x)$ is in (18). Simply by using Taylor expansion for $(1+x^{-\beta_2})^{\alpha_2+1}$ then $\lim_{x\to\infty} r_2(x) = 0$, so (17) is proved, and thus the proof is complete. □

2.4. Performance of the Failure Rate Graphs. If x_i^*, $i=1,2$, refers to the mode of the pdf $f_i(x)$, $x_1 = \min(x_1^*, x_2^*)$ and $x_2 = \max(x_1^*, x_2^*)$. Also, $r_i(x) = f_i(x)/R_i(x)$; we observe both $f_1(x)$ and $f_2(x)$ in the numerator of $r_i(x) \uparrow$ in $(0, x_1)$, with the denominator $\downarrow$ in the same interval. Finally, $r(x)\uparrow$ is in $(0,x_1)$. Moreover, as $x\to\infty$, $r(x)\to 0$. In such case of the interval (x_1,∞), two statuses appear.

(a) Unimodal Status. Here x^* defines the maximum point of the failure rate of the MIWIBD. In the interval (x_1, x^*) the difference Δ between $r_1(x)$ and $r_2(x)$ is small so that the first two terms of the derivative of the failure rate function $r'(x)$ in (14) dominate the third term and then the derivative of the failure rate function $r'(x)>0$. When the difference Δ increases to the point that the third term in the derivative of the failure rate function $r'(x)$ dominates the first two terms, then the derivative of the failure rate function $r'(x)<0$ in (x^*,∞). Summarizing, we have the failure rate of the MIWIBD to be $\uparrow$ in $(0,x^*)$ and $\downarrow$ in (x^*,∞), reaching zero as $x\to\infty$; see Figure 1(b).

(b) Bimodal Status. Here x^* and x^{**} refer to, respectively, the smallest and largest maximum points of the failure rate of the mixture. When the difference Δ between $r_1(x)$ and $r_2(x)$ in the interval (x_1,x^*) is small, where $x_1<x^*<x_2<x^{**}$, the third term in (14) is dominated by the first two terms and so $r'(x)>0$ in $(0,x^*)$. The difference Δ in the interval (x^*,x^{***}), where x^{***} is the local minimum point of $r(x)$, becomes larger to the point that the third term in $r'(x)$ dominates the first two terms resulting in $r'(x)<0$ in (x^*,x^{***}). In the interval (x^{***},x^{**}), the difference becomes small so that the third term in $r'(x)$ is dominated by the first two terms, and so $r'(x)>0$. Summarizing, we have the failure rate of the mixed model to be $\uparrow$ in $(0,x^*)$, decreasing in (x^*,x^{***}), $\uparrow$ in (x^{***},x^{**}), and $\downarrow$ again in (x^{**},∞), reaching 0 as x tends to ∞; see Figure 2(b).

We observe, from Figures 1(b) and 2(b), the shape of the model (unimodal and bimodal) is influenced by the parameters selected. Clearly, when $(\beta_1,\alpha_2,\beta_2)$ varied from 1.5, 2.0, and 3.0 to 0.75, 4.0, and 6.0 the model is varied from the unimodal case to the bimodal case.

3. Classical Method for Estimating the Parameters from the MIWIBD

Here, we define the classical method estimation of the maximum likelihood approach for the five-dimensional parameter vector Θ of the mixture density. Equation (1) is found on a random sample of size n. The MLE $\widehat{\Theta}$ is determined as a result of the likelihood equations

$$\frac{\partial L(\Theta)}{\partial\theta_i} = 0, \quad i=1,2,3,4,5, \quad (20)$$

or given by

$$\frac{\partial \log L(\Theta)}{\partial\theta_i} = 0, \quad i=1,2,3,4,5, \quad (21)$$

where

$$L(\Theta) = \prod_{j=1}^{n} f(x_j; \Theta) \quad (22)$$

explains the likelihood function formed under the assumption of identically independent distributions (iid) data $x_1, \ldots, x_n$. The likelihood function based on the mixture density in (1) is obtained by

$$L(\Theta) = \prod_{j=1}^{n} \left[p_1 f_1 (x_j; \Theta_1) + p_2 f_2 (x_j; \Theta_2) \right], \quad (23)$$

where $\Theta_1 = (\alpha_1, \beta_1)$ and $\Theta_2 = (\alpha_2, \beta_2)$.

By taking the derivative of the log-likelihood function $L^* = \log L(\Theta)$ with respect to the five parameters from the MIWIBD then, the derivatives from the first order of L^* become

$$\begin{aligned}
\frac{\partial L^*}{\partial p_1} &= \sum_{j=1}^{n} \omega (x_j; \Theta) = 0, \\
\frac{\partial L^*}{\partial \alpha_1} &= \sum_{j=1}^{n} p_1 \phi_1 (x_j; \Theta) \eta_1 (x_j; \Theta) = 0, \\
\frac{\partial L^*}{\partial \beta_1} &= \sum_{j=1}^{n} p_1 \phi_2 (x_j; \Theta) \eta_1 (x_j; \Theta) = 0, \\
\frac{\partial L^*}{\partial \beta_2} &= \sum_{j=1}^{n} p_2 \psi_1 (x_j; \Theta) \eta_2 (x_j; \Theta) = 0, \\
\frac{\partial L^*}{\partial \alpha_2} &= \sum_{j=1}^{n} p_2 \psi_2 (x_j; \Theta) \eta_2 (x_j; \Theta) = 0,
\end{aligned} \quad (24)$$

where $\omega(x_j; \Theta)$, $\phi_1(x_j; \Theta)$, $\phi_2(x_j; \Theta)$, $\eta_1(x_j; \Theta)$, $\eta_2(x_j; \Theta)$, $\psi_1(x_j; \Theta)$, and $\psi_2(x_j; \Theta)$ are given, respectively, by

$$\begin{aligned}
\omega (x_j; \Theta) &= \frac{f_1 (x_j; \Theta_1) - f_2 (x_j; \Theta_2)}{f (x_j; \Theta)}, \\
\phi_1 (x_j; \Theta) &= -\beta_1 \alpha_1^{-1} + \beta_1 \alpha_1^{-(\beta_1+1)} (x_j)^{-\beta_1}, \\
\phi_2 (x_j; \Theta) &= \beta_1^{-1} - \log (\alpha_1) - \log (x_j) \\
&\quad + (\alpha_1 x_j)^{-\beta_1} \log (\alpha_1 x_j), \\
\eta_1 (x_j; \Theta) &= \frac{f_1 (x_j; \Theta_1)}{f (x_j; \Theta)}, \\
\eta_2 (x_j; \Theta) &= \frac{f_2 (x_j; \Theta_2)}{f (x_j; \Theta)}, \\
\psi_1 (x_j; \Theta) &= \beta_2^{-1} \\
&\quad + \left((\alpha_2 + 1) x_j^{-\beta_2} \left(1 + x_j^{-\beta_2}\right)^{-1} - 1 \right) \log (x_j), \\
\psi_2 (x_j; \Theta) &= \alpha_2^{-1} - \log \left(1 + x_j^{-\beta_2}\right),
\end{aligned} \quad (25)$$

and $f(x_j; \Theta)$, $f_1(x_j; \Theta_1)$, and $f_2(x_j; \Theta_2)$ are as in (1)–(3), respectively. We can obtain the solutions for (24) to get the estimates of the five parameters from the MIWIBD and to solve them using Newton-Raphson method.

4. Bayesian Method by Using Lindley's Approximation for Estimating the Parameters from the MIWIBD

In Bayesian estimation the posterior distribution function is defined by multiplying the likelihood function with a prior distribution for $\Theta = (p_1, \alpha_1, \beta_1, \beta_2, \alpha_2)$. Hence, the likelihood function is given by (3) in Section 3. The parameters $p_1, \alpha_1, \alpha_2, \beta_1$, and β_2 are independent random variables for the prior distribution of $\Theta = (p_1, \alpha_1, \beta_1, \beta_2, \alpha_2)$ represented by $g(p_1, \alpha_1, \beta_1, \beta_2, \alpha_2)$ as follows:

$$\begin{gathered}
g(\Theta) \propto 1, \\
0 < p_1 < 1, \\
\alpha_i > 0, \\
\beta_i > 0.
\end{gathered} \quad (26)$$

Thus, the joint posterior density of the vector Θ is obtained by multiplying (22) and (26) as follows:

$$\begin{aligned}
&q(\Theta \mid \underline{x}) \\
&\quad \propto g(\Theta) \prod_{j=1}^{n} \left[p_1 f_1 (x_j; \Theta_1) + p_2 f_2 (x_j; \Theta_2) \right].
\end{aligned} \quad (27)$$

From (27), we observe that the posterior density of the vector Θ is proportional to the likelihood function mentioned in Section 3.

Lindley's [11] approximation under the squared error loss function is evaluated to get the Bayes estimator of $U \equiv U(\Theta)$, where $\Theta = (\theta_1, \theta_1, \ldots, \theta_m)'$ and $U \equiv U(\Theta)$ is a function of Θ. For the unknown five parameters' status, the approximation form reduces to the following:

$$\widetilde{U}_{\mathrm{BL}}(\Theta) = U(\Theta) + \frac{1}{2} \left[A + \sum_{i=1}^{5} B_i d_i \right], \quad (28)$$

where $i = 1, 2, \ldots, 5$,

$$\begin{aligned}
A &= \sum_{i=1}^{5} \sum_{j=5}^{5} U_{ij} (\Theta) \tau_{ij}, \\
d_i &= \sum_{j=1}^{5} U_j (\Theta) \tau_{ij}, \\
B_i &= \sum_{j=1}^{5} \tau_{jj} Q_{jji} (\Theta) + 2 \left[\sum_{j=2}^{5} \tau_{ij} Q_{iji} (\Theta) + E_i \right],
\end{aligned}$$

$$E_i = \tau_{23}Q_{23i} + \tau_{24}Q_{24i} + \tau_{25}Q_{25i} + \tau_{34}Q_{34i} + \tau_{35}Q_{35i} + \tau_{45}Q_{45i}, \tag{29}$$

where $Q(\Theta)$ is the logarithm of a posterior function for n observations, forming a random sample $\underline{X}$, from a density $f(\cdot)$ for $i, j, s = 1, 2, \ldots, 5$,

$$\begin{aligned} U_i\left(\widehat{\Theta}\right) &= \left.\frac{\partial U(\Theta)}{\partial \theta_i}\right|_{\Theta=\widehat{\Theta}}, \\ U_{ij}\left(\widehat{\Theta}\right) &= \left.\frac{\partial^2 U(\Theta)}{\partial \theta_i \partial \theta_j}\right|_{\Theta=\widehat{\Theta}}, \\ Q_{ij}\left(\widehat{\Theta}\right) &= \left.\frac{\partial^2 Q(\Theta)}{\partial \theta_i \partial \theta_j}\right|_{\Theta=\widehat{\Theta}}, \\ Q_{ijs}\left(\widehat{\Theta}\right) &= \left.\frac{\partial^3 Q(\Theta)}{\partial \theta_i \partial \theta_j \partial \theta_s}\right|_{\Theta=\widehat{\Theta}}, \\ \Sigma_{m\times m} &= \tau_{ij} = \left(-Q_{ij}\right)^{-1}_{m\times m}. \end{aligned} \tag{30}$$

All terms on (28) are to be computed at the posterior mode since the logarithm of the posterior density in (27) is defined by

$$Q(\Theta \mid \underline{x}) \equiv \log q \propto \sum_{j=1}^{n} \log\left[p_1 f_1\left(x_j;\Theta_1\right) + p_2 f_2\left(x_j;\Theta_2\right)\right]. \tag{31}$$

The mode of the posterior density can be obtained by solving the five nonlinear equations $i = 1, 2$, the same as that mentioned before in Section 3 in (24) since the noninformative previous ones $g(\Theta) \propto 1$, $0 < p_1 < 1$, $\alpha_i > 0$, and $\beta_i > 0$.

To evaluate Lindley's [11] approximation for the Bayes estimator of the vector of parameters form in (28), we define the elements of the matrix $(Q_{ij})_{5\times5}$, $i, j = 1, 2, \ldots, 5$, in (30). These elements are evaluated by (A.1) in Appendix A. Numerically by inverting the matrix $-Q_{ij}$, the elements τ_{ij}, $i, j = 1, 2, \ldots, 5$, of the matrix Σ are computed where the elements Q_{ijs}, $i, j, s = 1, 2, \ldots, 5$, are defined by (B.1) in Appendix B.

Now, for the MTBIIID, Lindley's [11] approximation for the Bayes estimator of the vector of five parameters of $p_1, \alpha_1, \alpha_2, \beta_1$, and β_2 is evaluated by equating $U(\Theta)$ in (28) to one of the five parameters, so that $U_{ij} = 0$ and $A = 0$, $i, j = 1, 2, \ldots, 5$.

$$\begin{aligned} \widetilde{p_1} &= \widehat{p}_1 + \frac{1}{2}\sum_{i=1}^{5} B_i \tau_{i1}, \\ \widetilde{\alpha_1} &= \widehat{\alpha}_1 + \frac{1}{2}\sum_{i=1}^{5} B_i \tau_{i2}, \\ \widetilde{\alpha_2} &= \widehat{\alpha}_2 + \frac{1}{2}\sum_{i=1}^{5} B_i \tau_{i3}, \\ \widetilde{\beta_1} &= \widehat{\beta}_1 + \frac{1}{2}\sum_{i=1}^{5} B_i \tau_{i4}, \\ \widetilde{\beta_2} &= \widehat{\beta}_2 + \frac{1}{2}\sum_{i=1}^{5} B_i \tau_{i5}, \end{aligned} \tag{32}$$

where B_i, $i = 1, \ldots, 5$, are obtained in (29) and τ_{ij}, $i, j = 1, 2, \ldots, 5$, are the elements of the inverse matrix Q_{ij}. In (31), the functions are calculated at the posterior mode.

5. Application

In this section, we apply the real data collected to fit the proposed mixture model. We use the data collected from Jeddah city for measuring the carbon monoxide level in different locations during the period of January–June 2009 with sample size 151.

Table 2 obtained the descriptive statistics for the carbon monoxide data.

The maximum likelihood estimates (MLEs) and Bayes estimates (Bs) for the MIWIBD are calculated in Table 3.

From Figure 3, the carbon monoxide data provides a suitable fit for the proposed mixture model under MLEs and Bayes estimates. In addition, we used Kolmogorov–Smirnov test (K-S) to fit the carbon monoxide data as shown in Table 4.

From Table 4, we observe that the values of the (K-S) test under the MLEs and Bayesian estimates give appropriate fit from the MIWIBD at 5% level of significance.

The Fisher information matrix $I(\Theta)$ is used to determine the approximate100$(1-\delta)$ confidence intervals (CIs) of the parameters Θ as $\widehat{\Theta} \pm \xi_{\delta/2}\sqrt{V(\widehat{\Theta})}$, where $V(\widehat{\Theta})$ are the variances of the parameters given from $I^{-1}(\widehat{\Theta})$ and $\xi_{\delta/2}$ is the upper $\xi_{\delta/2}$ percentile of the standard normal distribution. The variance-covariance matrix of Θ is calculated as

$$I^{-1}\left(\widehat{\Theta}\right) = \begin{pmatrix} 0.006571 & 0.000881 & -0.005074 & 0.052351 & -0.018985 \\ 0.000881 & 0.000697 & -0.000678 & 0.003241 & -0.002062 \\ -0.005074 & -0.000678 & 0.057469 & -0.025678 & 0.013121 \\ 0.052351 & 0.003241 & -0.025678 & 0.857329 & -0.255263 \\ -0.018985 & -0.002062 & 0.013121 & -0.255263 & 0.090989 \end{pmatrix}. \tag{33}$$

Table 2: Descriptive statistics for the carbon monoxide data of the MIWIBD.

Mean data	N	Mean	Mode	Median	Variance	St dev	Minimum	Maximum
Carbon monoxide level	151	2.0911	1.51	1.88	1.34	1.16	0.10	4.78

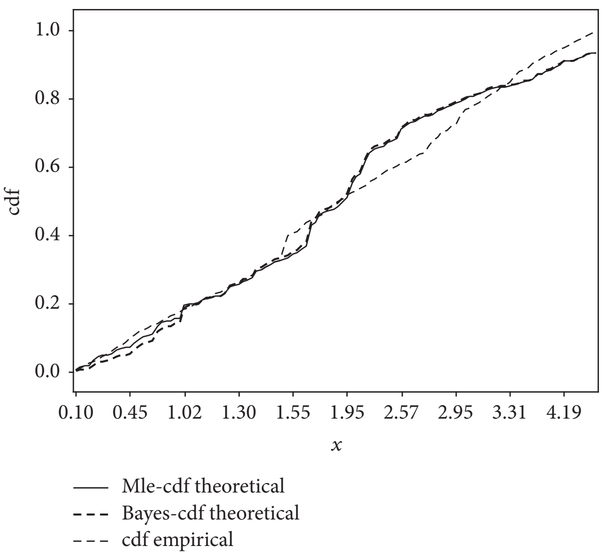


Figure 3: Empirical cdf for the fitted carbon data of the MIWIB model.

Table 3: MLEs and Bs for the carbon monoxide data of the MIWIBD.

Parameter	$\widehat{p}_1$	$\widehat{\alpha}_1$	$\widehat{\alpha}_2$	$\widehat{\beta}_1$	$\widehat{\beta}_2$
MLE	0.6995	0.5038	0.6461	2.7637	2.5551
Bayes estimate	0.6813	0.5058	0.7814	2.7275	2.6999

Table 4: MLEs and Bs (K-S) test for the carbon monoxide data of the MIWIBD.

(K-S) MLE	0.0978
(K-S) Bayes estimate	0.0995

The 90% and 95% CIs for the MLEs of the parameters are evaluated in Table 5.

6. Conclusion

In this paper, the MIWIBD are proposed and some important measures of the MIWIBD are discussed such as measures of locations and measures of dispersion. Also, numerical results for the mode and median of the MIWIBD are computed based on different choices of Θ and the performance of the failure rate functions of the MIWIBD is interpreted through the plots. In addition, the estimates of the vector of the unknown parameters of the MIWIBD are given. Further, the MIWIBD are fitted to the data from Jeddah city for measuring the carbon monoxide level in different locations. Finally, the expressions for Lindley's approximation matrix are shown in Appendix.

Appendix

A. Elements Q_{ij}

From (28) and the definitions in (31), the elements Q_{ij}, $i, j = 1, 2, 3, 4, 5$, are derived as follows:

$$Q_{11} = -\sum_{j=1}^{n} \omega^2 \left(x_j; \Theta\right),$$

$$Q_{12} = \sum_{j=1}^{n} \phi_1 \left(x_j; \Theta\right) \xi \left(x_j; \Theta\right) = Q_{21},$$

$$Q_{13} = \sum_{j=1}^{n} \phi_2 \left(x_j; \Theta\right) \xi \left(x_j; \Theta\right) = Q_{31},$$

$$Q_{14} = -\sum_{j=1}^{n} \psi_1 \left(x_j; \Theta\right) \xi \left(x_j; \Theta\right) = Q_{41},$$

$$Q_{15} = -\sum_{j=1}^{n} \psi_2 \left(x_j; \Theta\right) \xi \left(x_j; \Theta\right) = Q_{51},$$

$$Q_{22} = \sum_{j=1}^{n} p_1 \eta_1 \left(x_j; \Theta\right) \left[V_1 \left(x_j; \Theta\right) + \left[1 - p_1 \eta_1 \left(x_j; \Theta\right)\right] \phi_1^2 \left(x_j; \Theta\right)\right],$$

$$Q_{23} = \sum_{j=1}^{n} p_1 \eta_1 \left(x_j; \Theta\right) \left[V_2 \left(x_j; \Theta_1\right) + \left[1 - p_1 \eta_1 \left(x_j; \Theta\right)\right] \phi_1 \left(x_j; \Theta\right) \phi_2 \left(x_j; \Theta\right)\right] = Q_{32},$$

$$Q_{24} = -p_1 p_2 \sum_{j=1}^{n} \phi_1 \left(x_j; \Theta\right) \psi_1 \left(x_j; \Theta\right) \xi \left(x_j; \Theta\right) = Q_{42},$$

$$Q_{25} = -p_1 p_2 \sum_{j=1}^{n} \phi_1 \left(x_j; \Theta\right) \psi_2 \left(x_j; \Theta\right) \xi \left(x_j; \Theta\right) = Q_{52},$$

$$Q_{33} = \sum_{j=1}^{n} p_1 \eta_1 \left(x_j; \Theta\right) \left[V_3 \left(x_j; \Theta_1\right) + \left[1 - p_1 \eta_1 \left(x_j; \Theta\right)\right] \phi_2^2 \left(x_j; \Theta\right)\right],$$

Table 5: MLEs and Bs 90% CIs for the carbon monoxide data of the MIWIBD.

Parameter	$\widehat{p}_1$	$\widehat{\alpha}_1$	$\widehat{\alpha}_2$	$\widehat{\beta}_1$	$\widehat{\beta}_2$
90% CIs	0.5665, 0.8324	0.0001, 0.5468	0.1516, 1.1405	2.3705, 3.1568	1.0366, 4.0735
95% CIs	0.5406, 0.8583	0.4523, 0.5552	0.0551, 1.2370	2.2938, 3.2335	0.7403, 4.3698

$$\begin{aligned}
Q_{34} &= -p_1 p_2 \sum_{j=1}^{n} \phi_2(x_j;\Theta)\,\psi_1(x_j;\Theta)\,\xi(x_j;\Theta) \\
&= Q_{43}, \\
Q_{35} &= -p_1 p_2 \sum_{j=1}^{n} \phi_2(x_j;\Theta)\,\psi_2(x_j;\Theta)\,\xi(x_j;\Theta) \\
&= Q_{53}, \\
Q_{44} &= \sum_{j=1}^{n} p_2\eta_2(x_j;\Theta)\big[W_2(x_j;\Theta_2) \\
&\quad + \left[1 - p_2\eta_2(x_j;\Theta)\right]\psi_1^2(x_j;\Theta)\big], \\
Q_{45} &= \sum_{j=1}^{n} p_2\eta_2(x_j;\Theta)\big[W_3(x_j;\Theta_2) \\
&\quad + \left[1 - p_2\eta_2(x_j;\Theta)\right]\psi_1(x_j;\Theta)\,\psi_2(x_j;\Theta)\big] \\
&= Q_{54}, \\
Q_{55} &= \sum_{j=1}^{n} p_2\eta_2(x_j;\Theta)\big[W_4(x_j;\Theta_2) \\
&\quad + \left[1 - p_2\eta_2(x_j;\Theta)\right]\psi_2^2(x_j;\Theta)\big],
\end{aligned} \tag{A.1}$$

where $V_1(x_j;\Theta_1)$, $V_2(x_j;\Theta_1)$, $V_3(x_j;\Theta_1)$, $W_2^*(x_j;\Theta_2)$, $W_3^*(x_j;\Theta_2)$, $V_1(x_j;\Theta_1)$, $V_2(x_j;\Theta_1)$, $V_3(x_j;\Theta_1)$, $W_2^*(x_j;\Theta_2)$, $W_3^*(x_j;\Theta_2)$, and $W_4^*(x_j;\Theta_2)$, for $j = 1, 2, \ldots, n$, are given, respectively, by

$$\begin{aligned}
\xi(x_j;\Theta) &= \frac{f_1(x_j;\Theta_1)\, f_2(x_j;\Theta_2)}{f^2(x_j;\Theta)}, \\
V_1(x_j;\Theta_1) &= \beta_1\alpha_1^{-2} - \beta_1(\beta_1+1)\,\alpha_1^{-(\beta_1+2)} x_j^{-\beta_1}, \\
V_2(x_j;\Theta_1) &= -\alpha_1^{-1} + \alpha_1^{-(\beta_1+1)} x_j^{-\beta_1} - \beta_1\alpha_1^{-(\beta_1+1)} x_j^{-\beta_1} \\
&\quad \cdot \log(\alpha_1 x_j), \\
V_3(x_j;\Theta_1) &= -\left(\beta_1^{-2} + (\alpha_1 x_j)^{-\beta_1}\left(\log(\alpha_1 x_j)\right)^2\right), \\
W_2^*(x_j;\Theta_2) &= -\beta_2^{-2} + (\alpha_2+1)\, x_j^{-\beta_2}\left(\log(x_j)\right)^2 \\
&\quad \cdot \left(1 + x_j^{-\beta_2}\right)^{-1}\left[x_j^{-\beta_2}\left(1 + x_j^{-\beta_2}\right)^{-1} - 1\right], \\
W_3^*(x_j;\Theta_2) &= x_j^{-\beta_2}\left(1 + x_j^{-\beta_2}\right)^{-1}\log(x_j), \\
W_4^*(x_j;\Theta_2) &= -\alpha_2^{-2},
\end{aligned} \tag{A.2}$$

with $f(x_j;\Theta)$, $f_1(x_j;\Theta_1)$, $f_2(x_j;\Theta_2)$, $\omega(x_j;\Theta)$, $\phi_1(x_j;\Theta)$, $\phi_2(x_j;\Theta)$, $\eta_1(x_j;\Theta)$, $\eta_2(x_j;\Theta)$, $\psi_1(x_j;\Theta)$, and $\psi_2(x_j;\Theta)$ being as given in (1)–(3) and (25), respectively.

B. Elements Q_{ijs}

From (28) and the definitions in (31), the elements Q_{ijs}, $Q_{ijs} = Q_{isj} = Q_{jis} = Q_{jsi} = Q_{sij} = Q_{sji}$, for $i, j, s = 1, 2, \ldots, 5$, are derived as follows:

$$\begin{aligned}
Q_{111} &= 2\sum_{j=1}^{n} \omega^3(x_j;\Theta), \\
Q_{121} &= -2\sum_{j=1}^{n} \phi_1(x_j;\Theta)\,\xi(x_j;\Theta)\,\omega(x_j;\Theta) = Q_{112}, \\
Q_{131} &= -2\sum_{j=1}^{n} \phi_2(x_j;\Theta)\,\xi(x_j;\Theta)\,\omega(x_j;\Theta) = Q_{113}, \\
Q_{141} &= 2\sum_{j=1}^{n} \psi_1(x_j;\Theta)\,\xi(x_j;\Theta)\,\omega(x_j;\Theta) = Q_{114}, \\
Q_{151} &= 2\sum_{j=1}^{n} \psi_2(x_j;\Theta)\,\xi(x_j;\Theta)\,\omega(x_j;\Theta) = Q_{115}, \\
Q_{122} &= \sum_{j=1}^{n} \big[\phi_1^2(x_j;\Theta)\left[1 - 2p_1\eta_1(x_j;\Theta)\right] \\
&\quad + V_1(x_j;\Theta)\big]\,\xi(x_j;\Theta) = Q_{221}, \\
Q_{123} &= (n-r)\,\frac{R_2(x_r;\Theta_2)}{R^2(x_r;\Theta)}\bigg[\ell_1^*(x_r;\Theta_1) - 2p_1 \\
&\quad \cdot \frac{a_1(x_r;\Theta_1)\, b_1(x_r;\Theta_1)}{R(x_r;\Theta)}\bigg] + \sum_{j=1}^{r}\big[\phi_1(x_j;\Theta) \\
&\quad \cdot \phi_2(x_j;\Theta)\left[1 - 2p_1\eta_1(x_j;\Theta)\right] + V_2(x_j;\Theta_1)\big] \\
&\quad \cdot \xi(x_j;\Theta) = Q_{132} = Q_{231}, \\
Q_{124} &= \sum_{j=1}^{n} \phi_1(x_j;\Theta)\,\psi_1(x_j;\Theta)\left[1 - 2p_2\eta_2(x_j;\Theta)\right] \\
&\quad \cdot \xi(x_j;\Theta) = Q_{142} = Q_{241},
\end{aligned}$$

$$Q_{125} = \sum_{j=1}^{n} \phi_1 (x_j; \Theta) \psi_2 (x_j; \Theta) [1 - 2p_2\eta_2 (x_j; \Theta)] \cdot \xi (x_j; \Theta) = Q_{152} = Q_{251},$$

$$Q_{134} = \sum_{j=1}^{n} \phi_2 (x_j; \Theta) \psi_1 (x_j; \Theta) [1 - 2p_2\eta_2 (x_j; \Theta)] \cdot \xi (x_j; \Theta) = Q_{143} = Q_{341},$$

$$Q_{135} = \sum_{j=1}^{n} \phi_2 (x_j; \Theta) \psi_2 (x_j; \Theta) [1 - 2p_2\eta_2 (x_j; \Theta)] \cdot \xi (x_j; \Theta) = Q_{153} = Q_{351},$$

$$Q_{145} = -\sum_{j=1}^{n} [\psi_1 (x_j; \Theta) \psi_2 (x_j; \Theta) \cdot [1 - 2p_2\eta_2 (x_j; \Theta)] + W_3^* (x_j; \Theta_2)] \xi (x_j; \Theta) = Q_{154} = Q_{451},$$

$$Q_{331} = \sum_{j=1}^{n} [V_3 (x_j; \Theta_1) + [1 - 2p_1\eta_1 (x_j; \Theta)] \cdot \phi_2^2 (x_j; \Theta)] \xi (x_j; \Theta) = Q_{133},$$

$$Q_{441} = -\sum_{j=1}^{n} [W_2^* (x_j; \Theta_2) + [1 - 2p_2\eta_2 (x_j; \Theta)] \cdot \psi_1^2 (x_j; \Theta)] \xi (x_j; \Theta) = Q_{144},$$

$$Q_{551} = -\sum_{j=1}^{n} [W_4^* (x_j; \Theta_2) + [1 - 2p_2\eta_2 (x_j; \Theta)] \cdot \psi_2^2 (x_j; \Theta)] \xi (x_j; \Theta) = Q_{155},$$

$$Q_{222} = \sum_{j=1}^{n} [p_1 p_2 \phi_1 (x_j; \Theta) \cdot (3V_1 (x_j; \Theta_1) + [1 - 2p_1\eta_1 (x_j; \Theta)] \phi_1^2 (x_j; \Theta)) \cdot \xi (x_j; \Theta) + p_1 W_1 (x_j; \Theta_1) \eta_1 (x_j; \Theta)],$$

$$Q_{223} = \sum_{j=1}^{n} p_1\eta_1 (x_j; \Theta) [\phi_1 (x_j; \Theta) [1 - p_1\eta_1 (x_j; \Theta)] \cdot (V_3 (x_j; \Theta_1) + [1 - 2p_1\eta_1 (x_j; \Theta)] \phi_2^2 (x_j; \Theta)) + W_3 (x_j; \Theta_1) + (2\phi_2 (x_j; \Theta) V_2 (x_j; \Theta_1) \cdot [1 - p_1\eta_1 (x_j; \Theta)])] = Q_{232},$$

$$Q_{224} = -p_1 p_2 \sum_{j=1}^{n} \psi_1 (x_j; \Theta) (V_1 (x_j; \Theta_1) + (1 - 2p_1\eta_1 (x_j; \Theta)) \phi_1^2 (x_j; \Theta)) \xi (x_j; \Theta) = Q_{242},$$

$$Q_{225} = -p_1 p_2 \sum_{j=1}^{n} \psi_2 (x_j; \Theta) (V_1 (x_j; \Theta_1) + (1 - 2p_1\eta_1 (x_j; \Theta)) \phi_1^2 (x_j; \Theta)) \xi (x_j; \Theta) = Q_{252},$$

$$Q_{332} = \sum_{j=1}^{n} p_1\eta_1 (x_j; \Theta) [\phi_1 (x_j; \Theta) [1 - p_1\eta_1 (x_j; \Theta)] \cdot (V_3 (x_j; \Theta_1) + [1 - 2p_1\eta_1 (x_j; \Theta)] \phi_2^2 (x_j; \Theta)) + W_3 (x_j; \Theta_1) + (2\phi_2 (x_j; \Theta) V_2 (x_j; \Theta_1) \cdot [1 - p_1\eta_1 (x_j; \Theta)])] = Q_{233},$$

$$Q_{244} = -p_1 p_2 \sum_{j=1}^{n} \phi_1 (x_j; \Theta) [W_2^* (x_j; \Theta_2) + \psi_1^2 (x_j; \Theta) \cdot (1 - 2p_2\eta_2 (x_j; \Theta))] \xi (x_j; \Theta) = Q_{442},$$

$$Q_{255} = -p_1 p_2 \sum_{j=1}^{n} \phi_1 (x_j; \Theta) [W_4^* (x_j; \Theta_2) + \psi_2^2 (x_j; \Theta) \cdot (1 - 2p_2\eta_2 (x_j; \Theta))] \xi (x_j; \Theta) = Q_{552},$$

$$Q_{234} = -p_1 p_2 \sum_{j=1}^{n} \psi_1 (x_j; \Theta) [V_2 (x_j; \Theta_1) + (1 - 2p_1\eta_1 (x_j; \Theta)) \phi_1 (x_j; \Theta) \phi_2 (x_j; \Theta)] \xi (x_j; \Theta) = Q_{243} = Q_{342},$$

$$Q_{235} = -p_1 p_2 \sum_{j=1}^{n} \psi_2 (x_j; \Theta) [V_2 (x_j; \Theta_1) + (1 - 2p_1\eta_1 (x_j; \Theta)) \phi_1 (x_j; \Theta) \phi_2 (x_j; \Theta)] \xi (x_j; \Theta) = Q_{253} = Q_{352},$$

$$Q_{245} = -p_1 p_2 \sum_{j=1}^{n} \phi_1 (x_j; \Theta) [W_3^* (x_j; \Theta_2) + (1 - 2p_2\eta_2 (x_j; \Theta)) \psi_1 (x_j; \Theta) \psi_2 (x_j; \Theta)] \xi (x_j; \Theta^*) = Q_{254} = Q_{452},$$

$$Q_{333} = \sum_{j=1}^{n} [p_1 p_2 \phi_2 (x_j; \Theta) \cdot (3V_3 (x_j; \Theta_1) + [1 - 2p_1\eta_1 (x_j; \Theta)] \phi_2^2 (x_j; \Theta)) \cdot \xi (x_j; \Theta) + p_1 W_3 (x_j; \Theta_1) \eta_1 (x_j; \Theta)],$$

$$Q_{334} = -p_1 p_2 \sum_{j=1}^{n} \psi_1 (x_j; \Theta) [V_3 (x_j; \Theta_1) + [1 - 2p_1\eta_1 (x_j; \Theta)] \phi_2^2 (x_j; \Theta)] \xi (x_j; \Theta) = Q_{343},$$

$$Q_{335} = -p_1 p_2 \sum_{j=1}^{n} \psi_2(x_j;\Theta)\left[V_3(x_j;\Theta_1) + \left[1 - 2p_1\eta_1(x_j;\Theta)\right]\phi_2^2(x_j;\Theta)\right]\xi(x_j;\Theta) = Q_{353},$$

$$Q_{344} = -p_1 p_2 \sum_{j=1}^{n} \phi_2(x_j;\Theta)\left[W_2^*(x_j;\Theta_2) + \psi_1^2(x_j;\Theta) \cdot \left(1 - 2p_2\eta_2(x_j;\Theta)\right)\right]\xi(x_j;\Theta) = Q_{443},$$

$$Q_{355} = -p_1 p_2 \sum_{j=1}^{n} \phi_2(x_j;\Theta)\left[W_4^*(x_j;\Theta_2) + \psi_2^2(x_j;\Theta) \cdot \left(1 - 2p_2\eta_2(x_j;\Theta)\right)\right]\xi(x_j;\Theta) = Q_{553},$$

$$Q_{345} = -p_1 p_2 \sum_{j=1}^{n} \phi_2(x_j;\Theta)\left[W_3^*(x_j;\Theta_2) + \psi_1(x_j;\Theta) \cdot \psi_2(x_j;\Theta)\left(1 - 2p_2\eta_2(x_j;\Theta)\right)\right]\xi(x_j;\Theta) = Q_{354} = Q_{453},$$

$$Q_{444} = \sum_{j=1}^{n} p_2\eta_2(x_j;\Theta)\left[n_1(x_j;\Theta) + \psi_1(x_j;\Theta)\left[1 - p_2\eta_2(x_j;\Theta)\right]\left(3W_2^*(x_j;\Theta_2) + \psi_1^2(x_j;\Theta) \cdot \left(1 - 2p_2\eta_2(x_j;\Theta)\right)\right) + \psi_1^2(x_j;\Theta)\right],$$

$$Q_{445} = \sum_{j=1}^{n} p_2\eta_2(x_j;\Theta)\left[n_2(x_j;\Theta) + W_3^*(x_j;\Theta_2) + \psi_2(x_j;\Theta)\left(W_2^*(x_j;\Theta_2) + \psi_1^2(x_j;\Theta) \cdot \left[1 - 2p_2\eta_2(x_j;\Theta)\right]\right)\right] = Q_{454},$$

$$Q_{455} = \sum_{j=1}^{n} p_2\eta_2(x_j;\Theta)\left(1 - p_2\eta_2(x_j;\Theta)\right)\left[\psi_1(x_j;\Theta) \cdot W_4^*(x_j;\Theta) + 2\psi_2(x_j;\Theta)W_3^*(x_j;\Theta_2) + \psi_1(x_j;\Theta)\psi_2^2(x_j;\Theta)\left(1 - p_2\eta_2(x_j;\Theta)\right)\right] = Q_{554},$$

$$Q_{555} = \sum_{j=1}^{n} p_2\eta_2(x_j;\Theta)\left[n_3(x_j;\Theta) + \psi_2(x_j;\Theta)\left(1 - p_2\eta_2(x_j;\Theta)\right)\left(3W_4^*(x_j;\Theta_2) + \psi_2^2(x_j;\Theta) \cdot \left(1 - 2p_2\eta_2(x_j;\Theta)\right)\right)\right], \tag{B.1}$$

where

$$W_1(x_j;\Theta_1) = -2\beta_1\alpha_1^{-3} + \beta_1(\beta_1+1)(\beta_1+2) \cdot \alpha_1^{-(\beta_1+3)}x_j^{-\beta_1},$$

$$W_2(x_j;\Theta_1) = \alpha_1^{-2} - (2\beta_1+1)\alpha_1^{-(\beta_1+2)}x_j^{-\beta_1} + \beta_1(\beta_1+1)\alpha_1^{-(\beta_1+2)}x_j^{-\beta_1}\log(\alpha_1 x_j),$$

$$W_3(x_j;\Theta_1) = \beta_1\alpha_1^{-(\beta_1+1)}x_j^{-\beta_1}\left(\log(\alpha_1 x_j)\right)^2 - 2\alpha_1^{-(\beta_1+1)}x_j^{-\beta_1}\log(\alpha_1 x_j),$$

$$n_1(x_j;\Theta_2) = 2\beta_2^{-3} + (\alpha_2+1)\left(\log x_j\right)^3 \cdot x_j^{-\beta_2}\left(1 + x_j^{-\beta_2}\right)^{-1}\left(x_j^{-\beta_2}\left(1 + x_j^{-\beta_2}\right)^{-1} - 1\right)^2 + x_j^{-\beta_2}\log(x_j)\left(1 + x_j^{-\beta_2}\right)^{-1} \cdot \left((\alpha_2+1) - x_j^{-\beta_2}\left(1 + x_j^{-\beta_2}\right)^{-1}\right),$$

$$n_2(x_j;\Theta_2) = x_j^{-\beta_2}\left(\log x_j\right)^2\left(1 + x_j^{-\beta_2}\right)^{-1} \cdot \left[x_j^{-\beta_2}\left(1 + x_j^{-\beta_2}\right)^{-1} - 1\right],$$

$$n_3(x_j;\Theta_2) = 2\alpha_2^{-3}. \tag{B.2}$$

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] K. S. Sultan, M. A. Ismail, and A. S. Al-Moisheer, "Mixture of two inverse Weibull distributions: properties and estimation," *Computational Statistics & Data Analysis*, vol. 51, no. 11, pp. 5377–5387, 2007.

[2] B. S. Everitt and D. J. Hand, *Finite Mixture Distributions*, Chapman & Hall, London, UK, 1981.

[3] G. McLachlan and D. Peel, *Finite Mixture Models*, John Wiley & Sons, New York, NY, USA, 2000.

[4] A. Z. Keller, A. R. R. Kamath, and U. D. Perera, "Reliability analysis of CNC machine tools," *Reliability Engineering*, vol. 3, no. 6, pp. 449–473, 1982.

[5] B. G. Lindsay, *Mixture Models: Theory, Geometry and Applications*, The Institute of mathematical Statistics, Hayward, CA, USA, 1995.

[6] K. S. Sultan and A. S. Al-Moisheer, "Estimation of a discriminant function from a mixture of two inverse Weibull distributions," *Journal of Statistical Computation and Simulation*, vol. 83, no. 3, pp. 405–416, 2013.

[7] A. S. Al-Moisheer, "A Mixture of Two Burr Type III Distributions: Identifiability and Estimation under Type II Censoring," *Mathematical Problems in Engineering*, vol. 2016, Article ID 7035279, 12 pages, 2016.

[8] K. S. Sultan and A. S. Al-Moisheer, "Mixture of inverse weibull and lognormal distributions: Properties, estimation, and illustration," *Mathematical Problems in Engineering*, vol. 2015, Article ID 526786, 8 pages, 2015.

[9] A. M. Abd-Elfattah and A. H. Alharbey, "Bayesian estimation for Burr distribution type III based on trimmed samples," *ISRN Applied Mathematics*, vol. 2012, Article ID 250393, 18 pages, 2012.

[10] E. K. Al-Hussaini and K. S. Sultan, "Reliability and hazard based on finite mixture models," in *Handbook of Statistics*, N. Balakrishnan and C. R. Rao, Eds., vol. 20, pp. 139–183, Elsevier, Amsterdam, Netherlands, 2001.

[11] D. V. Lindley, "Approximate Bayesian methods," *Trabajos de Estadistica y de Investigacion Operativa*, vol. 31, no. 1, pp. 223–237, 1980.