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Research Article

Out-of-Plane Vibration of Curved Uniform and Tapered Beams with Additional Mass

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As there is a gap in literature about out-of-plane vibrations of curved and variable cross-sectioned beams, the aim of this study is to analyze the free out-of-plane vibrations of curved beams which are symmetrically and nonsymmetrically tapered. Out-of-plane free vibration of curved uniform and tapered beams with additional mass is also investigated. Finite element method is used for all analyses. Curvature type is assumed to be circular. For the different boundary conditions, natural frequencies of both symmetrical and unsymmetrical tapered beams are given together with that of uniform tapered beam. Bending, torsional, and rotary inertia effects are considered with respect to no-shear effect. Variations of natural frequencies with additional mass and the mass location are examined. Results are given in tabular form. It is concluded that (i) for the uniform tapered beam there is a good agreement between the results of this study and that of literature and (ii) for the symmetrical curved tapered beam there is also a good agreement between the results of this study and that of a finite element model by using MSC.Marc. Results of out-of-plane free vibration of symmetrically tapered beams for specified boundary conditions are addressed.

1. Introduction

Due to their importance and wide using areas in engineering, dynamics of curved beams have been prevalently investigated by many researchers. Particularly, vibration analysis of curved beams has been a remarkable research area in mechanics due to its various applications. For the complicated problems of many architectural and structural implementations, curved beams with variable cross-sections are generally main parts, such that beams can be used not only in the design of rib, curved continuous bridge, and ship, but also in gear, pump, turbine and so on. Kawakami et al. [1] present an approximate method to study the analysis for the in-plane and out-of-plane free vibration of horizontally curved beams with arbitrary shapes and variable cross-sections. It is stated that the characteristic equation for free vibration can be derived by applying the Green function, which is obtained as a discrete type solution of differential equations governing the flexural behavior of the curved beam under the action of a concentrated load. Krishnan and Suresh [2] investigate the effect of shear deformation and rotary inertia on natural and cross over frequencies of both curved uniform and nonuniform (varied cross-section) beams by using a simple cubic linear beam element. Results of free and forced in-plane vibrations of circular arches with variable cross-sections are given by Tong et al. [3] using the Kirchhoff assumptions for thin beams and taking the neutral axis as inextensible. Huang et al. [4] take into consideration out-of-plane dynamics of beams with arbitrarily varying curvature and cross-section by dynamic stiffness matrix method. Viola et al. [5, 6] investigate the in-plane linear free vibrations of nonuniform circular arches with damaged configurations by using analytical and generalized differential quadrature (GDQ) methods. Tornabene et al. [7] apply GDQ to solve linear dynamics of the arch with different geometrical and boundary conditions. Viola et al. [8, 9] present a solution of free harmonic vibration problem of multistepped and multi-damaged arches by using analytical and GDQ with domain decomposition technique (GDQE) methods. In-plane free vibration of circular arches is worked by Liu and Wu [10] using the generalized differential quadrature rule (GDQR). Arches with uniform, continuously varying, and stepped cross-sections are presented to illustrate the validity and accuracy of the GDQR. Karami and Malekzadeh [11] analyze in-plane free vibration of circular arches with varying cross-sections by developing a differential quadrature method. A boundary element method is developed by Sapountzakis [12] for the nonuniform torsional vibration problem of doubly symmetric composite bars of arbitrary variable cross-section. Chen [13] uses the differential quadrature element method (DQEM) for in-plane vibration analysis models of arbitrarily curved beam structures. Tufekci and Dogruer [14] analyze free out-of-plane vibrations of a circular arch with uniform cross-section with respect to effects of transverse shear and rotary inertia due to the both flexural and torsional vibrations. Yang et al. [15] investigate free in-plane vibration of uniform and nonuniform curved beams with variable curvatures, including the effects of the axis extensibility, shear deformation, and rotary inertia by using extended-Hamilton principle. Shin et al. [16] solve the problem of vibration of a circular arch with variable cross-section using differential transformation and generalized differential quadrature. Similarly, on one hand, Malekzadeh et al. [17–19] present in-plane free vibration of laminated/functionally graded circular arches according to the differential quadrature method. On the other hand, Malekzadeh et al. [20] investigate out-of-plane free vibration of functionally graded circular curved beams in thermal environment. Another work about functionally graded circular curved beams is held by Piovan et al. [21]. In order to obtain natural frequencies for "bare" and "loaded" curved beams for different end conditions, Wu et al. [22] present an effective approach for free in-plane vibration analysis of a curved beam with various arbitrary concentrated elements. Özyiğit and Işık [23] investigate in-plane vibration of curved beams with variable cross-sections carrying additional mass to compute frequencies of symmetrical and nonsymmetrical tapered beams with different mass locations. Ni et al. [24] pay attention to in-plane and out-of-plane free vibration and stability of a curved rod in flow. Vibration suppression of curved beams is regarded by Rostam et al. [25]. The performance of two curved beam element models based on coupled polynomial displacement fields is taken into account by Ishaquddin et al. [26] for out-of-plane vibration of arches. The coupled polynomial interpolation fields are derived independently for Timoshenko and Euler-Bernoulli beam elements using the force-moment equilibrium equations. Lee and Jeong [27] consider flexural and torsional free vibrations of horizontally curved beams on Pasternak foundations.

Since out-of-plane vibrations of curved and variable cross-sectioned beams are not widely studied, the purpose of this work is to analyze the free out-of-plane vibrations of curved beams which are symmetrically and nonsymmetrically tapered. In this analysis, the linear free out-of-plane vibrations of uniform and variable cross-section beams are considered by finite element method (FEM). The curvature of beams is circular and the cross-sections are taken circular and rectangular. The natural frequency is computed for different boundary conditions. An additional mass on beam is also considered and its effects on natural frequencies are investigated.

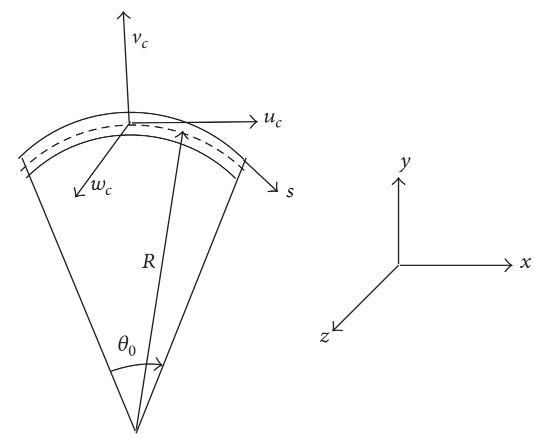


Figure 1: Curved beam element.

2. Modeling and Governing Equations

In applying finite element method to the approximate solution of curved beam problems, the following procedure can be considered [28–30]: Curved beam is modeled as a finite element as shown in Figure 1. X, Y, and Z are global coordinates, and u_c, v_c, and w_c are the tangential, radial, and out-of-plane displacements for the curved beam, respectively. Curved beam lies in $X-Y$ plane, s is tangential coordinate, and θ_0 is the arch angle of one finite element. The out-of-plane elastic and kinetic energy equations of the curved beam can be expressed as follows:

$$\begin{aligned} U_c &= \frac{1}{2}EI\int_S \kappa_c^2 ds + \frac{1}{2}GJ\int_S \varphi_c^2 ds \\ T_c &= \frac{1}{2}\rho A\int_S \dot{w}_c^2 ds + \frac{1}{2}\rho I\int_S \dot{\Psi}_c^2 ds + \frac{1}{2}\rho J\int_S \dot{\Phi}_c^2 ds, \end{aligned} \tag{1}$$

where A is the cross-sectional area, E is modulus of elasticity, G is modulus of rigidity, and I and J are mass moment and polar moment of inertia, respectively. Due to the fact that the beam is assumed to be of Bernoulli-Euler type, shear effect on the beam is neglected. In (1), $(\cdot)$ denotes differentiation with respect to time t. Out-of-plane curvature change, torsion, and slope terms are

$$\begin{aligned} \kappa_c &= \frac{\Phi_c}{R} - \frac{\partial^2 w_c}{\partial s^2}, \\ \varphi_c &= \frac{\partial \Phi_c}{\partial s} + \frac{1}{R}\frac{\partial w_c}{\partial s}, \\ \Psi_c &= \frac{\partial w_c}{\partial s}, \end{aligned} \tag{2}$$

where Φ_c is the torsional displacement of the curved element. Out-of-plane displacement vector for one finite element is

$$[V]_{ce}^T = \left[\Phi_{ci} \quad w_{ci} \quad \Psi_{ci} \quad \Phi_{ci+1} \quad w_{ci+1} \quad \Psi_{ci+1}\right]. \tag{3}$$

Three degrees of freedom are taken for each node of elements. By following the finite element procedure, the stiffness and inertia matrices are obtained for out-of-plane vibrations.

Table 1: Convergence analysis for the out-of-plane natural frequencies of the beam with midpoint mass location (see midpoint position as Position 1 in Table 11) under C-C end condition.

Mode	# of elements									
	10	20	30	40	50	60	70	80	90	100
1	0.798	0.789	0.787	0.786	0.786	0.786	0.786	0.786	0.786	0.786
2	6.076	6.015	6.003	6.000	5.997	5.996	5.995	5.995	5.995	5.994
3	9.556	9.447	9.425	9.417	9.413	9.411	9.410	9.409	9.409	9.408
4	21.190	20.977	20.937	20.923	20.917	20.913	20.911	20.910	20.909	20.908

Table 2: Convergence analysis for the out-of-plane natural frequencies of the beam with bare position (see bare position as Position 6 in Table 11) under C-C end condition.

Mode	# of elements									
	10	20	30	40	50	60	70	80	90	100
1	2.220	2.197	2.193	2.191	2.190	2.190	2.190	2.190	2.189	2.189
2	6.076	6.015	6.003	5.999	5.997	5.996	5.995	5.995	5.995	5.994
3	12.436	12.307	12.282	12.273	12.269	12.267	12.266	12.265	12.264	12.264
4	21.198	20.979	20.938	20.924	20.917	20.913	20.911	20.910	20.909	20.908

Table 3: The out-of-plane nondimensional natural frequency parameters of uniform and circularly curved beams with circular cross-sections under C-C end condition.

θ	Mode	Present study	Viola et al. [8]	Malekzadeh and Setoodeh [17]	Piovan et al. [21]	Ni et al. [24]
60°	1	19.610	19.40190	19.398	19.442	
	2	55.070	54.02958	54.014	54.093	
	3	108.947	105.64828	105.61	105.707	
	4	180.868	172.77355			
120°	1	4.490	4.451450	4.4515	4.471	
	2	12.970	12.82629	12.825	12.885	
	3	26.300	25.98937	25.984	26.064	
	4	44.205	43.57053			
180°	1	1.8256	1.804340	1.8048	1.817	1.8108
	2	5.2687	5.197995	5.1984	5.239	5.2359
	3	11.0489	10.91819	10.918	10.984	11.0046
	4	18.9317	18.72548			18.8837

3. Free Vibration Analysis: Results and Discussion

Matrix equation for the free vibrations of the curved beam starts with an equation of the form

$$[K]\{V\} + [M]\left\{\frac{d^2V}{dt^2}\right\} = \{0\}, \tag{4}$$

where $\{V\}$ is a global displacement vector and $[K]$ and $[M]$ are global stiffness and inertia matrices, respectively. The solution of (4) is assumed to be

$$\{V\} = \left\{\overline{V}\right\} e^{j\omega_n t}, \tag{5}$$

where $j = \sqrt{-1}$, ω_n is natural frequency, and $\{\overline{V}\}$ is displacement amplitude vector of all nodes. Then, one can obtain the eigenvalue equation giving the natural frequencies for both in-plane and out-of-plane vibrations

$$\left|[K] - \omega_n^2 [M]\right| = 0. \tag{6}$$

3.1. Convergence Study. A convergence study against the number of used elements is also put into consideration to see appropriate error reduction properties of the out-of-plane natural frequencies of the beam with or without mass under clamped end condition (C-C). Results of the convergence study are given in Tables 1 and 2. It can be noted that the experimental convergence starts with 10 elements and ends up with 100 elements. The number of elements for optimal convergence is in the range of 90 and 100.

3.2. Uniform Beams. According to the finite element model proposed, vibration of uniform circularly curved beams is compared with that of the literature. Results of the out-of-plane dimensionless frequencies of curved beams for clamped end condition (C-C) are given in Table 3. It can be noted that the results of the present study coincide with the results of the literature.

3.3. Curved Tapered Beams. Actually, many studies about in-plane vibrations of curved tapered beams can be found in literature [2, 3, 5–11, 16–20]. However, out-of-plane vibrations

Table 4: Comparison of out-of-plane nondimensional fundamental natural frequency parameters of symmetrical tapered curved beams under C-C end condition.

Mode	$\alpha = 0$		$\alpha = 0.1$		$\alpha = 0.2$		$\alpha = 0.3$		$\alpha = 0.4$	
	FEM	Marc	FEM	Marc	FEM	Marc	FEM	Marc	FEM	Marc
1	1.82656	1.71268	1.99231	1.95735	2.16707	2.06221	2.37049	2.23698	2.55085	2.35337
2	5.2687	4.96329	5.62739	5.29291	5.97692	5.94196	6.35269	6.15867	6.70673	6.5816
3	11.0489	10.90525	11.63925	11.25478	12.23346	11.99578	12.85946	12.82765	13.44632	13.31699
4	18.9317	18.73466	19.8881	19.60848	20.90173	20.62211	21.87971	21.42602	22.83602	22.57947

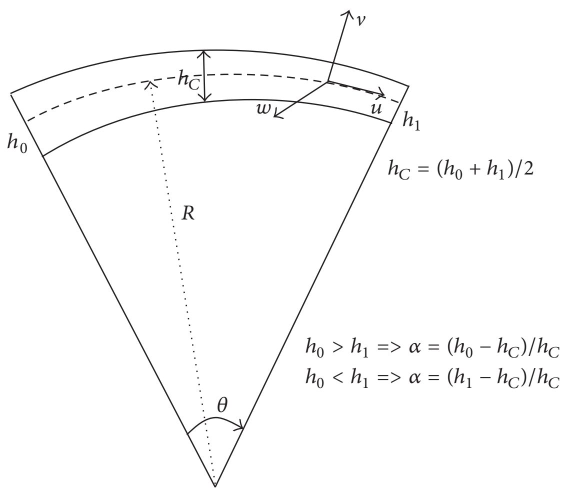


Figure 2: Unsymmetrical tapered beam.

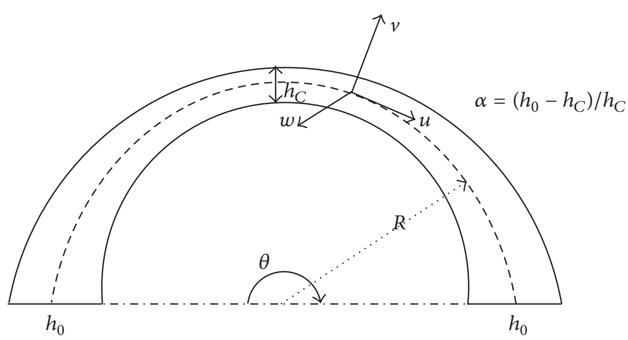
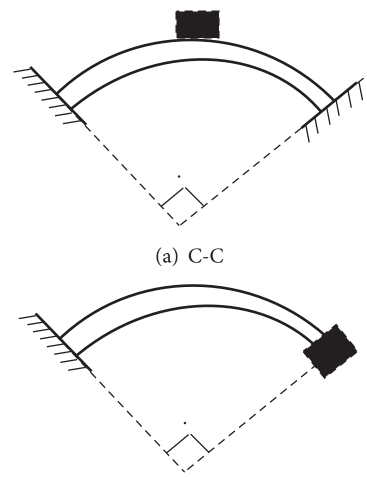


Figure 3: Symmetrical tapered beam.



(a) C-C

(b) C-F

Figure 4: Unsymmetrically tapered beams with additional mass.

of curved and variable cross-sectioned beams are not widely studied. The aim of this study is to analyze the free out-of-plane vibrations of curved beams which are symmetrically and nonsymmetrically tapered (see Figures 2 and 3).

It is assumed that (i) each beam possesses a constant width with R as radius of the curvature, θ as arc angle, and α as cross-section change parameter and (ii) the beam is circularly curved and its cross-section is rectangular with height of the cross-section as h_0 at the beginning, h_c at the crown, and h_1 at the end for *an unsymmetrical tapered beam* and as h_0 at both ends and h_c at the crown for *a symmetrical tapered beam*.

For the comparison of out-of-plane nondimensional fundamental natural frequency parameters, symmetrical tapered curved beam analysis under C-C end condition (see Figure 3) is assumed to be under plane strain condition and the general-purpose finite element code MSC.Marc (v2014, MSC Software, Santa Ana, CA, USA) is used. Four-node linear plane strain element (full integration 11) is considered for all five finite element beam models with constant 14400 elements, respectively. Results for the symmetrical curved tapered beam are given in Table 4. It is obvious that there is a good agreement between the results of this study and that of finite element model by MSC.Marc.

3.3.1. Unsymmetrical Tapered Beams. In the first part of this section, analysis of an unsymmetrical tapered beam or beam with varying cross-section, change of arc angle at different end conditions is considered. Cross-section change parameter is taken as 0.2 and 0.4 while h_c is same for both ($h_0 > h_1$). First natural frequency parameters for clamped (C-C), clamped-hinged (C-H), and hinged (H-H) end conditions are shown in Table 5. It is noted that (i) frequencies decrease with increasing of arc angle, (ii) as expected, the frequencies decrease from C-C to H-H end conditions, and (iii) increase of α causes an increase for frequencies at C-H and mostly a decrease at the other two end conditions.

In the second part of this section, an additional mass is considered on some points of tapered beams. Corresponding to C-C boundary conditions and 90° arc angle, a curved beam with a concentrated mass (10 kg) at the midpoint is given in Figure 4(a). In Figure 4(b), the same beam is under free right end (C-F) and mass location at the free end. Dimensionless

Table 5: Nondimensional fundamental natural frequency parameters of unsymmetrical tapered curved beams.

θ	C-C		C-H		H-H	
	$\alpha = 0.2$	$\alpha = 0.4$	$\alpha = 0.2$	$\alpha = 0.4$	$\alpha = 0.2$	$\alpha = 0.4$
30°	80.43	78.28	58.19	59.81	34.51	33.12
60°	19.69	19.20	14.04	14.56	7.82	7.51
90°	8.47	8.29	5.90	6.21	2.896	2.782
120°	4.58	4.50	3.09	3.32	1.186	1.139
150°	2.811	2.779	1.826	2.017	0.409	0.393
180°	1.888	1.877	1.190	1.350	0.020	0.021
270°	0.873	0.876	0.593	0.668	0.446	0.402
360°	0.635	0.629	0.477	0.513	0.041	0.041

Table 6: Nondimensional natural frequencies of unsymmetrical tapered curved beams with C-C end conditions.

Mode	$\alpha = 0.1$		$\alpha = 0.2$		$\alpha = 0.3$		$\alpha = 0.4$	
	Bare	Loaded*	Bare	Loaded*	Bare	Loaded*	Bare	Loaded*
1	8.51	4.76	8.47	4.75	8.40	4.74	8.29	4.72
2	24.16	24.08	24.00	23.73	23.73	23.13	23.33	22.32
3	48.04	38.59	47.73	38.78	47.17	39.05	46.34	39.34
4	79.94	79.44	79.42	77.59	78.48	74.85	77.09	71.48

*Loaded with additional mass (10 kg) at midpoint of the beam ($\theta = 90^\circ$).

Table 7: Nondimensional natural frequencies of unsymmetrical tapered curved beams with C-F end conditions.

Mode	$\alpha = 0.1$		$\alpha = 0.2$		$\alpha = 0.3$		$\alpha = 0.4$	
	Bare	Loaded*	Bare	Loaded*	Bare	Loaded*	Bare	Loaded*
1	1.84	0.87	2.04	0.92	2.25	0.97	2.48	1.00
2	8.43	5.97	8.80	6.16	9.14	6.32	9.46	6.44
3	24.28	19.77	24.56	19.85	24.74	19.82	24.81	19.69
4	48.13	41.66	48.25	41.57	48.15	41.29	47.81	40.77

*Loaded with additional mass (10 kg) at free end of the beam ($\theta = 90^\circ$).

natural frequency parameters of unsymmetrically tapered curved beams at C-C boundary conditions are presented in Table 6. The results represent both bare and mass loaded cases with the range of α as 0.1–0.4 and constant h_c.

In the third part of this section, similar computation for C-F beam with the additional mass at the free end is completed with respect to bare and loaded beam cases. Results of first four frequency parameters are shown in Table 7.

In the last part of this section, vibration for unsymmetrically tapered beams is considered for different arc angles. Again, according to bare and loaded beam cases, first natural frequencies are obtained for cross-section parameters 0.2 and 0.4, respectively, as given in Table 8.

Table 8: Nondimensional fundamental natural frequencies of unsymmetrical tapered curved beams with C-F end condition (10 kg additional mass at the free end).

θ	$\alpha = 0.2$		$\alpha = 0.4$	
	Bare	Loaded	Bare	Loaded
30°	16.41	7.167	20.21	7.529
60°	4.28	1.897	5.25	1.985
90°	2.04	0.925	2.48	0.961
120°	1.261	0.588	1.514	0.605
150°	0.906	0.435	1.071	0.442
180°	0.717	0.351	0.833	0.353
270°	0.481	0.226	0.533	0.226
360°	0.382	0.176	0.419	0.178

3.3.2. Symmetrical Tapered Beams. Although there are number of studies about in-plane vibration of curved and symmetrically tapered beams in literature [4, 5, 15], there is not any about out-of-plane vibration of that kind of structures. In this section, out-of-plane free vibration of symmetrically tapered beams at only C-C end conditions is put into consideration in such a way that (i) the beam is separated into 100 finite elements and (ii) a concentrated mass is located at the midpoint. The beam, namely, a half circled curved beam ($\theta = 180^\circ$), is shown in Figure 5.

In the first part of this section, the fundamental natural frequencies for different arc angles' α values are obtained regarding bare and loaded cases. Results are presented in Table 9. One can clearly say that while increasing arc angle

Table 9: Fundamental frequencies of symmetrical tapered beams under C-C end condition.

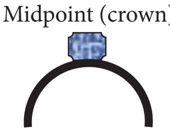
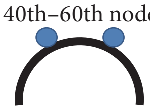
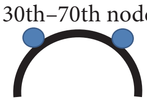
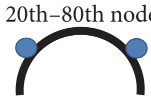
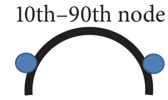
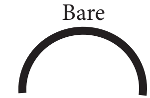
θ	$\alpha = 0.1$		$\alpha = 0.2$		$\alpha = 0.3$		$\alpha = 0.4$	
	Bare	Loaded*	Bare	Loaded*	Bare	Loaded*	Bare	Loaded*
20°	196.753	94.883	210.796	101.640	224.883	108.433	239.017	115.266
40°	48.603	23.436	52.149	25.144	55.701	26.858	59.261	28.580
60°	21.205	10.222	22.801	10.992	35.411	11.764	25.997	12.538
80°	11.651	5.613	12.562	6.053	13.471	6.493	14.380	6.934
100°	7.261	3.495	7.851	3.780	8.439	4.065	9.025	4.350
120°	4.903	2.357	5.315	2.557	5.726	2.756	6.136	2.955
140°	3.501	1.680	3.805	1.826	4.108	1.974	4.409	2.121
160°	2.608	1.249	2.840	1.361	3.071	1.473	3.301	1.585
180°	2.008	0.959	2.189	1.047	2.370	1.135	2.551	1.222

*Loaded with additional mass (50 kg) at midpoint of the beam.

Table 10: Nondimensional out-of-plane natural frequencies of half circled uniform and symmetrical tapered beams ($\theta = 180°$).

Type	Mode	Amount of the additional mass				
		Bare	25 kg	50 kg	75 kg	100 kg
Uniform	1	1.825	1.112	0.870	0.739	0.653
	2	*5.265*	*5.265*	*5.265*	*5.265*	*5.265*
	3	11.038	9.082	8.758	8.626	8.554
	4	*18.910*	*18.910*	*18.910*	*18.910*	*18.910*
Tapered $\alpha = 0.3$	1	2.370	1.449	1.135	0.963	0.851
	2	*6.353*	*6.353*	*6.353*	*6.353*	*6.353*
	3	12.860	10.457	10.066	9.908	9.822
	4	*21.880*	*21.880*	*21.880*	*21.880*	*21.880*

Table 11: The out-of-plane natural frequencies of the beam with different mass locations under C-C end condition.

Mode	Position 1 Midpoint (crown) 	Position 2 40th–60th node 	Position 3 30th–70th node 	Position 4 20th–80th node 	Position 5 10th–90th node 	Position 6 Bare 
1	0.786	0.868	1.123	1.620	2.127	2.189
2	5.994	2.921	2.342	2.870	4.948	5.994
3	9.408	11.534	9.621	6.209	7.725	12.264
4	20.908	14.013	20.491	14.923	11.729	20.908

Additional mass: 100 kg, $\alpha = 0.2$.

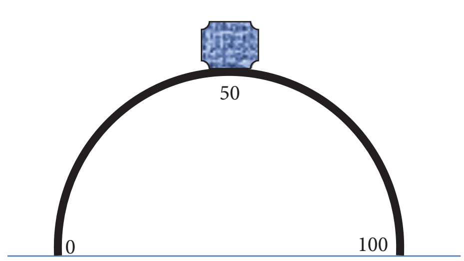


Figure 5: Symmetrically curved beam with additional mass is at the top (crown).

with additional mass decreases the frequencies, increasing of α with additional mass increases the frequencies.

In the second part of this section, again, the first five frequencies are computed for uniform and tapered ($\alpha = 0.3$) beams with the range of additional mass of 25–100 kg. As seen in Table 10, results indicate that the significant point is about the even modes (2–4). On one hand, these two modes keep their constancy independently of the amount of the additional mass. On the other hand, odd modes show regular decreasing behavior due to the increasing of load.

In the last part of this section, a change of location of concentrated mass is taken into account under bare beam case. As represented in Table 11, the following conditions are considered: (i) mass is considered at midpoint as Position 1 (100 kg) and (ii) the mass is separated into two equal pieces (50 kg each) and these pieces are located at two sides symmetrically as Positions 2–5, respectively. For the first natural frequency point of view, it is said that increasing position number from 1 to 6 increases fundamental frequency. For the rest of the modes, there is an up-and-down behavior due to the locations of additional masses. Also, when comparing

Positions 5 with 6, first and second natural frequencies become close to each other.

4. Conclusions

Due to the fact that there is a gap in literature about out-of-plane vibrations of curved and variable cross-sectioned beams, this study is presented to analyze the free out-of-plane vibrations of curved beams which are symmetrically and nonsymmetrically tapered. Results conclude that the finite element solution proposed here is suitable for vibration analyses of curved and tapered beams with or without additional mass.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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