Hindawi
Mathematical Problems in Engineering
Volume 2017, Article ID 9898726, 15 pages
https://doi.org/10.1155/2017/9898726

Research Article

Bifurcation Analysis for an SEIRS-V Model with Delays on the Transmission of Worms in a Wireless Sensor Network

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Received 10 February 2016; Accepted 23 May 2016; Published 18 January 2017

Academic Editor: Vladimir Turetsky

Hopf bifurcation for an SEIRS-V model with delays on the transmission of worms in a wireless sensor network is investigated. We focus on existence of the Hopf bifurcation by regarding the diverse delay as a bifurcation parameter. The results show that propagation of worms in the wireless sensor network can be controlled when the delay is suitably small under some certain conditions. Then, we study properties of the Hopf bifurcation by using the normal form theory and center manifold theorem. Finally, we give a numerical example to support the theoretical results.

1. Introduction

In recent years, wireless sensor networks have received extensive attention due to their vast potential in many application environments. However, security of wireless sensor networks still remains one of the most critical challenges because sensor nodes are often placed in a hostile or dangerous environment [1]. Many epidemiological models [2–6] have been proposed to study and predict the spread of viruses in wireless networks motivated by the pioneering work of Murray [7] and Kephart and White [8, 9]. In [10], Mishra and Keshri proposed the following SEIRS-V model to describe the propagation of worms in a wireless sensor network:

$$\begin{aligned}
\frac{dS(t)}{dt} &= A - \beta S(t) I(t) - (\zeta + p) S(t) + \delta R(t) \\
&\quad + \eta V(t), \\
\frac{dE(t)}{dt} &= \beta S(t) I(t) - (\zeta + \alpha) E(t), \\
\frac{dI(t)}{dt} &= \alpha E(t) - (\zeta + \varepsilon + \gamma) I(t), \\
\frac{dR(t)}{dt} &= \gamma I(t) - (\zeta + \delta) R(t), \\
\frac{dV(t)}{dt} &= pS(t) - (\zeta + \eta) V(t),
\end{aligned} \tag{1}$$

where $S(t)$, $E(t)$, $I(t)$, $R(t)$, and $V(t)$ denote the number of susceptible, exposed (infected, but not infectious), infectious, recovered, and vaccinated sensor nodes at time t, respectively. $A, p, \alpha, \beta, \gamma, \delta, \eta, \varepsilon$, and ζ are the positive parameters of system (1) and for the specific meanings of them one can refer to [10]. Considering the time delays in system (1), Zhang and Si [11] proposed the following delayed SEIRS-V system:

$$\begin{aligned}
\frac{dS(t)}{dt} &= A - \beta S(t) I(t) - (\zeta + p) S(t) + \delta R(t - \tau_2) \\
&\quad + \eta V(t - \tau_2), \\
\frac{dE(t)}{dt} &= \beta S(t) I(t) - (\zeta + \alpha) E(t),
\end{aligned}$$

$$\frac{dI(t)}{dt} = \alpha E(t) - (\zeta + \varepsilon) I(t) - \gamma I(t - \tau_1),$$
$$\frac{dR(t)}{dt} = \gamma I(t - \tau_1) - \zeta R(t) - \delta R(t - \tau_2),$$
$$\frac{dV(t)}{dt} = pS(t) - \zeta V(t) - \eta V(t - \tau_2), \tag{2}$$

where $\tau_1 \geq 0$ is the time delay due to the period that antivirus software uses to clean worms in the infected nodes; $\tau_2 \geq 0$ is the time delay due to the temporary immunity period of the recovered and the vaccinated nodes. For the convenience of analysis, Zhang and Si [11] let $\tau_1 = \tau_2$ and considered the following system:

$$\begin{aligned}
\frac{dS(t)}{dt} &= A - \beta S(t) I(t) - (\zeta + p) S(t) + \delta R(t - \tau) \\
&\quad + \eta V(t - \tau), \\
\frac{dE(t)}{dt} &= \beta S(t) I(t) - (\zeta + \alpha) E(t), \\
\frac{dI(t)}{dt} &= \alpha E(t) - (\zeta + \varepsilon) I(t) - \gamma I(t - \tau), \\
\frac{dR(t)}{dt} &= \gamma I(t - \tau) - \zeta R(t) - \delta R(t - \tau), \\
\frac{dV(t)}{dt} &= pS(t) - \zeta V(t) - \eta V(t - \tau).
\end{aligned} \tag{3}$$

Zhang and Si [11] investigated existence and properties of the Hopf bifurcation of system (3).

It should be pointed out that one of the significant features of worms in networks is its latent characteristic. Therefore, there exists a certain period before the exposed nodes develop themselves into the infectious ones. In addition, as far as we know, there have been some papers that deal with research of Hopf bifurcation of dynamical systems with multiple delays in recent years [12–15]. In [12], Xu et al. studied Hopf bifurcation of a ring of five neurons with delays. In [14], Bianca et al. investigated Hopf bifurcation of an economic growth model with two delays. Considering that there is a latent period of worms in the exposed nodes in system (3), we study the following system with delays:

$$\begin{aligned}
\frac{dS(t)}{dt} &= A - \beta S(t) I(t) - (\zeta + p) S(t) + \delta R(t - \tau_2) \\
&\quad + \eta V(t - \tau_2), \\
\frac{dE(t)}{dt} &= \beta S(t) I(t) - \alpha E(t - \tau_1) - \zeta E(t), \\
\frac{dI(t)}{dt} &= \alpha E(t - \tau_1) - (\zeta + \varepsilon) I(t) - \gamma I(t - \tau_2), \\
\frac{dR(t)}{dt} &= \gamma I(t - \tau_2) - \zeta R(t) - \delta R(t - \tau_2), \\
\frac{dV(t)}{dt} &= pS(t) - \zeta V(t) - \eta V(t - \tau_2),
\end{aligned} \tag{4}$$

where τ_1 is the time delay due to the latent period of worms in the exposed nodes and τ_2 is the time delay due to the period that the antivirus software uses to clean worms in the infected nodes and that due to the temporary immunity period of the recovered and the vaccinated nodes.

The structure of this paper is as follows. In Section 2, we obtain sufficient conditions for local stability of the positive equilibrium and existence of a Hopf bifurcation of system (4). In Section 3, we deal with the properties of the Hopf bifurcation by using the normal form theory and center manifold theorem. Some numerical simulations are carried out in Section 4 with the aim of verifying the obtained analytic results. Finally, conclusions and future work are summarized.

2. Hopf Bifurcation Analysis

By a direct computation, we know that if $R_0 = (\alpha\beta A(\zeta + \eta) + p\eta(\zeta + \alpha)(\zeta + \varepsilon + \gamma))/(\zeta + p)(\zeta + \alpha)(\zeta + \eta)(\zeta + \varepsilon + \gamma) > 1$, then system (4) has a unique positive equilibrium $D_*(S_*, E_*, I_*, R_*, V_*)$ in which

$$\begin{aligned}
S_* &= \frac{(\zeta + \alpha)(\zeta + \varepsilon + \gamma)}{\alpha\beta}, \\
E_* &= \frac{\zeta + \varepsilon + \gamma}{\alpha} I_*, \\
R_* &= \frac{\gamma}{\zeta + \delta} I_*, \\
V_* &= \frac{p(\zeta + \alpha)(\zeta + \varepsilon + \gamma)}{\alpha\beta(\zeta + \eta)}, \\
I_* &= \frac{\alpha\beta A(\zeta + \delta)(\zeta + \eta) + p\eta(\zeta + \alpha)(\zeta + \delta)(\zeta + \varepsilon + \gamma) - (\zeta + p)(\zeta + \alpha)(\zeta + \delta)(\zeta + \eta)(\zeta + \varepsilon + \gamma)}{\beta(\zeta + \alpha)(\zeta + \delta)(\zeta + \eta)(\zeta + \varepsilon + \gamma) - \alpha\beta\delta\gamma(\zeta + \eta)}.
\end{aligned} \tag{5}$$

The characteristic equation of system (4) at D_* is

$$\begin{aligned}
&\lambda^5 + A_4\lambda^4 + A_3\lambda^3 + A_2\lambda^2 + A_1\lambda + A_0 \\
&\quad + \left(B_4\lambda^4 + B_3\lambda^3 + B_2\lambda^2 + B_1\lambda + B_0\right)e^{-\lambda\tau_1} \\
&\quad + \left(C_4\lambda^4 + C_3\lambda^3 + C_2\lambda^2 + C_1\lambda + C_0\right)e^{-\lambda\tau_2} \\
&\quad + \left(D_3\lambda^3 + D_2\lambda^2 + D_1\lambda + D_0\right)e^{-2\lambda\tau_2} \\
&\quad + \left(E_3\lambda^3 + E_2\lambda^2 + E_1\lambda + E_0\right)e^{-\lambda(\tau_1+\tau_2)} \\
&\quad + \left(F_2\lambda^2 + F_1\lambda + F_0\right)e^{-\lambda(\tau_1+2\tau_2)} \\
&\quad + \left(G_2\lambda^2 + G_1\lambda + G_0\right)e^{-3\lambda\tau_2} \\
&\quad + \left(H_1\lambda + H_0\right)e^{-\lambda(\tau_1+3\tau_2)} = 0,
\end{aligned} \tag{6}$$

where

$$\begin{aligned}
A_0 &= -a_{11}a_{22}a_{33}a_{44}a_{55}, \\
A_1 &= a_{11}a_{22}a_{33}a_{44} + a_{11}a_{22}a_{55}\left(a_{33} + a_{44}\right) \\
&\quad + a_{33}a_{44}a_{55}\left(a_{11} + a_{22}\right), \\
A_2 &= -a_{55}\left(a_{11}a_{22} + a_{33}a_{44} + \left(a_{11} + a_{22}\right)\left(a_{33} + a_{44}\right)\right) \\
&\quad - \left(a_{11}a_{22}\left(a_{33} + a_{44}\right) + a_{33}a_{44}\left(a_{11} + a_{22}\right)\right), \\
A_3 &= a_{11}a_{22} + a_{33}a_{44} + \left(a_{11} + a_{22}\right)\left(a_{33} + a_{44}\right) \\
&\quad + a_{55}\left(a_{11} + a_{22} + a_{33} + a_{44}\right), \\
A_4 &= -\left(a_{11} + a_{22} + a_{33} + a_{44} + a_{55}\right), \\
B_0 &= a_{44}a_{55}b_{32}\left(a_{11}a_{23} - a_{13}a_{21}\right) - a_{11}a_{22}a_{33}a_{44}b_{22}, \\
B_1 &= b_{22}\left(a_{11}a_{22}\left(a_{33} + a_{44}\right) + a_{33}a_{44}\left(a_{11} + a_{22}\right)\right) \\
&\quad + b_{32}\left(\left(a_{44} + a_{55}\right)\left(a_{13}a_{21} - a_{23}a_{11}\right) - a_{23}a_{44}a_{55}\right), \\
B_2 &= a_{23}b_{32}\left(a_{44} + a_{55}\right) + b_{32}\left(a_{11}a_{23} - a_{13}a_{21}\right) \\
&\quad - b_{22}\left(a_{11}a_{22} + a_{33}a_{44} + \left(a_{11} + a_{22}\right)\left(a_{33} + a_{44}\right)\right), \\
B_3 &= b_{22}\left(a_{11} + a_{22} + a_{33} + a_{44}\right) - a_{23}b_{32}, \\
B_4 &= -b_{22}, \\
C_0 &= -a_{11}a_{22}a_{33}\left(a_{44}b_{55} + a_{55}b_{44}\right) - a_{11}a_{22}a_{44}a_{55}b_{33}, \\
C_1 &= b_{33}\left(a_{11}a_{22}\left(a_{44} + a_{55}\right) + a_{44}a_{55}\left(a_{11} + a_{22}\right)\right) \\
&\quad + \left(a_{11}a_{22} + a_{11}a_{33} + a_{22}a_{33}\right)\left(a_{44}b_{55} + a_{55}b_{44}\right) \\
&\quad + a_{11}a_{22}a_{33}\left(b_{44} + b_{55}\right), \\
C_2 &= -\left(a_{11}a_{22} + a_{11}a_{33} + a_{22}a_{33}\right)\left(b_{44} + b_{55}\right) \\
&\quad - \left(a_{44}b_{55} + a_{55}b_{44}\right)\left(a_{11} + a_{22} + a_{33}\right) \\
&\quad - b_{33}\left(a_{11}a_{22} + a_{44}a_{55} + \left(a_{11} + a_{22}\right)\left(a_{44} + a_{55}\right)\right), \\
C_3 &= b_{33}\left(a_{11} + a_{22} + a_{44} + a_{55}\right) + a_{44}b_{55} + a_{55}b_{44} \\
&\quad + \left(a_{11} + a_{22} + a_{33}\right)\left(b_{44} + b_{55}\right), \\
C_4 &= -\left(b_{22} + b_{44} + b_{55}\right), \\
D_0 &= -a_{11}a_{22}\left(a_{33}b_{44}b_{55} + a_{44}b_{33}b_{55} + a_{55}b_{33}b_{44}\right), \\
D_1 &= b_{33}b_{44}\left(a_{11}a_{22} + a_{11}a_{55} + a_{22}a_{55}\right) \\
&\quad + b_{33}b_{55}\left(a_{11}a_{22} + a_{11}a_{44} + a_{22}a_{44}\right) \\
&\quad + b_{44}b_{55}\left(a_{11}a_{22} + a_{11}a_{33} + a_{22}a_{33}\right), \\
D_2 &= -b_{33}b_{44}\left(a_{11} + a_{22} + a_{55}\right) \\
&\quad - b_{33}b_{55}\left(a_{11} + a_{22} + a_{44}\right) \\
&\quad - b_{44}b_{55}\left(a_{11} + a_{22} + a_{33}\right), \\
D_3 &= b_{33}b_{44} + b_{33}b_{55} + b_{44}b_{55}, \\
E_0 &= b_{32}\left(a_{44}b_{55} + a_{55}b_{44}\right)\left(a_{11}a_{23} - a_{13}a_{21}\right) \\
&\quad - a_{11}b_{22}\left(a_{44}a_{55}b_{33} + a_{33}a_{44}b_{55} + a_{33}a_{55}b_{44}\right), \\
E_1 &= a_{11}a_{33}b_{22}\left(b_{44} + b_{55}\right) \\
&\quad + b_{22}b_{33}\left(a_{11}a_{44} + a_{11}a_{55} + a_{44}a_{55}\right) \\
&\quad + \left(a_{11}b_{22} + a_{33}b_{22} - a_{23}b_{32}\right)\left(a_{44}b_{55} + a_{55}b_{44}\right) \\
&\quad + b_{32}\left(b_{44} + b_{55}\right)\left(a_{13}a_{21} - a_{11}a_{23}\right), \\
E_2 &= a_{23}b_{32}\left(b_{44} + b_{55}\right) - b_{22}b_{33}\left(a_{11} + a_{44} + a_{55}\right) \\
&\quad - b_{22}\left(a_{44}b_{55} + a_{55}b_{44} + \left(b_{44} + b_{55}\right)\left(a_{11} + a_{33}\right)\right), \\
E_3 &= b_{22}\left(b_{33} + b_{44} + b_{55}\right), \\
F_0 &= a_{21}a_{55}b_{14}b_{32}b_{43} + b_{32}b_{44}b_{55}\left(a_{11}a_{23} - a_{13}a_{21}\right) \\
&\quad - a_{11}b_{22}\left(a_{33}b_{44}b_{55} + a_{44}b_{33}b_{55} + a_{55}b_{33}b_{44}\right), \\
F_1 &= b_{22}b_{33}b_{55}\left(a_{11} + a_{44}\right) + b_{22}b_{44}b_{55}\left(a_{11} + a_{33}\right) \\
&\quad - \left(a_{21}b_{14}b_{32}b_{43} + a_{23}a_{32}b_{44}b_{55}\right), \\
F_2 &= -b_{22}\left(b_{33}b_{44} + b_{33}b_{55} + b_{44}b_{55}\right), \\
G_0 &= -a_{11}a_{22}b_{33}b_{44}b_{55}, \\
G_1 &= b_{33}b_{44}b_{55}\left(a_{11} + a_{22}\right), \\
G_2 &= -b_{33}b_{44}b_{55}, \\
H_0 &= a_{21}b_{14}b_{32}b_{43}b_{55} - a_{11}b_{22}b_{33}b_{44}b_{55}, \\
H_1 &= b_{22}b_{33}b_{44}b_{55},
\end{aligned} \tag{7}$$

with

$$
\begin{aligned}
a_{11} &= -\left(\beta I_* + \zeta + p\right),\\
a_{13} &= -\beta S_*,\\
a_{21} &= \beta I_*,\\
a_{22} &= -\zeta,\\
a_{23} &= \beta S_*,\\
a_{33} &= -(\zeta + \varepsilon),\\
a_{44} &= -\zeta,\\
a_{51} &= p,\\
a_{55} &= -\zeta,\\
b_{14} &= \delta,\\
b_{15} &= \eta,\\
b_{22} &= -\alpha,\\
b_{32} &= \alpha,\\
b_{33} &= -\gamma,\\
b_{43} &= \gamma,\\
b_{44} &= -\delta,\\
b_{55} &= -\eta.
\end{aligned}
\tag{8}
$$

Case 1. When $\tau_1 = \tau_2 = 0$, (6) becomes

$$\lambda^5 + A_{14}\lambda^4 + A_{13}\lambda^3 + A_{12}\lambda^2 + A_{11}\lambda + A_{10} = 0, \tag{9}$$

where

$$
\begin{aligned}
A_{10} &= A_0 + B_0 + C_0 + D_0 + E_0 + F_0 + G_0 + H_0,\\
A_{11} &= A_1 + B_1 + C_1 + D_1 + E_1 + F_1 + G_1 + H_1,\\
A_{12} &= A_2 + B_2 + C_2 + D_2 + E_2 + F_2 + G_2,\\
A_{13} &= A_3 + B_3 + C_3 + D_3 + E_3,\\
A_{14} &= A_4 + B_4 + C_4.
\end{aligned}
\tag{10}
$$

It is easy to get that $A_{14} = p+\alpha+\gamma+\delta+\varepsilon+\eta+5\zeta+\beta I_* > 0$. Thus, we have the following condition:

(H_1) If (11) holds, D_* is locally asymptotically stable:

$$
\begin{aligned}
\det{}_2 &= \begin{pmatrix} A_{14} & 1 \\ A_{12} & A_{13} \end{pmatrix} > 0,\\
\det{}_3 &= \begin{pmatrix} A_{14} & 1 & 0 \\ A_{12} & A_{13} & A_{14} \\ 0 & A_{11} & A_{12} \end{pmatrix} > 0,\\
\det{}_4 &= \begin{pmatrix} A_{14} & 1 & 0 & 0 \\ A_{12} & A_{13} & A_{14} & 1 \\ A_{10} & A_{11} & A_{12} & A_{13} \\ 0 & 0 & A_{10} & A_{11} \end{pmatrix} > 0,\\
\det{}_5 &= \begin{pmatrix} A_{14} & 1 & 0 & 0 & 0 \\ A_{12} & A_{13} & A_{14} & 1 & 0 \\ A_{10} & A_{11} & A_{12} & A_{13} & A_{14} \\ 0 & 0 & A_{10} & A_{11} & A_{12} \\ 0 & 0 & 0 & 0 & A_{10} \end{pmatrix} > 0.
\end{aligned}
\tag{11}
$$

Case 2. When $\tau_1 > 0, \tau_2 = 0$, (6) becomes

$$
\begin{aligned}
&\lambda^5 + A_{24}\lambda^4 + A_{23}\lambda^3 + A_{22}\lambda^2 + A_{21}\lambda + A_{20}\\
&\quad + \left(B_{24}\lambda^4 + B_{23}\lambda^3 + B_{22}\lambda^2 + B_{21}\lambda + B_{20}\right) e^{-\lambda\tau_1}\\
&= 0,
\end{aligned}
\tag{12}
$$

where

$$
\begin{aligned}
A_{20} &= A_0 + C_0 + D_0 + G_0,\\
A_{21} &= A_1 + C_1 + D_1 + G_1,\\
A_{22} &= A_2 + C_2 + D_2 + G_2,\\
A_{23} &= A_3 + C_3 + D_3,\\
A_{24} &= A_4 + C_4,\\
B_{20} &= B_0 + E_0 + F_0 + H_0,\\
B_{21} &= B_1 + E_1 + F_1 + H_1,\\
B_{22} &= B_2 + E_2 + F_2,\\
B_{23} &= B_3 + E_3,\\
B_{24} &= B_4.
\end{aligned}
\tag{13}
$$

Let $\lambda = i\omega_1$ $(\omega_1 > 0)$ be a root of (12). Then,

$$
\begin{aligned}
&\left(B_{21}\omega_1 - B_{23}\omega_1^3\right)\sin\tau_1\omega_1\\
&\quad + \left(B_{24}\omega_1^4 - B_{22}\omega_1^2 + B_{20}\right)\cos\tau_1\omega_1 = A_{22}\omega_1^2\\
&\quad - A_{24}\omega_1^4 - A_{20},\\
&\left(B_{21}\omega_1 - B_{23}\omega_1^3\right)\cos\tau_1\omega_1\\
&\quad - \left(B_{24}\omega_1^4 - B_{22}\omega_1^2 + B_{20}\right)\sin\tau_1\omega_1 = A_{23}\omega_1^3 - \omega_1^5\\
&\quad - A_{21}\omega_1.
\end{aligned}
\tag{14}
$$

Thus, we have

$$\omega_1^{10} + s_{24}\omega_1^8 + s_{23}\omega_1^6 + s_{22}\omega_1^4 + s_{21}\omega_1 + s_{20} = 0, \tag{15}$$

where

$$\begin{aligned} s_{20} &= A_{20}^2 - B_{20}^2, \\ s_{21} &= A_{21}^2 - B_{21}^2 - 2A_{20}A_{22} + 2B_{20}B_{22}, \\ s_{22} &= A_{22}^2 - B_{22}^2 + 2A_{20}A_{24} - 2A_{21}A_{23} - 2B_{20}B_{24} \\ &\quad + 2B_{21}B_{23}, \\ s_{23} &= A_{23}^2 - B_{23}^2 + 2A_{21} - 2A_{22}A_{24} + 2B_{22}B_{24}, \\ s_{24} &= A_{24}^2 - B_{24}^2 - 2A_{23}. \end{aligned} \tag{16}$$

Let $\omega_1^2 = v_1$; then, (15) becomes

$$v_1^5 + s_{24}v_1^4 + s_{23}v_1^3 + s_{22}v_1^2 + s_{21}v_1 + s_{20} = 0. \tag{17}$$

According to the analysis of roots of (17) in [16], we assume that

(H_{21}) Equation (17) has at least one positive root.

Then, there exists a positive root v_{10} for (17). Thus, we obtain $\omega_{10} = \sqrt{v_{10}}$. Further,

$$\begin{aligned} \tau_{10} &= \frac{1}{\omega_{10}} \\ &\quad \times \arccos \frac{p_{28}\omega_{10}^8 + p_{26}\omega_{10}^6 + p_{24}\omega_{10}^4 + p_{22}\omega_{10}^2 + p_{20}}{q_{28}\omega_{10}^8 + q_{26}\omega_{10}^6 + q_{284}\omega_{10}^4 + q_{22}\omega_{10}^2 + q_{20}}, \end{aligned} \tag{18}$$

where

$$\begin{aligned} q_{20} &= B_{20}^2, \\ q_{22} &= B_{21}^2 - 2B_{22}, \\ q_{24} &= B_{22}^2 + 2B_{20}B_{24} - B_{21}B_{23}, \\ q_{26} &= B_{23}^2 - 2B_{22}B_{24}, \\ q_{28} &= B_{24}^2, \\ p_{20} &= -A_{20}B_{20}, \\ p_{22} &= A_{20}B_{22} + A_{22}B_{20} - A_{21}B_{21}, \\ p_{24} &= A_{21}B_{23} - A_{20}B_{24} - A_{22}B_{22} + A_{23}B_{21} - A_{24}B_{20}, \\ p_{26} &= A_{22}B_{24} + A_{24}B_{22} - A_{23}B_{23} - B_{21}, \\ p_{28} &= B_{23} - A_{24}B_{24}. \end{aligned} \tag{19}$$

Differentiating (12) regarding τ_1, we have

$$\begin{aligned} \left[\frac{d\lambda}{d\tau_1}\right]^{-1} &= -\frac{5\lambda^4 + 4A_{24}\lambda^3 + 3A_{23}\lambda^2 + 2A_{22}\lambda + A_{21}}{\lambda\left(\lambda^5 + A_{24}\lambda^4 + A_{23}\lambda^3 + A_{22}\lambda^2 + A_{21}\lambda + A_{20}\right)} \\ &\quad + \frac{4B_{24}\lambda^3 + 3B_{23}\lambda^2 + 2B_{22}\lambda + B_{21}}{\lambda\left(B_{24}\lambda^4 + B_{23}\lambda^3 + B_{22}\lambda^2 + B_{21}\lambda + B_{20}\right)} - \frac{\tau_1}{\lambda}. \end{aligned} \tag{20}$$

Then,

$$\begin{aligned} &\operatorname{Re}\left[\frac{d\lambda}{d\tau_1}\right]^{-1}_{\lambda=i\omega_{10}} \\ &\quad = \frac{f_{21}'(v_{1*})}{\left(B_{21}\omega_{10} - B_{23}\omega_{10}^3\right)^2 + \left(B_{24}\omega_{10}^4 - B_{22}\omega_{10}^2 + B_{20}\right)^2}, \end{aligned} \tag{21}$$

where

$$f_{21}(v_1) = v_1^5 + s_{24}v_1^4 + s_{23}v_1^3 + s_{22}v_1^2 + s_{21}v_1 + s_{20} \tag{22}$$

and $v_{1*} = \omega_{10}^2$.

Thus, we can conclude that if

(H_{22}) $f_{21}'(v_{1*}) \neq 0$ holds, then $\operatorname{Re}[d\lambda/d\tau_1]^{-1}_{\lambda=i\omega_{10}} \neq 0$.

Summarizing the analysis above, we have the following.

Theorem 1. *For system (4), let τ_{10} be specified by (18). If the conditions (H_{21})-(H_{22}) are satisfied, then $D_*(S_*, E_*, I_*, R_*, V_*)$ is asymptotically stable when $\tau_1 \in [0, \tau_{10})$ and a Hopf bifurcation occurs at $D_*(S_*, E_*, I_*, R_*, V_*)$ when $\tau_1 = \tau_{10}$.*

Case 3. When $\tau_1 = 0$ and $\tau_2 > 0$, (6) becomes

$$\begin{aligned} &\lambda^5 + A_{34}\lambda^4 + A_{33}\lambda^3 + A_{32}\lambda^2 + A_{31}\lambda + A_{30} \\ &\quad + \left(B_{34}\lambda^4 + B_{33}\lambda^3 + B_{32}\lambda^2 + B_{31}\lambda + B_{30}\right)e^{-\lambda\tau_2} \\ &\quad + \left(C_{33}\lambda^3 + C_{32}\lambda^2 + C_{31}\lambda + C_{30}\right)e^{-2\lambda\tau_2} \\ &\quad + \left(D_{32}\lambda^2 + D_{31}\lambda + D_{30}\right)e^{-3\lambda\tau_2} = 0. \end{aligned} \tag{23}$$

Multiplying by $e^{\lambda\tau_2}$ on both sides of (23), we have

$$\begin{aligned} &B_{34}\lambda^4 + B_{33}\lambda^3 + B_{32}\lambda^2 + B_{31}\lambda + B_{30} \\ &\quad + \left(\lambda^5 + A_{34}\lambda^4 + A_{33}\lambda^3 + A_{32}\lambda^2 + A_{31}\lambda + A_{30}\right)e^{\lambda\tau_2} \\ &\quad + \left(C_{33}\lambda^3 + C_{32}\lambda^2 + C_{31}\lambda + C_{30}\right)e^{-\lambda\tau_2} \\ &\quad + \left(D_{32}\lambda^2 + D_{31}\lambda + D_{30}\right)e^{-2\lambda\tau_2} = 0. \end{aligned} \tag{24}$$

Let $\lambda = i\omega_2$ $(\omega_2 > 0)$ be the root of (24). Then,

$$\begin{aligned} &g_{31}(\omega_2)\cos\tau_2\omega_2 - g_{32}(\omega_2)\sin\tau_2\omega_2 + g_{33}(\omega_2) \\ &\quad = h_{31}(\omega_2)\sin 2\tau_2\omega_2 + h_{32}(\omega_2)\cos 2\tau_2\omega_2, \\ &g_{34}(\omega_2)\sin\tau_2\omega_2 + g_{35}(\omega_2)\cos\tau_2\omega_2 + g_{36}(\omega_2) \\ &\quad = h_{31}(\omega_2)\cos 2\tau_2\omega_2 - h_{32}(\omega_2)\sin 2\tau_2\omega_2, \end{aligned} \tag{25}$$

where

$$\begin{aligned}
g_{31}(\omega_2) &= A_{34}\omega_2^4 - (A_{32}+C_{32})\omega_2^2 + A_{30} + C_{30},\\
g_{32}(\omega_2) &= \omega_2^5 - (A_{33}-C_{33})\omega_2^3 + (A_{31}-C_{31})\omega_2,\\
g_{33}(\omega_2) &= B_{34}\omega_2^4 - B_{32}\omega_2^2 + B_{30},\\
g_{34}(\omega_2) &= A_{34}\omega_2^4 - (A_{32}-C_{32})\omega_2^2 + A_{30} - C_{30},\\
g_{35}(\omega_2) &= \omega_2^5 - (A_{33}+C_{33})\omega_2^3 + (A_{31}+C_{31})\omega_2,\\
g_{36}(\omega_2) &= B_{31}\omega_2 - B_{33}\omega_2^3,\\
h_{31}(\omega_2) &= -D_{31}\omega_2,\\
h_{32}(\omega_2) &= D_{32}\omega_2^2 - D_{30}.
\end{aligned} \tag{26}$$

According to the analysis in [11], we obtain the expressions of $\cos\tau_2\omega_2$ and $\sin\tau_2\omega_2$ when $\sin\tau_2\omega_2 = \sqrt{1-\cos^2\tau_2\omega_2}$ and we denote $f_{31}(\omega_2) = \cos\tau_2\omega_2$ and $f_{32}(\omega_2) = \sin\tau_2\omega_2$. Then, we obtain a function regarding ω_2:

$$f_{31}^2(\omega_2) + f_{32}^2(\omega_2) = 1. \tag{27}$$

Next, we suppose that

(H_{31}) Equation (27) has at least one positive root.

Then, there exists $\omega_{210} > 0$ which makes (24) have roots $\pm i\omega_{210}$. For ω_{210},

$$\tau_{210} = \frac{1}{\omega_{210}} \arccos f_{31}(\omega_{210}). \tag{28}$$

Similarly, one can also obtain expressions of $\cos\tau_2\omega_2$ and $\sin\tau_2\omega_2$ for $\sin\tau_2\omega_2 = -\sqrt{1-\cos^2\tau_2\omega_2}$ and we denote $f_{33}(\omega_2) = \cos\tau_2\omega_2$ and $f_{34}(\omega_2) = \sin\tau_2\omega_2$. Then,

$$f_{33}^2(\omega_2) + f_{34}^2(\omega_2) = 1. \tag{29}$$

Obviously, if (29) has a root $\omega_{220} > 0$, then (24) has roots $\pm i\omega_{220}$. For ω_{220},

$$\tau_{220} = \frac{1}{\omega_{220}} \arccos f_{33}(\omega_{220}). \tag{30}$$

Let

$$\tau_{20} = \min\{\tau_{210}, \tau_{220}\}, \tag{31}$$

and $\pm i\omega_{20}$ are the roots of (24) when $\tau_2 = \tau_{20}$. Differentiating (24) regarding τ_2, one can obtain

$$\begin{aligned}
&\left[\frac{d\lambda}{d\tau_2}\right]^{-1}\\
&= \frac{p_{30}(\lambda) + p_{31}(\lambda)e^{\lambda\tau_2} + p_{32}(\lambda)e^{-\lambda\tau_2} + p_{33}(\lambda)e^{-2\lambda\tau_2}}{q_{30}(\lambda) - q_{31}(\lambda)e^{\lambda\tau_2}}\\
&\quad - \frac{\tau_2}{\lambda},
\end{aligned} \tag{32}$$

with

$$\begin{aligned}
p_{30}(\lambda) &= 4B_{34}\lambda^3 + 3B_{33}\lambda^2 + 2B_{32}\lambda + B_{31},\\
p_{31}(\lambda) &= 5\lambda^4 + 4A_{34}\lambda^3 + 3A_{33}\lambda^2 + 2A_{32}\lambda + A_{31},\\
p_{32}(\lambda) &= 3C_{33}\lambda^2 + 2C_{32}\lambda + C_{31},\\
p_{33}(\lambda) &= 2D_{32}\lambda + D_{31},\\
q_{30}(\lambda) &= C_{33}\lambda^4 + (C_{32}+2D_{32})\lambda^3 + (C_{31}+2D_{31})\lambda^2\\
&\quad + (C_{30}+2D_{30})\lambda,\\
q_{31}(\lambda) &= \lambda^6 + A_{34}\lambda^5 + A_{33}\lambda^4 + A_{32}\lambda^3 + A_{31}\lambda^2\\
&\quad + A_{30}\lambda.
\end{aligned} \tag{33}$$

Define

$$\operatorname{Re}\left[\frac{d\lambda}{d\tau_2}\right]^{-1}_{\lambda=i\omega_{20}} = \frac{P_{3R}Q_{3R} + P_{3I}Q_{3I}}{Q_{3R}^2 + Q_{3I}^2}. \tag{34}$$

Obviously, if the condition

(H_{32}) $P_{3R}Q_{3R} + P_{3I}Q_{3I} \neq 0$ holds, then $\operatorname{Re}[d\lambda/d\tau_2]^{-1}_{\lambda=i\omega_{20}} \neq 0$.

Therefore, we can otain the following according to the analysis above.

Theorem 2. *For system (4), let τ_{20} be specified by (31). If the conditions (H_{31})-(H_{32}) are satisfied, then $D_*(S_*, E_*, I_*, R_*, V_*)$ is asymptotically stable when $\tau_2 \in [0, \tau_{20})$ and a Hopf bifurcation occurs at $D_*(S_*, E_*, I_*, R_*, V_*)$ when $\tau_2 = \tau_{20}$.*

Case 4 ($\tau_1 > 0$, $\tau_2 > 0$, and $\tau_2 \in (0, \tau_{20})$). Let $\lambda = i\omega_{1*}$ ($\omega_{1*} > 0$) be the root of (6); then, one can obtain

$$\begin{aligned}
&g_{41}(\omega_{1*})\sin\tau_1\omega_{1*} + g_{42}(\omega_{1*})\cos\tau_1\omega_{1*}\\
&\quad = h_{41}(\omega_{1*}),\\
&g_{41}(\omega_{1*})\cos\tau_1\omega_{1*} - g_{42}(\omega_{1*})\sin\tau_1\omega_{1*}\\
&\quad = h_{42}(\omega_{1*}),
\end{aligned} \tag{35}$$

where

$$\begin{aligned}
g_{41}(\omega_{1*}) = B_1\omega_{1*} - B_3\omega_{1*}^3 &\\
+ \left(E_1\omega_{1*} - E_3\omega_{1*}^3\right)\cos\tau_2\omega_{1*} &\\
- \left(E_0 - E_2\omega_{1*}^2\right)\sin\tau_2\omega_{1*} &\\
+ F_1\omega_{1*}\cos 2\tau_2\omega_{1*} &\\
- \left(F_0 - F_2\omega_{1*}^2\right)\sin 2\tau_2\omega_{1*} &\\
+ H_1\omega_{1*}\cos 3\tau_2\omega_{1*} - H_0\sin 3\tau_2\omega_{1*}, &
\end{aligned}$$

$$\begin{aligned}
g_{42}(\omega_{1*}) = {} & B_4\omega_{1*}^4 - B_2\omega_{1*}^2 + B_0 \\
& + \left(E_1\omega_{1*} - E_3\omega_{1*}^3\right)\sin\tau_2\omega_{1*} \\
& + \left(E_0 - E_2\omega_{1*}^2\right)\cos\tau_2\omega_{1*} \\
& + F_1\omega_{1*}\sin 2\tau_2\omega_{1*} \\
& + \left(F_0 - F_2\omega_{1*}^2\right)\cos 2\tau_2\omega_{1*} \\
& + H_1\omega_{1*}\sin 3\tau_2\omega_{1*} + H_0\cos 3\tau_2\omega_{1*}, \\
h_{41}(\omega_{1*}) = {} & A_2\omega_{1*}^2 - A_4\omega_{1*}^4 - A_0 \\
& + \left(C_3\omega_{1*}^3 - C_1\omega_{1*}\right)\sin\tau_2\omega_{1*} \\
& + \left(C_2\omega_{1*}^2 - C_4\omega_{1*}^4 - C_0\right)\cos\tau_2\omega_{1*} \\
& + \left(D_3\omega_{1*}^3 - D_1\omega_{1*}\right)\sin 2\tau_2\omega_{1*} \\
& + \left(D_2\omega_{1*}^2 - D_0\right)\cos 2\tau_2\omega_{1*} \\
& - G_1\omega_{1*}\sin 3\tau_2\omega_{1*} \\
& + \left(G_2\omega_{1*}^2 - G_0\right)\cos 3\tau_2\omega_{1*}, \\
h_{42}(\omega_{1*}) = {} & A_3\omega_{1*}^3 - \omega_{1*}^5 - A_1\omega_{1*} \\
& + \left(C_3\omega_{1*}^3 - C_1\omega_{1*}\right)\cos\tau_2\omega_{1*} \\
& - \left(C_2\omega_{1*}^2 - C_4\omega_{1*}^4 - C_0\right)\sin\tau_2\omega_{1*} \\
& + \left(D_3\omega_{1*}^3 - D_1\omega_{1*}\right)\cos 2\tau_2\omega_{1*} \\
& - \left(D_2\omega_{1*}^2 - D_0\right)\sin 2\tau_2\omega_{1*} \\
& - G_1\omega_{1*}\cos 3\tau_2\omega_{1*} \\
& - \left(G_2\omega_{1*}^2 - G_0\right)\sin 3\tau_2\omega_{1*}.
\end{aligned} \tag{36}$$

Thus, we can obtain

$$\begin{aligned}
& f_{40}(\omega_{1*}) + 2f_{41}(\omega_{1*})\cos\tau_2\omega_{1*} \\
& \quad + 2f_{42}(\omega_{1*})\sin\tau_2\omega_{1*} + 2f_{43}(\omega_{1*})\cos 2\tau_2\omega_{1*} \\
& \quad + 2f_{44}(\omega_{1*})\sin 2\tau_2\omega_{1*} + 2f_{45}(\omega_{1*})\cos 3\tau_2\omega_{1*} \\
& \quad + 2f_{46}(\omega_{1*})\sin 3\tau_2\omega_{1*} = 0,
\end{aligned} \tag{37}$$

where

$$\begin{aligned}
f_{40}(\omega_{1*}) = {} & \omega_{1*}^{10} + \left(A_4^2 - B_4^2 + C_4^2 - 2A_3\right)\omega_{1*}^8 + \left(A_3^2\right. \\
& - B_3^2 + C_3^2 + D_3^2 - E_3^2 - 2A_2A_4 + 2B_2B_4 - 2C_2C_4 \\
& \left. + 2A_1\right)\omega_{1*}^6 + \left(A_2^2 - B_2^2 + C_2^2 + D_2^2 - E_2^2 - F_2^2 + G_2^2\right. \\
& + 2A_0A_4 - 2A_1A_3 - 2B_0B_4 + 2B_1B_3 + 2C_0C_4 \\
& \left. - 2C_1C_3 - 2D_1D_3 + 2E_1E_3\right)\omega_{1*}^4 + \left(A_1^2 - B_1^2 + C_1^2\right. \\
& + D_1^2 - E_1^2 + G_1^2 - H_1^2 - 2A_0A_2 - 2C_0C_2 - 2D_0D_2 \\
& \left. + 2B_0B_2 + 2E_0E_2 + 2F_0F_2 - 2G_0G_2\right)\omega_{1*}^2 + A_0^2 \\
& + C_0^2 + D_0^2 + G_0^2 - B_0^2 - E_0^2 - F_0^2 - H_0^2, \\
f_{41}(\omega_{1*}) = {} & (A_4C_4 - C_3)\omega_{1*}^8 + (A_3C_3 - A_2C_4 \\
& - A_4C_2 - B_3E_3 + B_4E_2 + C_3D_3 - C_4D_2 + C_1)\omega_{1*}^6 \\
& + (A_0C_4 - A_1C_3 + A_2C_2 - A_3C_1 + A_4C_0 - C_1D_3 \\
& - C_3D_1 + D_2G_2 - D_3G_1 + C_2D_2 + C_4D_0 + B_1E_3 \\
& - B_2E_2 + B_3E_1 - B_4E_0 + E_3F_1 - E_2F_2)\omega_{1*}^4 \\
& + (A_1C_1 - A_0C_2 - A_2C_0 + C_1D_1 - C_0D_2 - C_2D_0 \\
& + D_1G_1 - D_0G_2 - D_2G_0 - B_1E_1 + B_2E_0 + B_0E_2 \\
& - E_1F_1 + E_0F_2 + E_2F_0 - F_1H_1 + F_2H_0)\omega_{1*}^2 - B_0E_0 \\
& - E_0F_0 - F_0H_0, \\
f_{42}(\omega_{1*}) = {} & -C_4\omega_{1*}^9 + (A_3C_4 - A_4C_3 + B_4E_3 \\
& - C_4D_3 + C_2)\omega_{1*}^7 + (A_2C_3 + A_4C_1 - A_3C_2 \\
& - A_1C_4 - B_2E_3 + B_3E_2 - B_4E_1 + C_2D_3 + C_4D_1 \\
& - C_3D_2 - D_3G_2 - E_3F_2 - C_0)\omega_{1*}^5 + (A_1C_2 \\
& + A_3C_0 - A_2C_1 - A_0C_3 + B_0E_3 - B_1E_2 + B_2E_1 \\
& - B_3E_0 - C_0D_3 + C_1D_2 - C_2D_1 + D_2G_1 + C_3D_0 \\
& - D_1G_2 - D_3G_0 - E_1F_2 + E_2F_1 + F_2H_1 - E_3F_0) \\
& \cdot\omega_{1*}^3 + (A_0C_1 - A_1C_0 - B_0E_1 - B_1E_0 + C_0D_1 \\
& - D_0G_1 - C_1D_0 - D_1G_0 - E_0F_1 + E_1F_0 - F_0H_0 \\
& + F_1H_0)\omega_{1*}, \\
f_{43}(\omega_{1*}) = {} & -D_3\omega_{1*}^8 + (A_3D_3 - A_4D_2 + B_4F_2 \\
& - C_4G_2 + D_1)\omega_{1*}^6 + (A_2D_2 - A_3D_1 + A_4D_0 \\
& - A_1D_3 - B_2F_2 + B_3F_1 - B_4F_0 - C_3G_1 + C_4G_0 \\
& + C_2G_2 - E_3H_1)\omega_{1*}^4 + (A_1D_1 - A_0D_2 - A_2D_0 \\
& + B_0F_2 + B_2F_0 - B_1F_1 - C_0G_2 + C_1G_1 - C_2G_0 \\
& - E_1H_1 + E_2H_0)\omega_{1*}^2 + A_0D_0 + C_0G_0 - B_0F_0 \\
& - E_0H_0, \\
f_{44}(\omega_{1*}) = {} & (D_2 - A_4D_3)\omega_{1*}^7 + (A_2D_3 - A_3D_2 \\
& + A_4D_1 + B_3F_2 - B_4F_1 - C_3G_2 + C_4G_1 - D_0)\omega_{1*}^5 \\
& + (A_1D_2 - A_2D_1 + A_3D_0 - A_0D_3 - B_1F_2 + B_2F_1
\end{aligned}$$

$$\begin{aligned}
&- B_3F_0 - C_2G_1 + C_1G_2 + C_3G_0 + E_2H_1 - E_3H_0) \\
&\cdot \omega_{1*}^3 + (A_0D_1 - A_1D_0 + C_0G_1 - C_1G_0 + B_1F_0 \\
&- B_0F_1 - E_0H_1 + E_1H_0)\,\omega_{1*}, \\
f_{45}(\omega_{1*}) &= (G_1 - A_4G_2)\,\omega_{1*}^6 + (A_2G_2 - A_3G_1 \\
&+ A_4G_0 + B_3H_1 - B_4H_0)\,\omega_{1*}^4 + (A_1G_1 - A_0G_2 \\
&- A_2G_0 - B_1H_1 + B_2H_0)\,\omega_{1*}^2 + A_0G_0 - B_0H_0, \\
f_{46}(\omega_{1*}) &= G_2\omega_{1*}^7 + (A_4G_1 - A_3G_2 - B_4H_1 - G_0) \\
&\cdot \omega_{1*}^5 + (A_1G_2 - A_2G_1 + A_3G_0 + B_2H_1 - B_3H_0) \\
&\cdot \omega_{1*}^3 + (A_0G_1 - A_1G_0 - B_0H_1 + B_1H_0)\,\omega_{1*}.
\end{aligned} \tag{38}$$

Suppose that

(H_{41}) Equation (37) has at least one positive root.

Then, there exists a positive root ω_{10}^* for (37) and (6) has roots $\pm i\omega_{10}^*$. For ω_{10}^*, we have

$$\tau_{10}^* = \frac{1}{\omega_{10}^*} \times \arccos \frac{g_{41}(\omega_{10}^*) \times h_{42}(\omega_{10}^*) + g_{42}(\omega_{10}^*) \times h_{41}(\omega_{10}^*)}{g_{41}^2(\omega_{10}^*) + g_{42}^2(\omega_{10}^*)}. \tag{39}$$

Differentiating (6) with respect to τ_1, one can obtain

$$\left[\frac{d\lambda}{d\tau_1}\right]^{-1} = \frac{P_{4*}(\lambda)}{Q_{4*}(\lambda)} - \frac{\tau_1}{\lambda}, \tag{40}$$

where

$$\begin{aligned}
P_{4*}(\lambda) &= p_{40}(\lambda) + p_{41}(\lambda)\,e^{-\lambda\tau_1} + p_{42}(\lambda)\,e^{-\lambda\tau_2} \\
&+ p_{43}(\lambda)\,e^{-2\lambda\tau_2} + p_{44}(\lambda)\,e^{-\lambda(\tau_1+\tau_2)} \\
&+ p_{45}(\lambda)\,e^{-\lambda(\tau_1+2\tau_2)} + p_{46}(\lambda)\,e^{-3\lambda\tau_2} \\
&+ p_{47}(\lambda)\,e^{-\lambda(\tau_1+3\tau_2)}, \\
Q_{4*}(\lambda) &= q_{41}(\lambda)\,e^{-\lambda\tau_1} + q_{42}(\lambda)\,e^{-\lambda(\tau_1+\tau_2)} \\
&+ q_{43}(\lambda)\,e^{-\lambda(\tau_1+2\tau_2)} + q_{44}(\lambda)\,e^{-\lambda(\tau_1+3\tau_2)},
\end{aligned} \tag{41}$$

with

$$\begin{aligned}
p_{40}(\lambda) &= 5\lambda^4 + 4A_4\lambda^3 + 3A_3\lambda^2 + 2A_2\lambda + A_1, \\
p_{41}(\lambda) &= 4B_4\lambda^3 + 3B_3\lambda^2 + 2B_2\lambda + B_1, \\
p_{42}(\lambda) &= -\tau_2C_4\lambda^4 + (4C_4 - \tau_2C_3)\,\lambda^3 \\
&+ (3C_3 - \tau_2C_2)\,\lambda^2 + (2C_2 - \tau_2C_1)\,\lambda \\
&+ C_1 - \tau_2C_0, \\
p_{43}(\lambda) &= -2\tau_2D_3\lambda^3 + (3D_3 - 2\tau_2D_2)\,\lambda^2 \\
&+ (2D_2 - 2\tau_2D_1)\,\lambda + D_1 - 2\tau_2D_0, \\
p_{44}(\lambda) &= -\tau_2E_3\lambda^3 + (3E_3 - \tau_2E_2)\,\lambda^2 \\
&+ (2E_2 - \tau_2E_1)\,\lambda + E_1 - \tau_2E_0, \\
p_{45}(\lambda) &= -2\tau_2F_2\lambda^2 + 2(F_2 - \tau_2F_1)\,\lambda + F_1 - 2\tau_2F_0, \\
p_{46}(\lambda) &= -3\tau_2G_2\lambda^2 + (2G_2 - 3\tau_2G_1)\,\lambda + G_1 - 3\tau_2G_0, \\
p_{47}(\lambda) &= -2\tau_2H_2\lambda + H_1 - 3\tau_2H_0, \\
q_{41}(\lambda) &= B_4\lambda^5 + B_3\lambda^4 + B_2\lambda^3 + B_1\lambda^2 + B_0\lambda, \\
q_{42}(\lambda) &= E_3\lambda^4 + E_2\lambda^3 + E_1\lambda^2 + E_0\lambda, \\
q_{43}(\lambda) &= F_2\lambda^3 + F_1\lambda^2 + F_0\lambda, \\
q_{44}(\lambda) &= H_1\lambda^2 + H_0\lambda.
\end{aligned} \tag{42}$$

Define

$$\operatorname{Re}\left[\frac{d\lambda}{d\tau_1}\right]_{\lambda=i\omega_{10}^*}^{-1} = \frac{P_{4R}Q_{4R} + P_{4I}Q_{4I}}{Q_{4R}^2 + Q_{4I}^2}. \tag{43}$$

Obviously, if the condition

(H_{42}) $P_{4R}Q_{4R} + P_{4I}Q_{4I} \neq 0$ holds, then $\operatorname{Re}[d\lambda/d\tau_1]_{\lambda=i\omega_{10}^*}^{-1} \neq 0$.

Thus, we have the following based on the analysis above.

Theorem 3. *For system (4), let τ_{10}^* be specified by (39) and $\tau_2 \in (0, \tau_{20})$. If the conditions (H_{41})-(H_{42}) are satisfied, then $D_*(S_*, E_*, I_*, R_*, V_*)$ is asymptotically stable when $\tau_1 \in [0, \tau_{10}^*)$ and a Hopf bifurcation occurs at $D_*(S_*, E_*, I_*, R_*, V_*)$ when $\tau_1 = \tau_{10}^*$.*

Case 5 ($\tau_1 > 0$, $\tau_2 > 0$, and $\tau_1 \in (0, \tau_{10})$). Multiplying by $e^{\lambda\tau_2}$, (6) becomes

$$\begin{aligned}
&C_4\lambda^4 + C_3\lambda^3 + C_2\lambda^2 + C_1\lambda + C_0 \\
&+ \left(E_3\lambda^3 + E_2\lambda^2 + E_1\lambda + E_0\right)e^{-\lambda\tau_1} \\
&+ \left(\lambda^5 + A_4\lambda^4 + A_3\lambda^3 + A_2\lambda^2 + A_1\lambda + A_0\right)e^{\lambda\tau_2} \\
&+ \left(D_3\lambda^3 + D_2\lambda^2 + D_1\lambda + D_0\right)e^{-\lambda\tau_2}
\end{aligned}$$

$$+\left(B_4\lambda^4+B_3\lambda^3+B_2\lambda^2+B_1\lambda+B_0\right)e^{\lambda(\tau_2-\tau_1)}$$
$$+\left(F_2\lambda^2+F_1\lambda+F_0\right)e^{-\lambda(\tau_1+\tau_2)}. \tag{44}$$

Let $\lambda = i\omega_{2*}$ ($\omega_{2*} > 0$) be a root of (44). Then,

$$\begin{aligned}
&g_{51}(\omega_{2*})\cos\tau_2\omega_{2*} - g_{52}(\omega_{2*})\sin\tau_2\omega_{2*} + g_{53}(\omega_{2*}) \\
&\quad = h_{51}(\omega_{2*})\sin 2\tau_2\omega_{2*} + h_{52}(\omega_{2*})\cos 2\tau_2\omega_{2*}, \\
&g_{54}(\omega_{2*})\sin\tau_2\omega_{2*} + g_{55}(\omega_{2*})\cos\tau_2\omega_{2*} + g_{56}(\omega_{2*}) \\
&\quad = h_{51}(\omega_{2*})\cos 2\tau_2\omega_{2*} - h_{52}(\omega_{2*})\sin 2\tau_2\omega_{2*},
\end{aligned} \tag{45}$$

where

$$\begin{aligned}
&g_{51}(\omega_{2*}) \\
&\quad = A_4\omega_{2*}^4 - (A_2+D_2)\omega_{2*}^2 + A_0 + D_0 \\
&\qquad + \left(B_4\omega_{2*}^4 - (B_2+F_2)\omega_{2*}^2 + B_0 + F_0\right)\cos\tau_1\omega_{2*} \\
&\qquad + \left((B_1+F_1)\omega_{2*} - B_3\omega_{2*}^3\right)\sin\tau_1\omega_{2*}, \\
&g_{52}(\omega_{2*}) \\
&\quad = \omega_{2*}^5 - (A_3-D_3)\omega_{2*}^3 + (A_1-D_1)\omega_{2*} \\
&\qquad + \left((B_2-F_2)\omega_{2*}^2 - B_4\omega_{2*}^4 - B_0 + F_0\right)\sin\tau_1\omega_{2*} \\
&\qquad + \left((B_1-F_1)\omega_{2*} - B_3\omega_{2*}^3\right)\cos\tau_1\omega_{2*}, \\
&g_{53}(\omega_{2*}) \\
&\quad = C_4\omega_{2*}^4 - C_2\omega_{2*}^2 + C_0 \\
&\qquad + \left(E_1\omega_{2*} - E_3\omega_{2*}^3\right)\sin\tau_1\omega_{2*} \\
&\qquad + \left(E_0 - E_2\omega_{2*}^2\right)\cos\tau_1\omega_{2*}, \\
&g_{54}(\omega_{2*}) \\
&\quad = A_4\omega_{2*}^4 - (A_2-D_2)\omega_{2*}^2 + A_0 - D_0 \\
&\qquad + \left(B_4\omega_{2*}^4 - (B_2-F_2)\omega_{2*}^2 + B_0 - F_0\right)\cos\tau_1\omega_{2*} \\
&\qquad + \left((B_1-F_1)\omega_{2*} - B_3\omega_{2*}^3\right)\sin\tau_1\omega_{2*}, \\
&g_{55}(\omega_{2*}) \\
&\quad = \omega_{2*}^5 - (A_3+D_3)\omega_{2*}^3 + (A_1+D_1)\omega_{2*} \\
&\qquad + \left((B_2+F_2)\omega_{2*}^2 - B_4\omega_{2*}^2 - B_0 - F_0\right)\sin\tau_1\omega_{2*} \\
&\qquad + \left((B_1+F_1)\omega_{2*} - B_3\omega_{2*}^3\right)\cos\tau_1\omega_{2*}, \\
&g_{56}(\omega_{2*}) \\
&\quad = C_1\omega_{2*} - C_3\omega_{2*}^3 + \left(E_1\omega_{2*} - E_3\omega_{2*}^3\right)\cos\tau_1\omega_{2*} \\
&\qquad - \left(E_0 - E_2\omega_{2*}^2\right)\sin\tau_1\omega_{2*}, \\
&h_{51}(\omega_{2*}) \\
&\quad = G_1\omega_{2*} + H_0\sin\tau_1\omega_{2*} - H_1\omega_{2*}\cos\tau_1\omega_{2*}, \\
&h_{52}(\omega_{2*}) \\
&\quad = G_0 - G_2\omega_{2*}^2 - H_0\cos\tau_1\omega_{2*} - H_1\omega_{2*}\sin\tau_1\omega_{2*}.
\end{aligned} \tag{46}$$

Similar to Case 3, we can obtain the expressions of $\cos\tau_2\omega_{2*}$ and $\sin\tau_2\omega_{2*}$ when $\sin\tau_2\omega_{2*} = \sqrt{1-\cos^2\tau_2\omega_{2*}}$ and we denote $f_{51}(\omega_{2*}) = \cos\tau_2\omega_{2*}$ and $f_{52}(\omega_{2*}) = \sin\tau_2\omega_{2*}$. Then, one can obtain

$$f_{51}^2(\omega_{2*}) + f_{52}^2(\omega_{2*}) = 1. \tag{47}$$

In order to give the main results in this paper, we suppose that

(H_{51}) Equation (47) has at least one positive root.

If the condition (H_{51}) holds, then there exists $\omega_{21*} > 0$ such that (6) has a pair of purely imaginary roots $\pm i\omega_{21*}$. For ω_{21*}, we have

$$\tau_{21*} = \frac{1}{\omega_{21*}} \times \arccos f_{51}(\omega_{21*}). \tag{48}$$

Similarly, one can obtain expressions of $\cos\tau_2\omega_{2*}$ and $\sin\tau_2\omega_{2*}$ when $\sin\tau_2\omega_{2*} = -\sqrt{1-\cos^2\tau_2\omega_{2*}}$ and we denote $f_{53}(\omega_{2*}) = \cos\tau_2\omega_{2*}$ and $f_{54}(\omega_{2*}) = \sin\tau_2\omega_{2*}$. Then, one can obtain

$$f_{53}^2(\omega_{2*}) + f_{54}^2(\omega_{2*}) = 1. \tag{49}$$

If (49) has one positive root ω_{22*}, then (6) has roots $\pm i\omega_{22*}$. For ω_{22*}, we have

$$\tau_{22*} = \frac{1}{\omega_{22*}} \times \arccos f_{53}(\omega_{22*}). \tag{50}$$

Let

$$\tau_{20}^* = \min\{\tau_{21*}, \tau_{22*}\} \tag{51}$$

and $\pm i\omega_{20}^*$ be the roots of (6) with $\tau_2 = \tau_{20}^*$. Differentiating (6) with respect to τ_2, we get

$$\left[\frac{d\lambda}{d\tau_2}\right]^{-1} = \frac{P_{5*}(\lambda)}{Q_{5*}(\lambda)} - \frac{\tau_2}{\lambda}, \tag{52}$$

where

$$\begin{aligned}
P_{5*}(\lambda) &= p_{50}(\lambda) + p_{51}(\lambda)e^{-\lambda\tau_1} + p_{52}(\lambda)e^{-\lambda\tau_2} \\
&\quad + p_{53}(\lambda)e^{-2\lambda\tau_2} + p_{54}(\lambda)e^{-\lambda(\tau_1+\tau_2)} \\
&\quad + p_{55}(\lambda)e^{-(\lambda\tau_1+2\tau_2)} + p_{56}(\lambda)e^{-3\lambda\tau_2} \\
&\quad + p_{57}(\lambda)e^{-\lambda(\tau_1+3\tau_2)},
\end{aligned} \tag{53}$$

$$\begin{aligned}
Q_{5*}(\lambda) &= q_{51}(\lambda)e^{-\lambda\tau_2} + q_{52}(\lambda)e^{-\lambda(\tau_1+\tau_2)} \\
&\quad + q_{53}(\lambda)e^{-2\lambda\tau_2} + q_{54}(\lambda)e^{-\lambda(\tau_1+\tau_2)} \\
&\quad + q_{55}(\lambda)e^{-3\lambda\tau_2} + q_{56}(\lambda)e^{-\lambda(\tau_1+3\tau_2)},
\end{aligned}$$

with

$$
\begin{aligned}
p_{50}(\lambda) &= 5\lambda^4 + 4A_4\lambda^3 + 3A_3\lambda^2 + 2A_2\lambda + A_1, \\
p_{51}(\lambda) &= -\tau_1 B_4\lambda^4 + (4B_4 - \tau_1 - B_3)\lambda^3 \\
&\quad + (3B_3 - \tau_1 B_2)\lambda^2 + (2B_2 - \tau_1 B_1) + B_1 \\
&\quad - \tau_1 B_0, \\
p_{52}(\lambda) &= 4C_4\lambda^3 + 3C_3\lambda^2 + 2C_2\lambda + C_1, \\
p_{53}(\lambda) &= 3D_3\lambda^2 + 2D_2\lambda + D_1, \\
p_{54}(\lambda) &= -\tau_1 E_3\lambda^3 + (3E_3 - \tau_1 E_2)\lambda^2 \\
&\quad + (2E_2 - \tau_1 E_1)\lambda + E_1 - \tau_1 E_0, \\
p_{55}(\lambda) &= -\tau_1 F_2\lambda^2 + (2F_2 - \tau_1 F_1)\lambda + F_1 - \tau_1 F_0, \\
p_{56}(\lambda) &= 2G_2\lambda + G_1, \\
p_{57}(\lambda) &= -\tau_1 H_1\lambda + H_1 - \tau_1 - H_0, \\
q_{51}(\lambda) &= C_4\lambda^5 + C_3\lambda^4 + C_2\lambda^3 + C_1\lambda^2 + C_0\lambda, \\
q_{52}(\lambda) &= E_3\lambda^4 + E_2\lambda^3 + E_1\lambda^2 + E_0\lambda, \\
q_{53}(\lambda) &= 2\left(D_3\lambda^4 + D_2\lambda^3 + D_1\lambda^2 + D_0\lambda\right), \\
q_{54}(\lambda) &= 2\left(F_2\lambda^3 + F_1\lambda^2 + F_0\lambda\right), \\
q_{55}(\lambda) &= 3\left(G_2\lambda^3 + G_1\lambda^2 + G_0\lambda\right), \\
q_{56}(\lambda) &= 3\left(H_1\lambda^2 + H_0\lambda\right).
\end{aligned} \tag{54}
$$

Define

$$
\operatorname{Re}\left[\frac{d\lambda}{d\tau_2}\right]^{-1}_{\lambda=i\omega_{20}^*} = \frac{P_{5R}Q_{5R} + P_{5I}Q_{5I}}{Q_{5R}^2 + Q_{5I}^2}. \tag{55}
$$

Obviously, if the condition

(H_{52}) $P_{5R}Q_{5R} + P_{5I}Q_{5I} \neq 0$ holds, then $\operatorname{Re}[d\lambda/d\tau_2]^{-1}_{\lambda=i\omega_{20}^*} \neq 0$.

Thus, we have the following based on the analysis above.

Theorem 4. *For system (4), let τ_{20}^* be specified by (51) and $\tau_1 \in (0, \tau_{10})$. If the conditions (H_{51})-(H_{52}) are satisfied, then $D_*(S_*, E_*, I_*, R_*, V_*)$ is asymptotically stable when $\tau_2 \in [0, \tau_{20}^*)$ and a Hopf bifurcation occurs at $D_*(S_*, E_*, I_*, R_*, V_*)$ when $\tau_2 = \tau_{20}^*$.*

3. Properties of the Hopf Bifurcation

We will investigate properties of the Hopf bifurcation of system (4) with respect to τ_1 and $\tau_2 \in (0, \tau_{20})$ in this section. Let $\tau_{2*} < \tau_{10}^*$ with $\tau_{2*} \in (0, \tau_{20})$ and $\tau_1 = \tau_{10}^* + \mu$, $\mu \in R$. By the transformation $u_1(t) = S(t) - S_*$, $u_2(t) = E(t) - E_*$, $u_3(t) = I(t) - I_*$, $u_4(t) = R(t) - R_*$, $u_5(t) = V(t) - V_*$, and $t \to (t/\tau_1)$, system (4) can be written as

$$
\dot{u}(t) = L_\mu u_t + F(\mu, u_t), \tag{56}
$$

where $u_t = (u_1(t), u_2(t), u_3(t), u_4(t), u_5(t))^T \in C([-1,0], R^5)$, and

$$
\begin{aligned}
L_\mu\phi &= (\tau_0 + \mu) \\
&\quad \cdot \left(A_{\max}\phi(0) + B_{\max 1}\phi\left(-\frac{\tau_{2*}}{\tau_{10}^*}\right) + B_{\max 2}\phi(-1)\right), \\
F(\mu, \phi) &= (\tau_{10}^* + \mu)\begin{pmatrix} -\beta\phi_1(0)\phi_3(0) \\ \beta\phi_1(0)\phi_3(0) \\ 0 \\ 0 \\ 0 \end{pmatrix},
\end{aligned} \tag{57}
$$

where

$$
\begin{aligned}
A_{\max} &= \begin{pmatrix} a_{11} & 0 & a_{13} & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 0 \\ 0 & a_{32} & a_{33} & 0 & 0 \\ 0 & 0 & 0 & a_{44} & 0 \\ a_{51} & 0 & 0 & 0 & a_{55} \end{pmatrix}, \\
B_{\max 1} &= \begin{pmatrix} 0 & 0 & 0 & b_{14} & b_{15} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & b_{33} & 0 & 0 \\ 0 & 0 & b_{43} & b_{44} & 0 \\ 0 & 0 & 0 & 0 & b_{55} \end{pmatrix}, \\
B_{\max 2} &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & b_{22} & 0 & 0 & 0 \\ 0 & b_{32} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.
\end{aligned} \tag{58}
$$

According to the Riesz representation theorem, we know that there exists a 5×5 matrix function $\eta(\theta, \mu) : [-1, 0] \to R^{5\times 5}$ such that

$$
L_\mu\phi = \int_{-1}^{0} d\eta(\theta, \mu)\phi(\theta), \quad \phi \in C. \tag{59}
$$

In fact, we choose

$$
\eta(\theta, \mu) = \begin{cases} (\tau_{10}^* + \mu)(A_{\max} + B_{\max 1} + B_{\max 2}), & \theta = 0, \\ (\tau_{10}^* + \mu)(B_{\max 1} + B_{\max 2}), & \theta \in \left[-\dfrac{\tau_{2*}}{\tau_{10}^*}, 0\right), \\ (\tau_{10}^* + \mu)B_{\max 2}, & \theta \in \left(-1, -\dfrac{\tau_{2*}}{\tau_{10}^*}\right), \\ 0, & \theta = -1. \end{cases} \tag{60}
$$

For $\phi \in C([-1,0], R^5)$, we define

$$A(\mu)\phi = \begin{cases} \dfrac{d\phi(\theta)}{d\theta}, & -1 \le \theta < 0, \\ \displaystyle\int_{-1}^{0} d\eta(\theta,\mu)\phi(\theta), & \theta = 0, \end{cases}$$
$$R(\mu)\phi = \begin{cases} 0, & -1 \le \theta < 0, \\ F(\mu,\phi), & \theta = 0. \end{cases} \quad (61)$$

Then, system (56) is equivalent to

$$\dot{u}(t) = A(\mu)u_t + R(\mu)u_t, \quad (62)$$

where $u_t(\theta) = u(t+\theta)$ for $\theta \in [-1,0]$.

Then, we define A^*:

$$A^*(\varphi) = \begin{cases} -\dfrac{d\varphi(s)}{ds}, & 0 < s \le 1, \\ \displaystyle\int_{-1}^{0} d\eta^T(s,0)\varphi(-s), & s = 0, \end{cases} \quad (63)$$

and a bilinear form

$$\langle\varphi(s),\phi(\theta)\rangle = \overline{\varphi}(0)\phi(0) - \int_{\theta=-1}^{0}\int_{\xi=0}^{\theta}\overline{\varphi}(\xi-\theta)\,d\eta(\theta)\phi(\xi)\,d\xi, \quad (64)$$

where $\eta(\theta) = \eta(\theta,0)$.

Let $\rho(\theta) = (1,\rho_2,\rho_3,\rho_4,\rho_5)^T e^{i\omega_{10}^*\tau_{10}^*\theta}$ be the eigenvector of $A(0)$ with $+i\omega_{10}^*\tau_{10}^*$ and let $\rho^*(s) = (1/K)(1,\rho_2^*,\rho_3^*,\rho_4^*,\rho_5^*)e^{i\omega_{10}^*\tau_{10}^* s}$ be the eigenvector of $A^*(0)$ with $-i\omega_{10}^*\tau_{10}^*$. Then, we obtain

$$\begin{aligned}
\rho_2 &= \frac{a_{21}\left(i\omega_{10}^* - a_{33} - b_{33}e^{-i\tau_{2*}\omega_{10}^*}\right)}{\left(i\omega_{10}^* - a_{22} - b_{22}e^{-i\tau_{10}^*\omega_{10}^*}\right)\left(i\omega_{10}^* - a_{33} - b_{33}e^{-\tau_{2*}\omega_{10}^*}\right) - a_{23}b_{32}e^{-i\tau_{10}^*\omega_{10}^*}}, \\
\rho_3 &= \frac{b_{32}e^{-i\tau_{10}^*\omega_{10}^*}}{i\omega_{10}^* - a_{33} - b_{33}e^{-\tau_{2*}\omega_{10}^*}}\rho_2, \\
\rho_4 &= \frac{b_{43}e^{-i\tau_{2*}\omega_{10}^*}}{i\omega_{10}^* - a_{44} - b_{44}e^{-i\tau_{2*}\omega_{10}^*}}\rho_3, \\
\rho_5 &= \frac{a_{51}}{\omega_{10}^* - a_{55} - b_{55}e^{-\tau_{2*}\omega_{10}^*}}, \\
\rho_2^* &= -\frac{i\omega_{10}^* + a_{11}}{a_{21}} + \frac{a_{51}b_{15}e^{i\tau_{2*}\omega_{10}^*}}{a_{21}\left(\omega_{10}^* + a_{55} + b_{55}e^{i\tau_{2*}\omega_{10}^*}\right)}, \\
\rho_3^* &= \frac{\left(i\omega_{10}^* + a_{11}\right)\left(i\omega_{10}^* + a_{22} + b_{22}e^{-i\tau_{10}^*}\omega_{10}^*\right)}{a_{21}b_{32}e^{-i\tau_{10}^*\omega_{10}^*}} - \frac{a_{51}b_{15}e^{i\tau_{2*}\omega_{10}^*}\left(i\omega_{10}^* + a_{22} + b_{22}e^{-i\tau_{10}^*\omega_{10}^*}\right)}{a_{21}b_{32}e^{-i\tau_{10}^*\omega_{10}^*}\left(i\omega_{10}^* + a_{55} + b_{55}e^{i\tau_{2*}\omega_{10}^*}\right)}, \\
\rho_4^* &= -\frac{b_{14}e^{i\tau_{2*}\omega_{10}^*}}{i\omega_{10}^* + a_{44} + b_{44}e^{i\tau_{2*}\omega_{10}^*}}, \\
\rho_5^* &= -\frac{b_{15}e^{-i\tau_{2*}\omega_{10}^*}}{i\omega_{140}^* + a_{55} + b_{55}e^{-i\tau_{2*}\omega_{10}^*}}.
\end{aligned} \quad (65)$$

From (64), we obtain

$$\begin{aligned}
\langle\rho^*,\rho\rangle = \frac{1}{\overline{K}}\Big[&1 + \rho_2\overline{\rho}_2^* + \rho_3\overline{\rho}_3^* + \rho_4\overline{\rho}_4^* + \rho_5\overline{\rho}_5^* \\
&+ \tau_{10}^* e^{-i\tau_{10}^*\omega_{10}^*}\rho_2\left(b_{22}\overline{\rho_2^*} + b_{32}\overline{\rho_3^*}\right) \\
&+ \tau_{2*}e^{-i\tau_{2*}\omega_{10}^*}\left(\rho_3\left(b_{33}\overline{\rho}_3^* + b_{43}\overline{\rho}_4^*\right)\right. \\
&\left.+ \rho_4\left(b_{14} + b_{44}\overline{\rho}_4^*\right) + \rho_5\left(b_{15} + b_{55}\overline{\rho}_5^*\right)\right)\Big].
\end{aligned} \quad (66)$$

Then, we can choose

$$\begin{aligned}
\overline{K} = 1 &+ \rho_2\overline{\rho}_2^* + \rho_3\overline{\rho}_3^* + \rho_4\overline{\rho}_4^* + \rho_5\overline{\rho}_5^* \\
&+ \tau_{10}^* e^{-i\tau_{10}^*\omega_{10}^*}\rho_2\left(b_{22}\overline{\rho_2^*} + b_{32}\overline{\rho_3^*}\right) \\
&+ \tau_{2*}e^{-i\tau_{2*}\omega_{10}^*}\left(\rho_3\left(b_{33}\overline{\rho}_3^* + b_{43}\overline{\rho}_4^*\right)\right. \\
&\left.+ \rho_4\left(b_{14} + b_{44}\overline{\rho}_4^*\right) + \rho_5\left(b_{15} + b_{55}\overline{\rho}_5^*\right)\right).
\end{aligned} \quad (67)$$

such that $\langle\rho^*,\rho\rangle = 1$ and $\langle\rho^*,\overline{\rho}\rangle = 0$.

Following the same algorithms introduced in [17] and the similar computation used in [18–22], we obtain

$$
\begin{aligned}
g_{20} &= 2\beta\tau_{10}^{*}\overline{K}\rho_3\left(\overline{\rho}_2^{*}-1\right),\\
g_{11} &= \beta\tau_{10}^{*}\overline{K}\left(\rho_3+\overline{\rho}_3\right)\left(\overline{\rho}_2^{*}-1\right),\\
g_{02} &= 2\beta\tau_{10}^{*}\overline{K}\,\overline{\rho}_3\left(\overline{\rho}_2^{*}-1\right),\\
g_{21} &= 2\beta\tau_{10}^{*}\overline{K}\left(\overline{\rho}_2^{*}-1\right)\Big(W_{11}^{(1)}(0)\,\rho_3+\frac{1}{2}W_{20}^{(1)}(0)\,\overline{\rho}_3\\
&\quad+W_{11}^{(3)}(0)+\frac{1}{2}W_{20}^{(3)}(0)\Big),
\end{aligned}
\tag{68}
$$

with

$$
\begin{aligned}
W_{20}(\theta) &= \frac{ig_{20}\rho(0)}{\tau_{10}^{*}\omega_{10}^{*}}e^{i\tau_{10}^{*}\omega_{10}^{*}\theta}+\frac{i\overline{g}_{02}\overline{\rho}(0)}{3\tau_{10}^{*}\omega_{10}^{*}}e^{-i\tau_{10}^{*}\omega_{10}^{*}\theta}+E_1e^{2i\tau_{10}^{*}\omega_{10}^{*}\theta},\\
W_{11}(\theta) &= -\frac{ig_{11}\rho(0)}{\tau_{10}^{*}\omega_{10}^{*}}e^{i\tau_{10}^{*}\omega_{10}^{*}\theta}+\frac{i\overline{g}_{11}\overline{\rho}(0)}{\tau_{10}^{*}\omega_{10}^{*}}e^{-i\tau_{10}^{*}\omega_{10}^{*}\theta}+E_2,\\
E_1 &= 2\begin{pmatrix}
a_{11}' & 0 & -a_{13} & -b_{14}e^{-2i\tau_{2*}\omega_{10}^{*}} & -b_{15}e^{-2i\tau_{2*}\omega_{10}^{*}}\\
-a_{21} & a_{22}' & -a_{23} & 0 & 0\\
0 & -a_{32}-b_{32}e^{-2i\tau_{10}^{*}\omega_{10}^{*}} & a_{33}' & 0 & 0\\
0 & 0 & -b_{43}e^{-2i\tau_{2*}\omega_{10}^{*}} & a_{44}' & 0\\
-a_{51} & 0 & 0 & 0 & a_{55}'
\end{pmatrix}^{-1}
\begin{pmatrix}-\beta\rho_3\\ \beta\rho_3\\ 0\\ 0\\ 0\end{pmatrix},\\
E_2 &= -\begin{pmatrix}
a_{11} & 0 & a_{13} & b_{14} & b_{15}\\
a_{21} & a_{22}+b_{22} & a_{23} & 0 & 0\\
0 & a_{32}+b_{32} & a_{33}+b_{33} & 0 & 0\\
0 & 0 & b_{43} & a_{44}+b_{44} & 0\\
a_{51} & 0 & 0 & 0 & a_{55}+b_{55}
\end{pmatrix}^{-1}
\begin{pmatrix}-\beta\left(\rho_3+\overline{\rho}_3\right)\\ \beta\left(\rho_3+\overline{\rho}_3\right)\\ 0\\ 0\\ 0\end{pmatrix},
\end{aligned}
\tag{69}
$$

where

$$
\begin{aligned}
a_{11}' &= 2i\omega_{10}^{*}-a_{11},\\
a_{22}' &= 2i\omega_{10}^{*}-a_{22}-b_{22}e^{-2i\omega_{10}^{*}\tau_{10}^{*}},\\
a_{33}' &= 2i\omega_{10}^{*}-a_{33}-b_{33}e^{-2i\omega_{10}^{*}\tau_{2*}},\\
a_{44}' &= 2i\omega_{10}^{*}-a_{44}-b_{44}e^{-2i\omega_{10}^{*}\tau_{2*}},\\
a_{55}' &= 2i\omega_{10}^{*}-a_{55}-b_{55}e^{-2i\omega_{10}^{*}\tau_{2*}}.
\end{aligned}
\tag{70}
$$

Consequently, we can compute the following values:

$$
\begin{aligned}
C_1(0) &= \frac{i}{2\tau_{10}^{*}\omega_{10}^{*}}\left(g_{11}g_{20}-2\left|g_{11}\right|^2-\frac{\left|g_{02}\right|^2}{3}\right)+\frac{g_{21}}{2},\\
\mu_2 &= -\frac{\operatorname{Re}\{C_1(0)\}}{\operatorname{Re}\{\lambda'(\tau_{10}^{*})\}},\\
\beta_2 &= 2\operatorname{Re}\{C_1(0)\},\\
T_2 &= -\frac{\operatorname{Im}\{C_1(0)\}+\mu_2\operatorname{Im}\{\lambda'(\tau_{10}^{*})\}}{\tau_{10}^{*}\omega_{10}^{*}}.
\end{aligned}
\tag{71}
$$

According to the analysis about properties of the Hopf bifurcation in [17], we have the following for system (4).

Theorem 5. *Let μ_2, β_2, and T_2 be specified by (71). μ_2 determines the direction of the Hopf bifurcation (supercritical if $\mu_2 > 0$, subcritical if $\mu_2 < 0$); β_2 determines the stability of the bifurcating periodic solutions (stable if $\beta < 0$, unstable if $\beta > 0$); $T_2 > 0$ determines the period of the bifurcating periodic solutions (increasing if $T_2 > 0$, decreasing if $T_2 < 0$).*

4. A Numerical Example

In this section we give a numerical example of system (4) to validate the analysis above. By extracting the same values from [11], we get the following system:

$$
\begin{aligned}
\frac{dS(t)}{dt} &= 2-0.3S(t)I(t)-0.323S(t)+0.3R(t-\tau_2)\\
&\quad+0.06V(t-\tau_2),\\
\frac{dE(t)}{dt} &= 0.3S(t)I(t)-0.003E(t)-0.25E(t-\tau_1),\\
\frac{dI(t)}{dt} &= 0.25E(t-\tau_1)-0.073I(t)-0.4I(t-\tau_2),
\end{aligned}
$$

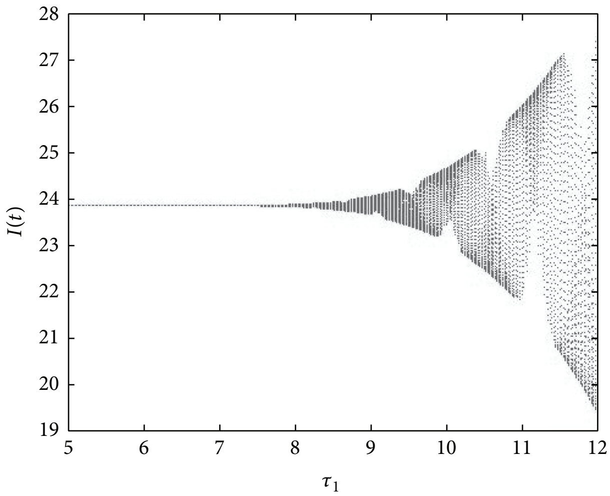

Figure 1: Bifurcation regarding τ_1 with $\tau_2 = 0$.

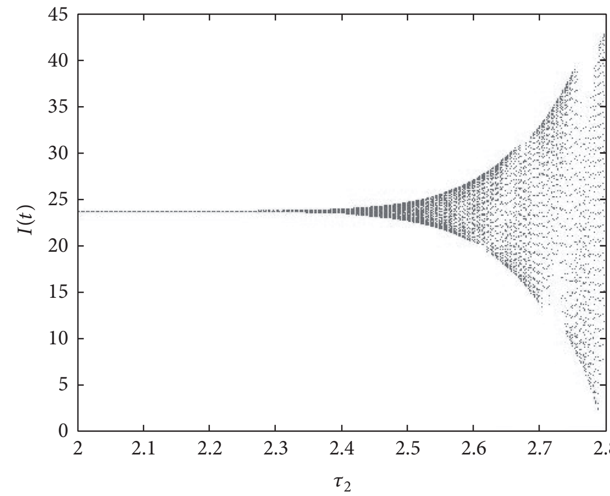

Figure 2: Bifurcation regarding τ_2 with $\tau_1 = 0$.

$$\begin{aligned}\frac{dR(t)}{dt} &= 0.4I(t-\tau_2) - 0.003R(t) - 0.3R(t-\tau_2),\\ \frac{dV(t)}{dt} &= 0.32S(t) - 0.003V(t) - 0.06V(t-\tau_2).\end{aligned} \tag{72}$$

It is easy to get $R_0 = 4.8750 > 1$ and system (72) has a unique positive equilibrium $D_*(1.5956, 44.7772, 23.6666, 31.2430, 8.1046)$. Also we get that the condition (H_1) holds.

Firstly, we can obtain $\omega_{10} = 0.2162$ and $\tau_{10} = 7.9833$ by some computations. According to Theorem 1, we can conclude that the positive equilibrium $D_*(1.5956, 44.7772, 23.6666, 31.2430, 8.1046)$ is asymptotically stable for $\tau_1 < \tau_{10} = 7.9833$. In this case, the propagation of worms can be predicted and controlled easily. However, once the value of τ_1 is above $\tau_{10} = 7.9833$, a Hopf bifurcation will occur which implies that the propagation of worms will be out of control. This property can be illustrated by the bifurcation diagram with respect to τ_1 in Figure 1. Similarly, we have $\omega_{20} = 0.4883$ and $\tau_{20} = 2.4273$. The corresponding bifurcation diagram with respect to τ_2 is as shown in Figure 2.

Secondly, we may obtain $\omega_{10}^* = 1.8164$ and $\tau_{10}^* = 3.5477$ for $\tau_1 > 0$ and $\tau_2 = 1.5 \in (0, \tau_{20})$ by computing. It follows from Theorem 3 that when $\tau_1 = \tau_{10}^* = 3.5477$, a Hopf bifurcation occurs, which can be shown by the bifurcation diagram regarding τ_1 and $\tau_2 = 1.5$ in Figure 3. Moreover, we have $C_1(0) = -1.2298 + 3.8550i$, $\mu_2 = 455.4815 > 0$, $\beta_2 = -2.4596 < 0$, and $T_2 = -22.5006 < 0$. Thus, It follows from Theorem 5 that the Hopf bifurcation is supercritical and the bifurcating periodic solutions are stable and decreasing. Since the bifurcating periodic solutions are stable, the numbers of every class of nodes in system (72) may coexist in an oscillatory mode, which is not welcome in networks.

Lastly, by computing, we obtain $\omega_{20}^* = 2.0979$ and $\tau_{20}^* = 1.5536$ for $\tau_2 > 0$ and $\tau_1 = 3.5 \in (0, \tau_{10})$. By Theorem 4, we know that the positive equilibrium $D_*(1.5956, 44.7772, 23.6666, 31.2430, 8.1046)$ is asymptotically stable for $\tau_2 < \tau_{20}^*$, which makes it possible for the propagation of worms to be predicted and controlled by taking effective measures. However, the positive equilibrium $D_*(1.5956, 44.7772, 23.6666, 31.2430, 8.1046)$ loses stability and a Hopf bifurcation occurs, when τ_2 passes through $\tau_{20}^* = 1.5536$. In this case, it is difficult to take measures to control the propagation of worms. The Hopf bifurcation phenomenon can be illustrated by Figure 4.

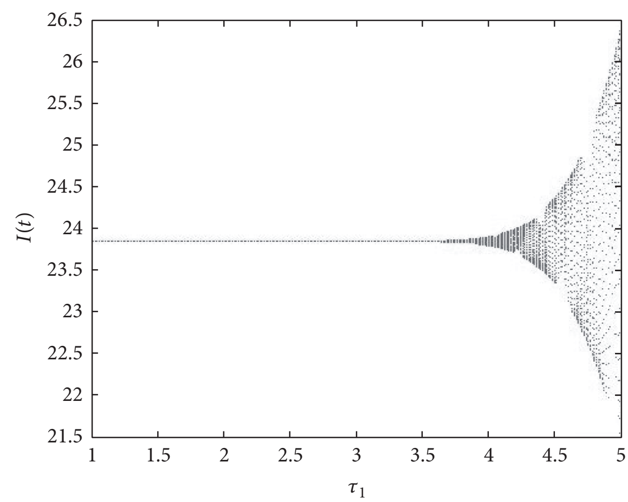

Figure 3: Bifurcation regarding τ_1 with $\tau_2 = 1.5$.

5. Conclusions

In the present paper, the generalization of the delayed SEIRS-V model describing worms spreading in a wireless sensor network investigated in [11] by inserting the latent period delay in the exposed sensor nodes has been considered. We find that not only τ_1 but also τ_2 can influence stability of system (4) and make system (4) undergo a Hopf bifurcation under some certain conditions. It has been shown that characteristics of the propagation of worms in system (4) can be easily predicted and eliminated when the value of delay is

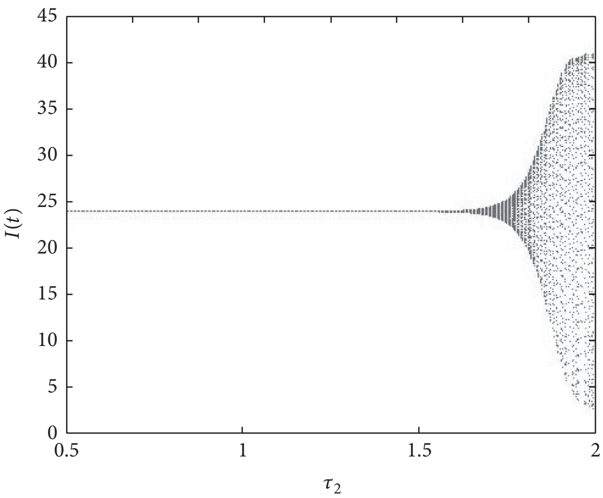

FIGURE 4: Bifurcation regarding τ_2 with $\tau_1 = 3.5$.

below the corresponding critical value and the propagation of worms in system (4) may be out of control once the delay passes through the corresponding critical value. Thus, we can conclude that the worms propagation can be controlled by postponing occurrence of a Hopf bifurcation. Subsequently, we use the normal form approach theory and center manifold theory introduced in [17] to deal with properties of the Hopf bifurcation when $\tau_1 > 0$ and $\tau_2 \in (0, \tau_{20})$.

According to the numerical example, we can conclude that the dynamics of system (4) is more complicated than that of the system considered in [11]. However, it should be pointed out that we assume that the exposed sensor nodes can not infect other nodes, which is not consistent with reality, because the exposed nodes can also infect other nodes through vulnerability seeking or other methods. Consequently, it is more realistic to investigate dynamics of the proposed model in this paper with graded infection rate. We leave this as our future work.

Competing Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

This research was supported by Natural Science Foundation of Anhui Province (nos. 1608085QF145 and 1608085QF151) and Natural Science Foundation of the Higher Education Institutions of Anhui Province (no. KJ2015A144).

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