Hindawi
Mathematical Problems in Engineering
Volume 2018, Article ID 1065451, 10 pages
https://doi.org/10.1155/2018/1065451

Research Article

The Robust Consensus of a Noisy Deffuant-Weisbuch Model

Jiangbo Zhang [1] and Yiyi Zhao [2]

[1] *School of Science, Southwest Petroleum University, Chengdu 610500, China*
[2] *School of Business Administration, Southwestern University of Finance and Economics, Chengdu 611130, China*

Correspondence should be addressed to Jiangbo Zhang; eridalin@163.com

Received 13 September 2018; Revised 11 November 2018; Accepted 7 December 2018; Published 30 December 2018

Academic Editor: Sergey Dashkovskiy

We construct a new opinion formation of the Deffuant-Weisbuch model with the interference of the outer noise, where there are finite n agents and the evolution is discrete-time. The opinion interaction occurs by one randomly chosen pair at each time step. The difference to the original Deffuant-Weisbuch model is that communications of any selected pairs will be affected by noises. The aim of this paper is to study the robust consensus of this noisy Deffuant-Weisbuch model. We first define the noise strength as the maximum noise absolute value. We will then show that when the noise strength is less than a certain threshold, this noisy model will achieve $T-$robust consensus when t is sufficiently large; next we prove that the noisy model achieves robust consensus with a positive probability; finally, we demonstrate these results and provide numerical relations among the noise strength and some model parameters.

1. Introduction

For grasping a deeper understanding of human social cognition and behaviours, sociologists and psychologists had been interested in understanding opinion formations and dynamics phenomena. This study had become of great interest in physics in the last decades [1, 2]. In a social system, the character of their mutual opinion interactions is one of the main research points. Sorts of interaction methods, the voter method [3], the network method [4–8], the influence method [9], the multilevel method [10, 11], and the self-organization method [12], are proposed. In the field of studying the interactions among agents, one general opinion update rule is realized by averaging neighbors opinions. Moreover, an individual often interacts with only those individuals whose opinions are close enough to his or her own. This update rule comes from a kind of social psychological phenomenon, *selective exposure*, a psychological concept broadly defined as "behaviors that bring the communication content within reach of ones sensory apparatus." These opinion models are called bounded confidence opinion models [13].

The most famous bounded confidence models are Hegselmann-Krause (HK) model [14] and Deffuant-Weisbuch (DW) model [15]. The fundamental difference between both models is materialized in the different definitions. In the DW model, two randomly chosen individuals meet and a pairwise averaging is implemented, while in the HK model, the communication takes place in more large groups and individuals move their own opinions to the average opinion of all individuals which lies in the area of confidence. Although opinion update rules are much different, it has been well established that they always approach to a limited state in which either perfect consensus is reached or the population splits into a set of opinion clusters with each of them holding the same opinion [16–18].

However, in real social phenomena, opinion agreement among individuals in a society is quite rare, whereas opinion disagreement is quite normal [19]. Opinion disagreement phenomena can be roughly classified as fragmentation and fluctuation. In fact, fluctuation had been studied in some physical areas widely such as electroacoustics [20], through simulations and calculations of the random variable variance [21].

For one special case of opinion fluctuation phenomena, some scholars [22–24] considered the existence of random noise among people communications. In detailed, [22, 23] studied the DW model for continuous-opinion dynamics with the influence of noises, where individuals are given the

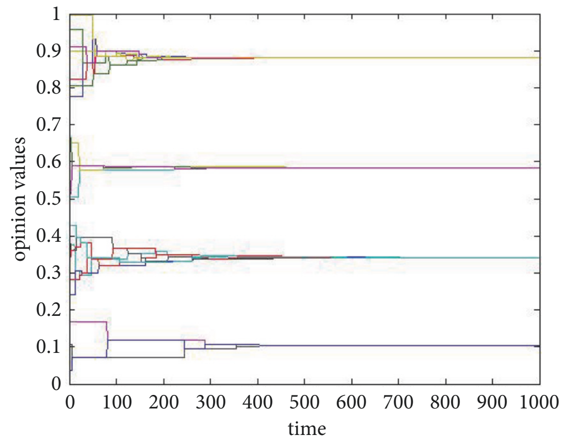


FIGURE 1: Opinion evolution of 20 agents for (1), where $n = 20$ and $\eta = 0.2$.

opportunity to change their opinion, with a given probability, to a randomly selected opinion inside the whole opinion space. Reference [24] studied the effects of noise in the continuous-time HK model and obtains approximate analytical results for critical conditions of opinion cluster formation. Reference [25] studied the noisy HK model and obtain that opinions reach robust consensus when the confidence bound satisfies a certain condition. Reference [26] demonstrated the HK model with the influence of opinion leaders and the noises.

Different from theirs, especially different from the probability's change in the noisy environment in [22, 23, 25, 26], we consider the opinion values for the DW model. Opinions will vibrate because of the existence of noises. Our main results are as follows:

(i) We propose a noisy DW model, where both selected agents are affected by the external noise if their distance is not larger than the confidence bound. Specially, the noises are confined in a certain interval;

(ii) We prove the robust consensus of this noisy DW model holds with a positive probability, when the noisy strength is small sufficiently. Besides, we provide an upper bound of the noise strength less which agent opinions will reach robust consensus;

(iii) The threshold of the noise strength and some relations among model parameters are demonstrated in finally.

This paper is organized as follows. Section 2 presents the original DW model, and then Section 3 gives a class of noisy DW models, and shows some basic facts. Following that, Section 4 shows that the noisy DW model achieves robust consensus when the noisy strength is sufficiently small and with a positive probability. Section 5 demonstrates how the opinion ranges change and compares the numerical relation among the threshold value of noise strength and some model parameters. Finally, Section 6 gives the conclusions.

2. DW Models

2.1. Review of the Original DW Model. The original DW model was introduced as a nonlinear extension of previous models of social influence [15]. We consider the discrete-time continuous opinion DW model with n individuals. The opinion $x_i(t)$ on a given topic is a real variable in the interval $[0, 1]$. We also assume that $x_i(0)$ for $i \in \mathscr{V} = \{1, 2, \dots, n\}$ are randomly distributed in this interval. Dynamics is introduced to reflect that individuals interact, discuss, and modify their opinions.

(1) At time t, a pair of agents (i, j) are equal randomly chosen;

(2) If their opinions satisfy $|x_i(t) - x_i(t)| \geq \eta$, then both keep unchanged;

(3) If their opinions satisfy $|x_i(t) - x_i(t)| < \eta$, then the averaged opinion of both is their opinion at next time $t + 1$;

(4) All others keep unchanged.

This rule can be rewritten as

$$\begin{aligned} x_i(t+1) &= x_i(t) + \frac{1}{2}\mathbb{1}_{\{|x_j(t)-x_i(t)|<\eta\}}\left(x_j(t) - x_i(t)\right), \\ x_j(t+1) &= x_j(t) + \frac{1}{2}\mathbb{1}_{\{|x_j(t)-x_i(t)|<\eta\}}\left(x_i(t) - x_j(t)\right), \\ x_k(t+1) &= x_k(t), \quad k \neq i, j. \end{aligned} \tag{1}$$

where $\mathbb{1}_S$ is the indicator function: $\mathbb{1}_S = 1$ if S is true and $\mathbb{1}_S = 0$ otherwise ([27]).

The theoretical results on this original DW model shows that agent opinions will aggregate into several limits a.s. An opinion evolution example based on the original DW model is shown in Figure 1.

2.2. The Noisy DW Model. In this section we propose a noisy DW model. Compared to the original DW model, any communicated agents are affected by noises, and the model is

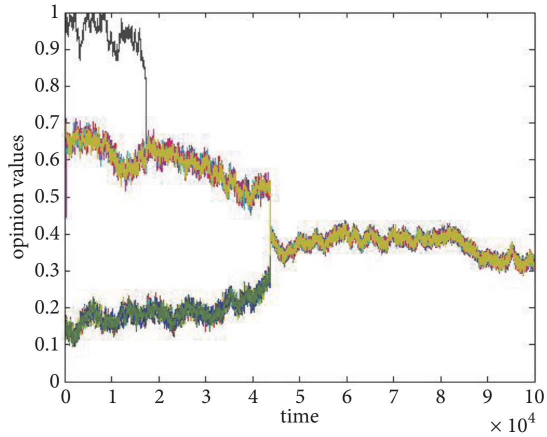


Figure 2: Opinion evolution of 20 agents for (3), where $n = 20$, $\eta = 0.2$, and $\delta = 0.01$.

$$\begin{aligned} x_i(t+1) &= x_i(t) + \mathbb{1}_{\{|x_j(t)-x_i(t)|<\eta\}}\left[\frac{1}{2}\left(x_j(t)-x_i(t)\right)+\xi_i(t+1)\right], \\ x_j(t+1) &= x_j(t) + \mathbb{1}_{\{|x_j(t)-x_i(t)|<\eta\}}\left[\frac{1}{2}\left(x_i(t)-x_j(t)\right)+\xi_j(t+1)\right], \\ x_k(t+1) &= x_k(t), \quad k \neq i, j, \end{aligned} \tag{2}$$

where $\{\xi_i(t)\}$ is independent bounded noise, satisfying

(i) $|\xi_i(t)| \le \delta$, a.s., $\delta > 0$ is the noisy strength;

(ii) $E[\xi_i(t)] = 0$;

(iii) $E[\xi_i^2(t)] \ge c\delta^2$, $(c \in (0, 1])$.

When $\delta = 0$, the model (2) is degenerated into the original DW model (1).

By the constraint of that $x_i(t)$ locating in $[0, 1]$, we adjust the model (2) into

$$\begin{aligned} x_i(t+1) &= \mathbb{1}_{\{\overline{x}_i(t)>1\}} + \overline{x}_i(t)\cdot \mathbb{1}_{\{\overline{x}_i(t)\in[0,1]\}} + 0 \cdot \mathbb{1}_{\{\overline{x}_i(t)<0\}}, \\ x_j(t+1) &= \mathbb{1}_{\{\overline{x}_j(t)>1\}} + \overline{x}_j(t)\cdot \mathbb{1}_{\{\overline{x}_j(t)\in[0,1]\}} + 0 \cdot \mathbb{1}_{\{\overline{x}_j(t)<0\}}, \\ x_k(t+1) &= x_k(t), \quad k \neq i, j, \end{aligned} \tag{3}$$

where the selection pair S_t is (i, j) at time t ($S_t = (i, j)$) and

$$\overline{x}_i(t) = x_i(t) + \mathbb{1}_{\{|x_j(t)-x_i(t)|<\eta\}}\left[\frac{1}{2}\left(x_j(t)-x_i(t)\right)+\xi_i(t+1)\right]. \tag{4}$$

An opinion evolution example based on the noisy DW model is shown in Figure 2. Compared with Figure 1, it seems that noises promote opinion aggregations and the aggregated group will be not divided in a certain condition.

3. Fundamental Definitions and Lemmas

In this section, we will give our definitions and some basic lemmas. For the original DW model (1), we first provide the following basic result [18].

Lemma 1. *For the original DW model (1), one of the following two results holds almost surely for any $i, j \in \mathscr{V}$:*

(i) $\lim_{t\to\infty}|x_i(t) - x_j(t)| = 0$,

(ii) $\lim_{t\to\infty}|x_i(t) - x_j(t)| > \eta$.

Remark 2. If $\lim_{t\to\infty}|x_i(t) - x_j(t)| > \eta$, then at last opinions will be divided into several clusters. Generally, the cluster number varies as model parameters, agent number, confidence bound, selection sequences, etc. change. It is found that when the confidence bound increases, the cluster number decreases a.s. until opinions reach consensus a.s., which can be defined as $\lim_{t\to\infty}x_i(t) = x^*$, $\forall i \in \mathscr{V}$ a.s.

However, for the noisy DW model, opinion consensus becomes extravagant hopes. Figure 2 shows that agent opinions always fluctuate before any large termination time. We provide the following definition, where the *robust consensus* is the main role in this work.

Definition 3. The opinion state $\{x_i(t), i \in \mathscr{V}\}$ is said to be fluctuating a.s. if $\mathbf{P}(\overline{\lim}_{t\to\infty} d(t) > 0) = 1$, where

$$d(t) = \max_{i,j\in\mathscr{V}}\left|x_i(t) - x_j(t)\right| \tag{5}$$

is the *opinion range* at time t.

Note that $d(t)$ also play the role of "aggregation degree" in social opinion system for one topic. Similarly, we can define $d_K(t) = \max_{i,j\in K}|x_i(t) - x_j(t)|$, where K is a subset of $\mathscr{V}$. Next, to study opinion fluctuation strength of the noisy DW model (3), we provide the definition of *robust consensus* as follows [28].

Definition 4. Given the initial opinions $x_i(0) \in [0,1]$ and the confidence bound $\eta \in (0,1)$, if there exists $T^* > 0$ and $T > 0$ and

$$d(t) \leq f(\delta) \tag{6}$$

holds a.s., where $T^* \leq t \leq T^* + T$, then we call the noisy DW model $T-$*robust consensus*. Generally, for any $t \geq T^*$, if

$$d(t) \leq f(\delta) \tag{7}$$

holds a.s., then we call the noisy DW model achieves *robust consensus*. Here $f : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$, $f(0) = 0$, and $f(\delta)$ approaches 0 when $\delta \longrightarrow 0$.

4. Robust Consensus of the Noisy DW Model

In this section, we will prove the *robust consensus* of the noisy DW model (3).

We first demonstrate how the fluctuation strength $d(t)$ changes as t increases (Figure 3). This example shows that when $t \geq 4 \times 10^4$, $d(t) < 0.2 = \eta$ always holds until the termination time.

Secondly, we will provide some lemmas for understanding how the opinions change for the noisy DW model (3).

Lemma 5. *If $d(0) \leq \eta$ and $\delta < (1/4)\eta$, then the noisy DW model (3) achieves $T-$*robust consensus.

Proof. We can prove it by the mathematical induction method.

[1] For $n = 2$, if $d(0) < \eta$ and $\delta < (1/4)\eta$, then

$$\begin{aligned} d(1) &= \left|x_1(1) - x_2(1)\right| \\ &\leq \left|\frac{1}{2}\left(x_1(0) + x_2(0) - x_2(0) - x_1(0)\right)\right| \\ &\quad + \left|\xi_1(0) - \xi_2(0)\right| \leq 2\delta < \eta \end{aligned} \tag{8}$$

holds if $S_0 = (1,2)$. Similarly, $d(t) < 2\delta$ holds for any $t \in \mathbb{N}$ if $S_k = (1,2)$, $k \leq t-1$. By the random of noise, $\{t_k : d(t_k) \leq 2\delta\}$ exists a.s. Defining $f(\delta) = 2\delta$, then the conclusion holds obviously.

[2] For $n = 3$, we assume $d(0) < \eta$ and $\delta < (1/4)\eta$. Without loss of generality, we set $x_1(0) \leq x_2(0) \leq x_3(0)$. If agent 1 and agent 2 are selected at time 0, then

$$\begin{aligned} &\left|x_1(1) - x_3(1)\right| \\ &\quad = \left|\frac{1}{2}\left(x_1(0) + x_2(0)\right) - x_3(0) + \xi_1(0)\right| \leq \eta + \delta \end{aligned} \tag{9}$$

while $|x_1(1) - x_2(1)| \leq 2\delta < \eta$ and $|x_2(1) - x_3(1)| \leq \eta + \delta$. Thus, if agent 1 and agent 2 always select each other and $\xi_1(t) + \xi_2(t) < 0$ for $t \in \mathbb{N}$, then there exists a finite time t_0 a.s., such that when $t < t_0$, $d(t) \leq \eta$. By the Borel-Cantelli (B-C) lemma [29], we can get that $P\{\bigcap_{t=1}^{\inf}\{\xi_1(t) + \xi_2(t) < 0\}\} = 0$. Further, because agent 3 always keeps unchanged,

$$\begin{aligned} &P\left\{\max\left\{\left|x_1(t) - x_3(t)\right|, \left|x_2(t) - x_3(t)\right|\right\} > \eta, \ \forall t \right. \\ &\quad \left. \geq 1\right\} = 0. \end{aligned} \tag{10}$$

Thus, there exists a finite time t_1 a.s., such that $d(t_1) \leq \eta$ and when $t_0 \leq t < t_1$, $d(t) > \eta$.

Similarly, by the B-C lemma, there exists a finite time t_2 a.s., such that $d(t_2) > \eta$ and when $t_1 \leq t < t_2$, $d(t) \leq \eta$. In a sum, there exist a stopping time sequence $\{T_{(1,2)}(k)\}$ a.s., such that $d(t) \leq \eta$, $t \in [T_{(1,2)}(2s-1), T_{(1,2)}(2s))$, $s \in \mathbb{N}^+$. Furthermore, among the interval sequence $\{[T_{(1,2)}(2s-1), T_{(1,2)}(2s))\}$, by the B-C lemma and the ergodicity of $[x_1(t), x_2(t)]$ or $[x_2(t), x_1(t)]$, there always exists a time interval subsequence, denoted by $\{[T^*_{(1,2)}(k), T^*_{(1,2)}(k+1))\}$, at which $x_1(t) \leq x_3(t) \leq x_2(t)$ or $x_2(t) \leq x_3(t) \leq x_1(t)$, $t \in [T^*_{(1,2)}(k), T^*_{(1,2)}(k+1))$, $k \in \mathbb{N}$. Because $|x_2(t) - x_1(t)| = |\xi_2(t-1) - \xi_1(t-1)| < 2\delta$,

$$\begin{aligned} d(t+1) &= \max_{i,j \in \mathcal{V}} \left|x_i(t+1) - x_j(t+1)\right| \\ &= \max\left\{\left|x_1(t+1) - x_2(t+1)\right|,\right. \\ &\quad \left.\left|x_3(t+1) - x_2(t+1)\right|, \left|x_1(t+1) - x_3(t+1)\right|\right\} \\ &\leq 2\delta. \end{aligned} \tag{11}$$

Denote $f(\delta) = 2\delta$. Thus, the system is $T-$robust consensus for the event that the selection pair is always $(1,2)$ for any time $t \in \mathbb{N}$.

According to the rotational symmetry of agent selection, if agent 1 and agent 3 always select each other, then the conclusion also follows. Generally, note that

$$\begin{aligned} \left|x_i(t) - x_j(t)\right| &= \left|\frac{1}{2}\left(x_i(t-1) + x_j(t-1)\right.\right. \\ &\quad \left.\left. - x_j(t-1) - x_i(t-1)\right) + \xi_i(t-1) - x_j(t-1)\right| \\ &\leq 2\delta \end{aligned} \tag{12}$$

if $S_{t-1} = (i,j)$ and $|x_j(t-1) - x_i(t-1)| < \eta$. Thus, if $d(t) = |x_1(t) - x_2(t)| < \eta$, $(1/2)(x_1(t) + x_2(t)) - \delta < x_3(t) < (1/2)(x_1(t) + x_2(t)) + \delta$, and $S_t = (1,2)$, then $d(t+1) < 2\delta$. Note that this event happens infinite often by the B-C lemma. Therefore, there exists a time subsequence $\{t_k\}$ a.s. in which $d(t_k) < 2\delta$. In a sum, the system is $T-$robust consensus for the general condition.

[3] Inductively, for any $n \in \mathbb{N}$, we assume $d(0) < \eta$ and $\delta < (1/4)\eta$.

Firstly, if agent i and agent j select each other at time 0, then

$$\begin{aligned} \left|x_i(1) - x_j(1)\right| &= \left|\frac{1}{2}\left(x_i(0) + x_j(0) - x_i(0) - x_j(0)\right)\right. \\ &\quad \left. + \xi_i(0) - \xi_j(0)\right| \leq \left|\xi_i(0)\right| + \left|\xi_j(0)\right| \leq 2\delta < \eta; \end{aligned} \tag{13}$$

Secondly, if agent i and agent j keep unchanged at time 0, then

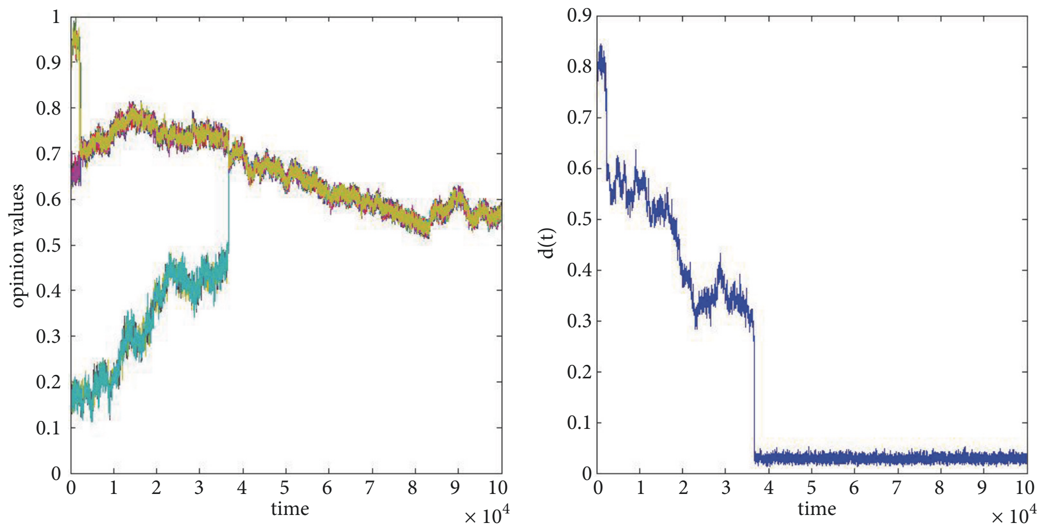


FIGURE 3: Opinion evolution of 20 agents for the noisy DW model (left) and the corresponding opinion range $d(t)$ (right), where $n = 20$, $\eta = 0.2$, and $\delta = 0.01$.

$$\left|x_i(1) - x_j(1)\right| = \left|x_i(0) - x_j(0)\right| < \eta; \tag{14}$$

Finally, if agent i selects another agent k, $k \neq j$, but agent j keeps unchanged at time 0, then

$$\begin{aligned}\left|x_i(1) - x_j(1)\right| &= \left|\frac{1}{2}\left(x_i(0) + x_k(0)\right) - x_j(0) + \xi_i(0)\right| \\ &\leq \eta + \delta.\end{aligned} \tag{15}$$

Note that agents i and k satisfy $|x_i(1)-x_k(1)| = |\xi_i(0)-\xi_k(0)| \leq 2\delta < (1/2)\eta$, and then by the similar method, the system is T−robust consensus for the event that any two agents are selected periodically. By the B-C lemma, we get that the no-periodical selection event is a zero event; thus the system is T−robust consensus for the general case. □

Figure 4 shows how the opinion range $d(t)$ changes as t increases. Note that $d(t) < 2\delta = 0.1$ is infinite often, $t \in \mathbb{N}$.

Besides, the ergodicity of any agent's opinion values can be shown in Figure 5. This is deduced by the randomness of the external noise $\{\xi_i(t)\}$, $i \in \mathcal{V}$, $t \in \mathbb{N}$.

Remark 6. In fact, Lemma 5 shows that when $d(0)$ is sufficiently small, the noisy DW model will be *T−robust consensus*. However, it is still a problem how opinions evolve if $d(0) > \eta$?

In the following, we study one special condition such that there are two agent subsets, where both satisfy the condition in Lemma 5 but their opinions' distances would be larger than η. We will show that the system (3) can also achieve *T−robust consensus* finally.

Lemma 7. *Suppose there exists a partition $\mathcal{V}_1, \mathcal{V}_2$ of $\mathcal{V}$, satisfying $d_{\mathcal{V}_1}(0), d_{\mathcal{V}_2}(0) < \eta$; then the system (3) achieves* T−robust consensus *if $\delta < (1/4)\eta$.*

Proof. [1] For $n = 3$, we assume $\mathcal{V}_1 = \{1, 2\}$ and $\mathcal{V}_2 = \{3\}$.

If $d_{\mathcal{V}_1,\mathcal{V}_2}(0) \triangleq \max_{i\in\mathcal{V}_1, j\in\mathcal{V}_2}|x_i(0) - x_j(0)| \leq \eta$, then we get that the system is T−robust consensus by Lemma 5.

If $d_{\mathcal{V}_1,\mathcal{V}_2}(0) > \eta$, for simplicity, $|x_3(0) - x_1(0)|, |x_3(0) - x_1(0)| > \eta$, then we can get that $x_3(1) = x_3(0) + \xi_3(0)$ if the selection pair is $(3, 3)$ at time 0. With positive probability, $(3, 3)$ happens to be infinite often and $\{x_3(t)\}$ is ergodic among $[0, 1]$ a.s. Similarly, among the time interval sequence $\{[T^*_{(1,2)}(k), T^*_{(1,2)}(k + 1))\}$, there is a finite time t_0 at which $d(t_0) < \eta$ a.s., and then we get that the system is T−robust consensus by Lemma 5.

[2] For $n > 3$, we fixed a partition of the agent set $\mathcal{V}$. Denote $i, j \in \mathcal{V}_1$, suppose the time subsequence $\{t_{(i,j)}(k)\}$ at which agent i and agent j select each other and $|x_i(t_{(i,j)}(k) - 1) - x_j(t_{(i,j)}(k) - 1)| < \eta$. In fact, with the same method used in Lemma 5, the time subsequence exists with the probability 1.

Note that $\{\sum_{s=1}^{t} \xi_i(s)\}$ is a random walk among $[0, 1]$, $i = 1, 2$. Therefore, the agent interval $[\min_{i\in\mathcal{V}_1} x_i(t), \max_{i\in\mathcal{V}_1} x_i(t)]$ can also change randomly among $[0, 1]$. Select $i \in \mathcal{V}_1$ and $w \in \mathcal{V}_2$, then the event $\{|x_i(t) - x_w(t)| < \eta\}$ happens i.o. by the B-C lemma. Similarly, the event that agent i and agent w select each other also happens i.o.. There also exists the time subsequence $\{t_{(i,w)}(k)\}$ a.s. such that agent i and agent w select each other and $|x_i(t_{(i,w)}(k) - 1) - x_w(t_{(i,w)}(k) - 1)| < \eta$. For the finiteness of the selection pair set $\{(i, w), i \in \mathcal{V}_1, w \in \mathcal{V}_2\}$, we get that there exists a time subsequence $\{t^*(k)\}$ at which $d(t^*(k)) \leq \eta$ i.o.. Consequently, we get that the system is T−robust consensus by Lemma 5. Thus, the conclusion follows. □

Finally, based on these lemmas, it is obviously to get the following theorem. Hence, we neglect its proof in the following.

Theorem 8. *For any initial states and $\delta < (1/4)\eta$, the noisy DW model (3) achieves* T−robust consensus *a.s..*

Besides, we further get the following theorem by weakening the probability condition.

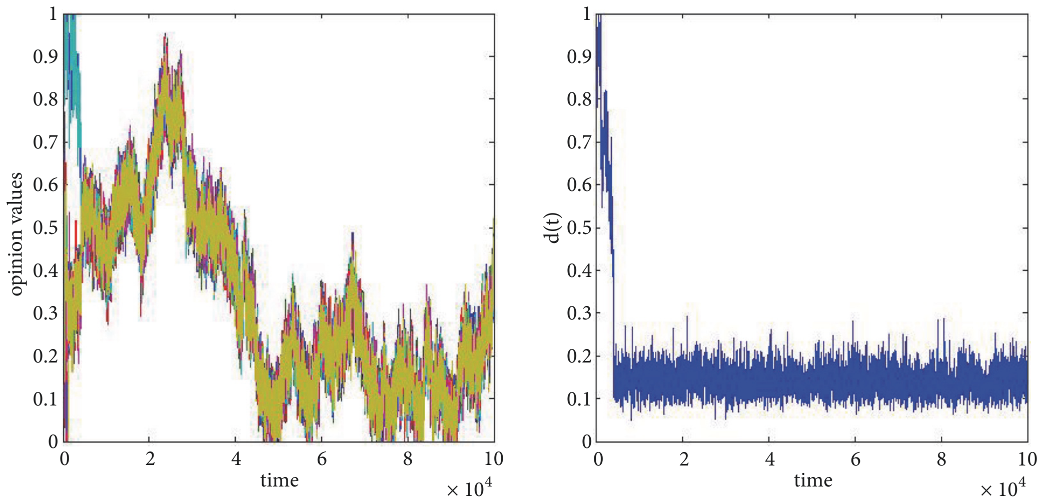


FIGURE 4: Opinion evolution of 20 agents for the noisy DW model (left) and the corresponding $d(t)$ (right), where $n = 20$, $\eta = 0.2$, and $\delta = 0.05$.

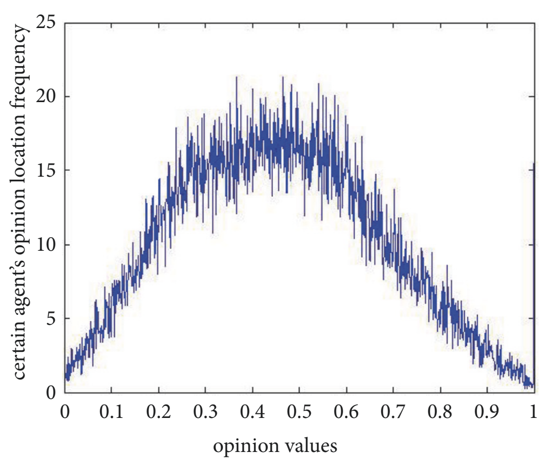


FIGURE 5: Agent 1's opinion locations in the opinion evolution process for the noisy DW model with parameters $n = 20$, $T = 10^5$, $\eta = 0.2$, and $\delta = 0.05$.

Theorem 9. *For any initial states and $\delta < (1/4)\eta$, the noisy DW model (3) achieves robust consensus with a positive probability.*

Proof. If $\min_{\{i,j\in\mathscr{V}\}}|x_i(0)-x_j(0)| > \eta$, then $x_i(1) = x_i(0)+\xi_i(0)$ if the selection pair is (i,i), $i \in \mathscr{V}$ at time 0. With the similar method used in Lemma 7, we only need to consider the initial condition $\min\{i, j \in \mathscr{V}\}|x_i(0) - x_j(0)| \le \eta$.

By Lemma 7, suppose there exists a finite partition $\mathscr{V}_1, \mathscr{V}_2, \ldots, \mathscr{V}_K$ of $\mathscr{V}$, satisfying $d_{\mathscr{V}_i}(0) < \eta$ for $i = 1, 2, \ldots, K$, then the system (3) achieves T*−robust consensus* if $\delta < (1/4)\eta$.

In the following we will prove that the system is also robust consensus. Denote $M = \max\{\lfloor \eta/2\delta \rfloor, 1\}$ and $\sharp\{(i, j)\}$ as the repeated times of the selection pair (i, j) in the following proof. Furthermore, the time interval sequence of the repeated selection pair of (i, j) is denoted by $\{T_{(i,j)}(k)\}$.

[1] For $n = 2$, all selection pairs are $(1, 1)$, $(2, 2)$ and $(1, 2)$. In fact, as δ approaches to 0, Lemmas 5 and 7 also hold. Consider the event

$$\mathscr{A}_2^0 = \left\{S_0 = (1,2) \text{ and then } \sharp\{(1,1)\}, \sharp\{(2,2)\} \le M\right\}. \quad (16)$$

Because $\min\{i, j \in \mathscr{V}\}|x_i(0) - x_j(0)| \le \eta$, in the event $\mathscr{A}_2$, note that $|x_1(t) - x_2(t)| \le \eta$ holds for any $t \in \mathbb{N}$. We get that $d(t) \le 2\delta$ if the selection pair is $(1, 2)$ at time $t - 1$ and $d(t) \le 2M\delta$ for other conditions. Thus, denote $f(\delta) = \max\{2M\delta, 2\delta\}$, thus, the system achieves robust consensus in the event $\mathscr{A}_2^0$.

Similarly, define

$$\mathscr{A}_2^1 = \left\{S_0 = (1,2) \text{ and then } \sharp\{(i,i)\} \le M + 1,\ \sharp_{s\in T_{(i,i)}(q)}\{\xi_i(s) < 0\}\right.$$

$$< M,\ \sharp_{s\in T_{(i,i)}(q)} \{\xi_i(s) > 0\} < M,\ \text{for}\ \sharp\{(i,i)\} = M + 1,\ i = 1,2,\ q \in \mathbb{N}\}. \quad (17)$$

Because $\sharp_{s\in T_{(i,i)}(q)}\{\xi_i(s) > 0\} < M$ and $\sharp\{(i,i)\} = M + 1$, there exists at least one moment $t' \in T_{(i,i)}(k)$ such that $\xi_i(t')$'s symbol is different from the ones in some other moments. Thus, agent i would not get out of the confidence range of another agent j, $i = 1$, $j = 2$, or $i = 2$, $j = 1$. Thus, with the similar method, the system achieves robust consensus in the event $\mathscr{A}_2^1$.

With a similar method, we can define an event sequence $\{\mathscr{A}_2^k\}$, where

$$\begin{aligned}\mathscr{A}_2^k = \{&S_0 = (1,2) \text{ and then } \sharp\{(i,i)\} \le M \\ &+ k,\ \sharp_{s\in T_{(i,i)}(q)}\{\xi_i(s) < 0\} \\ &< M,\ \sharp_{s\in T_{(i,i)}(q)}\{\xi_i(s) > 0\} < M,\ \text{for}\ \sharp\{(i,i)\} = M \\ &+ k,\ i = 1,2,\ q \in \mathbb{N}\}.\end{aligned} \quad (18)$$

Based on the randomness of $\xi_i(t)$, we get that

$$\begin{aligned}P\left\{\bigcup_{k=0}^{\infty}\mathscr{A}_2^k\right\} &= 1 - P\left\{\bigcap_{k=0}^{\infty}\{\mathscr{A}_2^k\}^c\right\} \ge 1 - P\left\{\bigcap_{k=0}^{\infty}\{S_0\right. \\ &= (1,2),\ \exists\sharp_{s\in T_{(i,i)}(q)}\{\xi_i(s) < 0\} \ge M \text{ for } \sharp\{(i,i)\} \\ &\left.= M + k\}\right\} > 0.\end{aligned} \quad (19)$$

Thus, the conclusion holds when $n = 2$.

[2] For $n > 2$, we can generate the analysis for $n = 2$ into $n > 2$. Similarly, we analyse the event sequence $\{\mathscr{A}_n^k\}$ where

$$\begin{aligned}\mathscr{A}_n^k = \{&d(T) \le \eta \text{ and then } \sharp\{(i,i)\} \le M \\ &+ k,\ \sharp_{s\in T_{(i,i)}(q)}\{\xi_i(s) < 0\} \\ &< M,\ \sharp_{s\in T_{(i,i)}(q)}\{\xi_i(s) > 0\} < M,\ \text{for}\ \sharp\{(i,i)\} = M \\ &+ k,\ i \in \mathscr{V},\ q \in \mathbb{N}\}.\end{aligned} \quad (20)$$

By Lemma 7, we get that the event $\{d(t) \le \eta\}$ happens i.o.; thus, $d(T) \le \eta$ happens with the probability larger than 0, $T < \infty$. Along with the same method used before, we can get that the conclusion holds for general cases. □

Remark 10. The main result, Theorem 8, shows that when δ is sufficiently small, the noisy DW model will achieve $T-$*robust consensus*. Furthermore, Theorem 9 tells us that the noisy DW model achieves *robust consensus* with a positive probability. However, it is nearly impossible to estimate the probability $P\{d(T) \le \eta\}$; thus we could not provide it in Theorem 9. Besides, it is still unknown on the relation of δ and η? We denote the threshold values of δ as δ^*. By the previous result, we get that $\delta^* \le (1/4)\eta$. For further study of the parameter δ^*, we will provide the experimental demonstrations in the next section.

5. Demonstrations for System Parameters

In this section, we exhibit the simulation results for the noisy DW model. The main aim is to study how the threshold of δ changes as the confidence bound η changes. In fact, both Lemma 7 and Theorem 8 imply that separated opinion subsets will merge together. Firstly, we demonstrate how the limited opinion range $\lim_{t\to\infty} d(t)$ changes as parameters increase. Secondly, the cluster number is studied as the noise strength δ changes. Finally, the parameter δ^* is simulated for further understanding the relation of the noise and opinion robust consensus. Specially, all simulations must be in finite steps and the termination time should be large enough. We will demonstrate them in the following subsections. All simulations are the averages of 1000 times experiments, respectively.

5.1. Opinion Range. First, we show how the opinion range $d(t)$ changes as parameters change. In Figure 6, we show how the fluctuation strength for each cluster varies as the confidence bound η and the noise strength δ change, respectively.

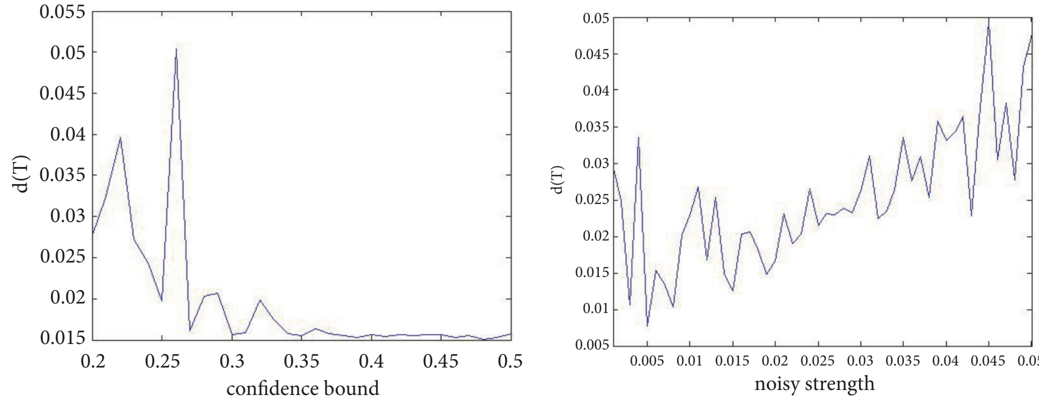
Secondly, we simulate the cluster number and average cluster width (Figure 7). It shows that when the noise strength is less than a certain value, the average cluster number is one. Specially, due to the finiteness of the termination time, the average cluster number is larger than one if the noise strength is small enough.

5.2. Noise Strength. Then, we demonstrate the threshold values less which opinions will achieve robust consensus when t is sufficiently large. In Figure 8, we show how the threshold values of δ change when the confidence bound η increases. It is worth noticing that the threshold values are nearly a linear relation with the confidence bound. Note that the average linear relation is nearly $\delta = (1/4)\eta$. However, because of the finiteness of the termination time T, we get that the time complexity is very high when η is very small. Thus, the estimated threshold values of δ may be no verifiable. Also, there exist some exceptional examples in Figure 7, and we will obtain the desirable results if we take a long but unrealistic termination time.

For Lemma 7 and Theorem 8, we consider the initial condition as follows: we assume the agent opinions are split into two parts, one in which opinions are located in $[0, \eta]$, another in which opinions are located in $[1 - \eta, 1]$.

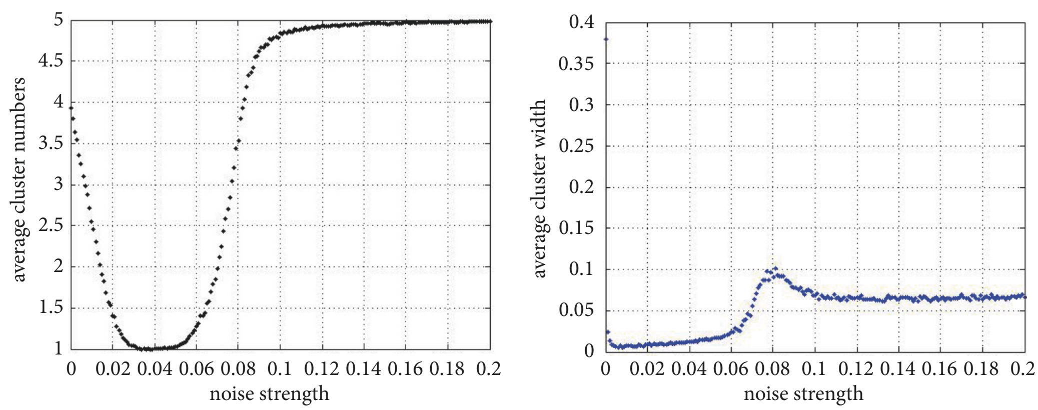
6. Conclusions

In this paper, we proposed a noisy DW model in which any selected pairs will be affected by external noises. We mainly proved that when the strength δ of noises $\{\xi_i(t)\}$ is small enough, opinions of this noisy DW model will reach $T-$*robust consensus*; that is to say, all opinions fluctuate but the fluctuation range's value is less if the noise strength is less for finite time intervals. Beside, opinions of this noisy DW model



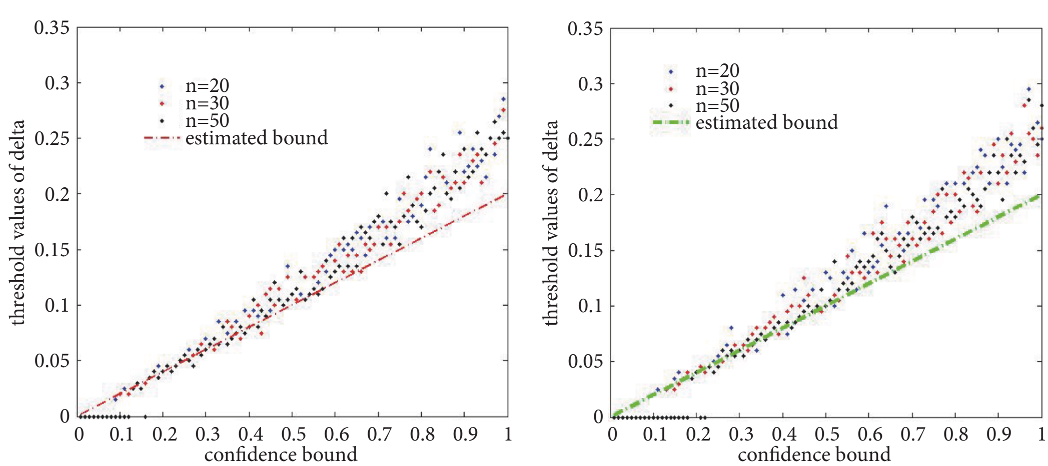
(a) How $d(T)$ changes as η increases
(b) How $d(T)$ changes as δ increases

FIGURE 6: Parameters are $n = 20$, the termination time $T = 10^5$. The noisy strength is $\delta = 0.05$ (a) and the confidence bound is $\eta = 0.2$. (b).



(a) How cluster number changes as δ increases
(b) How cluster width changes as δ increases

FIGURE 7: Parameters are $n = 20$, the termination time $T = 10^5$, and the confidence bound is $\eta = 0.2$.



(a) How δ^* changes as η increases
(b) How δ^* changes as δ increases

FIGURE 8: The termination time $T = 10^5$. Dots denote the threshold experimental values, and the dotted line is the approximated linear relationship of δ and η.

will achieve *robust consensus* with a positive probability. The definition of the *robust consensus* was introduced to describe the relation of opinion fluctuation and noise strength δ. We proposed some lemmas. For one, we described that the aggregated opinions will not be divided into different parts with distance larger than the confidence bound η, when the noisy strength δ is sufficiently small; For another, we described that the separated parts will merge together when δ is sufficiently small. Then we gave the theorem describing that the noisy DW model achieves robust consensus when δ is sufficiently small. Finally, we demonstrated threshold values of δ, where there exists a positive probability larger than which the aggregated agent opinions will be divided, and another positive probability larger than which separated parts will not merge together.

Data Availability

The simulation data are random values on the real interval $[0, 1]$.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

This work was supported by the NNSF of China under Grants 11601450 and 71503206 and the Social science project of Sichuan Education Department under Grant XQ15B10.

Supplementary Materials

Supplementary material contains five MATLAB programs on the noisy DW model. Program 1: The opinion evolution of the noisy DW model. Program 2: How the opinion fluctuation strength changes as the time increases. Program 3: How the opinion fluctuation strength changes as the model parameters change. Program 4: The clusters of opinion evolution. Program 5: How δ^* changes as the confidence bound increases. *(Supplementary Materials)*

References

[1] C. Castellano, S. Fortunato, and V. Loreto, "Statistical physics of social dynamics," *Reviews of Modern Physics*, vol. 81, no. 2, pp. 591–646, 2009.

[2] A. C. Martins, "Opinion particles: classical physics and opinion dynamics," *Physics Letters A*, vol. 379, no. 3, pp. 89–94, 2015.

[3] H.-X. Yang, W.-X. Wang, Y.-C. Lai, and B.-H. Wang, "Convergence to global consensus in opinion dynamics under a nonlinear voter model," *Physics Letters A*, vol. 376, no. 4, pp. 282–285, 2012.

[4] S. Gil and D. H. Zanette, "Coevolution of agents and networks: Opinion spreading and community disconnection," *Physics Letters A*, vol. 356, no. 2, pp. 89–94, 2006.

[5] G. Shi and K. H. Johansson, "Randomized optimal consensus of multi-agent systems," *Automatica*, vol. 48, no. 12, pp. 3018–3030, 2012.

[6] G. Shi, M. Johansson, and K. H. Johansson, "How agreement and disagreement evolve over random dynamic networks," *IEEE Journal on Selected Areas in Communications*, vol. 31, no. 6, pp. 1061–1071, 2013.

[7] G. Shi and K. H. Johansson, "The role of persistent graphs in the agreement seeking of social networks," *IEEE Journal on Selected Areas in Communications*, vol. 31, no. 9, pp. 595–606, 2013.

[8] Y. Lou and Y. Hong, "Target aggregation of multiple mobile agents by multiple leaders with markovian switching topologies," in *Automatica*, vol. 48, pp. 559–563, 2012.

[9] N. E. Friedkin and E. C. Johnsen, "Social influence networks and opinion change," in *Advances in Group Processes*, E. J. Lawler and M. W. Macy, Eds., vol. 33, pp. 1–29, JAI Press, 1999.

[10] Y. Zhao, G. Kou, Y. Peng, and Y. Chen, "Understanding influence power of opinion leaders in e-commerce networks: an opinion dynamics theory perspective," *Information Sciences*, vol. 426, pp. 131–147, 2018.

[11] G. Kou, Y. Zhao, Y. Peng, and Y. Shi, "Multi-Level Opinion Dynamics under Bounded Confidence," *PLoS ONE*, vol. 7, no. 9, 2012.

[12] A. Czirók and T. Vicsek, "Collective behavior of interacting self-propelled particles," *Physica A: Statistical Mechanics and its Applications*, vol. 281, no. 1, pp. 17–29, 2000.

[13] A. Mirtabatabaei and F. Bullo, "Opinion dynamics in heterogeneous networks: convergence conjectures and theorems," *SIAM Journal on Control and Optimization*, vol. 50, no. 5, pp. 2763–2785, 2012.

[14] R. Hegselmann and U. Krause, "Opninion dynamics and bounded confidence models, analysis and simulation," *Journal of Artificial Societes and Social Simulations (JASSS)*, vol. 5, 2002.

[15] G. Deffuant, D. Neau, F. Amblard, and G. Weisbuch, "Mixing beliefs among interacting agents," *Advances in Complex Systems (ACS)*, vol. 3, pp. 87–98, 2000.

[16] A. Nedic and B. Touri, "Multi-dimensional Hegselmann-Krause dynamics," in *Proceedings of the 51st IEEE Conference on Decision and Control, CDC 2012*, pp. 68–73, Hawaii, HI, USA, December 2012.

[17] S. Fortunato, V. Latora, A. Pluchino, and A. Rapisarda, "Vector opinion dynamics in a bounded confidence consensus model," *International Journal of Modern Physics C*, vol. 16, no. 10, pp. 1535–1551, 2005.

[18] J. Lorenz, "A stabilization theorem for dynamics of continuous opinions," *Physica A: Statistical Mechanics and its Applications*, vol. 355, no. 1, pp. 217–223, 2005.

[19] X. Chen, X. Zhang, Z. Wu, H. Wang, G. Wang, and W. Li, "Opinion evolution in different social acquaintance networks," *Chaos: An Interdisciplinary Journal of Nonlinear Science*, vol. 27, no. 11, Article ID 113111, 11 pages, 2017.

[20] H. Fastl, "Fluctuation strength and temporal masking patterns of amplitude-modulated broadband noise," *Hearing Research*, vol. 8, no. 1, pp. 59–69, 1982.

[21] K.-I. Goh, B. Kahng, and D. Kim, "Fluctuation-driven dynamics of the Internet topology," *Physical Review Letters*, vol. 88, no. 10, pp. 1087011–1087014, 2002.

[22] M. Pineda, R. Toral, and E. Hernández-García, "Noisy continuous-opinion dynamics," *Journal of Statistical Mechanics: Theory and Experiment*, vol. 2009, no. 8, p. 1, 2009.

[23] M. Pineda, R. Toral, and E. Hernández-García, "Diffusing opinions in bounded confidence processes," *The European Physical Journal D*, vol. 62, no. 1, pp. 109–117, 2011.

[24] M. Pineda, R. Toral, and E. Hernández-García, "The noisy Hegselmann-Krause model for opinion dynamics," *The European Physical Journal B*, vol. 86, article 490, pp. 1–10, 2013.

[25] W. Su, G. Chen, and Y. Hong, "Noise leads to quasi-consensus of Hegselmann–Krause opinion dynamics," *Automatica*, vol. 85, pp. 448–454, 2017.

[26] Y. Zhao, L. Zhang, M. Tang, and G. Kou, "Bounded confidence opinion dynamics with opinion leaders and environmental noises," *Computers & Operations Research*, vol. 74, pp. 205–213, 2016.

[27] H.-F. Chen, *Stochastic Approximation and Its Applications*, vol. 64, Kluwer Academic Publishers, Dordrecht, Netherlands, 2002.

[28] L. Wang and L. Guo, "Robust consensus and soft control of multi-agent systems with noises," *Journal of Systems Science & Complexity*, vol. 21, no. 3, pp. 406–415, 2008.

[29] Y. S. Chow and H. Teicher, *Probability Theory: Independence Interchangeability Martingales*, Springer-Verlag Press, Heidelberg, Berlin, Germany, 2nd edition, 1978.