Hindawi
Mathematical Problems in Engineering
Volume 2018, Article ID 1245656, 10 pages
https://doi.org/10.1155/2018/1245656

Research Article

Dynamic Multivariate Quantile Residual Life in Reliability Theory

M. Shafaei Noughabi,[1] M. Kayid ,[2] and A. M. Abouammoh[2]

[1]*Department of Mathematics and Statistics, University of Gonabad, Gonabad, Iran*
[2]*Department of Statistics and Operations Research, College of Science, King Saud University, Riyadh, Saudi Arabia*

Correspondence should be addressed to M. Kayid; el_kayid2000@yahoo.com

Received 26 December 2017; Revised 25 June 2018; Accepted 30 July 2018; Published 26 August 2018

Academic Editor: Inmaculada T. Castro

We extend the univariate α-quantile residual life function to multivariate setting preserving its dynamic feature. Principal attributes of this function are derived and their relationship to the dynamic multivariate hazard rate function is discussed. A corresponding ordering, namely, α-quantile residual life order, for random vectors of lifetimes is introduced and studied. Based on the proposed ordering, a notion of positive dependency is presented. Finally, a discussion about conditions characterizing the class of decreasing multivariate α-quantile residual life functions is pointed out.

1. Introduction

For a random lifetime X, the α-quantile residual life (α-QRL) function proposed by Haines and Singpurwalla (1974) describes the α-quantile of the well-known remaining lifetime of X given its survival at time $x > 0$. This function has been regarded as a prominent tool in reliability theory and survival analysis specially due to its potential advantages rather than the popular mean residual lifetime (MRL) function. Schmittlein and Morrison [1] discussed some of these advantages and applications of the median residual life function. Joe and Proschan [2], Gupta and Longford [3], Franco-Pereira and Uña-Álvarez [4], and Lillo [5] are among many authors who conducted their researches on the α-QRL function. Intuitively, we may deal with vectors of possibly dependent random lifetimes. In such situations, extending concerned concepts to multivariate setting allows us to treat the problems in the right way. The multivariate statistical methods play a crucial role in studying a wide variety of several complex engineering models. From many researchers who have extensively studied the multivariate lifetime measures, we refer to Johnson and Kotz [6], Arjas and Norros [7], Arnold and Zahedi [8], Baccelli and Makowski [9], Nair and Nair [10], Shaked and Shanthikumar [11, 12], Kulkarni and Rattihalli [13], and Hu et. al. [14]. Shaked and Shanthikumar [15] introduced and studied a dynamic version of the multivariate MRL function. This function is called *dynamic* in the sense that it is a measure conditioned on an observed history (which can consist of some failures) up to time x.

Recently, Shafaei Noughabi and Kayid [16] proposed a bivariate α-QRL (α-BQRL) function which characterizes the underlying distribution properly. Although this function is useful and applicable in statistics and reliability fields, it is *nondynamic*. In the areas of reliability theory, the dynamic residual life function authorizes engineers to track reliability of their systems at any time given any observed history. As an example, consider a machine having some belts working simultaneously. One engineer that tracks the machine may observe different types of histories. At arbitrary time x, he/she may observe that (i) all belts may be safely working or (ii) one or more of them may fail at x. The bivariate α-QRL (α-BQRL) function introduced by Shafaei Noughabi and Kayid [16] does not support histories of type (ii). This motivates us to extend the univariate α-quantile residual life function to multivariate setting preserving its dynamic feature. Now we are motivated to propose a dynamic measure which enables engineer to describe the belts lifetimes after observing any history, type (i) or (ii). For the components or subsystems survived until time x, the dynamic multivariate α-QRL (α-MQRL) function measures the α-quantile of the remaining lifetime conditioned on any possible history at this time. It can be regarded as a serious competitor for the multivariate MRL recommended by Shaked and Shanthikumar [15] and may even be preferred to that due to the comments of Schmittlein and Morrison [1].

The rest of the paper is arranged as follows. The next section provides some preparative material that we need to develop the results. We start our results with the dynamic α-BQRL function and its basic behaviour in Section 3. In that section, the concept has been generalized to multivariate setting. Section 4 deals with a new stochastic order for random vectors based on the proposed α-MQRL function. Also, a notion of positive dependency has been proposed and discussed. Section 5 investigates conditions defining the class of distributions with decreasing α-MQRL functions and provides some related results. Finally, in Section 6, we give a brief conclusion and some remarks of the current and future of this research.

Throughout the paper, we assume that the random vectors $\mathbf{X}$ and $\mathbf{Y}$ follow absolutely continuous distributions on the support $[0,\infty)^n$. Moreover, to distinguish nondynamic multivariate functions from dynamic we insert a tilde (~) sign for nondynamic ones. Also, to provide succinct notations, denote $(x_1,\ldots,x_{i-1},t,x_{i+1},\ldots,x_n)$ by $\mathbf{x}\langle i;t\rangle$.

2. Preliminaries

Let random lifetime X be distributed on $[0,\infty)$ according to continuous distribution F. Then, the well-known hazard rate and α-QRL functions are given, respectively, by

$$\lambda(x) = -\frac{d}{dx}\ln\overline{F}(x), \quad x \geq 0, \tag{1}$$

and

$$q_\alpha(x) = \overline{F}^{-1}\left(\overline{\alpha}\overline{F}(x)\right) - x, \quad x \geq 0, \tag{2}$$

in which $\overline{F}(x) = 1 - F(x)$ shows the reliability function, $\overline{\alpha} = 1-\alpha$, and $\overline{F}^{-1}(p) = \inf\{x; \overline{F}(x) = p\}$ is the inverse function of $\overline{F}$. These two functions are related in the way of the simple equation

$$\int_x^{x+q_\alpha(x)} \lambda(x)\,dx = -\ln\overline{\alpha}, \tag{3}$$

which directly can be translated to

$$1 + q'_\alpha(x) = \frac{\lambda(x)}{\lambda(x+q_\alpha(x))}. \tag{4}$$

It implies immediately that $q'_\alpha(x) \geq -1$. Moreover, when the hazard rate is increasing (decreasing) at all points of the support, the α-QRL exhibits a decreasing (increasing) form.

Let $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ follow the absolutely continuous reliability function $\overline{F}$. Johnson and Kotz [6] defined the multivariate hazard rate, as a vector

$$\widetilde{\Lambda}(\mathbf{x}) = -\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n}\right)\ln\overline{F}(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^{+n}. \tag{5}$$

Denote the ith element of this vector by $\widetilde{\lambda}_i(\mathbf{x})$. We generalize the α-BQRL function proposed by Shafaei Noughabi and Kayid [16] to define the α-MQRL given by

$$\widetilde{\mathbf{q}}_\alpha(\mathbf{x}) = \left(\widetilde{q}_{\alpha,1}(\mathbf{x}), \widetilde{q}_{\alpha,2}(\mathbf{x}), \ldots, \widetilde{q}_{\alpha,n}(\mathbf{x})\right), \quad \mathbf{x} \in \mathbb{R}^{+n}, \tag{6}$$

where

$$\widetilde{q}_{\alpha,i}(\mathbf{x}) = \inf\left\{t : \overline{F}^{(i)}(t;\mathbf{x}) = \overline{\alpha}\right\}, \tag{7}$$

with $\overline{F}^{(i)}(t;\mathbf{x}) = P(X_i - x_i > t \mid X_1 > x_1, \ldots, X_n > x_n)$. Applying straightforward algebra, it can be written as

$$\widetilde{q}_{\alpha,i}(\mathbf{x}) = \overline{F}_i^{-1}\left(\overline{\alpha}\overline{F}(\mathbf{x}); \mathbf{x}_{(-i)}\right) - x_i, \tag{8}$$

in which $\overline{F}_i^{-1}(p;\mathbf{x}_{(-i)}) = \inf\{x_i : \overline{F}(\mathbf{x}) = p\}$ and vector $\mathbf{x}_{(-i)}$ has dimension $n-1$ and is obtained by removing the ith element of $\mathbf{x}$. This version of α-MQRL gives a measure just for histories without experiencing any failure which violates its dynamicity. Nevertheless, it is sufficiently useful and applicable in reliability engineering and survival analysis to be studied in detail.

The next result investigates the relation of α-MQRL with the multivariate hazard rate function. The proof is straightforward and hence omitted (cf. Shafaei Noughabi and Kayid [16]).

Lemma 1. *Consider the reliability function $\overline{F}$ with the hazard rate function $\widetilde{\Lambda}(\mathbf{x})$and the α-MQRL function $\widetilde{\mathbf{q}}_\alpha(\mathbf{x})$; then for every $i = 1, 2, \ldots, n$ we have*

$$\int_{x_i}^{x_i+\widetilde{q}_{\alpha,1}(\mathbf{x})} \widetilde{\lambda}_i(\mathbf{x}\langle i;t\rangle)\,dt = -\ln\overline{\alpha}, \tag{9}$$

and in turn

$$1 + \frac{\partial}{\partial x_i}\widetilde{q}_{\alpha,i}(\mathbf{x}) = \frac{\widetilde{\lambda}_i(\mathbf{x})}{\widetilde{\lambda}_i\left(\mathbf{x}\langle i; x_i + \widetilde{q}_{\alpha,i}(\mathbf{x})\rangle\right)}. \tag{10}$$

3. Dynamic Multivariate α-Quantile Residual Life

Let $\overline{F}$ represent the reliability function of a bivariate random variable $\mathbf{X}$. For brief representation, denote $\lim_{\delta\to 0}(1/\delta)P(x_1 < X_1 \leq x_1+\delta, X_2 > x_2)$ by $P_{\delta 1}(X_1 = x_1, X_2 > x_2)$ and similarly $\lim_{\delta\to 0}(1/\delta)P(X_1 > x_1, x_2 < X_2 \leq x_2+\delta)$ by $P_{\delta 2}(X_1 > x_1, X_2 = x_2)$. The conditional hazard rate functions of $\mathbf{X}$ are (cf. Shaked and Shanthikumar [15] or Cox [17])

$$\begin{aligned}
\lambda_i(x) &= \lim_{\delta\to 0}\frac{1}{\delta}P\left(x < X_i \leq x+\delta \mid \mathbf{X} > x\mathbf{1}\right), \\
&\qquad\qquad i = 1,2, \ x \geq 0, \\
\lambda_1(x \mid x_2) & \\
&= \lim_{\delta\to 0}\frac{1}{\delta}P\left(x < X_1 \leq x+\delta \mid X_1 > x, \ X_2 = x_2\right), \\
&\qquad\qquad x \geq x_2 \geq 0,
\end{aligned} \tag{11}$$

and

$$\lambda_2(x \mid x_1) = \lim_{\delta \longrightarrow 0} \frac{1}{\delta} P(x < X_2 \le x + \delta \mid X_1 = x_1,\ X_2 > x), \quad x \ge x_1 \ge 0. \tag{12}$$

Intuitively $\lambda_i(x)$, $i = 1, 2$, are referred to initial hazard rate functions in the sense that they measure the hazard rate for components before any failure. The underlying distribution F can be characterized uniquely by these four functions (cf. Cox [17]). Shaked and Shanthikumar [15] applied the conditional hazard rate functions in description attributes of dynamic bivariate MRL. We define the dynamic α-BQRL functions by

$$\begin{aligned} q_{\alpha,i}(x) &= \inf\left\{t : \overline{F}^{(i)}(t; x) = \overline{\alpha}\right\}, \quad i = 1, 2,\ x \ge 0, \\ q_{\alpha,1}(x \mid x_2) &= \inf\left\{t : \overline{F}^{(1*)}(t; x, x_2) = \overline{\alpha}\right\}, \quad x \ge x_2 \ge 0, \end{aligned} \tag{13}$$

and

$$q_{\alpha,2}(x \mid x_1) = \inf\left\{t : \overline{F}^{(2*)}(t; x_1, x) = \overline{\alpha}\right\}, \quad x \ge x_1 \ge 0, \tag{14}$$

where

$$\begin{aligned} \overline{F}^{(i)}(t; x) &= P(X_i - x > t \mid X_1 > x,\ X_2 > x), \\ \overline{F}^{(1*)}(t; x, x_2) &= P(X_1 - x > t \mid X_1 > x,\ X_2 = x_2), \end{aligned} \tag{15}$$

and

$$\overline{F}^{(2*)}(t; x_1, x) = P(X_2 - x > t \mid X_1 = x_1,\ X_2 > x). \tag{16}$$

Simple calculations imply

$$q_{\alpha,i}(x) = \overline{F}_i^{-1}\left(\overline{\alpha}\overline{F}(x, x); x\right) - x, \quad i = 1, 2, \tag{17}$$

$$q_{\alpha,1}(x \mid x_2) = \overline{F}_1^{*-1}(\overline{\alpha} P_{\delta 2}(X_1 > x,\ X_2 = x_2); x_2) - x, \tag{18}$$

and

$$q_{\alpha,2}(x \mid x_1) = \overline{F}_2^{*-1}(\overline{\alpha} P_{\delta 1}(X_1 = x_1,\ X_2 > x); x_1) - x, \tag{19}$$

where

$$\begin{aligned} \overline{F}_1^{-1}(p; x) &= \inf\left\{t : \overline{F}(t, x) = p\right\}, \\ \overline{F}_1^{*-1}(p; x_2) &= \inf\left\{y : P_{\delta 2}(X_1 > y,\ X_2 = x_2) = p\right\}, \end{aligned} \tag{20}$$

and the expressions for $\overline{F}_2^{-1}(p; x)$ and $\overline{F}_2^{*-1}(p; x_1)$ are analogous.

Relations (17) to (19) give us the possibility of computing the quantiles of remaining life of the surviving components conditioning on the observed history up to time x. Thus, these functions may be relevant for engineers that deal with systems of multiple dependent objects. They can measure the remaining quantiles of surviving objects taking the effect of the observed history at any time x into account. The next result provides the relation between conditional bivariate hazard rate functions and dynamic α-BQRL.

Theorem 2. *Let $\tilde{q}_{\alpha,i}(x_1, x_2)$, $i = 1, 2$, have continuous differentiation functions with respect to their both coordinates. Then, we have*

$$1 + q'_{\alpha,1}(x) = \frac{\tilde{\lambda}_1(x, x)}{\tilde{\lambda}_1(x + q_{\alpha,1}(x), x)} + \frac{P_{\delta_2}(X_1 > x + q_{\alpha,1}(x \mid x),\ X_2 = x) - P_{\delta_2}(X_1 > x + q_{\alpha,1}(x),\ X_2 = x)}{P_{\delta_1}(X_1 = x + q_{\alpha,1}(x),\ X_2 > x)}, \tag{21}$$

$$1 + q'_{\alpha,2}(x) = \frac{\tilde{\lambda}_2(x, x)}{\tilde{\lambda}_2(x, x + q_{\alpha,2}(x))} + \frac{P_{\delta_1}(X_1 = x,\ X_2 > x + q_{\alpha,2}(x \mid x)) - P_{\delta_1}(X_1 = x,\ X_2 > x + q_{\alpha,2}(x))}{P_{\delta_2}(X_1 > x,\ X_2 = x + q_{\alpha,2}(x))}, \tag{22}$$

$$1 + q'_{\alpha,1}(x \mid x_2) = \frac{\lambda_1(x \mid x_2)}{\lambda_1(x + q_{\alpha,1}(x \mid x_2) \mid x_2)}, \tag{23}$$

and

$$1 + q'_{\alpha,2}(x \mid x_1) = \frac{\lambda_2(x \mid x_1)}{\lambda_2(x + q_{\alpha,2}(x \mid x_1) \mid x_1)}. \tag{24}$$

Proof. Due to the relation $q_{\alpha,1}(x) = \tilde{q}_{\alpha,1}(x, x)$, the differentiation of $q_{\alpha,1}(x)$ can be described as the sum of differentiations in two directions

$$\frac{\partial}{\partial x} q_{\alpha,1}(x) = \frac{\partial}{\partial x_1} \tilde{q}_{\alpha,1}(x_1, x)\Big|_{x_1 = x} + \frac{\partial}{\partial x_2} \tilde{q}_{\alpha,1}(x, x_2)\Big|_{x_2 = x}, \tag{25}$$

Taking $n = 2$ and applying (10) with $i = 1$ gives the first expression. By some algebra for the second differentiation, we have

$$1 + q'_{\alpha,1}(x) = \frac{\widetilde{\lambda}_1(x,x)}{\widetilde{\lambda}_1(x + q_{\alpha,1}(x), x)} + \frac{P_{\delta 2}(X_1 > x + q_{\alpha,1}(x \mid x),\ X_2 = x) - P_{\delta 2}(X_1 > x + q_{\alpha,1}(x),\ X_2 = x)}{P_{\delta 1}(X_1 = x + q_{\alpha,1}(x),\ X_2 > x)}, \quad (26)$$

which shows (21). Analogous statements indicate (22). To justify (23), we consider

$$\begin{aligned} q_{\alpha,1}(x \mid x_2) + x &= \inf\{y : P_{\delta 2}(X_1 > y,\ X_2 = x_2) \\ &= \overline{\alpha} P_{\delta 2}(X_1 > x,\ X_2 = x_2)\}. \end{aligned} \quad (27)$$

By differentiation from y with respect to x in the equation inside brackets and applying definition of $q_{\alpha,1}(x \mid x_2)$ the result follows. In a similar way, (24) is obtained, and this complete the proof. □

Suppose that **X** consists of two independent components X_1 and X_2. It is easily detectable that $q_{\alpha,1}(x \mid x_2) = q_{\alpha,1}(x)$ and $q_{\alpha,2}(x \mid x_1) = q_{\alpha,2}(x)$ for every x_1 and x_2, where $q_{\alpha,i}(x)$ shows the α-QRL defined by (2) for X_i. In the sequel, we will see that a form of positive dependency between X_1 and X_2 implies $q_{\alpha,1}(x) \geq q_{\alpha,1}(x \mid x_2)$ and $q_{\alpha,2}(x) \geq q_{\alpha,2}(x \mid x_1)$. Therefore, for X_1 and X_2 positively dependent in such way, (21) and (22), respectively, imply

$$1 + q'_{\alpha,1}(x) \geq \frac{\widetilde{\lambda}_1(x,x)}{\widetilde{\lambda}_1(x + q_{\alpha,1}(x), x)}, \quad (28)$$

and

$$1 + q'_{\alpha,2}(x) \geq \frac{\widetilde{\lambda}_2(x,x)}{\widetilde{\lambda}_2(x, x + q_{\alpha,2}(x))}. \quad (29)$$

Fix x_2 and suppose that $\lambda_1(x \mid x_2)$ is increasing in x and then (23) implies that $q_{\alpha,1}(x \mid x_2)$ decreases in x. Similar argument holds for $q_{\alpha,2}(x \mid x_1)$.

Example 3. The reliability function

$$\begin{aligned} \overline{F}(\mathbf{x}) &= \exp\{-\theta_1 x_1 - \theta_2 x_2 - \theta_{12} \max(x_1, x_2)\}, \\ &\qquad \theta_1, \theta_2, \theta_{12} \geq 0,\ x_1, x_2 \geq 0, \end{aligned} \quad (30)$$

proposed by Marshal and Olkin [18] reveals a simple bivariate structure. Direct calculations show

$$\begin{aligned} q_{\alpha,i}(x) &= -\frac{\ln \overline{\alpha}}{\theta_i + \theta_{12}}, \quad x \geq 0,\ i = 1,2, \\ q_{\alpha,1}(x \mid x_2) &= -\frac{\ln \overline{\alpha}}{\theta_1 + \theta_{12}}, \quad x > x_2 \geq 0, \end{aligned} \quad (31)$$

and

$$q_{\alpha,2}(x \mid x_1) = -\frac{\ln \overline{\alpha}}{\theta_2 + \theta_{12}}, \quad x > x_1 \geq 0, \quad (32)$$

exhibiting constant α-BQRL functions.

Next, we generalize the concept to multivariate setting. Let the nonnegative random vector $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ accommodate distribution F, and h_x captures history of events related to n components up to time x, i.e.,

$$h_x = \{\mathbf{X}_I = \mathbf{x}_I,\ \mathbf{X}_{I'} > x\mathbf{1}\}, \quad (33)$$

in which $I = \{i_1, i_2, \ldots, i_k\}$ shows indices of events up to x, I' is the complement of I with respect to $N = \{1, 2, \ldots, n\}$, and $\mathbf{1}$ is a vector of 1's with proper dimension. Note that $x\mathbf{1}$ is the multiplication of scalar x by a vector $\mathbf{1}$ (a vector with same elements 1 and proper dimension) which clearly reduces to a vector with the same elements x and dimension of $\mathbf{1}$. Fix the history h_x as above, and then for any component $j \in I'$, the conditional hazard rate function can be written as

$$\begin{aligned} \lambda_j(x \mid h_x) &= \lim_{\delta \to 0} \frac{1}{\delta} P(x < X_j \leq x + \delta \mid h_x), \\ &\qquad x \geq 0, \end{aligned} \quad (34)$$

which describes the probability of instant failure of component j at time x, given history h_x. For empty set I, we have initial hazard functions. Denote α-quantile of a random variable X with reliability function $\overline{F}$ by $Q_\alpha(X) = \overline{F}^{-1}(\overline{\alpha})$. Then for $j \in I'$, we define the α-MQRL function at time x by

$$q_{\alpha,j}(x \mid h_x) = Q_\alpha(X_j - x \mid h_x), \quad x \geq 0, \quad (35)$$

which can be simplified to

$$\begin{aligned} &q_{\alpha,j}(x \mid h_x) \\ &\quad = \inf\{y : P(X_j - x > y \mid X_I = x_I,\ X_{I'} \geq x\mathbf{1}) \\ &\quad = \overline{\alpha}\} \\ &\quad = \inf\{y : P_I(X_j > x + y,\ X_{I'} \geq x\mathbf{1},\ X_I = x_I) \\ &\quad = \overline{\alpha} P_I(X_I = x_I,\ X_{I'} \geq x\mathbf{1})\}. \end{aligned} \quad (36)$$

For a simple representation denote

$$\begin{aligned} &\lim_{\delta_i \to 0,\, i \in I} \left(\prod_{i \in I} \frac{1}{\delta_i} \right) \\ &\quad \cdot P(\mathbf{X}_{I'} > \mathbf{x}_{I'},\ x_i < X_i \leq x_i + \delta_i,\ i \in I) \end{aligned} \quad (37)$$

by $P_{\delta_I}(\mathbf{X}_I = \mathbf{x}_I, \mathbf{X}_{I'} > \mathbf{x}_{I'})$. Observing history $h_x = \{X_1 > x, \ldots, X_n > x\}$ at x, i.e., $I = \emptyset$, we have the initial α-MQRL functions that can be calculated as

$$\begin{aligned} q_{i,\alpha}(x) &= \inf\left\{ y : \overline{F}\left(x, \ldots, \underbrace{x+y}_{i\text{th element}}, \ldots, x \right) = \overline{\alpha}\overline{F}(x,\ldots,x) \right\} = \inf\left\{ y : \overline{F}(x,\ldots,x+y,\ldots,x) \right. \\ &\left. = \overline{\alpha}\overline{F}(x,\ldots,x) \right\} - x. \end{aligned} \tag{38}$$

Theorem 4. *Assume that $\tilde{q}_{\alpha,i}(x_1,\ldots,x_n)$, $i = 1,2,\ldots,n$, defined by (7), have continuous differentiation functions with respect to their coordinates. If $I = \emptyset$, we have a system of n equations*

$$\begin{aligned} 1 + q'_{\alpha,i}(x) = {} & \frac{\tilde{\lambda}_i(x,\ldots,x)}{\tilde{\lambda}_i(x,\ldots,x+q_{\alpha,i}(x),\ldots,x)} + \sum_{j=1, j\neq i}^{n} \left[\frac{P_{\delta j}\left(X_j = x,\ X_i > x + q_{\alpha,i}(x \mid h_i),\ X_r > x;\ r = 1,\ldots n,\ r \neq i,j\right)}{P_{\delta i}\left(X_1 > x,\ldots,X_i = x+q_{\alpha,i}(x),\ldots,X_n > x\right)} \right. \\ & \left. - \frac{P_{\delta j}\left(X_j = x,\ X_i > x + q_{\alpha,i}(x),\ X_r > x;\ r = 1,\ldots n,\ r \neq i,j\right)}{P_{\delta i}\left(X_1 > x,\ldots,X_i = x+q_{\alpha,i}(x),\ldots,X_n > x\right)} \right], \end{aligned} \tag{39}$$

for $i = 1,\ldots,n$. Also, if we fix the history at x by $h_x = \{\mathbf{X}_J = \mathbf{x}_J, \mathbf{X}_{J'} > x\mathbf{1}\}$, where $\mathbf{x}_J < x\mathbf{1}$, then for any survived component $i \in J'$ at x we can write

$$\begin{aligned} 1 + q'_{\alpha,i}(x \mid h_x) = {} & \frac{\lambda_i(x \mid h_x)}{\lambda_i^*\left(x + q_{\alpha,i}(x \mid h_x) \mid ch_{x+q_{\alpha,i}}(J' - \{i\})\right)} + \frac{\overline{\alpha}\sum_{k\in J'\neq i}^{n} P_{\delta(J\cup\{k\})}\left(\mathbf{X}_J = \mathbf{x}_J,\ X_k = x,\ \mathbf{X}_{J'-\{k\}} > x\mathbf{1}\right)}{P_{\delta(\{i\}\cup J)}\left(X_i = x + q_{\alpha,i}(x \mid h_x),\ \mathbf{X}_{J'} > x\mathbf{1},\ \mathbf{X}_J = \mathbf{x}_J\right)} \\ & - \frac{\sum_{k\in J'\neq i}^{n} P_{\delta(\{k\}\cup J)}\left(\mathbf{X}_J = \mathbf{x}_J,\ X_k = x,\ \mathbf{X}_{J'-\{k\}} > x\mathbf{1},\ X_i > x + q_{\alpha,i}(x \mid h_x)\right)}{P_{\delta(\{i\}\cup J)}\left(X_i = x + q_{\alpha,i}(x \mid h_x),\ \mathbf{X}_{J'} > x\mathbf{1},\ \mathbf{X}_J = \mathbf{x}_J\right)}, \end{aligned} \tag{40}$$

where

$$\begin{aligned} & \lambda_i^*\left(x + q_{\alpha,i}(x \mid h_x) \mid ch_{x+q_{\alpha,i}}(J' - \{i\})\right) \\ & = \frac{P_{\delta(\{i\}\cup J)}\left(X_i = x + q_{\alpha,i}(x \mid h_x),\ \mathbf{X}_{J'} > x\mathbf{1},\ \mathbf{X}_J = \mathbf{x}_J\right)}{P_{\delta J}\left(X_i > x + q_{\alpha,i}(x \mid h_x),\ \mathbf{X}_{J'} > x\mathbf{1},\ \mathbf{X}_J = \mathbf{x}_J\right)} \end{aligned} \tag{41}$$

represents instantaneous risk of failure of component i by time $x + q_{\alpha,i}(x \mid h_x)$, given history h_x up to time x and censored history of components J' except i after x.

Proof. As (38) shows, $q_{\alpha,i}(x)$ equals $\tilde{q}_{\alpha,i}(x,\ldots,x)$ defined by (7), so

$$q'_{\alpha,i}(x) = \sum_{j=1}^{n} \frac{\partial}{\partial x_j} \tilde{q}_{\alpha,i}\left(x\mathbf{1}\left\langle j; x_j \right\rangle\right)\Big|_{x_j = x}. \tag{42}$$

The ith term of this summation is

$$\begin{aligned} & \frac{\partial}{\partial x_i} \tilde{q}_{\alpha,i}\left(x\mathbf{1}\left\langle i; x_i \right\rangle\right)\Big|_{x_i = x} \\ & = \frac{\tilde{\lambda}_i(x,\ldots,x)}{\tilde{\lambda}_i(x,\ldots,x+q_{\alpha,i}(x),\ldots,x)} - 1. \end{aligned} \tag{43}$$

Other terms of it can be obtained by differentiation from both sides of the following equality with respect to x_j at point x, for $j = 1,\ldots,n,\ j \neq i$:

$$\begin{aligned} & \overline{F}\left(\left(x,\ldots,x+q_{\alpha,i}\left(x\mathbf{1}\left\langle j; x_j \right\rangle\right),\ldots,x_n\right)\left\langle x_j \right\rangle\right) \\ & = \overline{\alpha}\overline{F}\left(x\mathbf{1}\left\langle j; x_j \right\rangle\right), \end{aligned} \tag{44}$$

which proves (39). To show (40), we can differentiate from both sides of

$$\begin{aligned} & P_{\delta J}\left(X_i > x + q_{\alpha,i}(x \mid h_x),\ \mathbf{X}_{J'} > \mathbf{1}x,\ \mathbf{X}_J = \mathbf{x}_J\right) \\ & = \overline{\alpha} P_{\delta J}\left(\mathbf{X}_J = \mathbf{x}_J,\ \mathbf{X}_{J'} > \mathbf{1}x\right), \end{aligned} \tag{45}$$

with respect to x, and hence the proof is completed. □

4. Multivariate α-Quantile Residual Life Order

Stochastic orderings are very useful tools and have several applications in various areas such as probability, statistics, reliability engineering, and statistical decision theory. In literature, several concepts of stochastic orders between random variables have been given (cf. Shaked and Shanthikumar [19], as an excellent treatment of this topic). Consider two lifetime random variables X and Y with reliability functions $\overline{F}$ and $\overline{G}$ in the univariate context. Statisticians apply different ordering criteria in their investigations. As a simple one, X is said to be smaller than Y in the usual stochastic order, $X \leq_{st} Y$, if for every $x > 0$, $\overline{F}(x) \leq \overline{G}(x)$. In the multivariate framework, $\mathbf{X} \leq_{st} \mathbf{Y}$ when $E\varphi(\mathbf{X}) \leq E\varphi(\mathbf{Y})$ for all nondecreasing functions $\varphi : R^{+n} \longrightarrow R^{+}$ for which these expectations exist.

As a stronger ordering, X is said to be smaller than Y in the hazard rate order, $X \leq_{hr} Y$, if

$$\overline{F}(x_2)\overline{G}(x_1) \leq \overline{F}(x_1)\overline{G}(x_2), \tag{46}$$

for all $x_1 \leq x_2$. Provided that X and Y are equipped with the hazard rate functions λ_X and λ_Y respectively, condition (46) is equivalent with the inequality $\lambda_X(x) \geq \lambda_Y(x)$ for every $x \geq 0$.

To extend some reliability and ageing concepts for random vectors, we should be able to compare possibly different histories. In the simplest case, we consider two histories with same lengths. At every time x, the history $\overline{h}_x$ is referred to be more severe than h_x, denoted as $h_x \preceq \overline{h}_x$, if

(i) every failed component in h_x also be failed in $\overline{h}_x$;

(ii) for common failures in both h_x and $\overline{h}_x$, the failures in $\overline{h}_x$ are earlier than failures in h_x.

More formally,

$$\overline{h}_x^X = \{\mathbf{X}_I = \mathbf{x}_I,\ \mathbf{X}_J = \mathbf{x}_J,\ \mathbf{X}_{(I\cup J)'} > x\mathbf{1}\}, \tag{47}$$

and

$$h_x^Y = \{\mathbf{Y}_I = \mathbf{y}_I,\ \mathbf{Y}_{I'} > x\mathbf{1}\}, \tag{48}$$

where $I \cap J = \emptyset$, $0\mathbf{1} \leq \mathbf{x}_I \leq \mathbf{y}_I \leq x\mathbf{1}$ and $0\mathbf{1} \leq \mathbf{x}_J \leq x\mathbf{1}$. In view of these notations, $\mathbf{X}$ is defined to be smaller than $\mathbf{Y}$ in the hazard rate order, $\mathbf{X} \leq_{hr} \mathbf{Y}$, if for every $x \geq 0$

$$\lambda_j^X\left(x \mid \overline{h}_x^X\right) \geq \lambda_j^Y\left(x \mid h_x^Y\right), \tag{49}$$

whenever $h_x^Y \preceq \overline{h}_x^X$, $j \in (I \cup J)'$; that is, j shows a component survived in both histories and λ_j^X and λ_j^Y are the multivariate hazard rate functions defined in (34) corresponding to $\mathbf{X}$ and $\mathbf{Y}$, respectively. This order is not reflexive; that is, $\mathbf{X} \leq_{hr} \mathbf{X}$ is not necessarily the case and implies a kind of positive dependency, namely, *hazard increasing upon failure* (cf. Shaked and Shanthikumar [11, 20] and Belzunce et. al. [21]).

In many problems, we may deal with situations in which some of X_i's are identically zero. Without loss of generality, $\mathbf{X}$ can be rearranged such that just X_i, $i = 1, 2, \ldots, k$, is identically zero and the rest of the vectors are absolutely continuous. Then, we say $\mathbf{X} \leq_{hr} \mathbf{X}$ if (49) is true for $j > k$.

As a weaker order, X is said to be smaller than Y in the α-quantile residual life order, denoted as $X \leq_{\alpha-q} Y$, if for every $x \geq 0$

$$q_\alpha^X(x) \leq q_\alpha^Y(x). \tag{50}$$

Franco-Pereira et. al. [22] proved that $X \leq_{hr} Y$ if for any $\alpha \in (0, 1)$ we have $X \leq_{\alpha-q} Y$. Here, we define $\mathbf{X}$ to be smaller than $\mathbf{Y}$ in the α-quantile residual life if order $\mathbf{X} \leq_{\alpha-q} \mathbf{Y}$, if

$$q_{\alpha,j}^X\left(x \mid \overline{h}_x^X\right) \leq q_{\alpha,j}^Y\left(x \mid h_x^Y\right), \tag{51}$$

whenever $h_x^Y \preceq \overline{h}_x^X$ and for all components j alive in both histories. Like multivariate hazard rate order, it is not reflexive too. In fact $\mathbf{X} \leq_{\alpha-q} \mathbf{X}$ shows a positive dependency between components of $\mathbf{X}$. Situations in which some of X_i's are identically zero can be treated as explained for multivariate hazard rate order.

Theorem 5. *For two vectors* $\mathbf{X}$ *and* $\mathbf{Y}$, $\mathbf{X} \leq_{hr} \mathbf{Y}$ *if and only if for every* $\alpha \in (0, 1)$ *we have* $\mathbf{X} \leq_{\alpha-q} \mathbf{Y}$.

Proof. Firstly, we show that $\mathbf{X} \leq_{hr} \mathbf{Y}$ if and only if for every $x \geq 0$, $t \geq 0$, $h_x^Y \preceq \overline{h}_x^X$, and j are alive in both of them

$$P\left(X_j - x > t \mid \overline{h}_x^X\right) \leq P\left(Y_j - x > t \mid h_x^Y\right). \tag{52}$$

To achieve this, let $\overline{h}_x^X = \{\mathbf{X}_I = \mathbf{x}_I, \mathbf{X}_J = \mathbf{x}_J, \mathbf{X}_{(I\cup J)'} > x\mathbf{1}\}$ and $h_x^Y = \{\mathbf{Y}_I = \tilde{\mathbf{x}}_I, \mathbf{Y}_{I'} > x\mathbf{1}\}$. We notice that $\mathbf{X} \leq_{hr} \mathbf{Y}$ if

$$\begin{aligned}&\lim_{\epsilon\longrightarrow 0}\frac{1}{\epsilon}P_{\delta I}\left(x < Y_j < x+\epsilon, h_x^Y\right)P_{\delta(I\cup J)}\left(\overline{h}_x^X\right)\\ &\quad - \lim_{\epsilon\longrightarrow 0}\frac{1}{\epsilon}P_{\delta(I\cup J)}\left(x < X_j < x+\epsilon, \overline{h}_x^X\right)P_{\delta I}\left(h_x^Y\right)\\ &\leq 0,\end{aligned} \tag{53}$$

which is equivalent to the statement that

$$\frac{P_{\delta(I\cup J)}\left(\overline{h}_x^X\right)}{P_{\delta I}\left(h_x^Y\right)} \tag{54}$$

is decreasing in the jth survived component at x. Denote $\{\mathbf{Y}_I = \tilde{\mathbf{x}}_I, \mathbf{Y}_{I'} > x\mathbf{1}, Y_j > y\}$ by $h_x^Y[j, y]$ and assume similar notation $\overline{h}_x^X[j, x]$. Now it is easy to check that (54) is the case if

$$\frac{P_{\delta(I\cup J)}\left(\overline{h}_x^X[j, x+t]\right)}{P_{\delta(I\cup J)}\left(\overline{h}_x^X[j, x]\right)} \leq \frac{P_{\delta I}\left(h_x^Y[j, x+t]\right)}{P_{\delta I}\left(h_x^Y[j, x]\right)}, \tag{55}$$

for every $x \geq 0$, $t \geq 0$, which is apparently equivalent to (52). Thus, the result follows by definition of α-MQRL function in (35) and (52), and the proof is completed. □

Remark 6. Let x and two histories $h_x \preceq \overline{h}_x$ be fixed. It can be seen from Shaked and Shanthikumar [15] that $\mathbf{X} \leq_{hr} \mathbf{Y}$ implies

$$\left[(\mathbf{X} - x\mathbf{1})^+ \mid \overline{h}_x\right] \leq_{st} \left[(\mathbf{Y} - x\mathbf{1})^+ \mid h_x\right], \tag{56}$$

in which $(\mathbf{X} - x\mathbf{1})^+$ is the vector $((X_1 - x)^+, \ldots, (X_n - x)^+)$ and $(X_i - x)^+ = \max\{X_i - x, 0\}$, $i = 1, \ldots, n$. As can be seen from the proof of Theorem 5, we can write

$$\begin{aligned}\mathbf{X} \leq_{hr} \mathbf{Y} \Longleftrightarrow \left[(\mathbf{X} - x\mathbf{1})^+ \mid \overline{h}_x\right] &\leq_{st} \left[(\mathbf{Y} - x\mathbf{1})^+ \mid h_x\right]\\ &\text{for every } x \geq 0 \text{ and } h_x \preceq \overline{h}_x.\end{aligned} \tag{57}$$

Assume that the structure of a lifetime vector $\mathbf{X}$ satisfies the following rule. For an alive component, the more severe history it belongs to, the smaller α-MQRL values are expected; i.e., the α-MQRL is decreasing in history h_x. Intuitively, this structure describes a positive dependency between lifetimes. More precisely, we say that $\mathbf{X}$ is α-QRL decreasing upon failure (α-QRL-DF) if for every $x \geq 0$

$$q_{\alpha,i}^{X}\left(x \mid \overline{h}_x\right) \leq q_{\alpha,i}^{X}\left(x \mid h_x\right), \quad (58)$$

where $h_x \preceq \overline{h}_x$ and i is an alive component at time x in both histories. Apparently, it is equivalent to say that

$$\mathbf{X} \leq_{\alpha-q} \mathbf{X}. \quad (59)$$

Shaked and Shanthikumar [15] discussed a similar dependency based on the multivariate MRL, namely, MRL-DF property. The condition investigated in the following theorem provides a simpler investigation of α-QRL-DF property which is similar to characterizations for MRL-DF, weakened by failure (WBF), supportive lifetimes (SL), hazard rate increase upon failure (HIF), and multivariate totally positive of order 2 (MTP2) presented in Shaked and Shanthikumar [11, 15].

Theorem 7. *A sufficient and necessary condition for $\mathbf{X}$ to be α-QRL-DF is that*

$$\begin{aligned}&\left[\mathbf{X}_{I'} - x\mathbf{1} \mid \mathbf{X}_I = \mathbf{x}_I,\ X_j = x,\ \mathbf{X}_{I'-\{j\}} > x\mathbf{1}\right] \\ &\quad \leq_{\alpha-q} \left[\mathbf{X}_{I'} - x\mathbf{1} \mid \mathbf{X}_I = \mathbf{x}_I,\ \mathbf{X}_{I'} > x\mathbf{1}\right],\end{aligned} \quad (60)$$

for every $I \subset \{1,\ldots,n\}$, $j \in I'$, $x \geq 0$, and $\mathbf{0}\mathbf{1} \leq \mathbf{x}_I \leq x\mathbf{1}$.

Proof. First, we state (60) in terms of α-MQRL of $\mathbf{X}$. Rename vectors of the left hand side and right hand side of (60) by $\mathbf{X}^1$ and $\mathbf{X}^2$, respectively. Let I, j, and x be fixed and note that $\mathbf{X}^1$ has one zero value and therefore its dimension is one unit less than dimension of $\mathbf{X}^2$. Then, (60) is equivalent to

$$q_{\alpha,k}^{X^1}\left(y \mid \overline{h}_y^*\right) \leq q_{\alpha,k}^{X^2}\left(y \mid h_y^*\right), \quad (61)$$

whence $h_y^* \preceq \overline{h}_y^*$; $k \in I' - \{j\}$ shows a component alive at time y at both histories h_y^* and $\overline{h}_y^*$ and $y \geq 0$. On the other hand, we have

$$\begin{aligned}&q_{\alpha,k}^{X^1}\left(y \mid \overline{h}_y^*\right) \\ &\quad = q_{\alpha,k}^{X}\left(x + y \mid \left\{\mathbf{X}_I = \mathbf{x}_I,\ X_j = x, \overline{h}_y^*\right\}\right),\end{aligned} \quad (62)$$

and

$$q_{\alpha,k}^{X^2}\left(y \mid h_y^*\right) = q_{\alpha,k}^{X}\left(x + y \mid \left\{\mathbf{X}_I = \mathbf{x}_I, h_y^*\right\}\right) \quad (63)$$

Thus, we can write (61) in the form

$$\begin{aligned}&q_{\alpha,k}^{X}\left(x + y \mid \left\{\mathbf{X}_I = \mathbf{x}_I,\ X_j = x, \overline{h}_y^*\right\}\right) \\ &\quad \leq q_{\alpha,k}^{X}\left(x + y \mid \left\{\mathbf{X}_I = \mathbf{x}_I, h_y^*\right\}\right),\end{aligned} \quad (64)$$

for every $y \geq 0$, $h_y^* \preceq \overline{h}_y^*$ and $k \in I' - \{i\}$ which is alive in both histories. Clearly, (58) implies (64) which by the fact that I, j, and x are arbitrary proves the necessity part.

To show sufficiency, assume that (60) holds and $h_t \preceq \overline{h}_t$ are two arbitrary histories up to time $t \geq 0$. We need to show that

$$q_{\alpha,k}^{X}\left(t \mid \overline{h}_t\right) \leq q_{\alpha,k}^{X}\left(t \mid h_t\right) \quad (65)$$

for every component k survived at t in $\overline{h}_t$. Note that $\overline{h}_t$ requires at least one recorded failure with failure time strictly greater than maximum failure times of h_t, to be more sever than h_t. If both of these histories contain same failed components which occur in same times, denote this set of common failed components and their failure times by I and $\mathbf{x}_I$, respectively. If there are not such common failures, set $I = \emptyset$. Then, if $I \neq \emptyset$, let j be the first component (which its existence is guaranteed) failed after components I and x denote its failure time. When $I = \emptyset$, let j and x be the first failed component of $\overline{h}_t$ and its failure time, respectively. Now, two different cases are possible in history h_t about component j:

(i) it be alive at time t

(ii) it fails at a time $\mathring{x}$ such that $x \leq \mathring{x} \leq t$.

Based on these arrangements, $\overline{h}_t$ and h_t can be written in one of the following cases (a) or (b):

(a) $\overline{h}_t = \{\mathbf{X}_I = \mathbf{x}_I, X_j = x, \mathbf{X}_A = \mathbf{x}_A, \mathbf{X}_B = \mathbf{x}_B, \mathbf{X}_C > t\mathbf{1}\}$,
and
$h_t = \{\mathbf{X}_I = \mathbf{x}_I, \mathbf{X}_A = \mathring{\mathbf{x}}_A, \mathbf{X}_{B\cup C\cup\{j\}} > t\mathbf{1}\}$,
where $x\mathbf{1} \leq \mathbf{x}_A \leq \mathring{\mathbf{x}}_A \leq t\mathbf{1}$ and $x\mathbf{1} \leq \mathbf{x}_B \leq t\mathbf{1}$.

(b) $\overline{h}_t = \{\mathbf{X}_I = \mathbf{x}_I, X_j = x, \mathbf{X}_A = \mathbf{x}_A, \mathbf{X}_B = \mathbf{x}_B, \mathbf{X}_C > t\mathbf{1}\}$,
and
$h_t = \{\mathbf{X}_I = \mathbf{x}_I, X_j = \mathring{x}, \mathbf{X}_A = \mathring{\mathbf{x}}_A, \mathbf{X}_{B\cup C} > t\mathbf{1}\}$,
where $x\mathbf{1} \leq \mathbf{x}_A \leq \mathring{\mathbf{x}}_A \leq t\mathbf{1}$, $x\mathbf{1} \leq \mathbf{x}_B \leq t\mathbf{1}$, and $x \leq \mathring{x} \leq t$.

In either case, (60) implies that $q_{\alpha,k}^{X}(t \mid \overline{h}_t) \leq q_{\alpha,k}^{X}(t \mid h_t)$ for every alive component k at time t in history $\overline{h}_t$, and this completes the proof. □

Remark 8. A random vector $\mathbf{X}$ is said to hold positive dependency property SL if

$$\begin{aligned}&\left[\mathbf{X}_{I'} - x\mathbf{1} \mid \mathbf{X}_I = \mathbf{x}_I,\ X_j = x,\ \mathbf{X}_{I'-\{j\}} > x\mathbf{1}\right] \\ &\quad \leq_{hr} \left[\mathbf{X}_{I'} - x\mathbf{1} \mid \mathbf{X}_I = \mathbf{x}_I,\ \mathbf{X}_{I'} > x\mathbf{1}\right],\end{aligned} \quad (66)$$

while $I \subset \{1,\ldots,n\}$, $j \in I'$, $x \geq 0$, and $\mathbf{0}\mathbf{1} \leq \mathbf{x}_I \leq x\mathbf{1}$ (cf. Shaked and Shanthikumar [11] for more information). Now by Theorem 5, it is immediate that

$$\mathbf{X} \text{ is SL} \Longleftrightarrow \mathbf{X} \text{ is } \alpha\text{-QRL-DF for all } \alpha \in (0,1). \quad (67)$$

5. Decreasing α-MQRL Class of Life Distributions

The study of changes in the properties of any model, as the constituent components vary, is of great interest. A univariate random lifetime X is said to accommodate decreasing α-QRL if $q_\alpha(x)$ is decreasing in $x \geq 0$ or one of the following equivalent conditions hold:

$$\left[X - x' \mid X \geq x'\right] \leq_{\alpha-q} X, \tag{68}$$

$$\left[X - x' \mid X \geq x'\right] \leq_{\alpha-q} [X - x \mid X \geq x], \quad x' \geq x \geq 0. \tag{69}$$

To extend these conditions in the dynamic multivariate setting, we need to generalize the concept of comparing severity of two histories, discussed in Section 4, in the case that their lengths are not necessarily equal. Consider two histories h_x and $\overline{h}_{x'}$ with different lengths $x \leq x'$. Then, $\overline{h}_{x'}$ is referred to be more severe than h_x, notationally $h_x \preceq \overline{h}_{x'}$, if $h_x \preceq \overline{h}_x$, where $\overline{h}_x$ contains all information of $\overline{h}_{x'}$ over $[0, x]$.

Define a shift operator θ_x on a random vector $\mathbf{X}$ by $\theta_x \mathbf{X} = (\mathbf{X} - x\mathbf{1})^+$, $x \geq 0$. For a random vector $\mathbf{X}$, we extend the arguments (68) and (69), respectively, by

$$\left[\theta_x \mathbf{X} \mid h_x\right] \leq_{\alpha-q} \mathbf{X}, \quad x \geq 0, \tag{70}$$

for every arbitrary history h_x and

$$\left[\theta_{x''} \mathbf{X} \mid \overline{h}_{x''}\right] \leq_{\alpha-q} \left[\theta_{x'} \mathbf{X} \mid h_{x'}\right], \quad x'' \geq x' \geq 0, \tag{71}$$

whence $h_{x'} \preceq \overline{h}_{x''}$. These extensions are similar to conditions for multivariate decreasing mean residual life by Shaked and Shanthikumar [15]. Another possible extension of (69) is

$$\left[\theta_{x'} \mathbf{X} \mid h_{x'}\right] \leq_{\alpha-q} \left[\theta_x \mathbf{X} \mid h_x\right], \quad x' \geq x \geq 0, \tag{72}$$

where h_x and $h_{x'}$ coincide on the interval $[0, x]$. This means that all failed components in h_x are failed in $h_{x'}$ too with equal failure times. The condition (72) is similar to the multivariate increasing failure rate, proposed in Arjas [7], and a multivariate decreasing mean residual life notion of Shaked and Shanthikumar [15].

Here we need some further notations to prove the following result. For $a < b$, let $h_{[a,b]}$ represent a history which gives us those components alive at a and components failed in the interval $[a, b]$ along with their failure times. Then, the previously defined history h_t is same as $h_{[0,t]}$. We say that $\overline{h}_{[a+u,b+u]}$, $u \geq 0$, is more severe than $h_{[a,b]}$, denoted as $h_{[a,b]} \preceq \overline{h}_{[a+u,b+u]}$, if every alive component at $a+u$ in history $\overline{h}_{[a+u,b+u]}$ is alive at a in $h_{[a,b]}$ too and if a component fails in history $h_{[a,b]}$ at time x, it also fails in $\overline{h}_{[a+u,b+u]}$ at some time $x_0 + u$ such that $x_0 \leq x$.

Let $0 \leq a < b < c$ be fixed times. For two histories $h_{[a,b]}$, and $\overline{h}_{[b,c]}$, define the aggregated history $h_{[a,b]} \oplus \overline{h}_{[b,c]}$ to be a history over the interval $[a, c]$, which shows the same alive components at time a as $h_{[a,b]}$ shows at a and contains all of the information that $h_{[a,b]}$ and $\overline{h}_{[b,c]}$ have, respectively, on the intervals $[a, b]$ and $[c, d]$.

Theorem 9. *Three conditions (70), (71), and (72) are equivalent.*

Proof. It is readily detectable that

$$(71) \Longrightarrow (72) \Longrightarrow (70). \tag{73}$$

Thus, it is sufficient to show that (70) $\Longrightarrow$ (71). Let (70) be true. Thus, we have

$$q^X_{\alpha,j}\left(x + s \mid h_x \oplus \overline{h}^1_{[x,x+s]}\right) \leq q^X_{\alpha,j}\left(s \mid h^1_{[0,s]}\right), \tag{74}$$

for every $x, s \geq 0$, $h^1_{[0,s]} \preceq \overline{h}^1_{[x,x+s]}$, and h_x, for components j alive at $x + s$ in history $\overline{h}^1_{[x,x+s]}$. Now we must show that $\mathbf{X}$ satisfies (71), that is,

$$q^X_{\alpha,j}\left(x'' + t \mid \overline{\hbar}_{x''} \oplus \overline{h}^2_{[x'',x''+t]}\right) \leq q^X_{\alpha,j}\left(x' + t \mid \hbar_{x'} \oplus h^2_{[x',x'+t]}\right), \tag{75}$$

for every $x'' \geq x' \geq 0$, $t \geq 0$, $\hbar_{x'} \preceq \overline{\hbar}_{x''}$, and $h^2_{[x',x'+t]} \preceq \overline{h}^2_{[x'',x''+t]}$.

Let $t, x'', x', \hbar_{x'} \preceq \overline{\hbar}_{x''}$, and $h^2_{[x',x'+t]} \preceq \overline{h}^2_{[x'',x''+t]}$ be given. We set $s = x' + t \geq 0$ and $x = x'' - x' \geq 0$. Moreover, set $h^1_{[0,s]}$, $\overline{h}^1_{[x,x+s]}$, and h_x to be $\hbar_{x'} \oplus h^2_{[x',x'+t]}$, $\overline{\hbar}_{[x''-x',x'']} \oplus \overline{h}^2_{[x'',x''+t]}$, and $\overline{\hbar}_{[0,x''-x']}$. By these considerations, it can be easily checked that (74) implies (75), which completes the proof immediately. □

In the presence of Theorem 9, the following definition is well defined.

Definition 10. A random vector $\mathbf{X}$ is decreasing multivariate α-quantile residual life (D α-MQRL) if one of the conditions (70), (71), or (72) is the case.

If X satisfies condition (70), then apparently we have $\mathbf{X} \leq_{\alpha-q} \mathbf{X}$. This means that the D α-MQRL implies the positive dependency property α-QRL-DF.

Shaked and Shanthikumar [12] proposed $\mathbf{X}$ to be multivariate increasing failure rate (MIFR) when

$$\left[\theta_{x'} \mathbf{X} \mid h_{x'}\right] \leq_{hr} \mathbf{X}, \tag{76}$$

for every $x' \geq 0$ and history $h_{x'}$. It is a proper extension of the following condition:

$$\left[X - x' \mid X \geq x'\right] \leq_{hr} X. \tag{77}$$

When it holds, X is said to accommodate an increasing failure rate form.

Theorem 11. *The random vector* $\mathbf{X}$ *is MIFR if it is D* α*-MQRL for every* $\alpha \in (0, 1)$.

Proof. The proof is followed immediately by (70), (76), and Theorem 5. □

Example 12. Ross [23] considered a system composed of n possibly dependent components, labelled by $1, 2, \ldots, n$, starting their work at time zero. The system satisfies a Markovian property in the sense that the failure rate of an alive component at any time, namely, $t \geq 0$, just depends on the set of alive components at that time. Suppose that $I' \subset \{1, 2, \ldots, n\}$ denote the set of alive components at some time t. Then, for a component $k \in I'$, the failure rate function at t, $\lambda_k(t \mid h_t)$, reduces to

$$\lambda_k \left(t \mid I'\right) = \lim_{\Delta \longrightarrow 0} \frac{1}{\Delta} P\left(t \leq X_k < t + \Delta \mid I'\right). \quad (78)$$

Applying Theorem 1 from Shaked and Shanthikumar [12], this model follows MIFR property and in turn D α-MQRL for every $\alpha \in (0, 1)$ if

$$\lambda_k \left(t \mid J'\right) \geq \lambda_k \left(t \mid I'\right), \quad (79)$$

for every $J' \subset I' \subset \{1, 2, \ldots, n\}$, where they show alive components at t and $k \in J'$.

6. Conclusion

The dynamic α-MQRL measure proposed in this paper is useful in both theoretical and applied aspects of reliability theory and survival analysis. It has been shown that this measure is closely related to the conditional hazard rate functions. In the bivariate case, the α-QRL of a survived component at time $x > 0$, given that the other one has failed at some prior time, is decreasing when its corresponding conditional hazard rate is increasing. However, the behaviour of initial α-QRL functions is affected by the dependency structure of the components. When the corresponding conditional hazard rates are increasing, the positive dependency relieves the rate of descending of initial α-QRL functions, as results show that similar spirit holds in the multivariate case. A new multivariate stochastic order, namely, α-MQRL order, has been defined. The results reveal that α-MQRL order is weaker than the multivariate hazard rate order. However, the α-MQRL order for every $\alpha \in (0, 1)$ implies the multivariate hazard rate order. Like some other multivariate orders, this order is not reflexive too. In fact, the statement that a vector is less than itself implies a positive dependency structure between components. This dependency is weaker than the supportive lifetime property discussed in Shaked and Shanthikumar [11]. The class of multivariate distributions with decreasing α-MQRL functions has been defined. It has been shown that this class includes the class of distributions following increasing multivariate hazard rate functions. Nevertheless, the following topics are interesting and still remain as open problems:

(i) Find out how closely the α-MQRL functions characterize the corresponding distributions. Specially, in the bivariate case it leads us to solving the following functional equations in terms of $\overline{F}$:

$$\begin{aligned} \overline{F}\left(\varphi_1(x), x\right) &= \overline{\alpha}\overline{F}(x, x), \\ \overline{F}\left(x, \varphi_2(x)\right) &= \overline{\alpha}\overline{F}(x, x), \\ \frac{\partial}{\partial x_2}\overline{F}\left(\varphi_3(x \mid x_2), x_2\right) &= \overline{\alpha}\frac{\partial}{\partial x_2}\overline{F}(x, x_2), \\ \frac{\partial}{\partial x_1}\overline{F}\left(x_1, \varphi_4(x \mid x_1)\right) &= \overline{\alpha}\frac{\partial}{\partial x_1}\overline{F}(x_1, x) \end{aligned} \quad (80)$$

where $\varphi_i(x) = x + q_{\alpha,i}(x)$, $i = 1, 2$, $\varphi_3(x \mid x_2) = x + q_{\alpha,1}(x \mid x_2)$, and $\varphi_4(x \mid x_1) = x + q_{\alpha,2}(x \mid x_1)$.

(ii) Provide a nonparametric estimator of the α-MQRL functions when $\mathbf{X}_i$, $i = 1, 2, \ldots, n$, is a sample of n independent and identically distributed random vectors.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Authors' Contributions

All of the authors have equally made contributions. All authors read and approved the final manuscript.

Acknowledgments

The second and third authors would like to extend their sincere appreciation to the Deanship of Scientific Research at King Saud University for funding this research group (no. RGP-1435-036).

References

[1] D. C. Schmittlein and D. G. Morrison, "The Median Residual Lifetime: A Characterization Theorem and an Application," *Operations Research*, vol. 29, no. 2, pp. 392–399, 1981.

[2] H. Joe and F. Proschan, "Percentile residual life functions," *Operations Research*, vol. 32, no. 3, pp. 668–678, 1984.

[3] R. C. Gupta and E. S. Langford, "On the determination of a distribution by its median residual life function: a functional equation," *Journal of Applied Probability*, vol. 21, no. 1, pp. 120–128, 1984.

[4] A. M. Franco-Pereira and J. De Uña-Álvarez, "Estimation of a monotone percentile residual life function under random censorship," *Biometrical Journal*, vol. 55, no. 1, pp. 52–67, 2013.

[5] R. E. Lillo, "On the median residual lifetime and its aging properties: A characterization theorem and applications," *Naval Research Logistics (NRL)*, vol. 52, no. 4, pp. 370–380, 2005.

[6] N. L. Johnson and S. Kotz, "A vector multivariate hazard rate," *Journal of Multivariate Analysis*, vol. 5, no. 1, pp. 53–66, 1975.

[7] E. Arjas and I. Norros, "Life lengths and association: a dynamic approach," *Mathematics of Operations Research*, vol. 9, no. 1, pp. 151–158, 1984.

[8] B. C. Arnold and H. Zahedi, "On multivariate mean remaining life functions," *Journal of Multivariate Analysis*, vol. 25, no. 1, pp. 1–9, 1988.

[9] F. Baccelli and A. M. Makowski, "Multidimensional stochastic ordering and associated random variables," *Operations Research*, vol. 37, no. 3, pp. 478–487, 1989.

[10] K. R. M. Nair and N. U. Nair, "Bivariate Mean Residual Life," *IEEE Transactions on Reliability*, vol. 38, no. 3, pp. 362–364, 1989.

[11] M. Shaked and J. G. Shanthikumar, "Multivariate stochastic orderings and positive dependence in reliability theory," *Mathematics of Operations Research*, vol. 15, no. 3, pp. 545–552, 1990.

[12] M. Shaked and J. G. Shanthikumar, "Dynamic multivariate aging notions in reliability theory," *Stochastic Processes and Their Applications*, vol. 38, no. 1, pp. 85–97, 1991.

[13] H. V. Kulkarni and R. N. Rattihalli, "Nonparametric estimation of a bivariate mean residual life function," *Journal of the American Statistical Association*, vol. 97, no. 459, pp. 907–917, 2002.

[14] T. Hu, B.-E. Khaledi, and M. Shaked, "Multivariate hazard rate orders," *Journal of Multivariate Analysis*, vol. 84, no. 1, pp. 173–189, 2003.

[15] M. Shaked and J. G. Shanthikumar, "Dynamic multivariate mean residual life functions," *Journal of Applied Probability*, vol. 28, no. 03, pp. 613–629, 1991.

[16] M. S. Noughabi and M. Kayid, "Bivariate quantile residual life: a characterization theorem and statistical properties," *Statistical Papers*, pp. 1–12.

[17] D. R. Cox, "Regression models and life tables," *Journal of the Royal Statistical Society Series B*, vol. 34, pp. 187–202, 1972.

[18] A. W. Marshall and I. Olkin, "A multivariate exponential distribution," *Journal of the American Statistical Association*, vol. 62, no. 317, pp. 30–44, 1967.

[19] M. Shaked and J. G. Shanthikumar, *Stochastic Orders*, Springer Series in Statistics, Springer, New York, NY, USA, 2007.

[20] M. Shaked and J. G. Shanthikumar, "Multivariate conditional hazard rate and mean residual life functions and their applications," in *Reliability and Decision Making*, R. E. Barlow, C. A. Clarotti, and F. Spizzichino, Eds., Chapman, 1993.

[21] F. Belzunce, J.-A. Mercader, J.-M. Ruiz, and F. Spizzichino, "Stochastic comparisons of multivariate mixture models," *Journal of Multivariate Analysis*, vol. 100, no. 8, pp. 1657–1669, 2009.

[22] A. M. Franco-Pereira, R. E. Lillo, J. Romo, and M. Shaked, "Percentile residual life orders," *Applied Stochastic Models in Business and Industry*, vol. 27, no. 3, pp. 235–252, 2011.

[23] S. M. Ross, "A model in which component failure rates depend on the working set," *Naval Research Logistics Quarterly*, vol. 31, no. 2, pp. 297–300, 1984.