Hindawi
Mathematical Problems in Engineering
Volume 2018, Article ID 1345346, 14 pages
https://doi.org/10.1155/2018/1345346

Research Article

Research on Time-Space Fractional Model for Gravity Waves in Baroclinic Atmosphere

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Received 8 August 2018; Accepted 14 October 2018; Published 24 October 2018

Academic Editor: Ivo Petras

The research of gravity solitary waves movement is of great significance to the study of ocean and atmosphere. Baroclinic atmosphere is a complex atmosphere, and it is closer to the real atmosphere. Thus, the study of gravity waves in complex atmosphere motion is becoming increasingly essential. Deriving fractional partial differential equation models to describe various waves in the atmosphere and ocean can open up a new window for us to understand the fluid movement more deeply. Generally, the time fractional equations are obtained to reflect the nonlinear waves and few space-time fractional equations are involved. In this paper, using multiscale analysis and perturbation method, from the basic dynamic multivariable equations under the baroclinic atmosphere, the integer order mKdV equation is derived to describe the gravity solitary waves which occur in the baroclinic atmosphere. Next, employing the semi-inverse and variational method, we get a new model under the Riemann-Liouville derivative definition, i.e., space-time fractional mKdV (STFmKdV) equation. Furthermore, the symmetry analysis and the nonlinear self-adjointness of STFmKdV equation are carried out and the conservation laws are analyzed. Finally, adopting the $\exp(-\Phi(\xi))$ method, we obtain five different solutions of STFmKdV equation by considering the different cases of the parameters (η, σ). Particularly, we study the formation and evolution of gravity solitary waves by considering the fractional derivatives of nonlinear terms.

1. Introduction

With the intercross and penetration of different knowledge, Rossby solitary waves have been applied to many fields successfully, such as physical oceanography, atmospheric science, hydraulic engineering, and communication engineering. Particularly, Rossby solitary waves have important theoretical significance and research value in marine atmospheric science. They have largely determined the impact of the oceans on the atmosphere and other climate change. On the time scale, the energy of Rossby waves determines the ocean energy spectrum, which makes the energy spread from east to west to form and maintain a strong ocean boundary flow, such as Kuroshio, Gulfstream, and East Australian flow. Great achievements have been made in this regard.

As we know, Rossby solitary waves in the westerly shear flow were first found by Long [1]. He found that the amplitude of the Rossby waves satisfied KdV equation by the β-plane approximation

$$u_t + \mu u u_x + \delta u_{xxx} = 0. \tag{1}$$

With the development of Rossby waves theory, Wadati [2] derived the modified KdV equation

$$u_t + \mu u^2 u_x + \delta u_{xxx} = 0. \tag{2}$$

In view of the barotropic fluid and stratified fluid model, the KdV model and the mKdV model are also generated to describe the generation and evolution of Rossby solitary waves by Redekopp [3]. Apart from the KdV model and the mKdV model, for other initial disturbance, employing a different time and space stretching transform, Boussinesq equation was derived by Meng and Lv [4]. Afterwards, the Rossby parameters β along with the changes of latitude were discussed by Luo [5] and generalized β-plane approximation was obtained. In recent years, in the theoretical study of Rossby waves, many new wave equations were obtained to describe the generation and evolution of various types of fluctuations in the ocean [6, 7], such as NLS equation, ILW-Burgers equation, and ZK-Burgers equation. In the past, predecessors studied wave equations in the barotropic atmospheric environment by using the β-plane approximation. But we know that the basic dynamic equations for describing the baroclinic atmospheric movement are more in line with the actual situation and are very complicated. And the baroclinic problem in real atmosphere is inevitable [8]. In this paper, starting with the basic equations adopting the Bousinesq approximation [9] and under the baroclinic atmospheric environment, using multiscale analysis and turbulence method, we get a new model (mKdV) to describe the Rossby solitary waves. The advantages of basic equations are as follows:

(1) The equations are multivariate, and the physical meaning of each variable is clear;

(2) The baroclinic atmosphere problem is considered to help us understand the generation and evolution of isolated waves in the ocean.

In recent years, the study of integer partial differential equations has yielded many brilliant achievements [10–15]. Simultaneously, it has been found that fractional order partial differential equations also play an important role in many fields [16–22]. The fractional differentiation calculus [23, 24] was first developed by Liouville primary. Liouville expands the function into an exponential form and defines the q-order derivatives of this expanded form term by term. Afterwards, Riemann proposed a different definition that can be implemented to a power series with a negative power term. Finally, Ross and Oldham [25, 26] unified the two definitions, so that the application of fractional differential was further developed. Subsequently, a version of the Euler-Lagrange equations for problems of calculus of variation with fractional derivatives was formulated by Riewe in 1990s [27, 28]. Recently, Agrawal [29, 30] studied the fractional Euler-Lagrange equation deeper and a series of new methods have been put forward in his research, which provide a new idea for us to study fractional partial differential equations [31, 32]. The fact has shown that the new fractional equation is more suitable than the integer order equation due to the precise description of the nonlinear phenomena[33, 34]. At the same time, in the field of oceanography, the fractional partial differential equation can better describe the generation and evolution of solitary waves, which is more favorable for us to study the theory of fluctuation.

Similar to the study of integer order differential equation, the conservation laws of the fractional differential equation are an important branch. As we know, if the fractional differential equation is an Euler-Lagrange equation, then conservation laws can be found using Noethem's theorem by variational Lie point symmetries of this equation [35–37]. Lie symmetry analysis was proposed by Sophus Lie. The main idea of this method is that the infinitesimal transformation keeps the solution set of the partial differential equation unchanged. The Lie symmetry analysis offers an efficient and powerful tool for the study of conservation laws of fractional partial differential equation [38–42]. For this reason, the researchers are very interested in studying the symmetry analysis of fractional differential equations. As far as we know, in the past, the symmetric method was only used for time fraction partial differential equations (TFPDE), but has not been used to analyze space and time fraction partial differential equations (STFPDE) [43, 44]. In this paper, the Lie symmetry analysis was used to study the conservation law of the STFmKdV equation [45, 46].

By studying the work of predecessors, we can find that several methods have been used to solve nonlinear partial differential equations, such as the trial equation method [47, 48], Hirota bilinear method [49], binary nonlinearization method [50], Darboux transformation method [51], Jacobi iteration method [52], (G'/G)-expansion method [53–55], exp-function method [48], sub-equation method [56], and others [57, 58]. Therefore, it is an important task to find an accurate and effective method to solve the fractional differential equation.

This paper is organized as follows: In Section 2, using multiscale analysis and turbulence method, from the basic dynamic multivariable equations under the baroclinic atmosphere [59, 60], the integer order mKdV equation is derived. In Section 3, we use the semi-inverse method to derive the Lagrangian form of the mKdV equation [61, 62] firstly. Then the Lagrangian space and time operator of the mKdV equation have been transformed into the fractional domain of the left Riemann-Liouville fractional differential operator. Finally, applying the variational method, we derive the STFmKdV from this Euler-Lagrangian equation. In Section 4, we first study the symmetry analysis of the fractional equation to obtain the corresponding infinitesimal generator of the equation. Then we discuss the nonlinear self-adjointness of the STFmKdV equation and finally get the conservation vectors of the equation. In Section 5, based on the STFmKdV equation, employing the $\exp(-\Phi(\xi))$ method, and considering the different cases of the parameters(η, σ), we obtain five different solutions of the equation.

2. Derivation of the mKdV Equation

Using the sum of disturbance pressure gradient force and buoyancy force expressing the vertical pressure gradient force and gravity force, and adopting the Bousinesq approximation [60], the dimensionless basic dynamic equations of atmospheric motion are as follows [59]:

$$\frac{\partial u'}{\partial t'} + \frac{U}{fL}\left(u'\frac{\partial u'}{\partial x'} + v'\frac{\partial u'}{\partial y'} + w'\frac{\partial u'}{\partial z'}\right) = -\frac{1}{\rho_s}\frac{\partial p'}{\partial x'} + v',$$

$$\frac{\partial v'}{\partial t'} + \frac{U}{fL}\left(u'\frac{\partial v'}{\partial x'} + v'\frac{\partial v'}{\partial y'} + w'\frac{\partial v'}{\partial z'}\right) = -\frac{1}{\rho_s}\frac{\partial p'}{\partial y'} - u',$$

$$\frac{\partial w'}{\partial t'} + \frac{U}{fL}\left(u'\frac{\partial w'}{\partial x'} + v'\frac{\partial w'}{\partial y'} + w'\frac{\partial w'}{\partial z'}\right) = \frac{gL\delta\theta}{DfU\theta_0}\left(-\frac{1}{\rho_s}\frac{\partial p'}{\partial z'} + \theta'\right),$$

$$\frac{\partial\theta'}{\partial t'}+\frac{U}{fL}\left(u'\frac{\partial\theta'}{\partial x'}+v'\frac{\partial\theta'}{\partial y'}\right)+\frac{\sigma UD}{fL\delta\theta}w'=0,$$
$$\frac{\partial\left(\rho_s u'\right)}{\partial x'}+\frac{\partial\left(\rho_s v'\right)}{\partial y'}+\frac{\partial\left(\rho_s w'\right)}{\partial z'}=0. \tag{3}$$

where u', v' are the level of the air speed, w' is the vertical velocity, p' is the atmospheric pressure, θ is the temperature field, and f is the Coriolis parameter. θ_0 is the potential of environmental flow field and ρ_s is the density of environmental flow field; they are both the height functions.

Because the second term of the fourth formula in the left side is lesser, we get the following approximation:

$$\delta\theta\sim\frac{\sigma UD}{fL},$$
$$\frac{U}{fL}\sim o(1). \tag{4}$$

Let the parameter $\varepsilon = f^2/N^2(\varepsilon \ll 1)$, $N^2 = g\sigma/\theta_0$. By varying, (3) transforms to

$$\frac{\partial u'}{\partial t'}+u'\frac{\partial u'}{\partial x'}+v'\frac{\partial u'}{\partial y'}+w'\frac{\partial u'}{\partial z'}=-\frac{1}{\rho_s}\frac{\partial p'}{\partial x'}+v',$$
$$\frac{\partial v'}{\partial t'}+u'\frac{\partial v'}{\partial x'}+v'\frac{\partial v'}{\partial y'}+w'\frac{\partial v'}{\partial z'}=-\frac{1}{\rho_s}\frac{\partial p'}{\partial y'}-u',$$
$$\frac{\partial w'}{\partial t'}+u'\frac{\partial w'}{\partial x'}+v'\frac{\partial w'}{\partial y'}+w'\frac{\partial w'}{\partial z'}$$
$$=\varepsilon^{-1}\left(-\frac{1}{\rho_s}\frac{\partial p'}{\partial x'}+\theta'\right),$$
$$\frac{\partial\theta'}{\partial t'}+u'\frac{\partial\theta'}{\partial x'}+v'\frac{\partial\theta'}{\partial y'}+w'=0,$$
$$\frac{\partial\left(\rho_s u'\right)}{\partial x'}+\frac{\partial\left(\rho_s v'\right)}{\partial y'}+\frac{\partial\left(\rho_s w'\right)}{\partial z'}=0. \tag{5}$$

We introduce the multiscale variables (omitting the sign at the top right corner of the variables)

$$t=\varepsilon^{3/2}t',$$
$$x=\varepsilon^{1/2}x',$$
$$y=y',$$
$$z=z', \tag{6}$$

so long time and space scales are defined as

$$\frac{\partial}{\partial t}=\varepsilon^{3/2}\frac{\partial}{\partial t'},$$
$$\frac{\partial}{\partial x}=\varepsilon^{1/2}\frac{\partial}{\partial x'},$$
$$\frac{\partial}{\partial y}=\frac{\partial}{\partial y'},$$
$$\frac{\partial}{\partial z}=\frac{\partial}{\partial z'}. \tag{7}$$

Further, according to the small parameter ε, u', v', w', p', θ' in (5) can be expended:

$$u'=-\int_0^y\left(U(\varsigma,z)-c+\varepsilon\lambda\right)d\varsigma+\varepsilon^{1/2}u_0+\varepsilon u_1$$
$$+\varepsilon^{3/2}u_2+\varepsilon^2u_3+\cdots,$$
$$v'=\varepsilon v_0+\varepsilon^{3/2}v_1+\varepsilon^2v_2+\varepsilon^{5/2}v_3+\cdots,$$
$$w'=\varepsilon w_0+\varepsilon^{3/2}w_1+\varepsilon^2w_2+\varepsilon^{5/2}w_3+\cdots,$$
$$\theta'=\Theta(y,z)+\varepsilon^{1/2}\theta_0+\varepsilon\theta_1+\varepsilon^{3/2}\theta_2+\varepsilon^2\theta_3+\cdots,$$
$$p'=P(y,z)+\varepsilon^{1/2}p_0+\varepsilon p_1+\varepsilon^{3/2}p_2+\varepsilon^2p_3+\cdots, \tag{8}$$

where U, P, Θ are the function of y, z. U is the speed of the basic flow, P is the air pressure, and Θ is the temperature field. Obviously, the zonal flow is in the following forms:

$$U=\begin{cases}U_y=0, & \text{as } 0\le y<y_0,\\ \text{costant}, & \text{as } y\ge y_0,\end{cases} \tag{9}$$

and the boundary conditions of (3) are

$$p'=0,$$
$$\text{as } y=0,$$
$$p'\longrightarrow 0,$$
$$\text{as } y\longrightarrow\infty. \tag{10}$$

Substituting (7) and (8) into (3), we get each order form for ε as follows:

$$O\left(\varepsilon^0\right):\begin{cases}\dfrac{1}{\rho_s}\dfrac{\partial p}{\partial y}-\displaystyle\int_0^y(U-c)\,d\varsigma=0,\\ \dfrac{1}{\rho_s}\dfrac{\partial p}{\partial z}-\Theta=0.\end{cases} \tag{11}$$

Assuming $|1/\rho_s^2|\ll 1$, we have

$$\frac{\partial}{\partial z}\left(\int_0^y U d\varsigma\right)=\Theta_y. \tag{12}$$

Next, we write the first-, second-, and third-order approximation for ε as the following form:

$$\int_0^y(U-c)\,d\varsigma\frac{\partial u_i}{\partial x}+(U-c+1)\,v_i+\Theta y w_i-\frac{1}{\rho_s}\frac{\partial p_i}{\partial x}$$
$$=Au_i,$$

$$\begin{aligned}
&\frac{1}{\rho_s}\frac{\partial p_i}{\partial y}+u_i=Av_i,\\
&\frac{1}{\rho_s}\frac{\partial p_i}{\partial z}-\theta_i=Aw_i,\\
&\int_0^y (U-c)\,d\varsigma\frac{\partial \theta_i}{\partial x}-\Theta y v_i-w_i=A\theta_i,\\
&\frac{\partial \rho_s u_i}{\partial x}+\frac{\partial \rho_s v_i}{\partial y}+\frac{\partial \rho_s w_i}{\partial z}=0,\\
&\qquad i=0,1,2,3\cdots
\end{aligned} \tag{13}$$

where

$$Au_0=Av_0=Aw_0=A\theta_0=0. \tag{14}$$

By eliminating u_0,v_0,w_0,θ_0 in (13), we can obtain the equation for p_0

$$L_{y,z}\left(\frac{\partial p_0}{\partial x}\right)=0, \tag{15}$$

where

$$\begin{aligned}
\Omega&=U_y-1+U_z^2,\\
\Omega_y&=\frac{\partial\Omega}{\partial y},\\
\Omega_z&=\frac{\partial\Omega}{\partial z},\\
L_{y,z}&=\frac{\partial^2}{\partial y^2}-\left(U_y-1\right)\frac{\partial^2}{\partial z^2}+2U_z\frac{\partial^2}{\partial y\partial z}\\
&\quad+\left[U_{zz}-\frac{\Omega_y}{\Omega}-U_z\frac{\Omega_z}{\Omega}\right]\frac{\partial}{\partial y}\\
&\quad+\left[\left(U_y-1\right)\frac{\Omega_z}{\Omega}-U_z\frac{\Omega_y}{\Omega}\right]\frac{\partial}{\partial z}\\
&\quad-\frac{1}{U}\left[U_{zz}-\frac{\Omega_y}{\Omega}-U_z\frac{\Omega_z}{\Omega}\right].
\end{aligned} \tag{16}$$

Clearly, (15) is a variable separable equation. Assume its solution is

$$p_0=\widetilde{p_0}\left(y,z\right)A\left(t,x\right), \tag{17}$$

and under a certain definite solution condition, we can get $\widetilde{p_0}$. Further, all the solutions of (15) can be obtained:

$$\begin{aligned}
u_0&=\widetilde{u_0}\left(y,z\right)A\left(t,x\right),\\
v_0&=\widetilde{v_0}\left(y,z\right)A_x\left(t,x\right),\\
w_0&=\widetilde{w_0}\left(y,z\right)A_x\left(t,x\right),\\
\theta_0&=\widetilde{\theta_0}\left(y,z\right)A\left(t,x\right).
\end{aligned} \tag{18}$$

Further, we know that

$$\begin{aligned}
Au_1&=u_0\frac{\partial u_0}{\partial x}+v_0\frac{\partial u_0}{\partial y}+w_0\frac{\partial u_0}{\partial z},\\
Av_1&=0,\\
Aw_1&=0,\\
A\theta_1&=u_0\frac{\partial \theta_0}{\partial x}+v_0\frac{\partial \theta_0}{\partial y}.
\end{aligned} \tag{19}$$

Similarly, we cannot get the equation for $A(x,t)$. So, we eliminate u_1,v_1,w_1,θ_1 in (13) and (19), and we can obtain the equation of p_1

$$L_{y,z}\left(\frac{\partial p_1}{\partial x}\right)=\ell_{1y,z}\left(Au_1\right)+\ell_{2y,z}\left(A\theta_1\right), \tag{20}$$

where

$$\begin{aligned}
\ell_{1y,z}&=-\left[-\frac{\partial}{\partial y}-U_{zz}-U_z\frac{\partial}{\partial z}-\frac{1}{U-c}\right.\\
&\quad\left.-\frac{1}{\triangle}\left(\triangle_y+U_z\triangle_z\right)\right],\\
\ell_{2y,z}&=\frac{-1}{U-c}\left[\frac{\partial}{\partial y}+U_{zz}+U_z\frac{\partial}{\partial z}\right.\\
&\quad\left.-\frac{1}{\triangle}\left(\triangle_y+U_z\triangle_z\right)\right]
\end{aligned} \tag{21}$$

Substituting (19) into (20) and observing both ends of (20), we can get the solutions for the variables as follows:

$$\begin{aligned}
u_1&=\widetilde{u_1}\left(y,z\right)A^2\left(t,x\right),\\
v_1&=\widetilde{v_1}\left(y,z\right)A\left(t,x\right)A_x\left(t,x\right),\\
w_1&=\widetilde{w_1}\left(y,z\right)A\left(x,t\right)A_x\left(t,x\right),\\
\theta_1&=\widetilde{\theta_1}\left(y,z\right)A^2\left(t,x\right),\\
p_1&=\widetilde{p_1}\left(y,z\right)A^2\left(t,x\right).
\end{aligned} \tag{22}$$

Next, we have a further discussion,

$$\begin{aligned}
Au_2&=\frac{\partial u_0}{\partial t}+\alpha\frac{\partial u_0}{\partial x}+u_0\frac{\partial u_1}{\partial x}+u_1\frac{\partial u_0}{\partial x}+v_0\frac{\partial u_1}{\partial y}\\
&\quad+v_1\frac{\partial u_0}{\partial y}+w_0\frac{\partial u_1}{\partial z}+w_1\frac{\partial u_0}{\partial z},\\
Av_2&=\int_0^y (U-c)\,d\varsigma\frac{\partial v_0}{\partial x},\\
Aw_2&=0,\\
A\theta_2&=\frac{\partial \theta_0}{\partial t}+u_0\frac{\partial \theta_1}{\partial x}+u_1\frac{\partial \theta_2}{\partial x}+v_0\frac{\partial \theta_1}{\partial y}+v_1\frac{\partial \theta_0}{\partial y},
\end{aligned} \tag{23}$$

Simplify (13) and (23) to the following form:

$$L_{y,z}\left(\frac{\partial p_2}{\partial x}\right) \equiv \ell_{3y,z}\left(\frac{\partial A v_2}{\partial x}\right) + \ell_{1y,z}\left(A u_2\right) + \ell_{2y,z}\left(A\theta_2\right), \tag{24}$$

where

$$\begin{aligned}
\ell_{1y,z} &= -\left[-\frac{\partial}{\partial y} - U_{zz} - U_z\frac{\partial}{\partial z} - \frac{1}{U-c} - \frac{1}{\triangle}\left(\triangle_y + U_z\triangle_z\right)\right] \\
\ell_{2y,z} &= \frac{-1}{U-c}\left[\frac{\partial}{\partial y} + U_{zz} + U_z\frac{\partial}{\partial z} - \frac{1}{\triangle}\left(\triangle_y + U_z\triangle_z\right)\right] \\
\ell_{3y,z} &= \frac{-1}{U-c}\left[-U_z\frac{\partial}{\partial y} + \left(U_y - 1\right)\frac{\partial}{\partial z} + \frac{1}{\triangle}\left(U_z\triangle_y + \left(U_y - 1\right)\triangle_z\right)\right].
\end{aligned} \tag{25}$$

We know that the homogeneous part in (24) is the same as (15). Substituting (18) and (22) into (19) and (23), and according to (13)×$\partial p_0/\partial x$-(20)×$\partial p_2/\partial x$, we can get

$$\begin{aligned}
A_{u_2} &= \widetilde{u_0}A_t + \lambda A_x + \left(3\widetilde{u_0}\widetilde{u_1} + \widetilde{v_0}\widetilde{u_1}_y + \widetilde{v_1}\widetilde{u_0}_y + \widetilde{w_0}\widetilde{u_1}_z + \widetilde{w_1}\widetilde{u_0}_z\right)A^2A_x, \\
A_{v_2} &= \int_0^y (U-c)\,d\varsigma\widetilde{v_0}A_{xx}, \\
A_{w_2} &= 0, \\
A_{\theta_2} &= \widetilde{\theta_0}A_t + \left(2\widetilde{u_0}\widetilde{\theta_1} + \widetilde{u_1}\widetilde{\theta_0} + \widetilde{v_0}\widetilde{\theta_1}_y + \widetilde{v_1}\widetilde{\theta_0}_y\right)A^2A_x,
\end{aligned} \tag{26}$$

$$\begin{aligned}
&\frac{d}{dy}\left[\frac{\partial p_2}{\partial x}\frac{d\widetilde{p_0}}{dy} - \widetilde{p_0}\frac{d}{dy}\left(\frac{\partial p_2}{\partial x}\right)\right] - \left[L_{y,z}\left(\widetilde{p_0}\right)\frac{\partial p_2}{\partial x} - L_{y,z}\left(\frac{\partial p_2}{\partial x}\right)\widetilde{p_0}\right] = -\widetilde{p_0}\left\{\left[\ell_{1y,z}\left(\widetilde{u_0}\right) + \ell_{2y,z}\left(\widetilde{\theta_0}\right)\right]\right. \\
&\cdot A_t + \ell_{1y,z}\lambda A_x + \left[\ell_{1y,z}\left(3\widetilde{u_0}\widetilde{u_1} + \widetilde{v_0}\widetilde{u_1}_y + \widetilde{v_1}\widetilde{u_0}_y + \widetilde{w_0}\widetilde{u_1}_z + \widetilde{w_1}\widetilde{u_0}_z\right) + \ell_{2y,z}\left(2\widetilde{u_0}\widetilde{\theta_1} + \widetilde{u_1}\widetilde{\theta_0} + \widetilde{v_0}\widetilde{\theta_1}_y\right.\right. \\
&\left.\left.\left. + \widetilde{v_1}\widetilde{\theta_0}_y\right)\right]A^2A_x + \left[\ell_{3y,z}(U-c)\widetilde{v_0}\right]A_{xxx}\right\}.
\end{aligned} \tag{27}$$

Integrating (27) over the domain $[0, y_0)$ leads to

$$\begin{aligned}
&\int\left[\widetilde{p_0}\left(y_0,z\right)\frac{d}{dy}\left(\frac{\partial p_2}{\partial x}\right) - \left(\frac{\partial p_2}{\partial x}\right)\frac{d\widetilde{p_0}\left(y_0,z\right)}{dy}\right]dz + \int\int_0^{y_0}\left[L\left(\widetilde{p_0}\left(y_0,z\right)\right)\left(\frac{\partial p_2}{\partial x}\right) - L\left(\frac{\partial p_2}{\partial x}\right)\widetilde{p_0}\left(y_0,z\right)\right]dy\,dz \\
&= \int\int_0^{y_0} -\widetilde{p_0}\left\{\left[\ell_{1y,z}\left(\widetilde{u_0}\right) + \ell_{2y,z}\left(\widetilde{\theta_0}\right)\right]A_t + \ell_{1y,z}\left(\widetilde{u_0}\right)\lambda A_x\right. \\
&+ \left[\ell_{1y,z}\left(3\widetilde{u_0}\widetilde{u_1} + \widetilde{v_0}\widetilde{u_1}_y + \widetilde{v_1}\widetilde{u_0}_y + \widetilde{w_0}\widetilde{u_1}_z + \widetilde{w_1}\widetilde{u_0}_z\right) + \ell_{2y,z}\left(2\widetilde{u_0}\widetilde{\theta_1} + \widetilde{u_1}\widetilde{\theta_0} + \widetilde{v_0}\widetilde{\theta_1}_y + \widetilde{v_1}\widetilde{\theta_0}_y\right)\right]A^2A_x \\
&\left. + \left[\ell_{3y,z}(U-c)\widetilde{v_0}\right]A_{xxx}\right\}dy\,dz.
\end{aligned} \tag{28}$$

We know that the two ends of (28) are identically zero, so we can write the simple form as the following form:

$$A_t + a_0\lambda A_x + a_1A^2A_x + a_2A_{xxx} = 0, \tag{29}$$

where

$$\begin{aligned}
a_0 &= \frac{\ell_{1y,z}\left(\widetilde{u_0}\right)}{\ell_{1y,z}\left(\widetilde{u_0}\right) + \ell_{2y,z}\left(\widetilde{\theta_0}\right)}, \\
a_1 &= \frac{\ell_{1y,z}\left(3\widetilde{u_0}\widetilde{u_1} + \widetilde{v_0}\widetilde{u_1}_y + \widetilde{v_1}\widetilde{u_0}_y + \widetilde{w_0}\widetilde{u_1}_z + \widetilde{w_1}\widetilde{u_0}_z\right) + \ell_{2y,z}\left(2\widetilde{u_0}\widetilde{\theta_1} + \widetilde{u_1}\widetilde{\theta_0} + \widetilde{v_0}\widetilde{\theta_1}_y + \widetilde{v_1}\widetilde{\theta_0}_y\right)}{\ell_{1y,z}\left(\widetilde{u_0}\right) + \ell_{2y,z}\left(\widetilde{\theta_0}\right)}, \\
a_2 &= \frac{\ell_{3y,z}(U-c)\widetilde{v_0}}{\ell_{1y,z}\left(\widetilde{u_0}\right) + \ell_{2y,z}\left(\widetilde{\theta_0}\right)}
\end{aligned} \tag{30}$$

Remark. According to the above study, we obtain a new model in baroclinic atmosphere. Based on the nonlinear term A^2A_x, the new model is called the generalized mKdV equation. Compared to the model which is obtained in the barotropic atmosphere, the new mKdV equation is more likely to describe the movement of solitary waves.

3. Formulation of the STFmKdV Equation

First we introduce the definition of Riemann-Liouville fractional derivatives and Caputo fractional derivatives[41, 63].

Definition 1 (Riemann-Liouville fractional derivative [41, 63]). $f(t)$ is a function defined in the $[a,b]$, for any nonnegative real α, satisfying $n-1 \le \alpha < n$,

$$\begin{aligned} {}_a^R D_t^\alpha f(t) &:= D^n {}_a^R D_t^{\alpha-n} f(t) \\ &= \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(\tau)}{(t-\tau)^{\alpha+1-n}} d\tau, \\ &\qquad \forall t \in [a,b], \end{aligned} \tag{31}$$

and this formula is called the α-order left Riemann-Liouville fractional derivative. And

$$\begin{aligned} {}_t^R D_b^\alpha f(t) &:= (-1)^n D^n {}_t^R D_b^{\alpha-n} f(t) \\ &= \frac{(-1^n)}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_t^b \frac{f(\tau)}{(\tau-t)^{\alpha+1-n}} d\tau, \\ &\qquad \forall t \in [a,b], \end{aligned} \tag{32}$$

and this formula is the α-order right Riemann-Liouville fractional derivative.

Remark 2. We note that if the function $f(t)$ is n-order continuous derivable on the interval of $[a,b]$, when α tends to n, the left fractional derivative is the traditional n-order derivative. In addition, if the function $f(t)$ is n-order continuous derivable on the interval of $[a,b]$, when α tends to n, the right fractional derivative is the traditional $n(n-1)$-order derivative multiplied by $(-1)^n$.

Definition 3 (Caputo fractional derivative [41, 63]). $f(t)$ is a function defined in the $[a,b]$, for any nonnegative real α, satisfying $n-1 \le \alpha < n$,

$$\begin{aligned} {}_a^C D_t^\alpha f(t) &:= {}_a^R D_t^{\alpha-n} D^n f(t) \\ &= \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}} d\tau, \\ &\qquad \forall t \in [a,b], \end{aligned} \tag{33}$$

and this formula is the α-order left Caputo fractional derivative.

$$\begin{aligned} {}_t^C D_b^\alpha f(t) &:= (-1)^n {}_t^R D_b^{(\alpha-n)} D^n f(t) \\ &= \frac{(-1)^n}{\Gamma(n-\alpha)} \int_t^b \frac{f^{(n)}(\tau)}{(\tau-t)^{\alpha+1-n}} d\tau, \\ &\qquad \forall t \in [a,b], \end{aligned} \tag{34}$$

and this formula is the α-order right Caputo fractional derivative.

Lemma 4 (see [61, 62]). *Riemann-Liouville fractional derivative and Caputo fractional derivative have the following relationship:*

$${}_a^R D_t^\alpha f(t) = {}_a^C D_t^\alpha f(t) + \sum_{j=0}^{n-1} \frac{f^i(a)(t-a)^{i-\alpha}}{\Gamma(1+i-\alpha)}. \tag{35}$$

In this section, based on the generalized mKdV equation obtained in Section 2, we use semi-inverse method and variational method[63] to construct the generalized STFmKdV equation under the Riemann-Liouville derivative definition. Let $a_{0,3} = 1$, so that (29) transforms to the following form:

$$A_t + \lambda A_x + a_1 A^2 A_x + a_2 A_{xxx} = 0, \tag{36}$$

where a_1, a_2 are arbitrary constants, $A(x,t)$ denotes the amplitude of the Rossby waves, $x \in R$ is the space variable in the propagation of the field, and $t \in T(= [0,T_0])$ is the time variable. The main steps are arranged as follows[62].

First of all, we introduce $u(x,t)$ as a potential function. Let $A(x,t) = u_x(x,t)$, so that (36) can be written as

$$u_{xt} + \lambda u_{xx} + a_1 u_x^2 u_{xx} + a_2 u_{xxx} = 0. \tag{37}$$

The Lagrangian form of (36) can be defined using the semi-inverse method. μA is considered as a fixed function. The functional of the potential equation (37) can be represented by

$$\begin{aligned} J(u) &= \int_R dx \int_T dt \left\{ u(x,t) \right. \\ &\quad \left. \cdot \left[c_1 u_{xt} + c_2 \lambda u_{xx} + c_3 a_1 u_x^2 u_{xx} + c_4 a_2 u_{xxxx} \right] \right\}, \end{aligned} \tag{38}$$

where c_1, c_2, c_3, and c_4 are Lagrangian multipliers to be determined later. Integrating (38) by parts and taking $u_t|_R = u_x|_R = u_x|_T = u_{xx}|_R = u_{xxx}|_R = 0$ lead to

$$\begin{aligned} J(u) &= \int_R dx \int_T dt \left\{ -c_1 u_x u_t - c_2 \lambda u_x^2 - \frac{1}{3} c_3 a_1 u_x^4 \right. \\ &\quad \left. + c_4 a_2 u_{xx}^2 \right\}. \end{aligned} \tag{39}$$

Secondly, by applying the variation of this functional with respect to $u(x,t)$, and integrating each term by parts, optimizing the variation $\delta J(u) = 0$, we have

$$2c_1 u_{xt} + 2c_2 \lambda u_{xx} + 4c_3 a_1 u_x^2 u_{xx} + 2c_4 a_2 u_{xxxx} = 0. \tag{40}$$

Equation (40) is equivalent to (37), so we can get

$$\begin{aligned} c_1 &= \frac{1}{2}, \\ c_2 &= \frac{1}{2}, \\ c_3 &= \frac{1}{4}, \\ c_4 &= \frac{1}{2}. \end{aligned} \tag{41}$$

A functional relation (39) is given to produce the direct Lagrange form of the mKdV equation.

$$L(u_t, u_x, u_{xx}, \cdots) = -\frac{1}{2}u_x u_t - \frac{1}{2}\lambda u_x^2 - \frac{1}{12}a_1 u_x^4 + \frac{1}{2}a_2 u_{xx}^2. \quad (42)$$

Thirdly, according to the Lagrangian form of the integer order, we can obtain the Lagrangian form of the space-time fractional order similarly.

$$F\left(D_t^\alpha * D_x^\beta u, D_x^\beta u, D_x^{2\beta} u, \cdots\right) = -\frac{1}{2}D_t^\alpha u * D_x^\beta u - \frac{1}{2}\lambda\left(D_x^\beta u\right)^2 - \frac{1}{12}a_1\left(D_x^\beta u\right)^4 + \frac{1}{2}a_2\left(D_x^{2\beta}u\right)^2, \quad (43)$$

where $0 \le \alpha, \beta < 1$. The fractional derivative $D_t^\alpha u(x,t)$ or $D_x^\beta u(x,t)$ in terms of the left Riemann-Liouville fractional derivative is defined by

$$D_\varsigma^\gamma f(\varsigma) = \frac{1}{\Gamma(k-\gamma)}\frac{d^k}{d\varsigma^k}\left[\int ds\,(\varsigma - s)^{k-\gamma-1} f(s)\right], \quad k-1 \le \gamma \le k, \ \varsigma = \varsigma(t,x). \quad (44)$$

Then, the STFmKdV-Burgers equation takes the following form:

$$J_F(u) = \int_R (dx)^\beta \int_T (dt)^\alpha F^*\left(D_t^\alpha * D_x^\beta u, D_x^\beta u, D_x^{2\beta} u\right). \quad (45)$$

The variation of (45) with respect to $u(x,t)$ leads to

$$\delta J_F(u) = \int_R (dx)^\beta \int_T (dt)^\alpha \left[\frac{\partial F^*}{\partial u}\delta u + \frac{\partial F^*}{\partial D_t^\alpha u}\delta D_t^\alpha u + \frac{\partial F^*}{\partial D_x^\beta u}\delta D_x^\beta u + \frac{\partial F^*}{\partial D_x^{2\beta} u}\delta D_x^{2\beta} u\right]. \quad (46)$$

Adopting the fractional integration rule and the right Riemann-Liouville fractional derivative, $\delta J_{F^*}(u)$ is written as

$$\delta J_F(u) = \int_R (dx)^\beta \int_T (dt)^\alpha \left[\frac{\partial F}{\partial u} - D_t^\alpha\left(\frac{\partial F}{\partial D_t^\alpha u}\right) - D_x^\beta\left(\frac{\partial F}{\partial D_x^\beta u}\right) + D_x^{2\beta}\left(\frac{\partial F}{\partial D_x^{2\beta} u}\right)\right]\delta u. \quad (47)$$

Optimizing the variation of the functional, i.e., $\delta J_{F^*}(u) = 0$, the Euler-Lagrange form for the STFmKdV equation leads to

$$\frac{\partial F}{\partial u} - D_t^\alpha\left(\frac{\partial F}{\partial D_t^\alpha u}\right) - D_x^\beta\left(\frac{\partial F}{\partial D_x^\beta u}\right) + D_x^{2\beta}\left(\frac{\partial F}{\partial D_x^{2\beta} u}\right) = 0. \quad (48)$$

Substituting (43) into (48), we have

$$D_t^\alpha D_x^\beta u + \lambda D_x^\beta\left(D_x^\beta u\right) + a_1\left(D_x^\beta u\right)^2 D_x^{2\beta} u + a_2 D_x^{2\beta}\left(D_x^{2\beta} u\right) = 0. \quad (49)$$

Substituting $D_x^\beta u(x,t) = A(x,t)$ into (49), we have the STFmKdV equation

$$D_t^\alpha A + \lambda D_x^\beta A + a_1 A^2 D_x^\beta A + a_2 D_x^{3\beta} A = 0. \quad (50)$$

In this paper, in order to make the content more complete, then we will study the conservation laws and the solutions of the STFmKdV equation.

4. Lie Symmetry Analysis and Conservation Laws of the STFmKdV Equation

In this section, we employ Lie symmetry analysis to discuss the conservation laws[64, 65] of STFmKdV equaiton which does not contain dissipation item. The details are as follows.

4.1. Lie Symmetry Analysis. The STFPDE of a function $f(x,t)$ with two independent variables considered in the Riemann-Liouville sense is defined as the following form:

$$D_t^\alpha u = \frac{\partial^\alpha u}{\partial t^\alpha} = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)}\dfrac{\partial^n}{\partial t^n}\displaystyle\int_0^t (t-s)^{n-\alpha-1} f(x,s)\,ds, & n-1 < \alpha < n \in N, \\ \dfrac{\partial^n f(x,t)}{\partial t^n}, & \alpha = n \in N. \end{cases} \quad (51)$$

First, we define a Lie group of point transformations under one parameter

$$x^* = x + \varepsilon\xi(x,t,A) + o\left(\varepsilon^2\right),$$
$$t^* = t + \varepsilon\tau(x,t,A) + o\left(\varepsilon^2\right), \quad (52)$$
$$u^* = u + \varepsilon\eta(x,t,A) + o\left(\varepsilon^2\right),$$

where ξ, τ, η are infinitesimal functions and ε is a small continuous parameter. The associated Lie algebra is spanned by

$$X = \xi(x,t,A)\frac{\partial^\beta}{\partial x^\beta} + \tau(x,t,A)\frac{\partial^\alpha}{\partial t^\alpha} + \eta(x,t,A)\frac{\partial}{\partial A}. \quad (53)$$

The prolonged generator can be defined

$$Pr^{(\alpha,\beta,2)}X = \xi(x,t,A)\frac{\partial^\beta}{\partial x^\beta} + \tau(x,t,A)\frac{\partial^\alpha}{\partial t^\alpha} + \eta(x,t,A)\frac{\partial}{\partial A} + \eta^{\alpha,t}\frac{\partial}{\partial\left(D_t^\alpha A\right)} + \eta^{\beta,x}\frac{\partial}{\partial\left(D_x^\beta A\right)} + \eta^{\beta,xx}\frac{\partial}{\partial\left(D_x^{2\beta} A\right)}. \quad (54)$$

On the basis of the infinitesimal invariance criterion, one can get

$$Pr^{(n)}X[F]\Big|_{F=0}=0, \tag{55}$$

where

$$F=D_t^{\alpha}A+\lambda D_x^{\beta}A+a_1A^2D_x^{\beta}A+a_2D_x^{3\beta}A. \tag{56}$$

The operators $\eta^{\alpha,t},\eta^{\beta,x},\eta^{\beta,xx}$ are fractional extended symmetry operators defined as follows[64]:

$$\begin{aligned}\eta^{\alpha,t}&=D_t^{\alpha}(\eta)+\xi D_t^{\alpha}(A_x)-D_t^{\alpha}(\xi A_x)\\&\quad+D_t^{\alpha}(A(D_t\tau))-D_t^{\alpha+1}(\tau A)+\tau D_t^{\alpha+1}A,\\\eta^{\beta,x}&=D_x^{\beta}(\eta)+D_x^{\beta}(A(D_x\xi))-D_x^{\beta+1}(\xi A)\\&\quad+\xi D_x^{\beta+1}(A)+\tau D_x^{\beta}(A_t)-D_x^{\beta}(\tau A_t),\\\eta^{\beta,xx}&=D_x^{\beta}\left(\eta^{\beta,x}\right)-A_{xx}D_x^{\beta}(\xi)-A_{xt}D_x^{\beta}(\tau),\end{aligned} \tag{57}$$

where the symbols D_t,D_x represent the total derivative operators defined by

$$\begin{aligned}D_t&=\partial_t+A_t\partial_A+A_{tt}\partial_{A_t}+A_{xt}\partial_{A_x}+\cdots,\\D_x&=\partial_x+A_x\partial_A+A_{xx}\partial_{A_x}+A_{xt}\partial_{A_t}+\cdots.\end{aligned} \tag{58}$$

Second, the conserved vectors of the STFmKdV equation are investigated as follows. Applying the second prolongation $Pr^{(\alpha,\beta,2)}X$, we get

$$\eta^{\alpha,t}+\lambda\eta^{\beta,x}+2a_1A\eta D_x^{\beta}A+a_1A^2\eta^{\beta,x}+a_2\eta^{\beta,xxx}=0 \tag{59}$$

Substituting (57) and (58) into (59), and equating the coefficients of alike partial derivatives of u, we can obtain the determining equations

$$\begin{aligned}&\xi_t=\xi_A=\tau_x=\tau_A=\eta_{AA}=0,\\&\alpha\tau_t-3\beta\xi_x=0,\\&\binom{\alpha}{n}\partial_t^{\alpha}\eta_A-\binom{\alpha}{n+1}D_t^{n+1}\tau=0,\\&\binom{\beta}{n}\partial_x^{\beta}\eta_A-\binom{\beta}{n+1}D_t^{n+1}\xi=0,\\&\partial_t^{\alpha}\eta-A\partial_t^{\alpha}\eta_A+\lambda\left(\partial_t^{\alpha}\eta-A\right)\partial_t^{\alpha}\eta_A+a_0A^2\eta_x+a_1\eta_{xxx}\\&\quad=0.\end{aligned} \tag{60}$$

Solving these equations, the infinitesimals can be derived in the following form:

$$\begin{aligned}\xi&=\frac{c_1}{\beta}x+c_2,\\\tau&=\frac{3c_1}{\alpha}t+c_3,\\\eta&=\frac{3c_1(\beta-1)}{\beta}A.\end{aligned} \tag{61}$$

where c_1,c_2,c_3 are arbitrary constants. Thus, the corresponding infinitesimal generator can be written

$$X=\frac{x}{\beta}\frac{\partial}{\partial x}+\frac{3t}{\alpha}\frac{\partial}{\partial t}+\frac{3A(\beta-1)}{\beta}\frac{\partial}{\partial A} \tag{62}$$

4.2. Nonlinear Self-Adjointness. The concept of nonlinear self adjoint is proposed in the application of new conservation theorem to the conservation laws of equations. This concept is extended to the space-time fractional partial differential equations. The lagrangian form of nondissipative STFmKdV equation is given by

$$\mathscr{L}=v(x,t)\left[D_t^{\alpha}A+\lambda D_x^{\beta}A+a_1A^2D_x^{\beta}A+a_2D_x^{3\beta}A\right], \tag{63}$$

where $v(x,t)$ is a new dependent variable. The adjoint equation of the STFmKdV equation is defined by

$$F^*\equiv\frac{\delta\mathscr{L}}{\delta A}=0, \tag{64}$$

where $\delta/\delta A$ is the Euler-Lagrange operator. According to (63) and (64), the adjoint equation of STFmKdV equation can be obtained as follows:

$$\begin{aligned}F^*&=\left(D_t^{\alpha}\right)^*v+\left(a_1A^2-\lambda\right)\left(D_x^{\beta}\right)^*v-a_2\left(D_x^{3\beta}\right)^*v\\&=0.\end{aligned} \tag{65}$$

Here, $(D_t^{\alpha})^*,(D_x^{\beta})^*$ are the adjoint operators of D_t^{α},D_x^{β}. For the Riemann-Liouville fractional differential operators, the corresponding adjoint operators[41] have the following form:

$$\begin{aligned}\left({}_0D_t^{\alpha}\right)^*&=(-1)_t^n\,I_T^{n-\alpha}\left(D_t^n\right)={}_t^CD_T^{\alpha},\\\left({}_0D_x^{\beta}\right)^*&=(-1)_x^m\,I_R^{m-\beta}\left(D_x^m\right)={}_x^CD_R^{\beta},\end{aligned} \tag{66}$$

where $I_T^{n-\alpha},I_R^{m-\beta}$ are the right fractional integral operators of order $n-\alpha,m-\beta$, defined by

$$\begin{aligned}I_T^{n-\alpha}f(x,t)&=\frac{1}{\Gamma(n-\alpha)}\int_t^T\frac{f(x,\tau)}{(\tau-t)^{\alpha+1-n}}d\tau,\\&\qquad n=[\alpha]+1,\\I_R^{m-\beta}f(x,t)&=\frac{1}{\Gamma(m-\beta)}\int_x^R\frac{f(\varsigma,t)}{(\varsigma-x)^{\beta+1-m}}d\varsigma,\\&\qquad m=[\beta]+1.\end{aligned} \tag{67}$$

For nonlinear self-adjointness, let us assume $v=\psi(x,t,A)$, where $\psi(x,t,A)\neq0$.

Substituting $v=\psi(x,t,A)$ into (65)

$$\begin{aligned}&\left(D_t^{\alpha}\right)^*\psi+\left(a_1A^2-\lambda\right)\left(D_x^{\beta}\right)^*\psi-a_2\Big[\left(D_x^{3\beta}\right)^*\psi\\&\quad+3\left(D_x^{2\beta}\right)^*\psi_AA_x+3\left(D_x^{\beta}\right)^*\psi_{AA}\left(A_x\right)^2\\&\quad+3\left(D_x^{\beta}\right)^*\psi_{Ax}A_{xx}+3\psi_{AA}A_xA_{xx}+\psi_AA_{xxx}\\&\quad+\psi_{AAA}{A_x}^3\Big]\equiv F^*.\end{aligned} \tag{68}$$

By using the method of undetermined coefficients, we obtain the following cases[64, 65]:

$$\begin{aligned} &\textit{case 1}: \quad 0<\beta<1, \\ &\qquad v(x,t)=C_0 \\ &\qquad\qquad \text{for } D_t^\alpha = {}_0D_t^\alpha, \\ &\textit{case 2}: \quad 0<\beta<1, \\ &\qquad v(x,t)=C_1x+C_2 \\ &\qquad\qquad \text{for } D_t^\alpha = {}_0^cD_t^\alpha. \end{aligned} \tag{69}$$

Here, C_0, C_1, C_2 are arbitrary constants. These solutions of $v(x,t)$ are introduced in the Lagrangian form for the construction of conserved vectors in the next subsection.

4.3. Conservation Laws. It is well known that $C = (C^t, C^x)$ is called the conserved vector that satisfies the conservation equation

$$D_t\left(C^t\right) + D_x\left(C^x\right) = 0, \tag{70}$$

Noether's theorem is widely adopted to construct the conserved vectors. Using this method, the conserved vectors are obtained by using the Noether operators for the Lagrangian form. According to[40], the fractional Noether operator for the variable x, t has been given.

Before constructing the conserved vectors, we should consider the Lie characteristic function for the Lie symmetry infinitesimal generator $X = \xi(\partial^\beta/\partial x^\beta) + \tau(\partial^\alpha/\partial t^\alpha) + \eta(\partial/\partial A)$. The Lie characteristic function is defined by

$$W = \eta - \xi A_x - \tau A_t = \frac{3A(\beta-1)}{\beta} - \frac{x}{\beta}A_x - \frac{3t}{\alpha}A_t, \tag{71}$$

Then, the fractional Noether operator for t of the STFmKdV-Burgers equation is

$$C^t = \sum_{k=0}^{n-1} (-1)^k D_t^{\alpha-1-k}(W) D_t^k \left(\frac{\partial F^*}{\partial\left(D_t^\alpha A\right)}\right), \tag{72}$$

Similarly, the fractional Noether operator for x of the STFmKdV-Burgers equation is

$$C^x = \sum_{k=0}^{m-1} (-1)^k D_x^{\beta-1-k}(W) D_x^k \left(\frac{\partial F^*}{\partial\left(D_x^\beta A\right)}\right), \tag{73}$$

Substituting the Lagrangian form (65) and (71) into (72) with case 1 in (69), let $C_0 = 1$; the t-component of the conserved vectors can be obtained:

$$C^t = \frac{3(\beta-1)}{\beta} I_t^{1-\alpha}(A) - \frac{x}{\beta} I_t^{1-\alpha}(A_x) - \frac{3}{\alpha} I_t^{1-\alpha}(tA_t), \quad as\ 0<\alpha<1, \tag{74}$$

$$\begin{aligned} C^t = {} & \frac{3(\beta-1)}{\beta} D_t^{\alpha-1}(A) - \frac{x}{\beta} D_t^{\alpha-1}(A_x) \\ & - \frac{3}{\alpha} D_t^{\alpha-1}(tA_t), \quad as\ 1<\alpha<2, \end{aligned} \tag{75}$$

The x-component of the conserved vectors is

$$\begin{aligned} C^x = {} & \left[\frac{3(\beta-1)}{\beta} D_x^{\beta-1}(A) - \frac{1}{\beta} D_x^{\beta-1}(xA_x) \right. \\ & \left. - \frac{3t}{\alpha} D_x^{\beta-1}(A_t)\right], \quad as\ 1<\alpha<2, \end{aligned} \tag{76}$$

In case 2 in (69), let $C_1 = 1, C_2 = 0$; the t-component of the conserved vectors can be obtained as follows:

$$\begin{aligned} C^t = {} & x \times \left[\frac{3(\beta-1)}{\beta} I_t^{1-\alpha}(A) - \frac{x}{\beta} I_t^{1-\alpha}(A_x) \right. \\ & \left. - \frac{3}{\alpha} I_t^{1-\alpha}(tA_t)\right], \quad as\ 0<\alpha<1, \end{aligned} \tag{77}$$

$$\begin{aligned} C^t = {} & x \times \left[\frac{3(\beta-1)}{\beta} D_t^{\alpha-1}(A) - \frac{x}{\beta} D_t^{\alpha-1}(A_x) \right. \\ & \left. - \frac{3}{\alpha} D_t^{\alpha-1}(tA_t)\right], \quad as\ 1<\alpha<2, \end{aligned} \tag{78}$$

The x-component of the conserved vectors is

$$\begin{aligned} C^x = {} & x \times \left[\frac{3(\beta-1)}{\beta} D_x^{\beta-1}(A) - \frac{1}{\beta} D_x^{\beta-1}(xA_x) \right. \\ & \left. - \frac{3t}{\alpha} D_x^{\beta-1}(A_t)\right] - \left[\frac{3(\beta-1)}{\beta} I_x^{2-\beta}(A) \right. \\ & \left. - \frac{1}{\beta} I_x^{2-\beta}(xA_x) - \frac{3t}{\alpha} I_x^{2-\beta}(A_t)\right]. \end{aligned} \tag{79}$$

5. The Exact Solutions of the STFmKdV Equation

In this part, we deal with the exact solutions [66–68] of (50). Using $\exp(-\Phi(\xi))$ method, the main steps of this method to solve the space time fractional partial equation can be given as follows.

Firstly, we introduce the fractional complex transform:

$$\begin{aligned} A(x.t) &= U(\xi), \\ \xi &= \frac{x^\beta}{\Gamma(1+\beta)} - \frac{\nu t^\alpha}{\Gamma(1+\alpha)}, \end{aligned} \tag{80}$$

where ν is the wave speed. Substitute (80) into the STFmKdV equation as follows:

$$D_t^\alpha A + \lambda D_x^\beta A + a_1 A^2 D_x^\beta A + a_2 D_x^{3\beta} A = 0 \tag{81}$$

Equation (81) transforms to an ordinary differential equation

$$-\nu U' + \lambda U' + a_1 U^2 U' + a_2 U''' = 0. \tag{82}$$

Integrating (82) with respect to ξ and setting the integration constant to zero, we have the following equation:

$$(-\nu+\lambda)U + \frac{a_1}{3}U^3 + a_2 U'' = 0. \tag{83}$$

Secondly, balancing the highest order derivative term and the highest order nonlinear term in (83), we get the balancing number $n = 1$. Thus the solution of (83) takes the form

$$U(\xi) = k_0 + k_1 e^{-\Phi(\xi)}, \quad (84)$$

where k_0, k_1 are constants to be determined later. And $\Phi(\xi)$ satisfies the following auxiliary ordinary differential equation:

$$\Phi'(\xi) = \exp(-\Phi(\xi)) + \eta \exp(\Phi(\xi)) + \sigma. \quad (85)$$

Substitute (84) and (81) into (83) and collect all terms with the same degree of $\exp(-\Phi(\xi))^n$ together. Equating each coefficient of the same degree of $\exp(-\Phi(\xi))^n$ to zero, a set of algebraic equations for $k_0, k_1, \eta, \sigma, \nu$ can be obtained.

$$\begin{aligned}
e^{0\Phi(\xi)}&: \quad (\lambda - \nu) a_0 + \eta\sigma a_1^2 + \frac{a_0^4}{3} = 0,\\
e^{-\Phi(\xi)}&: \quad (\lambda - \nu) a_1 + a_0^3 a_1 + \left(\sigma^2 + 2\eta\right) a_1^2 = 0,\\
e^{-2\Phi(\xi)}&: \quad a_0 62 a_1^2 + 3a_1^2\sigma = 0,\\
e^{-3\Phi(\xi)}&: \quad \frac{a_0 a_1^3}{3} + 2a_1^2 = 0.
\end{aligned} \quad (86)$$

Solving the algebraic equation system, we can get the solutions. Let $\rho = \pm\sqrt{\sigma}$; the solutions of (86) are expressed by

$$\begin{aligned}
a_0 &= \sqrt{3}\rho,\\
a_1 &= \frac{2\sqrt{3}}{\rho},\\
\nu &= \lambda + \frac{\sqrt{3}\left(4\eta + \rho^4\right)}{\rho},
\end{aligned} \quad (87)$$

Different solutions of the auxiliary equation (85) have been given in [68], so we get the solutions of (81). Different cases are discussed as follows.

Case 1 (hyperbolic function solutions). When $\sigma^2 - 4\eta > 0$, $\eta \neq 0$,

$$A_1(x,t) = \sqrt{3}\rho + \frac{2\eta}{\sqrt{\sigma^2 - 4\eta}\tanh\left\{\left(\sqrt{\sigma^2 - 4\eta}/2\right)\left[x^\beta/\Gamma(1+\beta) - \left[\lambda + \sqrt{3}(4\eta + \rho^4)/\rho\right]t^\alpha/\Gamma(1+\alpha) + c\right]\right\} + \sigma} \quad (88)$$

Case 2 (trigonometric function solutions). When $\sigma^2 - 4\eta < 0$, $\eta \neq 0$,

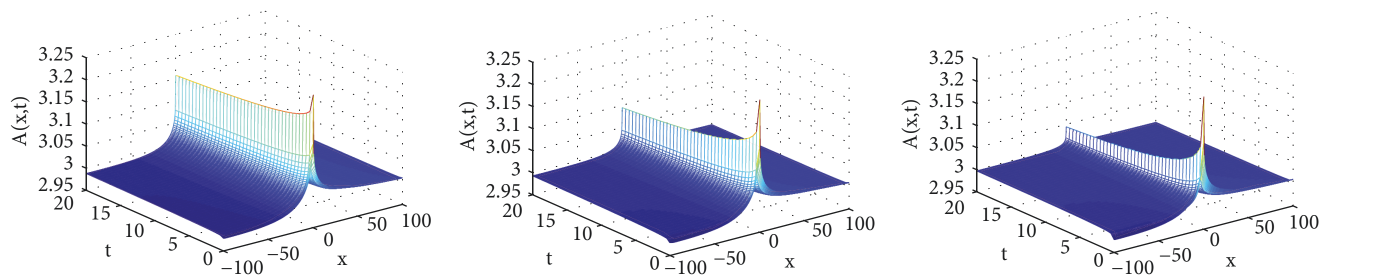
$$A_2(x,t) = \sqrt{3}\rho + \frac{2\eta}{\sqrt{4\eta - \sigma^2}\tan\left\{\left(\sqrt{4\eta - \sigma^2}/2\right)\left[x^\beta/\Gamma(1+\beta) - \left[\lambda + \sqrt{3}(4\eta + \rho^4)/\rho\right]t^\alpha/\Gamma(1+\alpha) + c\right]\right\} - \sigma}, \quad (89)$$

Case 3 (hyperbolic function solutions). When $\sigma^2 - 4\eta > 0$, $\eta = 0$, $\sigma \neq 0$,

$$\begin{aligned}
A_3(x,t) &= \sqrt{3}\rho\\
&+ \frac{\sigma}{\cosh\left\{\sigma\left[x^\beta/\Gamma(1+\beta) - \left[\lambda + \sqrt{3}(4\eta + \rho^4)/\rho\right]t^\alpha/\Gamma(1+\alpha) + c\right]\right\} + \sinh\left\{\sigma\left[x^\beta/\Gamma(1+\beta) - \left[\lambda + \sqrt{3}(4\eta + \rho^4)/\rho\right]t^\alpha/\Gamma(1+\alpha) + c\right]\right\} - 1},
\end{aligned} \quad (90)$$

Case 4 (rational function solutions). When $\sigma^2 - 4\eta = 0$, $\eta \neq 0$, $\sigma \neq 0$,

$$A_4(x,t) = \sqrt{3}\rho - \frac{\sigma^2\left\{x^\beta/\Gamma(1+\beta) - \left[\lambda + \sqrt{3}\left(4\eta + \rho^4\right)/\rho\right]t^\alpha/\Gamma(1+\alpha) + c\right\}}{2\sigma\left\{x^\beta/\Gamma(1+\beta) - \left[\lambda + \sqrt{3}(4\eta + \rho^4)/\rho\right]t^\alpha/\Gamma(1+\alpha) + c\right\} + 4}, \quad (91)$$



(a) $\alpha = 0.25, \beta = 0.25$ (b) $\mu = 0.5, \beta = 0.25$ (c) $\mu = 0.85, \beta = 0.25$

Figure 1: The effect of the variation of the dissipation coefficient on the amplitude if the space and time derivative are determined.

Case 5. When $\sigma^2 - 4\eta = 0,\ \eta = 0,\ \sigma = 0,$

$$A_5(x,t) = \frac{1}{x^{\beta}/\Gamma(1+\beta) - \left[\lambda + \sqrt{3}\left(4\eta + \rho^4\right)/\rho\right] t^{\alpha}/\Gamma(1+\alpha) + c}. \quad (92)$$

We take the solution of Case 1 as an example to discuss the influence of fractional derivatives α and β on the wave, respectively.

Figure 1 shows that when the space and time derivative are determined, the amplitude of the wave decreases as time increases.

In Figures 2 and 3, we investigate how the parameters α, β affect the nonlinear solitary waves; the corresponding physical interpretations can be given as follows:

(1) From Figure 2, we can see that the amplitude of solitary wave increases with increase of β value, while the width of the wave decreases.

(2) Figure 3 shows that when the values of α increase, the amplitude of solitary waves has an increasing tread; however, the amplitude of the wave declines more and more quickly.

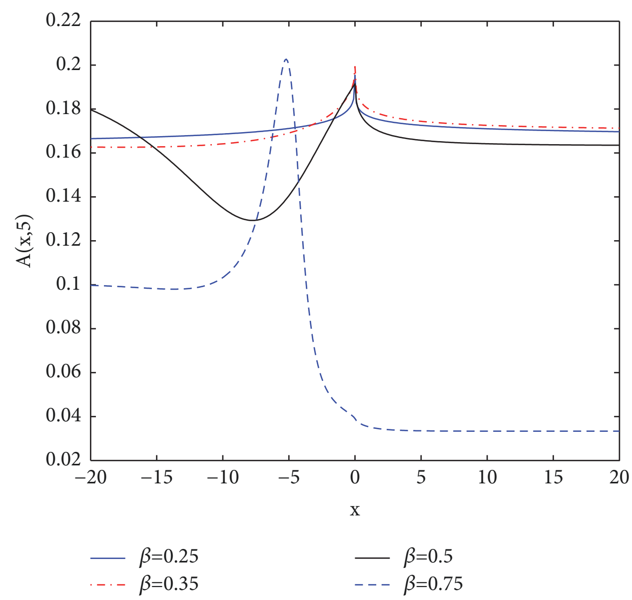


Figure 2: The amplitude of solitary wave $A(x, 10)$ at $t = 10, \beta = 0.5$, for different value of α.

6. Conclusions

In this paper, using multiscale analysis and turbulence method, from the basic dynamic multivariable equations under the baroclinic atmosphere, the integer order mKdV equation is derived. In Section 3, we use the semi-inverse method and variational method to derive the STFmKdV equation under the Riemann-Liouville definition. In Section 4, we extend the symmetry analysis of the fractional equation to obtain the corresponding infinitesimal generator of the equation. Then we discuss the nonlinear self-adjointness of the STFmKdV equation and finally get the conservation vectors of the equation. In Section 5, based on the STFmKdV equation, employing the $\exp(-\Phi(\xi))$ method, and considering the different cases of the parameters (η, σ), we obtain five different solutions of the equation.

Note 5. In this paper, we only study the space-time fractional order equation under the Riemann-Liouville derivative definition. In future studies, we can also consider the fractional equation under the Caputo derivative definition.

Note 6. In future studies, we can also consider the conservation laws of fractional equations under the Caputo derivative definition.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Acknowledgments

This project was supported by National Natural Science Foundation of China (No. 41576023), Nature Science Foundation of Shandong Province of China (No. ZR2018MA017), and China Postdoctoral Science Foundation funded project

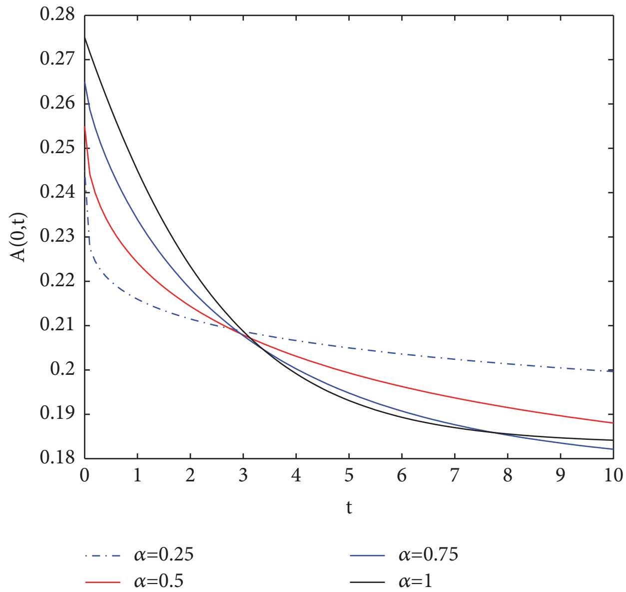


FIGURE 3: The amplitude of solitary wave $A(0,t)$ at $x = 0, \alpha = 0.5$, for different value of β.

(No. 2017M610436). The authors sincerely thank Xin Chen, Min Guo, Qiulan Zhao, and Jianxin Liu for their contribution to the research.

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