Hindawi
Mathematical Problems in Engineering
Volume 2018, Article ID 2538658, 11 pages
https://doi.org/10.1155/2018/2538658

Research Article

Random Attractor of Reaction-Diffusion Hopfield Neural Networks Driven by Wiener Processes

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Received 12 July 2017; Accepted 6 November 2017; Published 16 January 2018

Academic Editor: Fazal M. Mahomed

This paper studies the global existence and uniqueness of the mild solution for reaction-diffusion Hopfield neural networks (RDHNNs) driven by Wiener processes by applying a Schauder fixed point theorem and a priori estimate; then the random attractor for this system is also studied by constructing proper random dynamical system.

1. Introduction

It is well known that the dynamics of Hopfield neural networks have been deeply investigated because they have been successfully employed in many areas such as pattern recognition, associate memory, and combinatorial optimization. The diffusion effect cannot be avoided in the neural networks when electrons are moving in asymmetric electromagnetic field. So, the dynamical behavior of reaction-diffusion Hopfield neural networks (RDHNNs, for short) has been receiving much attention, recently [1–11].

But in a more realistic model, in order to describe the propagation of an electric potential in a neuron, it is sensible to include some noise in the system. In fact, a neural network can be stabilized or destabilized by certain stochastic inputs. Many scholars have been devoted to the stochastic RDHNNs such as [2, 12, 13]. Giving a deep insight into these literatures, we will find that most of these literatures consider the RDHNNs with finite dimensional Wiener processes; there are few results on the RDHNNs driven by infinite dimensional Wiener processes. However, since the neurons can be regarded as long thin cylinders, which act like electrical cables, the infinite dimensional Wiener processes are more favorable than standard Brownian motion.

On the other hand, attractor plays an important role in the long time behavior for dynamical systems. The random attractor extends the concept of a strange attractor from deterministic to stochastic system. There has been great interest in random attractors for stochastic partial differential equations in recent decades. The random attractors are compact invariant random sets attracting all the orbits of attraction basin. They provide crucial geometric information about their asymptotic regime as $t \to \infty$. They can help us understand the chaotic behavior of the stochastic DRDHNNs and reduce the complexity, as well as providing the statistical properties of this system. When the global existence and uniqueness of the solution can be assured, many scholars pay much attention to the global stability, boundedness, and even synchronization of the RDHNNs [1–3, 11]. However, to the best of our knowledge, there is no result on the attractor for RDHNNs, let alone random attractor for stochastic RDHNNs. We hope this work can lay a solid foundation for the future research.

So, this paper is devoted to the random attractor for the following RDHNNs driven by Wiener processes:

$$\mathrm{d}u_i(t,\mathbf{x}) = \left(\sum_{j=1}^{l}\frac{\partial}{\partial x_j}\left(G_{ij}(\mathbf{x})\frac{\partial u_i}{\partial x_j}\right) - b_i u_i + \sum_{j=1}^{n} c_{ij} f_j(u_j) + I_{\mathrm{i}}\right)\mathrm{d}t + \sigma_i \mathrm{d}W_i(t,\mathbf{x}),$$

$$u_i(t,\mathbf{x}) = 0, \quad t \geq t_0, \ \mathbf{x} \in \partial\mathcal{O},$$
$$u_i(t_0,\mathbf{x}) = \phi_i(\mathbf{x}), \quad \mathbf{x} \in \mathcal{O} \in \mathbb{R}^l, \ i = 1,2,\ldots,n, \tag{1}$$

where $u_i(t,\mathbf{x})$ denote the potential of the cell i at t and $\mathbf{x} \in \mathbb{R}^l$. b_i are positive constants and denote the rate with which the ith unit will reset its potential to the resting state in isolation when it is disconnected from the network and external inputs at t. c_{ij} are connection weights of the neural network. f_i are the active functions of the neural network, which are continuous. σ_i are the intensity of the noise. I_i denote the ith component of an external input source introduced from outside the network to the ith neuron, which are constant numbers. $\mathcal{O}$ denotes an open bounded and connected subset of $\mathbb{R}^l$ with a sufficiently regular boundary $\partial\mathcal{O}$. ∇ is the gradient. Initial data ϕ_i are $\mathcal{F}_0$-measurable and belong to $L^2(\mathcal{O})$, a.e. $\omega \in \Omega$.

For convenience, we rewrite system (1) in the vector form

$$\begin{aligned} \mathrm{d}\mathbf{u} &= \left(\nabla\cdot(G(\mathbf{x})\circ\nabla\mathbf{u}) - \mathbf{B}\mathbf{u} + \mathbf{C}\tilde{f}(\mathbf{u}) + \mathbf{I}\right)\mathrm{d}t \\ &\quad + \sigma\mathrm{d}\mathbf{W}, \\ \mathbf{u}(t,\mathbf{x}) &= 0, \quad t \geq t_0, \ \mathbf{x} \in \partial\mathcal{O}, \\ \mathbf{u}(t_0,\mathbf{x}) &= \boldsymbol{\phi}(\mathbf{x}), \end{aligned} \tag{2}$$

where $\mathbf{C} = (c_{ij})_{n\times n}$, $\sigma = \mathrm{diag}\{\sigma_1,\sigma_2,\ldots,\sigma_n\}$ $\mathbf{I} = (I_1, I_2, \ldots, I_n)^T$, $\mathbf{u} = (u_1,u_2,\ldots,u_n)^T$, $\nabla\mathbf{u} = (\nabla u_1, \nabla u_2, \ldots, \nabla u_n)^T$, $\mathbf{W} = (W_1,W_2,\ldots,W_n)^T$, $\tilde{\mathbf{f}}(\mathbf{u}) = (f_1(u_1), f_2(u_2), \ldots, f_n(u_n))^T$ is the diagonal map, ∇ is the gradient operator, $\mathbf{B} = \mathrm{diag}(b_1,b_2,\ldots,b_n)$, $\boldsymbol{\phi}(\mathbf{x}) = (\phi_1(\mathbf{x}),\phi_2(\mathbf{x}),\ldots,\phi_n(\mathbf{x}))^T$, $\mathbf{G}(\mathbf{x}) = (G_{ij})_{n\times l}$, $\mathbf{G}\circ\nabla\mathbf{u} = (G_{ij}(\partial u_i/\partial x_j))_{n\times l}$ is the Hadamard product between matrix $\mathbf{G}$ and $\nabla\mathbf{u}$, and $\nabla\cdot$ is the divergence operator.

We will also use the following notations in the paper.

(i) $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space with filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (see [1, 2, 14–18]).

(ii) $\mathbf{W}(t,\mathbf{x}) = \mathbf{W}(t,\mathbf{x})(\omega)$ is a space-time Wiener process with values in the separable Hilbert space K with $E\mathbf{W} = \mathbf{0}$ and $E(\mathbf{W},\mathbf{u})_K(\mathbf{W},\mathbf{v})_K = (t\wedge s)(Q\mathbf{u},\mathbf{v})_X$, $\forall s,t \geq 0$, $\mathbf{u},\mathbf{v} \in X$, $(\cdot,\cdot)_X$ denotes the inner product of X, $\mathbf{W}(t,\mathbf{x}) = \sum_{n=1}^{\infty}\sqrt{\alpha_n}\beta_n(t)e_n(\mathbf{x})$, where $t \wedge s = \min\{t,s\}$ and the Hilbert-Schmidt operator Q is a positive definite, nuclear, symmetric, self-adjoint operator having a finite trace $\mathrm{tr}\,Q \triangleq \sum_{n=1}^{\infty}\alpha_n < +\infty$ with eigenvalues α_n, $\{e_n\}_{n=1}^{\infty}$ is an orthogonal basis of X, and $\{\beta_n\}_{n=1}^{\infty}$ is a sequence of mutually independent standard Brownian motions in $(\Omega,\mathcal{F},\mathbb{P})$ (see [2, 14]).

(iii) $\mathcal{L}_2(X_0,U)$ is the space of all Hilbert-Schmidt operators from $X_0 \triangleq Q^{1/2}(X)$ into U; when equipped with the norm $\|\Phi\|_2 \triangleq \sqrt{\mathrm{tr}(\Phi Q\Phi^*)}$ it becomes a Hilbert space, where $\Phi \in \mathcal{L}_2(X_0,U)$; Φ^* denotes the adjoint of Φ.

2. Preliminaries and Notations

In this paper, we introduce the following Hilbert spaces: $U = \{L^2(\mathcal{O})\}^n$ and $V = \{H^1(\mathcal{O})\}^n$; according to [14, 19–21], $V \subset U = U' \subset V'$; U', V' denote the dual of the spaces U, V respectively, the injection continues, and the embedding is compact; $\|\cdot\|$, $|||\cdot|||$ represent the usual norm in U, V, respectively. Let us define the operator as follows:

$$\begin{gathered} A : \Pi(A) \in U \longrightarrow U \\ A\mathbf{u} = \nabla\cdot(\mathbf{G}\circ\nabla\mathbf{u}) - \mathbf{B}\mathbf{u} \end{gathered} \tag{3}$$

and $\Pi(A)$ is the domain of A defined as $\Pi(A) = \{H^2(\mathcal{O})\}^n \cap \{H_0^1(\mathcal{O})\}^n$. A is the infinitesimal generator of an analytic semigroup $T(t)$.

Defining the Nemytskii operator as follows:

$$\mathbf{f}(\mathbf{u})(\mathbf{x}) = \tilde{f}(\mathbf{u}(\mathbf{x})), \quad \forall\mathbf{x} \in \mathcal{O}. \tag{4}$$

With these notations, we rewrite system (2) in the more abstract form

$$\begin{aligned} \mathrm{d}\mathbf{u} &= (A\mathbf{u} + \mathbf{C}\mathbf{f}(\mathbf{u}) + \mathbf{I})\,\mathrm{d}t + \sigma\mathrm{d}\mathbf{W}, \\ \mathbf{u}(t_0) &= \boldsymbol{\phi}. \end{aligned} \tag{5}$$

We recall that

$$\mathbf{W}_{\mathbf{A}}(t) = \sigma\int_{-\infty}^{t} e^{A(t-s)}\mathrm{d}\mathbf{W}(s) \tag{6}$$

is the solution of the Ornstein-Uhlenbeck process

$$\mathrm{d}\mathbf{u} = A\mathbf{u}\mathrm{d}t + \sigma\mathrm{d}\mathbf{W}. \tag{7}$$

The regularity of (6) has been proved in [14, 22] and $\mathbf{W}_{\mathbf{A}}(t)$ has a α-Hölder continuous version with respect to t, $\alpha < 1/4$. Furthermore, by the law of large number

$$\begin{gathered} \lim_{t\to\pm\infty}\frac{\|\mathbf{W}_{\mathbf{A}}(t)\|}{|t|} = 0, \\ \lim_{t\to\pm\infty}\frac{\|\ln\mathbf{W}_{\mathbf{A}}(t)\|}{|t|} = 0, \\ \text{a.s.} \end{gathered} \tag{8}$$

(H1) We assume that $\|\mathbf{f}(\mathbf{u})\| \leq k_1 + k_2\|\mathbf{u}\|$, $\|\mathbf{f}(\mathbf{u}) - \mathbf{f}(\mathbf{v})\| \leq k_3\|\mathbf{u}-\mathbf{v}\|$, $\forall\mathbf{u},\mathbf{v} \in U$.

(H2) We assume that there exists a positive number $\beta(\mathcal{O})$ such that the following Poincaré inequality is valid:

$$\|\mathbf{u}\| \leq \beta^{-1}|||\mathbf{u}|||, \quad \forall\mathbf{u} \in V. \tag{9}$$

(H3) We assume that there exists $\alpha > 0$ such that $G_{ij}(\mathbf{x}) \geq \alpha/nl$, $i = 1,2,\ldots,n$.

(H4) Let $k_4 > \sqrt{2}k_2k_5$, with $k_4 = \min\{b_1,b_2,\ldots,b_n\}$ and $k_5 = \max\{|c_{ij}|\}$.

We also need the following propositions in the following sections.

Proposition 1. *Consider the following equation:*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{u} = A\mathbf{u}, \quad t \geq 0,$$
$$\mathbf{u}(0) = \phi. \tag{10}$$

Suppose (H2), (H3) hold; let $\mathbf{u}(t) = T(t)\phi$ denote the mild solution of (10); then $T(t)$ is a contraction map in U.

Proof. We recall that the solution of this linear equation is $\mathbf{u}(t) = e^{At}\phi$, so $T(t) = e^{At}$. Now we take the inner product of (10) with $\mathbf{u}(t)$ in U, by employing the Gaussian theorem and condition (H3), we get

$$(A\mathbf{u}, \mathbf{u}) \leq -\alpha\|\|\mathbf{u}\|\|^2 - k_4\|\mathbf{u}\|^2, \quad \forall \mathbf{u} \in V, \tag{11}$$

where $(\cdot,\cdot)$ is the inner product in U (see [1–3]), and we also have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{u}(t)\|^2 + \alpha\|\|\mathbf{u}(t)\|\|^2 + k_4\|\mathbf{u}\|^2 \leq 0. \tag{12}$$

By (H2), one obtains

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{u}(t)\|^2 + 2\left(\alpha\beta^2 + k_4\right)\|\mathbf{u}(t)\|^2 \leq 0. \tag{13}$$

By Gronwall-Bellman inequality, we have

$$\|\mathbf{u}(t)\|^2 \leq e^{-2(\alpha\beta^2+k_4)t}\|\phi\|^2; \tag{14}$$

by the definition of $\|T(t)\|$ and uniform boundedness principle, we have

$$\|T(t)\| \leq e^{-(\alpha\beta^2+k_4)t} \leq 1. \tag{15}$$

So $T(t)$ is a contraction map. □

Remark 2 (see [14]). Let $T(t)$ be a strongly continuous semigroup and A be the generator of $T(t)$; then we have

$$T(t_1)\phi - T(t_2)\phi = \int_{t_2}^{t_1} T(s)A\phi\,\mathrm{d}s = \int_{t_2}^{t_1} AT(s)\phi\,\mathrm{d}s, \quad \forall\phi \in \Pi(A). \tag{16}$$

Proposition 3 (see [2]). *Let $\Psi = (\psi_{ij})_{n\times n}$ and $\mathbf{u} = (u_1, u_2, \ldots, u_n)^T$; then $|\Psi\mathbf{u}|_{\mathbb{R}^n} \leq \sqrt{n}\|\Psi\|_F\|\mathbf{u}\|_{\mathbb{R}^n}$, where $\|\mathbf{u}\|_{\mathbb{R}^n}$ represents the usual norm of $\mathbb{R}^n$ and $\|\Psi\|_F$ represents the Frobenius norm of a matrix $\Psi \in \mathbb{R}^{n\times n}$; that is, $\|\Psi\|_F = (\sum_{i=1}^n\sum_{j=1}^n \psi_{ij}^2)^{1/2}$.*

Let $(X; d)$ be a complete separable metric space. We shall recall the notions of random dynamical system and random attractor.

Definition 4 (see [22]). Let (X, d) be a complete separable metric space. A metric dynamical system (MDS) $\theta \triangleq (\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t, t \in R\})$ is a family of measure-preserving transformations $\theta_t : \Omega \to \Omega$, $t \in \mathbb{R}$ such that

(i) $\theta_0 = id$, $\theta_t \bullet \theta_s = \theta_{t+s}$ for all $t, s \in \mathbb{R}$; $\bullet$ is the composition operator;

(ii) the map $(t,\omega) \to \theta_t(\omega)$ is $\mathcal{B}(\mathbb{R}) \otimes \mathcal{F}$ measurable, and $\theta_t\mathbb{P} = \mathbb{P}$ for all $t \in \mathbb{R}$, where $\mathcal{B}(\mathbb{R})$ is the Borel σ field of $\mathbb{R}$.

Definition 5 (see [22–25]). A random dynamical system (RDS) over θ_t is a measurable map $\boldsymbol{\varphi}$, such that

(1) $\boldsymbol{\varphi}(0,\omega)x = x$, $x \in X$, $\omega \in \Omega$;

(2) $\boldsymbol{\varphi}(s+t,\omega)x = \boldsymbol{\varphi}(s,\theta_t\omega) \bullet \boldsymbol{\varphi}(t,\omega)x$, $s, t \in \mathbb{R}$, $x \in X$, $\omega \in \Omega$;

(3) $(t, x) \to \boldsymbol{\varphi}(t,\omega)x$ is continuous a.e. $\omega \in \Omega$;

(4) $\omega \to \boldsymbol{\varphi}(t,\omega)x$ is $\mathcal{F}$-measurable for all $(t,x) \in \mathbb{R} \times X$.

Definition 6 (see [26, 27]). A stochastic flow is a family of mappings $S(t,s;\omega) : X \to X$, $-\infty < s \leq t < \infty$, parameterized by ω, such that

$$(t, s, x, \omega) \longrightarrow S(t, s; \omega)x \tag{17}$$

is $\mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(X) \otimes \mathcal{F}$-measurable and

$$S(t, r; \omega) \bullet S(r, s; \omega)x = S(t, s; \omega)x, \quad s \leq r \leq t, \ x \in X,$$
$$S(t, s; \omega)x = S(t-s, 0; \theta_s\omega)x, \quad \omega \in \Omega. \tag{18}$$

S is said to be a continuous stochastic flow, if $x \to S(t,s;\omega)x$ is continuous.

Definition 7 (see [28, 29]). A map $K : \Omega \to 2^X$ is said to be a closed random set if $K(\omega)$ is closed for a.e. $\omega \in \Omega$ and the map $\omega \to \mathrm{dist}(x, K(\omega))$ is a.e. measurable for all $x \in X$; dist denotes the Hausdorff semidistance defined as $\mathrm{dist}(x, \mathbb{A}) = \inf_{y\in\mathbb{A}} d(x,y)$ and $\mathrm{dist}(\mathbb{A},\mathbb{B}) = \sup_{x\in\mathbb{A}} d(x,\mathbb{B})$, $\mathbb{A}, \mathbb{B} \in 2^X$.

Definition 8 (see [29, 30]). A map $K : \Omega \to 2^X$ is said to be a compact random set if $K(\omega)$ is compact for a.e. $\omega \in \Omega$ and the map $\omega \to \mathrm{dist}(x, K(\omega))$ is a.s. measurable.

Definition 9 (see [31, 32]). A random set $K(\omega)$ is called an absorbing set in X, if for all $\mathbb{B} \in X$ and a.e. $\omega \in \Omega$, there exists $t_{\mathbb{B}}(\omega) > 0$ such that $t \geq t_{\mathbb{B}}(\omega) > 0$

$$\boldsymbol{\varphi}(t, \theta_{-t}\omega)\mathbb{B} \subset K(\omega). \tag{19}$$

Lemma 10 (Schauder fixed point theorem [33–38]). *If $\mathbb{B}$ is a closed bounded convex subset of a space X and $\mathfrak{L} : \mathbb{B} \to \mathbb{B}$ is completely continuous, then $\mathfrak{L}$ has a fixed point in $\mathbb{B}$.*

Lemma 11 (see [39, 40]). *Suppose $\boldsymbol{\varphi}$ is a RDS on a Polish space X, and there exists a compact set $K(\omega)$ absorbing every bounded deterministic set $\mathbb{B} \in X$. Then the set*

$$\mathfrak{A}(\omega) = \overline{\bigcup_{\mathbb{B}\in X}\bigcap_{T\geq 0}\overline{\bigcup_{t\geq T}\phi(t, \theta_{-t}\omega)\mathbb{B}}} \tag{20}$$

is a random attractor for $\boldsymbol{\varphi}$.

3. Existence and Uniqueness of the Solutions

Let

$$\mathbf{v}(t) = \mathbf{u}(t) - \mathbf{W_A}(t), \quad (21)$$

where $\mathbf{W_A}(t)$ has been defined in the previous section. Then, from (5) and (6), $\mathbf{v}(t)$ satisfies the equation

$$\begin{aligned} d\mathbf{v} &= (\mathbf{A}\mathbf{v} + \mathbf{Cf}(\mathbf{v} + \mathbf{W_A}) + \mathbf{I})\, dt, \\ \mathbf{v}(t_0) &= \boldsymbol{\psi}, \end{aligned} \quad (22)$$

where $\boldsymbol{\psi} = \boldsymbol{\phi} - \mathbf{W_A}(t_0)$. Let us rewrite (23) in the integral form

$$\mathbf{v}(t,\omega) = T(t)\boldsymbol{\psi} + \int_{t_0}^{t} T(t-s)(\mathbf{Cf}(\mathbf{v} + \mathbf{W_A}) + \mathbf{I})\, ds. \quad (23)$$

Definition 12. If $\mathbf{v}$ satisfies (23), we say that the $\mathbf{u}(t)$ is a mild solution of (1).

Let $X_{T^*} \triangleq C([t_0, t_0 + T^*]; U)$, when equipped with the norm $\|\mathbf{u}\|_{X_{T^*}} = (\sup_{t\in[t_0,t_0+T^*]} \|\mathbf{u}(t)\|^2)^{1/2}$, it becomes a Banach space. Let

$$\begin{aligned} \Sigma(m, T^*) = \{&\mathbf{v} \in C([t_0, t_0 + T^*]; U) : \|\mathbf{v}(t)\| \le m\ \forall t \\ &\in [t_0, t_0 + T^*]\} \end{aligned} \quad (24)$$

and consider an initial data $\boldsymbol{\psi}$ which is $\mathcal{F}_0$-measurable and belong to U, a.e. $\omega \in \Omega$. From now on, we are going to discuss mild solution of equation (22) by Schauder fixed point method in the space X_{T^*} for some $T^* > 0$.

Theorem 13. *Assume that $\boldsymbol{\psi}$ is $\mathcal{F}_0$-measurable and belongs to U, a.e. $\omega \in \Omega$. If conditions (H1)–(H3) hold, then for any $\|\boldsymbol{\psi}\| < m$, there exists a stopping time $T^* > 0$, such that (1) has a mild solution in $\Sigma(m, T^*)$.*

Proof. We split our proof into the following steps.

Step 1. $\Sigma(m, T^*)$ is a nonempty closed bounded convex set.

Let $\mathbf{v}_1, \mathbf{v}_2$ belong to $\Sigma(m, T^*)$; then for $0 \le \lambda \le 1$

$$\begin{aligned} \|\lambda\mathbf{v}_1 + (1-\lambda)\mathbf{v}_2\|_{X_{T^*}} &\le \lambda\|\mathbf{v}_1\|_{X_{T^*}} + (1-\lambda)\|\mathbf{v}_2\|_{X_{T^*}} \\ &\le m \end{aligned} \quad (25)$$

so $\lambda\mathbf{v}_1 + (1-\lambda)\mathbf{v}_2 \in \Sigma(m, T^*)$; hence $\Sigma(m, T^*)$ is a convex set.

Step 2. $\mathcal{T}$ maps $\Sigma(m, T^*)$ into $\Sigma(m, T^*)$.

Take any $\mathbf{v}$ in $\Sigma(m, T^*)$ and define $\mathbf{z} = \mathcal{T}\mathbf{v}$ by

$$\mathbf{z}(t) = T(t)\boldsymbol{\psi} + \int_{t_0}^{t} T(t-s)(\mathbf{Cf}(\mathbf{v} + \mathbf{W_A}) + \mathbf{I})\, ds. \quad (26)$$

By the triangle inequality

$$\begin{aligned} \|\mathbf{z}\| \le\ &\|T(t)\boldsymbol{\psi}\| \\ &+ \left\|\int_{t_0}^{t} T(t-s)(\mathbf{Cf}(\mathbf{v} + \mathbf{W_A}) + \mathbf{I})\, ds\right\|. \end{aligned} \quad (27)$$

By Proposition 1, it follows that

$$\|\mathbf{z}(t)\| \le \|\boldsymbol{\psi}\| + \int_{t_0}^{t} \|\mathbf{Cf}(\mathbf{v} + \mathbf{W_A}) + \mathbf{I}\|\, ds. \quad (28)$$

By using Proposition 3 and norm inequality, as well as condition (H1), we have

$$\begin{aligned} \|\mathbf{Cf}(\mathbf{v} + \mathbf{W_A})\| &\le \sqrt{n}\|\mathbf{C}\|_F \|\mathbf{f}(\mathbf{v} + \mathbf{W_A})\| \\ &\le \sqrt{n}\|\mathbf{C}\|_F (k_1 + k_2\|\mathbf{v} + \mathbf{W_A}\|) \\ &\le \sqrt{n}\|\mathbf{C}\|_F (k_1 + k_2(\|\mathbf{v}\| + \|\mathbf{W_A}\|)) \\ &\le \sqrt{n}\|\mathbf{C}\|_F (k_1 + k_2\|\mathbf{v}\|_{X_{T^*}} + k_2\|\mathbf{W_A}\|_{X_{T^*}}); \end{aligned} \quad (29)$$

therefore

$$\begin{aligned} \|\mathbf{z}(t)\|_{X_{T^*}} \le\ &\|\boldsymbol{\psi}\| \\ &+ \int_{t_0}^{t} (k_5\|\mathbf{W_A}\|_{X_{T^*}} + k_6 + k_7\|\mathbf{v}\|_{X_{T^*}})\, ds \\ \le\ &\|\boldsymbol{\psi}\| + (k_6\mu + k_7 + k_8 m)\, t, \end{aligned} \quad (30)$$

where $\mu = \|\mathbf{W_A}\|_{X_{T^*}}$, $k_6 = \sqrt{n}k_2\|\mathbf{C}\|_F$, $k_7 = \|\mathbf{I}\| + \sqrt{n}\|\mathbf{C}\|_F k_1$, and $k_8 = \sqrt{n}k_2\|\mathbf{C}\|_F$.

Hence $\|\mathbf{z}(t)\| \le m$ for all $t \in [t_0, t_0 + T^*]$, provided that

$$\|\boldsymbol{\psi}\| + (k_6\mu + k_7 + k_8 m)(t - t_0) < m. \quad (31)$$

It is clear that, for any $\|\boldsymbol{\psi}\| < m$, there exists a T^* satisfying (31).

Step 3. $\mathcal{T}$ is a equicontinuous map of $\Sigma(m, T^*)$.

Let $t_1, t_2 > 0$, then by Remark 2 and Propositions 1 and 3, we have

$$\begin{aligned} &\|\mathcal{T}\mathbf{v}(t_1) - \mathcal{T}\mathbf{v}(t_2)\| \\ &\quad = \|(T(t_1) - T(t_2))\boldsymbol{\psi}\| \\ &\qquad + \left\|\int_{t_2}^{t_1} T(t-s)(\mathbf{Cf}(\mathbf{v} + \mathbf{W_A}) + \mathbf{I})\, ds\right\| \\ &\quad \le \int_{t_2}^{t_1} \|T(s)A\boldsymbol{\psi}\|\, ds \\ &\qquad + \int_{t_2}^{t_1} \|T(t-s)(\mathbf{Cf}(\mathbf{v} + \mathbf{W_A}) + \mathbf{I})\|\, ds \\ &\quad \le \int_{t_2}^{t_1} \|A\boldsymbol{\psi}\|\, ds + \int_{t_2}^{t_1} (\|\mathbf{Cf}(\mathbf{v} + \mathbf{W_A})\| + \|\mathbf{I}\|)\, ds \\ &\quad \le \int_{t_2}^{t_1} \|A\boldsymbol{\psi}\|\, ds + (k_6\mu + k_7 + k_8 m)|t_1 - t_2| \\ &\quad \le (\|A\boldsymbol{\psi}\| + k_6\mu + k_7 + k_8 m)|t_1 - t_2|, \\ &\qquad\qquad \forall v \in \Sigma(m, T^*) \end{aligned} \quad (32)$$

so $\mathcal{T}\Sigma(m, T^*)$ is an equicontinuous set.

By the Arzelá-Ascoli theorem, for any bounded set $\mathscr{B} \subset \Sigma(m, T^*)$, the closure of $\mathscr{T}\mathscr{B}$ is compact, so $\mathscr{T}$ is a completely continuous map, then by Steps 1–3 and Lemma 10, $\mathscr{T}$ has a fixed point in $\Sigma(m, T^*)$, which is a mild solution of (1). □

Theorem 14. *Suppose (H1)–(H3) hold, then for any $\|\psi\| < m$, (1) has at most one mild solution in $[t_0, t_0 + T^*]$.*

Proof. Let $\mathbf{v}_1, \mathbf{v}_2 \in \Sigma(m, T^*)$ be two solutions of system (23) on $[0; T]$ with $\mathbf{v}_1(0) = \mathbf{v}_2(0) = \psi$, and set $\mathbf{z}_i = \mathscr{T} v_i$, $i = 1, 2$, and $\mathbf{z} = \mathbf{z}_1 - \mathbf{z}_2$. Then

$$\begin{aligned} \mathbf{z}(t) &= T(t)\left(\mathbf{v}_1(0) - \mathbf{v}_2(0)\right) \\ &+ \int_{t_0}^{t} T(t-s)\,\mathbf{C}\left(\mathbf{f}\left(\mathbf{v}_1 + \mathbf{W_A}\right) - \mathbf{f}\left(\mathbf{v}_2 + \mathbf{W_A}\right)\right) \mathrm{d}s. \end{aligned} \tag{33}$$

Following the method in Step 3 of Theorem 13, we have

$$\|\mathbf{z}(t)\|_{X_{T^*}} \le \sqrt{n}\,\|\mathbf{C}\|_F\, k_3 T^* \left\|\mathbf{v}_1 - \mathbf{v}_2\right\|_{X_{T^*}}. \tag{34}$$

We take a stopping time T^* such that

$$\sqrt{n}\,\|\mathbf{C}\|_F\, k_3 T^* < 1 \tag{35}$$

which implies that $\mathbf{v}_1(t) = \mathbf{v}_2(t)$ and $t \in [t_0, t_0 + T^*]$. By the combination of Theorem 13, there is one unique mild solution $\mathbf{u}(t)$ for (1). □

Theorem 15. *Suppose (H1)–(H3) hold. If $\mathbf{v} \in C([t_0, t_0 + T]; U)$ is a solution of system (23), then*

$$\|\mathbf{v}(t)\| \le \|\psi\| + \sqrt{\delta} \tag{36}$$

for some constant $\mu_\infty > 0$, where $\delta = e^{|\eta|T}(\|\mathbf{I}\|^2 + \sqrt{n}\|\mathbf{C}\|_F k_1^2 + 2k_2^2\sqrt{n}\|\mathbf{C}\|_F \mu_\infty^2)$ and $\gamma = 2\alpha\beta^2 + 2k_4 - \sqrt{n}\|\mathbf{C}\|_F(2k_2^2 + 1) - 1$.

Proof. By employing the method in [41, 42], let $\{\psi^m\}_{m=1}^{\infty}$ be a sequence in $\{C^\infty(\mathscr{O})\}^n$ such that

$$\psi^m \longrightarrow \psi \in U. \tag{37}$$

Let $\{\mathbf{W}_\mathbf{A}^m\}$ be a sequence of regular process such that

$$\mathbf{W}_\mathbf{A}^m(t) = \int_{t_0}^{t} T(t-s)\,\mathrm{d}\mathbf{W}^m(s) \longrightarrow \mathbf{W_A}(t) \tag{38}$$

in $\{C([t_0, t_0 + T] \times \mathscr{O})\}^n$ a.e. $\omega \in \Omega$.

Let $\mathbf{v}^m$ be the solution of

$$\begin{aligned} \mathbf{v}^m(t) &= T(t)\,\psi^m \\ &+ \int_{t_0}^{t} T(t-s)\left(\mathbf{Cf}\left(\mathbf{v}^m + \mathbf{W}_A^m\right) + \mathbf{I}\right) \mathrm{d}s. \end{aligned} \tag{39}$$

By using the method in the Theorem 13, it is easy to see that $\mathbf{v}^m$ does exist on an interval of time $[t_0, t_0 + T^m]$ such that $T^m \to T^*$ a.s. and that $\mathbf{v}^m$ converges to $\mathbf{v}$ in $C([t_0, t_0 + T]; U)$ [39]. Moreover $\mathbf{v}^m$ is regular and satisfies

$$\mathrm{d}\mathbf{v}^m = \left(A\mathbf{v}^m + \mathbf{Cf}\left(\mathbf{v}^m + \mathbf{W}_A^m\right) + \mathbf{I}\right) \mathrm{d}t. \tag{40}$$

Taking the inner product of (40) with $\mathbf{v}^m$ in U and employing the result of (11), we find that

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{v}^m\|^2 &= \left(\mathbf{v}^m, \frac{\mathrm{d}\mathbf{v}^m}{\mathrm{d}t}\right) \\ &= \left(\mathbf{v}^m, A\mathbf{v}^m + \mathbf{Cf}\left(\mathbf{v}^m + \mathbf{W}_A^m\right) + \mathbf{I}\right) \\ &\le -\left(\alpha\beta^2 + k_4\right)\|\mathbf{v}^m\|^2 \\ &\quad + \left(\mathbf{v}^m, \mathbf{Cf}\left(\mathbf{v}^m + \mathbf{W}_A^m\right)\right) + \left(\mathbf{v}^m, \mathbf{I}\right). \end{aligned} \tag{41}$$

By the Cauchy-Schwartz inequality and Young inequality, we get

$$\left(\mathbf{v}^m, \mathbf{I}\right) \le \|\mathbf{v}^m\|\,\|\mathbf{I}\| \le \frac{1}{2}\|\mathbf{v}^m\|^2 + \frac{1}{2}\|\mathbf{I}\|^2. \tag{42}$$

By using Young inequality and Proposition 3, as well as condition (H1), one obtains

$$\begin{aligned} &\left(\mathbf{v}^m, \mathbf{Cf}\left(\mathbf{v}^m + \mathbf{W}_A^m\right)\right) \le \|\mathbf{v}^m\|\,\left\|\mathbf{Cf}\left(\mathbf{v}^m + \mathbf{W}_A^m\right)\right\| \\ &\le \sqrt{n}\,\|\mathbf{C}\|_F\,\|\mathbf{v}^m\|\,\left\|\mathbf{f}\left(\mathbf{v}^m + \mathbf{W}_A^m\right)\right\| \\ &\le \frac{1}{2}\sqrt{n}\,\|\mathbf{C}\|_F\left(\|\mathbf{v}^m\|^2 + \left\|\mathbf{f}\left(\mathbf{v}^m + W_A^m\right)\right\|^2\right) \\ &\le \frac{1}{2}\sqrt{n}\,\|\mathbf{C}\|_F\left(\|\mathbf{v}^m\|^2 + 2k_1^2 + 2k_2^2\left\|\mathbf{v}^m + \mathbf{W}_A^m\right\|^2\right) \\ &\le \frac{1}{2}\sqrt{n}\,\|\mathbf{C}\|_F\left(\left(4k_2^2 + 1\right)\|\mathbf{v}^m\|^2 + 2k_1^2 + 4k_2^2\left\|\mathbf{W}_A^m\right\|^2\right). \end{aligned} \tag{43}$$

By (H2) and (H3), we deduce that

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{v}^m\|^2 \\ &\le -2\left(\alpha\beta^2 + k_4\right)\|\mathbf{v}^m\|^2 \\ &\quad + \sqrt{n}\,\|\mathbf{C}\|_F\left(\left(4k_2 + 1\right)\|\mathbf{v}^m\|^2 + 2k_1^2 + 4k_2^2\left\|\mathbf{W}_A^m\right\|^2\right) \\ &\quad + \|\mathbf{v}^m\|^2 + \|\mathbf{I}\|^2 \le -\eta\|\mathbf{v}^m\|^2 + \kappa, \end{aligned} \tag{44}$$

where $\gamma = 2\alpha\beta^2 + 2k_4 - \sqrt{n}\|\mathbf{C}\|_F(4k_2^2 + 1) - 1$ and $\kappa = \|\mathbf{I}\|^2 + 2\sqrt{n}\|\mathbf{C}\|_F k_1^2 + 4k_2^2\sqrt{n}\|\mathbf{C}\|_F\|\mathbf{W}_A^m\|^2$. By the classical Gronwall inequality, then we have

$$\begin{aligned} \|\mathbf{v}^m(t)\|^2 &\le e^{-\gamma t}\|\psi^m\|^2 + \int_{t_0}^{t} e^{-\gamma s}\kappa\,\mathrm{d}s \le \|\psi^m\|^2 \\ &\quad + e^{|\gamma|T}\left(\|\mathbf{I}\|^2 + 2\sqrt{n}\,\|\mathbf{C}\|_F\,k_1^2 + 4k_2^2\sqrt{n}\,\|\mathbf{C}\|_F\,\mu_{m,\infty}^2\right) \\ &\quad \cdot T \end{aligned} \tag{45}$$

with $\mu_{m,\infty} = \sup_{t\in[t_0,t_0+T]}\|\mathbf{W}_A^m(t)\|$, for a.e. $\omega \in \Omega$.

Taking the limit as $m \to \infty$, we see that a.s.

$$\|\mathbf{v}(t)\|^2 \le \|\psi\|^2 + \delta. \tag{46}$$

It follows that

$$\|\mathbf{v}(t)\| \le \|\psi\| + \sqrt{\delta} \tag{47}$$

thus we complete the proof. □

It is easy to derive the following from Theorems 13, 14, and 15.

Theorem 16. *Let ϕ be given which is $\mathcal{F}_0$-measurable and bounded in U, a.s.; then there exists a unique mild solution $\mathbf{u}$ of (2), with $\mathbf{u} \in C([t_0, t_0 + T]; U)$, $\forall T < \infty$.*

4. Existence of the Random Attractor

We refer $\omega = \mathbf{W}(t)$ and $\Omega = \{\omega \in C(\mathbb{R}, \mathbb{R}^n) \mid \boldsymbol{\omega}(0) = \mathbf{0}\}$, with $\mathbb{P}$ being the product measure of two Wiener measures on the negative and positive parts of Ω. In this case, there exists a Wiener shift θ_t defined as $\theta_t\omega(s) = \omega(t+s) - \omega(s)$, $s, t \in \mathbb{R}$, which is an ergodic transformation.

Let $\mathbf{v}(t, \omega; t_0, \boldsymbol{\psi})$ be the solution of (23) with $\mathbf{v}(0) = \boldsymbol{\psi}$ and $\mathbf{u}(t, \omega; t_0, \boldsymbol{\phi})$ is the mild solution of (1) with $\mathbf{u}(0) = \boldsymbol{\phi}$. Then, from Theorem 15, we know that the map $\boldsymbol{\psi} \to \mathbf{v}(t, \omega; t_0, \boldsymbol{\psi})$ is continuous, by $\mathbf{v}(t) = \mathbf{u}(t) - \mathbf{W}_A(t)$; then one can define the maps $S(t, s; \omega)$ and $\boldsymbol{\varphi}(t, \omega)$ by

$$S(t, t_0; \omega)\boldsymbol{\phi} = \mathbf{v}(t, \omega; t_0, \boldsymbol{\psi}) + \mathbf{W}_{\mathbf{A}}(t, \omega),$$
$$\boldsymbol{\varphi}(t, \omega)\boldsymbol{\phi} = S(t, 0; \omega)\boldsymbol{\phi} = \mathbf{v}(t, 0; \omega)\boldsymbol{\psi} + \mathbf{W}_{\mathbf{A}}(t, \omega). \quad (48)$$

By the result of Theorem 15, S is a continuous stochastic flow and $\boldsymbol{\varphi}$ is a continuous RDS associated with (1).

4.1. Absorbing Sets in U. Let us define the operator as follows:

$$\mathcal{A} : \Pi(\mathcal{A}) \in U \longrightarrow U$$
$$\mathcal{A}\mathbf{u} = -\nabla \cdot (G(\mathbf{x}) \circ \nabla \mathbf{u}) \quad (49)$$

and $\Pi(\mathcal{A}) = \Pi(A) = \{H^2(\mathcal{O})\}^n \cap \{H_0^1(\mathcal{O})\}^n$. We can infer from (10) and (49) that

$$A\mathbf{u} = -\mathcal{A}\mathbf{u} - \mathbf{B}\mathbf{u}. \quad (50)$$

The associated bilinear operator with $\mathcal{A}$ is defined as $a(\mathbf{u}, \mathbf{v}) = (\mathcal{A}\mathbf{u}, \mathbf{v})$. By utilizing the result of [2], we have

$$a(\mathbf{u}, \mathbf{u}) = \int_{\mathcal{O}} G : (\nabla \mathbf{u} \circ \nabla \mathbf{u})\, dx \geq \alpha \|\|\mathbf{u}\|\|^2, \quad (51)$$

where $\mathbf{A} : \mathbf{B}$ is the Frobenius inner product of two $n \times m$ matrixes, defined as $\mathbf{A} : \mathbf{B} \triangleq \sum_i^n \sum_j^m a_{ij} b_{ij}$.

Let $t_0 < -1$ and $\boldsymbol{\psi} \in U$ be given, and let $\mathbf{v}$ be the solution of equation (23). Taking the inner product of (22) with $\mathbf{v}$ in U we find that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{v}\|^2 = \left(\mathbf{v}, \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}\right) = (\mathbf{v}, A\mathbf{v}) + \mathbf{Cf}(\mathbf{v} + \mathbf{W}_{\mathbf{A}}) + \mathbf{I}$$
$$\leq -a(\mathbf{v}, \mathbf{v}) - k_4\|\mathbf{v}\|^2 + (\mathbf{v}, \mathbf{Cf}(\mathbf{v} + \mathbf{W}_{\mathbf{A}})) + (\mathbf{v}, \mathbf{I}). \quad (52)$$

By the Cauchy-Schwartz inequality and Young inequality

$$(\mathbf{v}, \mathbf{I}) \leq \|\mathbf{v}\|\|\mathbf{I}\| \leq \frac{k_4}{4}\|\mathbf{v}\|^2 + \frac{1}{k_4}\|\mathbf{I}\|^2. \quad (53)$$

By using Young inequality and (H1)

$$(\mathbf{v}, \mathbf{Cf}(\mathbf{v} + \mathbf{W}_{\mathbf{A}})) \leq \|\mathbf{v}\|\|\mathbf{Cf}(\mathbf{v} + \mathbf{W}_{\mathbf{A}})\|$$
$$\leq k_5\|\mathbf{v}\|\|\mathbf{f}(\mathbf{v} + \mathbf{W}_{\mathbf{A}})\|$$
$$\leq \frac{k_4}{4}\|\mathbf{v}\|^2 + \frac{k_5^2}{k_4}\|\mathbf{f}(\mathbf{v} + \mathbf{W}_{\mathbf{A}})\|^2$$
$$\leq \left(\frac{k_4}{4} + \frac{4k_2^2k_5^2}{k_4}\right)\|\mathbf{v}\|^2 + \frac{2k_5^2}{k_4}\left(k_1^2 + 2k_2^2\|\mathbf{W}_{\mathbf{A}}\|^2\right). \quad (54)$$

We deduce from (51)–(54) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{v}\|^2 + 2a(\mathbf{v}, \mathbf{v}) \leq -\eta\|\mathbf{v}\|^2 + P_1(t, \omega), \quad (55)$$

where $\eta = k_4 - 2k_2^2k_5^2/k_4 > 0$ and $P_1(t, \omega) = (2k_5^2/k_4)(k_1 + 2k_2\|\mathbf{W}_{\mathbf{A}}\|^2) + (2/k_4)\|\mathbf{I}\|^2$. By (H4), we know that $\eta > 0$; using the classical Gronwall inequality, for $t_0 \leq -1$, we have

$$\|\mathbf{v}(-1)\|^2 \leq e^{-\eta(-1-t_0)}\|\mathbf{v}(t_0)\|^2 + \int_{t_0}^{-1} e^{\eta(-1-s)}P_1(s, \omega)\,\mathrm{d}s$$
$$= e^{-\eta(-1-t_0)}\|\mathbf{u}(t_0)\|^2 + e^{-\eta(-1-t_0)}\|\mathbf{W}_{\mathbf{A}}(t_0)\|^2 + \int_{t_0}^{-1} e^{-\eta(-1-s)}P_1(s, \omega)\,\mathrm{d}s. \quad (56)$$

Theorem 17. *Under conditions (H1)–(H4), the RDS $\boldsymbol{\varphi}$ defined by (48) admits a random absorbing set in U.*

Proof. Let the random variable

$$r_1^2(\omega) = 2 + \int_{-\infty}^{-1} e^{-\eta(-1-s)}P_1(s, \omega)\,\mathrm{d}s. \quad (57)$$

By (8), we know that $P_1(s, \omega)$ has at most polynomial growth for $s \to -\infty$, so $r(\omega)$ is finite. On the other hand, $\|\mathbf{W}_{\mathbf{A}}(t_0)\|^2$ also has at most polynomial growth for t_0, as t_0 tending to $-\infty$, according to (8). So, we can choose $\bar{t}$ such that

$$e^{-\eta(-1-t_0)}\left(\|\boldsymbol{\phi}\|^2 + \|\mathbf{W}_{\mathbf{A}}(t_0)\|^2\right) \leq 2, \quad t_0 \leq \bar{t}. \quad (58)$$

By (56), (57), and (58), for all $\rho > 0$, there exists $t(\omega) \leq -1$, such that for all $t_0 \leq t(\omega)$ and $\boldsymbol{\phi} \in U$, with $\|\boldsymbol{\phi}\| \leq \rho$, the following inequality is satisfied:

$$\|\mathbf{v}(-1, \omega; t_0, \boldsymbol{\psi})\|^2 \leq r_1^2(\omega). \quad (59)$$

By (48) and (59), the stochastic flow of system (1) satisfies

$$\|S(-1, t_0; \omega)\boldsymbol{\phi}\| \leq r_1(\omega) + \|\mathbf{W}_{\mathbf{A}}(t_0, \omega)\| \triangleq r_2(\omega). \quad (60)$$

By using the relationship between stochastic flow and RDS

$$\begin{aligned}\varphi(t,\theta_{-t}\omega)B_U(\mathbf{0},\rho) &= S(t,0;\theta_{-t}\omega)B_U(\mathbf{0},\rho)\\ &= S(0,-t;\omega)B_U(\mathbf{0},\rho)\\ &= S(0,-1;\omega)S(-1,-t;\omega)B_U(\mathbf{0},\rho)\\ &\subset S(0,-1;\omega)B_U(\mathbf{0},r_2(\omega)),\quad \forall\rho>0,\end{aligned} \tag{61}$$

where $B_U(\mathbf{0},\rho)$ denotes the ball of U centered at $\mathbf{0}$ with radius ρ, let $K_1(\omega) = S(0,-1;\omega)B_U(\mathbf{0},r_2(\omega))$; there exists $t_B(\omega) > 0$ such that $t \geq t_B(\omega) > 0$

$$\varphi(t,\theta_{-t}\omega)B_U(\mathbf{0},\rho) \subset K_1(\omega),\quad \text{a.e. } \omega\in\Omega. \tag{62}$$

By Definition 9, we get the conclusion. □

We need the following auxiliary proposition in the following sections.

Theorem 18. *There exists a random variable $r_3(\omega), r_4(\omega) > 0$, such that, for all $\rho > 0$, there exist $t(\omega) \leq -1$, such that for all $t_0 \leq t(\omega)$ and all $\phi\in U$, with $\|\phi\| \leq \rho$, the solution $\mathbf{v}(t,\omega;t_0,\psi)$ of system (22) satisfies the inequality*

$$\begin{aligned}\int_{-1}^{0}\|\mathbf{v}(s,\omega;t_0,\psi)\|^2\,ds &\leq r_3(\omega),\\ \int_{-1}^{0}\|a(\mathbf{v},\mathbf{v})\|\,ds &\leq r_4(\omega).\end{aligned} \tag{63}$$

Proof. From (55), we can get

$$\int_{-1}^{0}a(\mathbf{v},\mathbf{v})\,ds \leq \frac{1}{2}\|\mathbf{v}(-1)\|^2 + \frac{1}{2}\int_{-1}^{0}P_1(s,\omega)\,ds. \tag{64}$$

Integrating (55) over $[-1,0]$, we can also get

$$\int_{-1}^{0}\|\mathbf{v}\|^2\,ds \leq \frac{1}{\eta}\|\mathbf{v}(-1)\|^2 + \frac{1}{\eta}\int_{-1}^{0}P_1(s,\omega)\,ds. \tag{65}$$

Let $r_3(\omega) = (1/2)\|\mathbf{v}(-1)\|^2 + (1/2)\int_{-1}^{0}P_1(s,\omega)ds$, $r_4(\omega) = (1/\eta)\|\mathbf{v}(-1)\|^2 + (1/\eta)\int_{-1}^{0}P_1(s,\omega)ds$, by Theorem 17, $\|\mathbf{v}(-1)\|$ is finite, and by using (8) again, we know that $r_3(\omega)$, $r_4(\omega)$ is finite. □

4.2. Absorbing Set in V. We now prove the existence of an absorbing set in V. Multiplying (22) by $\mathscr{A}\mathbf{v}$ and integrating over $\mathscr{O}$, by (50), we have

$$\begin{aligned}\left(\mathscr{A}\mathbf{v},\frac{d\mathbf{v}}{dt}\right) &= \left(\mathscr{A}\mathbf{v}, A\mathbf{v} + C\mathbf{f}(\mathbf{v}+\mathbf{W}_\mathbf{A}) + \mathbf{I}\right)\\ &\leq -\|\mathscr{A}\mathbf{v}\|^2 - (\mathscr{A}\mathbf{v},\mathbf{B}\mathbf{v})\\ &\quad + \|\mathscr{A}\mathbf{v}\|\,\|\mathbf{C}\mathbf{f}(\mathbf{v}+\mathbf{W}_\mathbf{A})\| + (\mathscr{A}\mathbf{v},\mathbf{I}).\end{aligned} \tag{66}$$

Using the Dirichlet boundary condition and the Green first identity

$$\left(\mathscr{A}\mathbf{v},\frac{d\mathbf{v}}{dt}\right) = a\left(\mathbf{v},\frac{d}{dt}\mathbf{v}\right) = \frac{1}{2}\frac{d}{dt}a(\mathbf{v},\mathbf{v}). \tag{67}$$

By Gauss formula [3]

$$(\mathscr{A}\mathbf{v},\mathbf{B}\mathbf{v}) = \int_{\mathscr{O}}(BG):(\nabla\mathbf{v}\circ\nabla\mathbf{v})\,d\mathbf{x} \geq k_4\alpha|||\mathbf{v}|||^2. \tag{68}$$

We can also infer from the divergence theorem

$$(\mathscr{A}\mathbf{v},\mathbf{I}) = -\sum_{i=1}^{n}\int_{\partial\mathscr{O}}I_i(G_i\nabla u_i)\cdot ds = 0. \tag{69}$$

Using the Young inequality again, we get

$$\|\mathscr{A}\mathbf{v}\|\,\|\mathbf{C}\mathbf{f}(\mathbf{v}+\mathbf{W}_\mathbf{A})\| \leq \|\mathscr{A}\mathbf{v}\|^2 + \frac{1}{4}\|\mathbf{C}\mathbf{f}(\mathbf{v}+\mathbf{W}_\mathbf{A})\|^2; \tag{70}$$

we can deduce from Proposition 3 and condition (H1)

$$\begin{aligned}\|\mathbf{C}\mathbf{f}(\mathbf{v}+\mathbf{W}_\mathbf{A})\|^2 &\leq n\|\mathbf{C}\|_F^2\|\mathbf{f}(\mathbf{v}+\mathbf{W}_\mathbf{A})\|^2\\ &\leq n\|\mathbf{C}\|_F^2\left(4k_2^2\|\mathbf{v}\|^2 + 2k_1^2 + 4k_2^2\|\mathbf{W}_\mathbf{A}\|^2\right).\end{aligned} \tag{71}$$

It follows from (66)–(71) that

$$\frac{d}{dt}a(\mathbf{v},\mathbf{v}) \leq -k_9 a(\mathbf{v},\mathbf{v}) + P_2(t,\omega), \tag{72}$$

where $k_9 = 2k_4$

$$P_2(t,\omega) = n\|\mathbf{C}\|_F^2\left(2k_2^2\|\mathbf{v}\|^2 + k_1^2 + 2k_2^2\|\mathbf{W}_\mathbf{A}\|^2\right). \tag{73}$$

Integrating (72) over $[s,0]$, we have

$$a(\mathbf{v}(0),\mathbf{v}(0)) \leq a(\mathbf{v}(s),\mathbf{v}(s)) + \int_{s}^{0}P_2(s,\omega)\,ds; \tag{74}$$

integrating (72) again in s over $[-1,0]$ and using (51), we have

$$\begin{aligned}|||\mathbf{v}(0)|||^2 &\leq \frac{1}{\alpha}\int_{-1}^{0}a(\mathbf{v}(s),\mathbf{v}(s))\,ds\\ &\quad + \frac{1}{\alpha}\int_{-1}^{0}P_2(s,\omega)\,ds.\end{aligned} \tag{75}$$

Theorem 19. *Under conditions (H1)–(H4), the RDS φ defined by (48) admits an absorbing set in V.*

Proof. Put the random variable

$$r_5^2(\omega) \triangleq \frac{1}{\alpha}\int_{-1}^{0}a(\mathbf{v}(s),\mathbf{v}(s))\,ds + \frac{1}{\alpha}\int_{-1}^{0}P_2(s,\omega)\,ds. \tag{76}$$

By Theorem 18, r_5 is a finite number; then by employing (71), for all $\rho > 0$, there exist $t(\omega) \leq -1$, for all $t_0 \leq t(\omega)$ and all $\phi\in V$, with $\|\phi\| \leq \rho$; the solution $\mathbf{v}(0,\omega;t_0,\psi)$ of system (23) satisfies the inequality

$$|||\mathbf{v}(0,\omega;t_0,\phi - \mathbf{W}_\mathbf{A}(0,\omega))|||^2 \leq r_5^2(\omega) \tag{77}$$

which also means the stochastic flow of (1) satisfies

$$|||S(0,t_0;\omega)\phi||| \leq r_5(\omega) + |||\mathbf{W}_\mathbf{A}(t_0,\omega)||| \triangleq r_6(\omega). \tag{78}$$

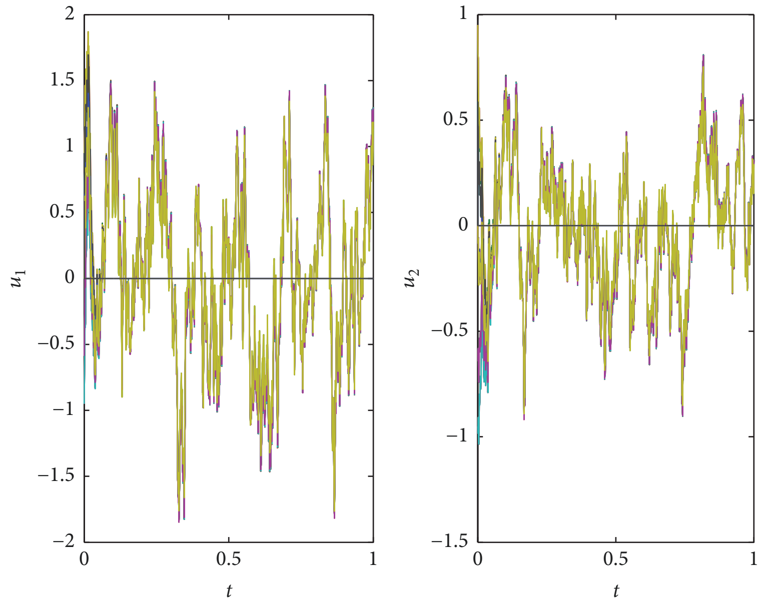

Figure 1: Frequency of Example 1.

By using the definition of RDS and the stochastic flow, we have

$$\boldsymbol{\varphi}\left(t,\theta_{-t}\omega\right)=S\left(t,0;\theta_{-t}\omega\right)=S\left(0,-t;\omega\right). \tag{79}$$

By (78) and (79), $\forall\|\boldsymbol{\phi}\|\leq\rho$, for sufficiently large t

$$\|\boldsymbol{\varphi}\left(t,\theta_{-t}\omega\right)\boldsymbol{\phi}\|\leq r_6\left(\omega\right). \tag{80}$$

Let $K_2(\omega)=B_V(\mathbf{0},r_6(\omega))$, where $B_V(\mathbf{0},r_6)$ denotes the ball of V centered at $\mathbf{0}$ with radius r_6, so for all $B_V(\mathbf{0},\rho)\in U$, there exists $t_B(\omega)>0$ such that $t\geq t_B(\omega)>0$

$$\boldsymbol{\varphi}\left(t,\theta_{-t}\omega\right)B_V\left(0,\rho\right)\subset K_2\left(\omega\right),\quad \text{a.e. } \omega\in\Omega. \tag{81}$$

□

Theorem 20. *Assume that (H1)–(H4) hold; then the RDS $\boldsymbol{\varphi}$ generated by (1) possesses a random attractor in U.*

Proof. By Theorem 19, $\boldsymbol{\varphi}(t;\omega)\boldsymbol{\phi}$ is absorbed by the random set $K_2(\omega)$. Since the embedding of V to U is compact, so the bounded set $K_2(\omega)$ is compact. A combination of Lemma 11 and Theorems 17 and 19, we get the existence of random attractor for the RDS $\boldsymbol{\varphi}$. □

If the active function $\mathbf{f}$ is a bounded Lipschitz function, then we can choose a sufficiently large k_1 such that $k_2=0$, so (H4) is satisfied automatically; then we get the following theorem.

Corollary 21. *Under conditions (H1)–(H3) and the fact that there exists a constant k_1 such that $\|\mathbf{f}(\mathbf{u})\|\leq k_1$, the RDS $\boldsymbol{\varphi}$ defined by (48) admits a random attractor in U.*

5. Example and Simulation

Example 1.

$$\begin{aligned}
\dot{u}_1\left(t,\mathbf{x}\right)&=\Delta u_1-2u_1-2\tanh\left(u_1\left(t-1,x\right)\right)\\
&\quad-\tanh\left(u_2\left(t-1,x\right)\right)+1.3\frac{\partial^2 W}{\partial t\partial x},\\
\dot{u}_2\left(t,\mathbf{x}\right)&=\Delta u_2-2u_2+\tanh\left(u_1\left(t-1,x\right)\right)\\
&\quad-3\tanh\left(u_2\left(t-1,x\right)\right)+1.5\frac{\partial^2 W}{\partial t\partial x},\\
u_i\left(t,0\right)&=u_i\left(t,20\right)=0,\quad t\geq 0,\\
u_1\left(0,\mathbf{x}\right)&=\sin\left(0.2\pi\mathbf{x}\right),\\
u_2\left(0,\mathbf{x}\right)&=\sin\left(0.2\pi\mathbf{x}\right),\\
&\qquad x\in\mathcal{O}=\left[0,20\right].
\end{aligned} \tag{82}$$

Proof. In this case $U=\{L^2(\mathcal{O})\}^2$ and $\mathbf{f}(\mathbf{u})$ is a bounded global Lipschitz continuous function with $k_1=20$, $k_2=0$, and $k_3=1$, $\|S(t)\|\leq 1$, and Δ is the Laplace operator. $EW=0$ and W is the cylindrical Wiener process with $E[W(s,x)\wedge W(t,y)]=(t\wedge s)(x\wedge y)$, $\forall s,t\geq 0$, $x,y\in[0,20]$, E is the expectation, and $a\wedge b=\min\{a,b\}$. Under these assumptions, by Theorem 15, this system has a global mild solution. Meanwhile $k_4\geq 2\sqrt{2}k_2k_5$, so according to Theorem 20, this system possesses a random attractor.

We simulate this example by using the Matlab; for detailed information see Figures 1–3. A Crank-Nicolson method in time and second-order center differences in space are used to discrete this model. A Newton iterative method is used to solve the discretized nonlinear equation. For more theories

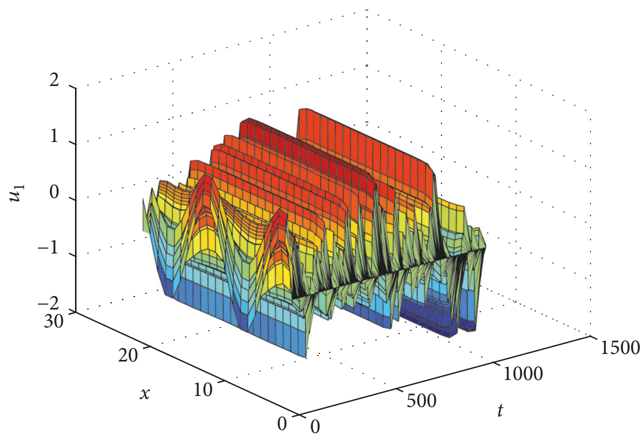

FIGURE 2: Simulation of u_1 in Example 1.

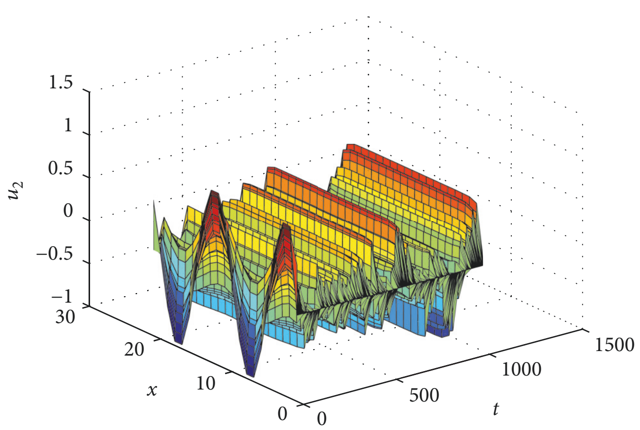

FIGURE 3: Simulation of u_2 in Example 1.

about the numerical theory about SPDE, we refer to [42, 43]. □

The method used in this article can be extended to other systems, such as the biological systems and fluid mechanical systems [44–48]. We can also apply this method to the system driven by other types of noise, such as the G-Brown motion [49]. By the way, some control technique may be used to stabilize these systems with random attractor [50–54]. We will study these problems in the future.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

The work was supported in part by the National Natural Science Foundation of China under Grant 11372051, Grant 11201035, and Grant 11475029, in part by the Fund of the China Academy of Engineering Physics under Grant 2015B0202045, in part by the Shandong Provincial Natural Science Foundation under Grant ZR2017BA014, and in part by Development Program for Defense Department of China under Grant C1520110002.

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