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Research Article

The Existence and Stability of Two Periodic Solutions on a Class of Riccati's Equation

Hua Ni

Faculty of Science, Jiangsu University, Jiangsu, Zhenjiang 212013, China

Correspondence should be addressed to Hua Ni; nihua979@126.com

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The existence of two periodic solutions of the Riccati's equation when the coefficients are ω-periodic and have different signs is obtained. One of these solutions is unstable on R and the other one is attractive on some region. Finally, an example is presented.

1. Introduction

The nonlinear Riccati type first-order differential equation

$$\frac{dx}{dt} = a(t)x^2 + b(t)x + c(t) \tag{1}$$

plays an important role in fluid mechanics and the theory of elastic vibration. There are a lot of research about this equation [1–6]: in [1, 2], the sufficient conditions of the existence of periodic solutions of the system were given; also in [1], the stability of periodic solutions of (1) was obtained, and there is no globally asymptotic stable periodic solution; [3] studied some special types of Riccati equations and got the general solution and the existence of periodic solutions of (1); [4] studied (1) with characteristic multiplier; [5] studied the high dimensional Riccati equation and obtained some sufficient conditions of the existence of periodic solutions of the equation; [6] obtained some criteria for the existence of periodic solutions of (1).

An extensive study of the set of periodic solutions of (1) was initiated in [7] and continued in [8–12]. In those papers the coefficients are real. The complex ones were considered in [13, 14]. The problem of nonexistence of periodic solutions was investigated in [9, 12, 15–19]. There are some papers (e.g., [20–24]) where stability and asymptotic behaviour of solutions were considered.

Recently, M. R. Mokhtarzadeh, M. R. Pournaki, and A. Razani [25] dealt with scalar Riccati differential equations and assumed that a, b, and c are ω-periodic continuous real functions on R and give certain conditions to guarantee the existence of at least one periodic solution for (1); Pawel Wilczynski [26] gave a few sufficient conditions for the existence of two periodic solutions of the Riccati ordinary differential equation in the plane and gave also examples of the equation without periodic solutions for the Riccati ordinary differential equation.

Consider a class of Riccati equation as follows:

$$\frac{dx}{dt} = p(t)x^2 + q(t), \tag{2}$$

where $p(t), q(t)$ are $\omega-$periodic continuous functions, about the existence of periodic solutions of (2); there are two results.

Proposition 1 (see [1]). *Consider (2), suppose*

$$p(t)q(t) < 0, \tag{3}$$

and then (2) has two $\omega-$periodic continuous solutions $\gamma_1(t), \gamma_2(t)$, and

$$\gamma_1(t) > 0 > \gamma_2(t). \tag{4}$$

Proposition 2 (see [6]). *Consider the following equation:*

$$\frac{dx}{dt} = x^2 + q(t), \tag{5}$$

where $q(t)$ is an $\omega-$periodic continuous function, if

$$\overline{q} + R^2(t) < 0, \tag{6}$$

and then (5) has two ω*−periodic continuous solutions* $\gamma_1(t)$, $\gamma_2(t)$, *and*

$$\gamma_1(t) > R(t) > \gamma_2(t), \tag{7}$$

where $\overline{q} = (1/\omega)\int_0^{\omega} q(t)dt$, $r(t) = q(t) - \overline{q}$, $R'(t) = r(t)$, $(1/\omega)\int_0^{\omega} R(t)dt = 0$.

There are also articles on the periodic solutions of Riccati equation (2)(see [27, 28]).

It is well known that scholars often use the fixed point theory to study the existence of periodic solutions on differential equation (see [29–31]).

Stimulated by the works of [29–31], in this paper, we consider (2), and by using the fixed point theory, we obtain the existence of two periodic continuous solutions of Riccati type equation: one is attractive on some region and unstable on another region, and another is unstable. We give the ranges of the size of the two periodic continuous solutions: one is positive, another is negative; they are symmetrical about $x = 0$, and following are our main results.

Conclusion. Consider (2), $p(t), q(t)$ are ω−periodic continuous functions, suppose that the following conditions hold:

$$\begin{aligned} &(H_1)\ \ p(t) < 0, \\ &(H_2)\ \ q(t) > 0, \end{aligned} \tag{8}$$

and then (2) has two ω−periodic continuous solutions, $\gamma_1(t)$, $\gamma_2(t)$, and

$$\begin{aligned} &\sqrt{-\sup_{t\in[0,\omega]}\frac{q(t)}{p(t)}} \le \gamma_1(t) \le \sqrt{-\inf_{t\in[0,\omega]}\frac{q(t)}{p(t)}}, \\ &-\sqrt{-\inf_{t\in[0,\omega]}\frac{q(t)}{p(t)}} \le \gamma_2(t) \le -\sqrt{-\sup_{t\in[0,\omega]}\frac{q(t)}{p(t)}}; \end{aligned} \tag{9}$$

meanwhile, we get the stability of two periodic solutions of (2).

Then, we consider (1) and give two results about the existence of two periodic solutions on (1). These conclusions generalize the relevant conclusions of related papers.

2. Some Lemmas, Definitions, and Abbreviations

Lemma 3 (see [32]). *Consider the following (10)*

$$\frac{dx}{dt} = a(t)x + b(t), \tag{10}$$

where $a(t), b(t)$ *are* ω*−periodic continuous functions. If* $\int_0^{\omega} a(t)dt \neq 0$*,then (10) has a unique* ω*−periodic continuous solution* $\eta(t)$, $\mathrm{mod}(\eta) \subset \mathrm{mod}(a(t), b(t))$, *and* $\eta(t)$ *can be written as follows.*

$$\eta(t) = \begin{cases} \displaystyle\int_{-\infty}^{t} e^{\int_s^t a(\tau)d\tau} b(s)\,ds, & \displaystyle\int_0^{\omega} a(t)\,dt < 0 \\ \displaystyle-\int_{t}^{+\infty} e^{\int_s^t a(\tau)d\tau} b(s)\,ds, & \displaystyle\int_0^{\omega} a(t)\,dt > 0 \end{cases} \tag{11}$$

Lemma 4 (see [33]). *Suppose that an* ω*−periodic function sequence* $\{f_n(t)\}$ *is convergent uniformly on any compact set of R,* $f(t)$ *is an* ω*−periodic function, and* $\mathrm{mod}(f_n) \subset \mathrm{mod}(f)(n = 1, 2, \cdots)$*; then* $\{f_n(t)\}$ *is convergent uniformly on R.*

Lemma 5 (see [34]). *Suppose V is a metric space, C is a convex closed set of V, its boundary is* ∂C*; if* $T : V \longrightarrow V$ *is a continuous compact mapping, such that* $T(\partial C) \subset C$*, then T has a fixed point on C.*

Definition 6 (see [33, page 43]). Suppose $f(t)$ is an ω−periodic continuous function on R; then

$$a(f, \lambda) = \int_0^{\omega} f(t)e^{-i\lambda t}dt \tag{12}$$

must exists,$a(f, \lambda)$ is called the Fourier coefficient of $f(t)$, the λ such that $a(f, \lambda) \neq 0$ is called the Fourier index of $f(t)$. There is a countable set Λ_f,when $\lambda \in \Lambda_f$, $a(f, \lambda) \neq 0$, as long as $\lambda \notin \Lambda_f$, there must be $a(f, \lambda) = 0$, and Λ_f is called the exponential set of $f(t)$.

Definition 7 (see [33, page 47]). A set of real numbers composed of linear combinations of integer coefficients of elements in Λ_f is called a module or a frequency module of $f(t)$, which is denoted as $\mathrm{mod}(f)$; that is,

$$\mathrm{mod}(f) = \left\{\mu \mid \mu = \sum_{j=1}^{N} n_j\lambda_j, n_j, N \in Z^+, N \ge 1, \lambda_j \in \Lambda_f\right\}. \tag{13}$$

For the sake of convenience, suppose that $f(t)$ is an ω-periodic continuous function on R; we denote

$$\begin{aligned} f_M &= \sup_{t\in[0,\omega]} f(t), \\ f_L &= \inf_{t\in[0,\omega]} f(t). \end{aligned} \tag{14}$$

3. Periodic Solutions of Riccati's Type Equation

Theorem 8. *Consider (2),* $p(t), q(t)$ *are* ω*−periodic continuous functions, and suppose that the following conditions hold:*

$$\begin{aligned} &(H_1)\ \ p(t) < 0, \\ &(H_2)\ \ q(t) > 0, \end{aligned} \tag{15}$$

and then (2) has two ω*−periodic continuous solutions.*

(1) One ω*−periodic continuous solution is* $\gamma_1(t)$,

$$\sqrt{-\left(\frac{q}{p}\right)_M} \le \gamma_1(t) \le \sqrt{-\left(\frac{q}{p}\right)_L}, \tag{16}$$

and $\gamma_1(t)$ *is attractive if given initial value on* $D_1 = \{x(t_0) \mid x(t_0) > 1/\zeta(t_0) + \gamma_1(t_0)\}$*, and unstable if given initial value on*

$D_2 = \{x(t_0) \mid x(t_0) \le 1/\zeta(t_0) + \gamma_1(t_0)\}$, where $x(t_0)$ is any given initial value of (2) and

$$\zeta(t) = \int_t^{+\infty} e^{-\int_s^t 2p(\tau)\gamma_1(\tau)d\tau} p(s)\, ds. \quad (17)$$

(2) Another ω–periodic continuous solution is $\gamma_2(t)$,

$$\gamma_2(t) = \frac{1}{\zeta(t)} + \gamma_1(t), \\ -\sqrt{-\left(\frac{q}{p}\right)_L} \le \gamma_2(t) \le -\sqrt{-\left(\frac{q}{p}\right)_M}, \quad (18)$$

and $\gamma_2(t)$ is unstable on R.

Proof. By $(H_1), (H_2)$, (2) can be turned into

$$\frac{dx}{dt} = p(t)\left(x - \sqrt{-\frac{q(t)}{p(t)}}\right)\left(x + \sqrt{-\frac{q(t)}{p(t)}}\right). \quad (19)$$

(1) Suppose

$$S = \{\varphi(t) \in C(R,R) \mid \varphi(t+\omega) = \varphi(t)\}. \quad (20)$$

Given any $\varphi(t), \psi(t) \in S$, the distance is defined as follows:

$$\rho(\varphi,\psi) = \sup_{t\in[0,\omega]} |\varphi(t) - \psi(t)|, \quad (21)$$

and thus (S,ρ) is a complete metric space. Take a convex closed set of S as follows:

$$B = \left\{\varphi(t) \in S \mid \sqrt{-\left(\frac{q}{p}\right)_M} \le \varphi(t) \le \sqrt{-\left(\frac{q}{p}\right)_L},\ \operatorname{mod}(\varphi) \subset \operatorname{mod}(p,q)\right\}. \quad (22)$$

Given any $\varphi(t) \in B$ and considering the following equation

$$\begin{aligned}\frac{dx}{dt} &= p(t)\left(x - \sqrt{-\frac{q(t)}{p(t)}}\right)\left(\varphi(t) + \sqrt{-\frac{q(t)}{p(t)}}\right) \\ &= p(t)\left(\varphi(t) + \sqrt{-\frac{q(t)}{p(t)}}\right)x \\ &\quad - p(t)\left(\varphi(t) + \sqrt{-\frac{q(t)}{p(t)}}\right)\sqrt{-\frac{q(t)}{p(t)}},\end{aligned} \quad (23)$$

by (H_1) and (22), we get that

$$\begin{aligned}2p_L\sqrt{-\left(\frac{q}{p}\right)_L} &\le p(t)\left(\varphi(t) + \sqrt{-\frac{q(t)}{p(t)}}\right) \\ &\le 2p_M\sqrt{-\left(\frac{q}{p}\right)_M} < 0,\end{aligned} \quad (24)$$

and hence we have

$$\int_0^{\omega} p(t)\left(\varphi(t) + \sqrt{-\frac{q(t)}{p(t)}}\right) dt < 0. \quad (25)$$

Since $p(t), \varphi(t), q(t)$ are ω–periodic continuous functions, it follows that

$$p(t)\left(\varphi(t) + \sqrt{-\frac{q(t)}{p(t)}}\right), \\ p(t)\left(\varphi(t) + \sqrt{-\frac{q(t)}{p(t)}}\right)\sqrt{-\frac{q(t)}{p(t)}} \quad (26)$$

are ω–periodic continuous functions; by (25), according to Lemma 3, (23) has a unique positive ω–periodic continuous solution as follows

$$\begin{aligned}\eta(t) = &-\int_{-\infty}^{t} e^{\int_s^t p(\tau)(\varphi(\tau)+\sqrt{-q(\tau)/p(\tau)})d\tau} p(s) \\ &\cdot \left(\varphi(s) + \sqrt{-\frac{q(s)}{p(s)}}\right)\sqrt{-\frac{q(s)}{p(s)}} ds,\end{aligned} \quad (27)$$

and

$$\begin{aligned}\operatorname{mod}(\eta) \subset \operatorname{mod}\Bigg(&p(t)\left(\varphi(t) + \sqrt{-\frac{q(t)}{p(t)}}\right), p(t) \\ &\cdot \left(\varphi(t) + \sqrt{-\frac{q(t)}{p(t)}}\right)\sqrt{-\frac{q(t)}{p(t)}}\Bigg).\end{aligned} \quad (28)$$

By (22), it follows that

$$\begin{aligned}&\operatorname{mod}\left(p(t)\left(\varphi(t) + \sqrt{-\frac{q(t)}{p(t)}}\right)\right) \subset \operatorname{mod}(p,q), \\ &\operatorname{mod}\left(p(t)\left(\varphi(t) + \sqrt{-\frac{q(t)}{p(t)}}\right)\sqrt{-\frac{q(t)}{p(t)}}\right) \\ &\quad \subset \operatorname{mod}(p,q),\end{aligned} \quad (29)$$

and hence we have

$$\operatorname{mod}(\eta) \subset \operatorname{mod}(p,q). \quad (30)$$

By (22), (24), and (27), we get

$$\begin{aligned}\eta(t) &\geq -\sqrt{-\left(\frac{q}{p}\right)_M}\int_{-\infty}^{t} e^{\int_s^t p(\tau)(\varphi(\tau)+\sqrt{-q(\tau)/p(\tau)})d\tau} p(s) \\ &\cdot\left(\varphi(s)+\sqrt{-\frac{q(s)}{p(s)}}\right)ds \\ &= \sqrt{-\left(\frac{q}{p}\right)_M}\int_{-\infty}^{t} e^{\int_s^t p(\tau)(\varphi(\tau)+\sqrt{-q(\tau)/p(\tau)})d\tau} d\left(\int_s^t p(\tau)\right. \\ &\left.\cdot\left(\varphi(\tau)+\sqrt{-\frac{q(\tau)}{p(\tau)}}\right)d\tau\right) \\ &= \sqrt{-\left(\frac{q}{p}\right)_M}\left[e^{\int_s^t p(\tau)(\varphi(\tau)+\sqrt{-q(\tau)/p(\tau)})d\tau}\right]_{-\infty}^{t} \\ &= \sqrt{-\left(\frac{q}{p}\right)_M}\left[1-e^{\int_{-\infty}^t p(\tau)(\varphi(\tau)+\sqrt{-q(\tau)/p(\tau)})d\tau}\right] \\ &\geq \sqrt{-\left(\frac{q}{p}\right)_M}\left[1-e^{2\int_{-\infty}^t p_M\sqrt{-(q/p)_M}d\tau}\right] = \sqrt{-\left(\frac{q}{p}\right)_M},\end{aligned} \tag{31}$$

and

$$\begin{aligned}\eta(t) &\leq -\sqrt{-\left(\frac{q}{p}\right)_L}\int_{-\infty}^{t} e^{\int_s^t p(\tau)(\varphi(\tau)+\sqrt{-q(\tau)/p(\tau)})d\tau} p(s) \\ &\cdot\left(\varphi(s)+\sqrt{-\frac{q(s)}{p(s)}}\right)ds \\ &= \sqrt{-\left(\frac{q}{p}\right)_L}\int_{-\infty}^{t} e^{\int_s^t p(\tau)(\varphi(\tau)+\sqrt{-q(\tau)/p(\tau)})d\tau} d\left(\int_s^t p(\tau)\right. \\ &\left.\cdot\left(\varphi(\tau)+\sqrt{-\frac{q(\tau)}{p(\tau)}}\right)d\tau\right) \\ &= \sqrt{-\left(\frac{q}{p}\right)_L}\left[e^{\int_s^t p(\tau)(\varphi(\tau)+\sqrt{-q(\tau)/p(\tau)})d\tau}\right]_{-\infty}^{t} \\ &= \sqrt{-\left(\frac{q}{p}\right)_L}\left[1-e^{\int_{-\infty}^t p(\tau)(\varphi(\tau)+\sqrt{-q(\tau)/p(\tau)})d\tau}\right] \\ &\leq \sqrt{-\left(\frac{q}{p}\right)_L}\left[1-e^{2\int_{-\infty}^t p_L\sqrt{-(q/p)_L}d\tau}\right] = \sqrt{-\left(\frac{q}{p}\right)_L},\end{aligned} \tag{32}$$

and hence, $\eta(t) \in B$.

Define a mapping as follows

$$\begin{aligned}(T\varphi)(t) = &-\int_{-\infty}^{t} e^{\int_s^t p(\tau)(\varphi(\tau)+\sqrt{-q(\tau)/p(\tau)})d\tau} p(s) \\ &\cdot\left(\varphi(s)+\sqrt{-\frac{q(s)}{p(s)}}\right)\sqrt{-\frac{q(s)}{p(s)}}ds,\end{aligned} \tag{33}$$

and thus, given any $\varphi(t) \in B$, $(T\varphi)(t) \in B$; hence $T : B \longrightarrow B$.

Now, we prove that the mapping T is a compact operator.

Consider any sequence $\{\varphi_n(t)\} \subset B (n = 1, 2, \cdots)$; then we have the following.

$$\begin{gathered}\sqrt{-\left(\frac{q}{p}\right)_M} \leq \varphi_n(t) \leq \sqrt{-\left(\frac{q}{p}\right)_L}, \\ \operatorname{mod}(\varphi_n) \subset \operatorname{mod}(p,q).\ (n = 1, 2, \cdots)\end{gathered} \tag{34}$$

On the other hand, $(T\varphi_n)(t) = x_{\varphi_n}(t)$ satisfies

$$\begin{aligned}\frac{dx_{\varphi_n}(t)}{dt} = &\ p(t)\left(\varphi_n(t)+\sqrt{-\frac{q(t)}{p(t)}}\right)x_{\varphi_n}(t) \\ &- p(t)\left(\varphi_n(t)+\sqrt{-\frac{q(t)}{p(t)}}\right)\sqrt{-\frac{q(t)}{p(t)}},\end{aligned} \tag{35}$$

thus we have

$$\begin{gathered}\left|\frac{dx_{\varphi_n}(t)}{dt}\right| \leq 4p_L\left(\frac{q}{p}\right)_L, \\ \operatorname{mod}\left(x_{\varphi_n}(t)\right) \subset \operatorname{mod}(p,q),\end{gathered} \tag{36}$$

and hence $\{dx_{\varphi_n}(t)/dt\}$ is uniformly bounded; therefore, $\{x_{\varphi_n}(t)\}$ is uniformly bounded and equicontinuous on R. By the theorem of Ascoli-Arzela, for any sequence $\{x_{\varphi_n}(t)\} \subset B$, there exists a subsequence (also denoted by $\{x_{\varphi_n}(t)\}$) such that $\{x_{\varphi_n}(t)\}$ is convergent uniformly on any compact set of R. Also combined with Lemma 4, $\{x_{\varphi_n}(t)\}$ is convergent uniformly on R; that is to say, T is relatively compact on B.

Next, we prove that T is a continuous operator.

Suppose $\{\varphi_k(t)\} \subset B, \varphi(t) \in B$, and

$$\varphi_k(t) \longrightarrow \varphi(t), \quad (k \longrightarrow \infty) \tag{37}$$

and by (33),we have

$$\begin{aligned}&\left|(T\varphi_k)(t) - (T\varphi)(t)\right| \\ &= \left|-\int_{-\infty}^{t} e^{\int_s^t p(\tau)(\varphi_k(\tau)+\sqrt{-q(\tau)/p(\tau)})d\tau} p(s)\left(\varphi_k(s)\right.\right. \\ &\left.+\sqrt{-\frac{q(s)}{p(s)}}\right)\sqrt{-\frac{q(s)}{p(s)}}ds \\ &+\int_{-\infty}^{t} e^{\int_s^t p(\tau)(\varphi(\tau)+\sqrt{-q(\tau)/p(\tau)})d\tau} p(s)\left(\varphi(s)\right. \\ &\left.\left.+\sqrt{-\frac{q(s)}{p(s)}}\right)\sqrt{-\frac{q(s)}{p(s)}}ds\right| \\ &= \left|\int_{-\infty}^{t} e^{\int_s^t p(\tau)(\varphi_k(\tau)+\sqrt{-q(\tau)/p(\tau)})d\tau} p(s)\left(\varphi_k(s)\right.\right. \\ &\left.+\sqrt{-\frac{q(s)}{p(s)}}\right)\sqrt{-\frac{q(s)}{p(s)}}ds\end{aligned}$$

$$
\begin{aligned}
&- \int_{-\infty}^{t} e^{\int_s^t p(\tau)(\varphi(\tau)+\sqrt{-q(\tau)/p(\tau)})d\tau} p(s)\Bigg(\varphi(s) \\
&+ \sqrt{-\frac{q(s)}{p(s)}}\Bigg)\sqrt{-\frac{q(s)}{p(s)}}ds\Bigg| \\
&= \Bigg|\int_{-\infty}^{t} e^{\int_s^t p(\tau)(\varphi_k(\tau)+\sqrt{-q(\tau)/p(\tau)})d\tau} p(s)(\varphi_k(s) \\
&- \varphi(s))\sqrt{-\frac{q(s)}{p(s)}}ds \\
&+ \int_{-\infty}^{t} \Big(e^{\int_s^t p(\tau)(\varphi_k(\tau)+\sqrt{-q(\tau)/p(\tau)})d\tau} \\
&- e^{\int_s^t p(\tau)(\varphi(\tau)+\sqrt{-q(\tau)/p(\tau)})d\tau}\Big) p(s)\Bigg(\varphi(s) \\
&+ \sqrt{-\frac{q(s)}{p(s)}}\Bigg)\sqrt{-\frac{q(s)}{p(s)}}ds\Bigg| \\
&= \Bigg|\int_{-\infty}^{t} e^{\int_s^t p(\tau)(\varphi_k(\tau)+\sqrt{-q(\tau)/p(\tau)})d\tau} p(s)(\varphi_k(s) \\
&- \varphi(s))\sqrt{-\frac{q(s)}{p(s)}}ds \\
&+ \int_{-\infty}^{t} e^{\xi}\left(\int_s^t p(\tau)(\varphi_k(\tau)-\varphi(\tau))d\tau\right) p(s)\Bigg(\varphi(s) \\
&+ \sqrt{-\frac{q(s)}{p(s)}}\Bigg)\sqrt{-\frac{q(s)}{p(s)}}ds\Bigg|,
\end{aligned} \tag{38}
$$

where ξ is between $e^{\int_s^t p(\tau)(\varphi_k(\tau)+\sqrt{-q(\tau)/p(\tau)})d\tau}$ and $e^{\int_s^t p(\tau)(\varphi(\tau)+\sqrt{-q(\tau)/p(\tau)})d\tau}$; thus ξ is between $2p_L\sqrt{-(q/p)_L}(t-s)$ and $2p_M\sqrt{-(q/p)_M}(t-s)$, and hence we have

$$
\begin{aligned}
&\left|(T\varphi_k)(t)-(T\varphi)(t)\right| \le \Bigg|\int_{-\infty}^{t} e^{2p_M\sqrt{-(q/p)_M}(t-s)}|p(s)| \\
&\cdot \sqrt{-\frac{q(s)}{p(s)}}ds \\
&+ \int_{-\infty}^{t} e^{2p_M\sqrt{-(q/p)_M}(t-s)}\left(\int_s^t |p(\tau)|d\tau\right)|p(s)| \\
&\cdot \Bigg(\varphi(s)+\sqrt{-\frac{q(s)}{p(s)}}\Bigg)\sqrt{-\frac{q(s)}{p(s)}}ds\Bigg|\rho(\varphi_k,\varphi) \\
&\le \Bigg|-\int_{-\infty}^{t} e^{2p_M\sqrt{-(q/p)_M}(t-s)} p_L\sqrt{-\left(\frac{q}{p}\right)_L}ds \\
&+ 2\int_{-\infty}^{t} e^{2p_M\sqrt{-(q/p)_M}(t-s)}(t-s)p_L^2\left(-\left(\frac{q}{p}\right)_L\right)ds\Bigg| \\
&\cdot \rho(\varphi_k,\varphi) = \Bigg(\frac{p_L\sqrt{-(q/p)_L}}{2p_M\sqrt{-(q/p)_M}} \\
&+ \frac{p_L^2(-(q/p)_L)}{2\left(p_M\sqrt{-(q/p)_M}\right)^2}\Bigg)\rho(\varphi_k,\varphi).
\end{aligned} \tag{39}
$$

By (37), it follows that

$$
(T\varphi_k)(t) \longrightarrow (T\varphi)(t), \quad (k \longrightarrow \infty) \tag{40}
$$

and, therefore, T is continuous; by (33), it is easy to see that $T(\partial B) \subset B$, and according to Lemma 5, T has at least a fixed point on B; the fixed point is the ω−periodic continuous solution $\gamma_1(t)$ of (2), and

$$
\sqrt{-\left(\frac{q}{p}\right)_M} \le \gamma_1(t) \le \sqrt{-\left(\frac{q}{p}\right)_L}. \tag{41}
$$

Let

$$
y(t) = x(t) - \gamma_1(t), \tag{42}
$$

where $x(t)$ is the unique solution of (2) with initial value $x(t_0) = x_0$, and $\gamma_1(t)$ is the periodic solution of (2); differentiating both sides of (42) along the solution of (2), we get

$$
\begin{aligned}
\frac{dy}{dt} &= \frac{dx(t)}{dt} - \frac{d\gamma_1(t)}{dt} = p(t)\left(x^2(t)-\gamma_1^2(t)\right) \\
&= p(t)(x(t)+\gamma_1(t))(x(t)-\gamma_1(t)) \\
&= p(t)(x(t)-\gamma_1(t)+2\gamma_1(t))(x(t)-\gamma_1(t)) \\
&= 2p(t)\gamma_1(t)(x(t)-\gamma_1(t)) \\
&\quad + p(t)(x(t)-\gamma_1(t))^2 \\
&= 2p(t)\gamma_1(t)y + p(t)y^2.
\end{aligned} \tag{43}
$$

This is Bernoulli's equation; let $u(t) = y^{-1}(t)$, and it can be turned into the following equation

$$
\frac{du}{dt} = -2p(t)\gamma_1(t)u - p(t). \tag{44}
$$

Note that

$$
\begin{aligned}
0 &< -2p_M\sqrt{-\left(\frac{q}{p}\right)_M} \le -2p(t)\gamma_1(t) \\
&\le -2p_L\sqrt{-\left(\frac{q}{p}\right)_L},
\end{aligned} \tag{45}
$$

according to Lemma 3, (44) has a unique $\omega-$periodic continuous solution as follows

$$\zeta(t)=\int_t^{+\infty} e^{-\int_s^t 2p(\tau)\gamma_1(\tau)d\tau} p(s)\,ds, \quad (46)$$

it is easy to know

$$\begin{aligned}
&\int_t^{+\infty}\left|e^{-\int_s^t 2p(\tau)\gamma_1(\tau)d\tau} p(s)\right| ds \\
&= \int_t^{+\infty} e^{-\int_s^t 2p(\tau)\gamma_1(\tau)d\tau} \left|p(s)\right| ds \\
&\le -\int_t^{+\infty} e^{2p_M\sqrt{-(q/p)_M}(s-t)} p_L ds \\
&= \frac{p_L}{2p_M\sqrt{-(q/p)_M}},
\end{aligned} \quad (47)$$

and thus the infinite integral $\int_t^{+\infty}|e^{-\int_s^t 2p(\tau)\gamma_1(\tau)d\tau}p(s)|ds$ is convergent; thereby, the infinite integral $\zeta(t) = \int_t^{+\infty} e^{-\int_s^t 2p(\tau)\gamma_1(\tau)d\tau}p(s)ds$ is convergent.

In addition,

$$\begin{aligned}
\zeta(t) &\ge \int_t^{+\infty} e^{-\sqrt{-(q/p)_M}\int_s^t 2p(\tau)d\tau} p(s)\,ds \\
&= \frac{1}{2\sqrt{-(q/p)_M}}\left[e^{-2\sqrt{-(q/p)_M}\int_s^t p(\tau)d\tau}\right]_t^{+\infty} \\
&= \frac{1}{2\sqrt{-(q/p)_M}}\left[e^{-2\sqrt{-(q/p)_M}\int_{+\infty}^t p(\tau)d\tau}-1\right] \\
&= \frac{1}{2\sqrt{-(q/p)_M}}\left[e^{2\sqrt{-(q/p)_M}\int_t^{+\infty} p(\tau)d\tau}-1\right] \\
&\ge \frac{1}{2\sqrt{-(q/p)_M}}\left[e^{2\sqrt{-(q/p)_M}\int_t^{+\infty} p_L d\tau}-1\right] \\
&= -\frac{1}{2\sqrt{-(q/p)_M}},
\end{aligned} \quad (48)$$

and

$$\begin{aligned}
\zeta(t) &\le \int_t^{+\infty} e^{-\sqrt{-(q/p)_L}\int_s^t 2p(\tau)d\tau} p(s)\,ds \\
&= \frac{1}{2\sqrt{-(q/p)_L}}\left[e^{-2\sqrt{-(q/p)_L}\int_s^t p(\tau)d\tau}\right]_t^{+\infty} \\
&= \frac{1}{2\sqrt{-(q/p)_L}}\left[e^{-2\sqrt{-(q/p)_L}\int_{+\infty}^t p(\tau)d\tau}-1\right] \\
&= \frac{1}{2\sqrt{-(q/p)_L}}\left[e^{2\sqrt{-(q/p)_L}\int_t^{+\infty} p(\tau)d\tau}-1\right] \\
&\le \frac{1}{2\sqrt{-(q/p)_L}}\left[e^{2\sqrt{-(q/p)_L}\int_t^{+\infty} p_M d\tau}-1\right] \\
&= -\frac{1}{2\sqrt{-(q/p)_L}},
\end{aligned} \quad (49)$$

and thus we have

$$-\frac{1}{2\sqrt{-(q/p)_M}} \le \zeta(t) \le -\frac{1}{2\sqrt{-(q/p)_L}}. \quad (50)$$

By (46), we know (44) has a unique $\omega-$periodic continuous solution $\zeta(t)$, and by the transformations $u(t) = y^{-1}(t)$, $y(t) = x(t) - \gamma_1(t)$,we know (2) has another $\omega-$periodic continuous solution $\gamma_2(t)$ as follows

$$\gamma_2(t) = \frac{1}{\zeta(t)} + \gamma_1(t). \quad (51)$$

Since $\gamma_1(t), \gamma_2(t)$ are periodic solutions of (2), we have

$$\begin{aligned}
&\frac{d\gamma_2(t)}{dt} \\
&= p(t)\left(\gamma_2(t)-\sqrt{-\frac{q(t)}{p(t)}}\right)\left(\gamma_2(t)+\sqrt{-\frac{q(t)}{p(t)}}\right),
\end{aligned} \quad (52)$$

$$\begin{aligned}
&\frac{d\gamma_1(t)}{dt} \\
&= p(t)\left(\gamma_1(t)-\sqrt{-\frac{q(t)}{p(t)}}\right)\left(\gamma_1(t)+\sqrt{-\frac{q(t)}{p(t)}}\right).
\end{aligned} \quad (53)$$

Since $\gamma_1(t)$ is a periodic solution of (2), we only consider its maximum and minimum values in a cycle; suppose $\gamma_1(t_1^*)$ is the minimum value of $\gamma_1(t)$, $\gamma_1(t_2^*)$ is the maximum value of $\gamma_1(t)$, t_1^* is the minimum value point of $\gamma_1(t)$, and t_2^* is the maximum value point of $\gamma_1(t)$, where $0 \le t_1^*, t_2^* \le \omega$; then we have

$$\begin{aligned}
\frac{d\gamma_1(t_1^*)}{dt} &= 0, \\
\frac{d\gamma_1(t_2^*)}{dt} &= 0,
\end{aligned} \quad (54)$$

thus it follows that

$$\begin{aligned}
\gamma_1(t_1^*) &= \sqrt{-\frac{q(t_1^*)}{p(t_1^*)}}, \\
\gamma_1(t_2^*) &= \sqrt{-\frac{q(t_2^*)}{p(t_2^*)}},
\end{aligned} \quad (55)$$

and it is easy to see that

$$\begin{aligned}
\gamma_1(t_1^*) &= \sqrt{-\frac{q(t_1^*)}{p(t_1^*)}} \ge \sqrt{-\left(\frac{q}{p}\right)_M}, \\
\gamma_1(t_2^*) &= \sqrt{-\frac{q(t_2^*)}{p(t_2^*)}} \le \sqrt{-\left(\frac{q}{p}\right)_L}.
\end{aligned} \quad (56)$$

From (46), let

$$\frac{d(1/\zeta(t))}{dt} = -\zeta^{-2}\frac{d\zeta(t)}{dt} = -\zeta^{-2}\left(-2p(t)\gamma_1(t)\zeta(t) - p(t)\right) = 0, \tag{57}$$

then we get that the possible extremum of $1/\zeta(t)$ satisfying

$$\frac{1}{\zeta(t^*)} = -2\gamma_1(t^*), \quad (0 \le t^* \le \omega) \tag{58}$$

so

$$\frac{1}{\zeta(t_1^*)} = -2\gamma_1(t_1^*), \quad \frac{1}{\zeta(t_2^*)} = -2\gamma_1(t_2^*), \tag{59}$$

thus we have

$$\zeta(t_1^*) = -\frac{1}{2\gamma_1(t_1^*)}, \quad \zeta(t_2^*) = -\frac{1}{2\gamma_1(t_2^*)}, \tag{60}$$

and it is easy to see that t_1^*, t_2^* are also extreme points of $\zeta(t)$, $\zeta(t_1^*)$ is the maximum value of $\zeta(t)$, $\zeta(t_2^*)$ is the minimum value of $\zeta(t)$, and thus $1/\zeta(t_1^*)$ is the minimum value of $1/\zeta(t)$ and $1/\zeta(t_2^*)$ is the maximum value of $1/\zeta(t)$. By (51), it follows that

$$\gamma_2(t_1^*) = \gamma_1(t_1^*) + \frac{1}{\zeta(t_1^*)} = -\gamma_1(t_1^*) \le -\sqrt{-\left(\frac{q}{p}\right)_M}, \tag{61}$$

$$\gamma_2(t_2^*) = \gamma_1(t_2^*) + \frac{1}{\zeta(t_2^*)} = -\gamma_1(t_2^*) \ge -\sqrt{-\left(\frac{q}{p}\right)_L}, \tag{62}$$

$\gamma_2(t_1^*), \gamma_2(t_2^*)$ are two possible extremums of $\gamma_2(t)$; moreover, let

$$\frac{d\gamma_2}{dt} = p(t)\gamma_2^2 + q(t) = 0, \tag{63}$$

and then we get the following equation that all possible extreme points t^* of function $\gamma_2(t)$ satisfy

$$\gamma_2(t^*) = \pm\sqrt{\frac{q(t^*)}{p(t^*)}}, \quad (0 \le t^* \le \omega). \tag{64}$$

Take the negative sign of (64); since they are the possible extremums of $\gamma_2(t)$, by (61), (62), and (64), we get

$$-\sqrt{-\left(\frac{q}{p}\right)_L} \le \gamma_2(t) \le -\sqrt{-\left(\frac{q}{p}\right)_M}. \tag{65}$$

(2) We prove the stability of two periodic solutions $\gamma_1(t)$ and $\gamma_2(t)$ of (2).

First, we prove the stability of the periodic solution $\gamma_1(t)$ of (2).

It is easy to know that the unique solution $u(t)$ of (44) with initial value $u(t_0) = u_0$ is

$$u(t) = e^{-\int_{t_0}^t 2p(s)\gamma_1(s)ds}u_0 - \int_{t_0}^t e^{-\int_s^t 2p(\tau)\gamma_1(\tau)d\tau}p(s)\,ds \tag{66}$$

$$= e^{-\int_{t_0}^t 2p(s)\gamma_1(s)ds}u_0 - \int_{t_0}^{+\infty} e^{-\int_s^t 2p(\tau)\gamma_1(\tau)d\tau}p(s)\,ds + \int_t^{+\infty} e^{-\int_s^t 2p(\tau)\gamma_1(\tau)d\tau}p(s)\,ds$$

$$= e^{-\int_{t_0}^t 2p(s)\gamma_1(s)ds}u_0 - e^{-\int_{t_0}^t 2p(s)\gamma_1(s)ds}\int_{t_0}^{+\infty} e^{-\int_s^{t_0} 2p(\tau)\gamma_1(\tau)d\tau}p(s)\,ds + \int_t^{+\infty} e^{-\int_s^t 2p(\tau)\gamma_1(\tau)d\tau}p(s)\,ds \tag{67}$$

$$= e^{-\int_{t_0}^t 2p(s)\gamma_1(s)ds}\left[u_0 - \zeta(t_0)\right] + \zeta(t). \tag{68}$$

By (42) and $u(t) = y^{-1}(t)$, the unique solution $y(t)$ of (43) with initial value

$$y(t_0) = \frac{1}{u(t_0)} = x(t_0) - \gamma_1(t_0) \tag{69}$$

is

$$y(t) = \frac{1}{e^{-\int_{t_0}^t 2p(s)\gamma_1(s)ds}\left[u_0 - \zeta(t_0)\right] + \zeta(t)} = \frac{1}{e^{-\int_{t_0}^t 2p(s)\gamma_1(s)ds}\left[1/\left(x(t_0) - \gamma_1(t_0)\right) - \zeta(t_0)\right] + \zeta(t)}. \tag{70}$$

By (42), we have

$$\left|x(t) - \gamma_1(t)\right| = \left|\frac{1}{e^{-\int_{t_0}^t 2p(s)\gamma_1(s)ds}\left[1/\left(x(t_0) - \gamma_1(t_0)\right) - \zeta(t_0)\right] + \zeta(t)}\right|. \tag{71}$$

By (45), we have

$$e^{-\int_{t_0}^t 2p(s)\gamma_1(s)ds} \longrightarrow +\infty \quad (t \longrightarrow +\infty). \tag{72}$$

Following we will discuss the sign of $1/(x(t_0) - \gamma_1(t_0)) - \zeta(t_0)$ in three cases:

(i) If $1/(x(t_0) - \gamma_1(t_0)) - \zeta(t_0) < 0$,that is

$$\frac{1}{\zeta(t_0)} + \gamma_1(t_0) < x(t_0) < \gamma_1(t_0), \tag{73}$$

by (50), (71), and (72), it follows that

$$\left|x(t) - \gamma_1(t)\right| \longrightarrow 0, \quad (t \longrightarrow +\infty), \tag{74}$$

and, therefore, the ω–periodic solution $\gamma_1(t)$ of (2) is attractive if given the initial value $1/\zeta(t_0) + \gamma_1(t_0) < x(t_0) < \gamma_1(t_0)$.

(ii) If

$$\frac{1}{x(t_0)-\gamma_1(t_0)}-\zeta(t_0)>0, \tag{75}$$

from (71), (68), (72), and (69), we have

$$|x(t)-\gamma_1(t)|=\frac{1}{|u(t)|}, \tag{76}$$

$$u(+\infty)\longrightarrow+\infty, \tag{77}$$

$$u(t_0)=\frac{1}{x(t_0)-\gamma_1(t_0)}. \tag{78}$$

Now, we discuss $u(t_0)$ in two cases.

(I) If $1/(x(t_0)-\gamma_1(t_0))>0$, then $x(t_0)>\gamma_1(t_0)$; thus we have

$$u(t_0)=u_0>0, \tag{79}$$

from (66), (79), when $t>t_0$, it follows that

$$u(t)>0, \tag{80}$$

by (76), (77), we have

$$|x(t)-\gamma_1(t)|\longrightarrow 0,\quad (t\longrightarrow+\infty), \tag{81}$$

and thus the ω–periodic solution $\gamma_1(t)$ of (2) is attractive if given the initial value

$$x(t_0)>\gamma_1(t_0). \tag{82}$$

If $x(t_0)=\gamma_1(t_0)$, then $x(t)=\gamma_1(t)$, (81) also holds; by (i) and (I) of (ii), the unique ω–periodic solution $\gamma_1(t)$ of (2) is attractive if given the initial value

$$x(t_0)\in D_1=\left\{x(t_0)\mid x(t_0)>\frac{1}{\zeta(t_0)}+\gamma_1(t_0)\right\}. \tag{83}$$

(II) If $1/(x(t_0)-\gamma_1(t_0))<0$, then

$$x(t_0)<\gamma_1(t_0), \tag{84}$$

thus $u(t_0)=1/(x(t_0)-\gamma_1(t_0))<0$, by (77), and according to zero point theorem, there exists a $t^*>t_0$, such that

$$\begin{aligned} u(t^*) &= e^{-\int_{t_0}^{t^*}(2a(s)\gamma_1(s)+b(s))ds}\left[\frac{1}{x(t_0)-\gamma_1(t_0)}-\zeta(t_0)\right] \\ &\quad+\zeta(t^*)=0, \end{aligned} \tag{85}$$

therefore, when $t\longrightarrow t^*$, we have

$$\begin{aligned} &e^{-\int_{t_0}^{t^*}(2a(s)\gamma_1(s)+b(s))ds}\left[\frac{1}{x(t_0)-\gamma_1(t_0)}-\zeta(t_0)\right] \\ &\quad+\zeta(t^*)\longrightarrow 0, \end{aligned} \tag{86}$$

thus

$$|x(t)-\gamma_1(t)|\longrightarrow+\infty,\quad (t\longrightarrow t^*), \tag{87}$$

by (75) and (84), it follows that

$$x(t_0)<\frac{1}{\zeta(t_0)}+\gamma_1(t_0), \tag{88}$$

and thus the periodic solution $\gamma_1(t)$ of (2) is unstable if initial value $x(t_0)<1/\zeta(t_0)+\gamma_1(t_0)$.

(iii) If $1/(x(t_0)-\gamma_1(t_0))-\zeta(t_0)=0$, that is, $x(t_0)=1/\zeta(t_0)+\gamma_1(t_0)$, at this time, the unique solution $x(t)$ of (2) with initial value $x(t_0)=1/\zeta(t_0)+\gamma_1(t_0)$ is just the periodic solution $\gamma_2(t)$,

$$|x(t)-\gamma_1(t)|=\frac{1}{|\zeta(t)|}>0, \tag{89}$$

and $\gamma_1(t)$ is also unstable.

By (II) of (ii) and (iii), we get that if given the initial value

$$x(t_0)\in D_2=\left\{x(t_0)\mid x(t_0)\le\frac{1}{\zeta(t_0)}+\gamma_1(t_0)\right\}, \tag{90}$$

$\gamma_1(t)$ is unstable.

Next, we prove the stability of the periodic solutions $\gamma_2(t)$ of (2).

By (51), it follows that

$$\begin{aligned} |x(t)-\gamma_2(t)| &= \left|x(t)-\gamma_1(t)-\frac{1}{\zeta(t)}\right| \\ &\ge \frac{1}{|\zeta(t)|}-|x(t)-\gamma_1(t)|, \end{aligned} \tag{91}$$

where $x(t)$ is the unique solution of (2) with initial value $x(t_0)=x_0$. From the above proof, we know that when $x(t_0)\in D_1$,$|x(t)-\gamma_1(t)|\longrightarrow 0,(t\longrightarrow+\infty)$, that is to say, given any $\varepsilon>0$, there is a $T>0$, such that $|x(t)-\gamma_1(t)|<\varepsilon$ as $t\ge t_0+T$, so, when $t\ge t_0+T$, we have

$$|x(t)-\gamma_2(t)|>\frac{1}{|\zeta(t)|}-\varepsilon, \tag{92}$$

and, therefore, it follows that

$$|x(t)-\gamma_2(t)|\ge\frac{1}{|\zeta(t)|}. \tag{93}$$

Note that $|\zeta(t)|$ is bounded and positive on R, and thus $\gamma_2(t)$ is unstable if $x(t_0)\in D_1$.

When $x(t_0)\in D_2$, there are two cases.

(I) If $x(t_0)<1/\zeta(t_0)+\gamma_1(t_0)$, by (87), there exists a $t^*>t_0$, such that

$$|x(t)-\gamma_1(t)|\longrightarrow+\infty,\quad (t\longrightarrow t^*) \tag{94}$$

since $|\zeta(t)|$ is bounded and positive on R, we have

$$\begin{aligned} |x(t)-\gamma_2(t)| &= \left|x(t)-\gamma_1(t)-\frac{1}{\zeta(t)}\right| \\ &\ge |x(t)-\gamma_1(t)|-\frac{1}{|\zeta(t)|}\longrightarrow+\infty, \\ &\qquad (t\longrightarrow t^*) \end{aligned} \tag{95}$$

and thus $\gamma_2(t)$ is unstable.

(II) If $x(t_0) = 1/\zeta(t_0) + \gamma_1(t_0)$, by (89),

$$|x(t) - \gamma_1(t)| = |\gamma_2(t) - \gamma_1(t)| = \frac{1}{|\zeta(t)|} > 0, \quad (96)$$

$\gamma_2(t)$ is also unstable.

Thus $\gamma_2(t)$ is unstable if $x(t_0) \in D_2$.

Therefore, the ω−periodic solution $\gamma_2(t)$ of (2) is unstable on $D_1 \cup D_2 = R$.

This is the end of the proof of Theorem 8. □

Theorem 9. *Under the conditions of Theorem 8, (2) has exactly two ω−periodic continuous solutions: $\gamma_1(t)$ and $\gamma_2(t)$.*

Proof. The proof of the existence of $\gamma_1(t)$ and $\gamma_2(t)$ is seen in Theorem 8; now, we prove that (2) has exactly two ω−periodic continuous solutions: $\gamma_1(t)$ and $\gamma_2(t)$.

We know that if $x(t_0) = \gamma_1(t_0)$, the unique solution of (2) is $\gamma_1(t)$, and if $x(t_0) = \gamma_2(t_0) = 1/\zeta(t_0) + \gamma_1(t_0)$,the unique solution of (2) is $\gamma_2(t)$.

(i) If $x(t_0) < \gamma_2(t_0) = 1/\zeta(t_0) + \gamma_1(t_0)$, by (87), the unique solution $x(t)$ of (2) satisfies

$$|x(t)| \longrightarrow +\infty, \quad (t \longrightarrow t^*) \quad (97)$$

and thus $x(t)$ cannot be a periodic solution.

(ii) If $x(t_0) > \gamma_2(t_0) = 1/\zeta(t_0) + \gamma_1(t_0)$, we know that $\gamma_1(t)$ is attractive; thus the unique solution $x(t)$ of (2) is satisfied

$$|x(t) - \gamma_1(t)| \longrightarrow 0, \quad (t \longrightarrow +\infty) \quad (98)$$

and hence $x(t)$ cannot be a periodic solution; otherwise, there is a certain $\delta > 0$ such that

$$|x(t) - \gamma_1(t)| \geq \delta > 0 \quad (99)$$

for any $t \in R$.

Therefore, (2) has exactly two ω−periodic continuous solutions, $\gamma_1(t)$ and $\gamma_2(t)$.

This is the end of the proof of Theorem 9. □

Theorem 10. *Consider (2), $p(t), q(t)$ are ω−periodic continuous functions, suppose that the following conditions hold:*

$$\begin{aligned} &(H_1) \ \ p(t) > 0, \\ &(H_2) \ \ q(t) < 0, \end{aligned} \quad (100)$$

and then (2) has two ω−periodic continuous solutions.

(1) One ω−periodic continuous solution is $\gamma_1(t)$,

$$\sqrt{-\left(\frac{q}{p}\right)_M} \leq \gamma_1(t) \leq \sqrt{-\left(\frac{q}{p}\right)_L}, \quad (101)$$

and $\gamma_1(t)$ is unstable on R.

(2) Another ω-periodic continuous solution is $\gamma_2(t)$,

$$\begin{aligned} &\gamma_2(t) = \frac{1}{\zeta(t)} + \gamma_1(t), \\ &-\sqrt{-\left(\frac{q}{p}\right)_L} \leq \gamma_2(t) \leq -\sqrt{-\left(\frac{q}{p}\right)_M}, \end{aligned} \quad (102)$$

and $\gamma_2(t)$ is attractive if given initial value on $D_1 = \{x(t_0) \mid x(t_0) < \gamma_1(t_0)\}$, and it is unstable if given initial value on $D_2 = \{x(t_0) \mid x(t_0) \geq \gamma_1(t_0)\}$, where $x(t_0)$ is any given initial value of (2), and

$$\zeta(t) = -\int_{-\infty}^{t} e^{-\int_s^t 2p(\tau)\gamma_1(\tau)d\tau} p(s)\, ds. \quad (103)$$

Proof. By (H_1), (H_2), (2) can be turned into

$$\frac{dx}{dt} = p(t)\left(x - \sqrt{-\frac{q(t)}{p(t)}}\right)\left(x + \sqrt{-\frac{q(t)}{p(t)}}\right). \quad (104)$$

(1) Suppose

$$S = \{\varphi(t) \in C(R,R) \mid \varphi(t+\omega) = \varphi(t)\}. \quad (105)$$

Given any $\varphi(t), \psi(t) \in S$, the distance is defined as follows:

$$\rho(\varphi, \psi) = \sup_{t\in[0,\omega]} |\varphi(t) - \psi(t)|, \quad (106)$$

and thus (S, ρ) is a complete metric space. Take a convex closed set of S as follows

$$\begin{aligned} B = \Bigg\{\varphi(t) \in S \mid \sqrt{-\left(\frac{q}{p}\right)_M} &\leq \varphi(t) \\ \leq \sqrt{-\left(\frac{q}{p}\right)_L}, \ \operatorname{mod}(\varphi) &\subset \operatorname{mod}(p,q)\Bigg\}. \end{aligned} \quad (107)$$

Given any $\varphi(t) \in B$, consider the following equation

$$\begin{aligned} \frac{dx}{dt} &= p(t)\left(x - \sqrt{-\frac{q(t)}{p(t)}}\right)\left(\varphi(t) + \sqrt{-\frac{q(t)}{p(t)}}\right) \\ &= p(t)\left(\varphi(t) + \sqrt{-\frac{q(t)}{p(t)}}\right)x \\ &\quad - p(t)\left(\varphi(t) + \sqrt{-\frac{q(t)}{p(t)}}\right)\sqrt{-\frac{q(t)}{p(t)}}. \end{aligned} \quad (108)$$

By (H_1) and (107), we get that

$$\begin{aligned} 0 < 2p_L\sqrt{-\left(\frac{q}{p}\right)_M} &\leq p(t)\left(\varphi(t) + \sqrt{-\frac{q(t)}{p(t)}}\right) \\ &\leq 2p_M\sqrt{-\left(\frac{q}{p}\right)_L}, \end{aligned} \quad (109)$$

and hence we have

$$\int_0^{\omega} p(t)\left(\varphi(t) + \sqrt{-\frac{q(t)}{p(t)}}\right) dt > 0. \quad (110)$$

Since $p(t), \varphi(t), q(t)$ are ω−periodic continuous functions, it follows that

$$\begin{aligned} &p(t)\left(\varphi(t) + \sqrt{-\frac{q(t)}{p(t)}}\right), \\ &p(t)\left(\varphi(t) + \sqrt{-\frac{q(t)}{p(t)}}\right)\sqrt{-\frac{q(t)}{p(t)}} \end{aligned} \quad (111)$$

are ω–periodic continuous functions; by (110), according to Lemma 3, (108) has a unique positive ω–periodic continuous solution as follows

$$\eta(t)=\int_t^{+\infty}e^{\int_s^t p(\tau)(\varphi(\tau)+\sqrt{-q(\tau)/p(\tau)})d\tau}p(s)\cdot\left(\varphi(s)+\sqrt{-\frac{q(s)}{p(s)}}\right)\sqrt{-\frac{q(s)}{p(s)}}ds, \tag{112}$$

and

$$\operatorname{mod}(\eta)\subset\operatorname{mod}\left(\varphi(\tau)+\sqrt{-\frac{q(\tau)}{p(\tau)}},p(s)\left(\varphi(s)+\sqrt{-\frac{q(s)}{p(s)}}\right)\sqrt{-\frac{q(s)}{p(s)}}\right). \tag{113}$$

By (107), it follows that

$$\begin{aligned}&\operatorname{mod}\left(\varphi(\tau)+\sqrt{-\frac{q(\tau)}{p(\tau)}}\right)\subset\operatorname{mod}(p,q),\\&\operatorname{mod}\left(p(s)\left(\varphi(s)+\sqrt{-\frac{q(s)}{p(s)}}\right)\sqrt{-\frac{q(s)}{p(s)}}\right)\subset\operatorname{mod}(p,q),\end{aligned} \tag{114}$$

and hence we have

$$\operatorname{mod}(\eta)\subset\operatorname{mod}(p,q). \tag{115}$$

By (109), (107), and (112), we get

$$\begin{aligned}\eta(t)&\geq\sqrt{-\left(\frac{q}{p}\right)_M}\int_t^{+\infty}e^{\int_s^t p(\tau)(\varphi(\tau)+\sqrt{-q(\tau)/p(\tau)})d\tau}p(s)\left(\varphi(s)+\sqrt{-\frac{q(s)}{p(s)}}\right)ds\\&=-\sqrt{-\left(\frac{q}{p}\right)_M}\int_t^{+\infty}e^{\int_s^t p(\tau)(\varphi(\tau)+\sqrt{-q(\tau)/p(\tau)})d\tau}d\left(\int_s^t p(\tau)\cdot\left(\varphi(\tau)+\sqrt{-\frac{q(\tau)}{p(\tau)}}\right)d\tau\right)\\&=-\sqrt{-\left(\frac{q}{p}\right)_M}\left[e^{\int_s^t p(\tau)(\varphi(\tau)+\sqrt{-q(\tau)/p(\tau)})d\tau}\right]_t^{+\infty}\\&=-\sqrt{-\left(\frac{q}{p}\right)_M}\left[e^{\int_{+\infty}^t p(\tau)(\varphi(\tau)+\sqrt{-q(\tau)/p(\tau)})d\tau}-1\right]\\&=-\sqrt{-\left(\frac{q}{p}\right)_M}\left[e^{-\int_t^{+\infty} p(\tau)(\varphi(\tau)+\sqrt{-q(\tau)/p(\tau)})d\tau}-1\right]\\&\geq-\sqrt{-\left(\frac{q}{p}\right)_M}\left[e^{-2\int_t^{+\infty}p_L\sqrt{-(q/p)_M}d\tau}-1\right]=\sqrt{-\left(\frac{q}{p}\right)_M},\end{aligned} \tag{116}$$

and

$$\begin{aligned}\eta(t)&\leq\sqrt{-\left(\frac{q}{p}\right)_L}\int_t^{+\infty}e^{\int_s^t p(\tau)(\varphi(\tau)+\sqrt{-q(\tau)/p(\tau)})d\tau}p(s)\left(\varphi(s)+\sqrt{-\frac{q(s)}{p(s)}}\right)ds\\&=-\sqrt{-\left(\frac{q}{p}\right)_L}\int_t^{+\infty}e^{\int_s^t p(\tau)(\varphi(\tau)+\sqrt{-q(\tau)/p(\tau)})d\tau}d\left(\int_s^t p(\tau)\cdot\left(\varphi(\tau)+\sqrt{-\frac{q(\tau)}{p(\tau)}}\right)d\tau\right)\\&=-\sqrt{-\left(\frac{q}{p}\right)_L}\left[e^{\int_s^t p(\tau)(\varphi(\tau)+\sqrt{-q(\tau)/p(\tau)})d\tau}\right]_t^{+\infty}\\&=-\sqrt{-\left(\frac{q}{p}\right)_L}\left[e^{\int_{+\infty}^t p(\tau)(\varphi(\tau)+\sqrt{-q(\tau)/p(\tau)})d\tau}-1\right]\\&=-\sqrt{-\left(\frac{q}{p}\right)_L}\left[e^{-\int_t^{+\infty} p(\tau)(\varphi(\tau)+\sqrt{-q(\tau)/p(\tau)})d\tau}-1\right]\\&\leq-\sqrt{-\left(\frac{q}{p}\right)_L}\left[e^{-2\int_t^{+\infty}p_M\sqrt{-(q/p)_L}d\tau}-1\right]=\sqrt{-\left(\frac{q}{p}\right)_L},\end{aligned} \tag{117}$$

and, hence, $\eta(t)\in B$.

Define a map as follows

$$(T\varphi)(t)=\int_t^{+\infty}e^{\int_s^t p(\tau)(\varphi(\tau)+\sqrt{-q(\tau)/p(\tau)})d\tau}p(s)\cdot\left(\varphi(s)+\sqrt{-\frac{q(s)}{p(s)}}\right)\sqrt{-\frac{q(s)}{p(s)}}ds, \tag{118}$$

and thus if given any $\varphi(t)\in B$, then $(T\varphi)(t)\in B$, and hence $T:B\longrightarrow B$.

Now, we prove that the mapping T is a compact operator.

Consider any sequence $\{\varphi_n(t)\}\subset B(n=1,2,\cdots)$, then it follows that

$$\begin{aligned}&\sqrt{-\left(\frac{q}{p}\right)_M}\leq\varphi_n(t)\leq\sqrt{-\left(\frac{q}{p}\right)_L},\\&\operatorname{mod}(\varphi_n)\subset\operatorname{mod}(p,q).\,(n=1,2,\cdots)\end{aligned} \tag{119}$$

On the other hand, $(T\varphi_n)(t)=x_{\varphi_n}(t)$ satisfies

$$\frac{dx_{\varphi_n}(t)}{dt}=p(t)\left(\varphi_n(t)+\sqrt{-\frac{q(t)}{p(t)}}\right)x_{\varphi_n}(t)-p(t)\left(\varphi_n(t)+\sqrt{-\frac{q(t)}{p(t)}}\right)\sqrt{-\frac{q(t)}{p(t)}}, \tag{120}$$

thus we have

$$\begin{aligned}&\left|\frac{dx_{\varphi_n}(t)}{dt}\right|\leq4p_M\left(-\frac{q}{p}\right)_M,\\&\operatorname{mod}\left(x_{\varphi_n}(t)\right)\subset\operatorname{mod}(p,q),\end{aligned} \tag{121}$$

and hence $\{dx_{\varphi_n}(t)/dt\}$ is uniformly bounded; therefore, $\{x_{\varphi_n}(t)\}$ is uniformly bounded and equicontinuous on R. By the theorem of Ascoli-Arzela, for any sequence $\{x_{\varphi_n}(t)\} \subset B$, there exists a subsequence (also denoted by $\{x_{\varphi_n}(t)\}$) such that $\{x_{\varphi_n}(t)\}$ is convergent uniformly on any compact set of R. Also combined with Lemma 4, $\{x_{\varphi_n}(t)\}$ is convergent uniformly on R; that is to say, T is relatively compact on B.

Next, we prove that T is a continuous operator.

Suppose $\{\varphi_k(t)\} \subset B, \varphi(t) \in B$, and

$$\varphi_k(t) \longrightarrow \varphi(t), \quad (k \longrightarrow \infty) \tag{122}$$

by (118), we have

$$\begin{aligned}
&\left|(T\varphi_k)(t) - (T\varphi)(t)\right| \\
&= \left| \int_t^{+\infty} e^{\int_s^t p(\tau)(\varphi_k(\tau)+\sqrt{-q(\tau)/p(\tau)})d\tau} p(s) \left(\varphi_k(s) + \sqrt{-\frac{q(s)}{p(s)}} \right) \sqrt{-\frac{q(s)}{p(s)}} ds \right. \\
&\quad \left. - \int_t^{+\infty} e^{\int_s^t p(\tau)(\varphi(\tau)+\sqrt{-q(\tau)/p(\tau)})d\tau} p(s) \left(\varphi(s) + \sqrt{-\frac{q(s)}{p(s)}} \right) \sqrt{-\frac{q(s)}{p(s)}} ds \right| \\
&= \left| \int_t^{+\infty} e^{\int_s^t p(\tau)(\varphi_k(\tau)+\sqrt{-q(\tau)/p(\tau)})d\tau} p(s) (\varphi_k(s) - \varphi(s)) \sqrt{-\frac{q(s)}{p(s)}} ds \right. \\
&\quad + \int_t^{+\infty} \left(e^{\int_s^t p(\tau)(\varphi_k(\tau)+\sqrt{-q(\tau)/p(\tau)})d\tau} - e^{\int_s^t p(\tau)(\varphi(\tau)+\sqrt{-q(\tau)/p(\tau)})d\tau} \right) p(s) \left(\varphi(s) \right. \\
&\quad \left. \left. + \sqrt{-\frac{q(s)}{p(s)}} \right) \sqrt{-\frac{q(s)}{p(s)}} ds \right| \\
&= \left| \int_t^{+\infty} e^{\int_s^t p(\tau)(\varphi_k(\tau)+\sqrt{-q(\tau)/p(\tau)})d\tau} p(s) (\varphi_k(s) - \varphi(s)) \sqrt{-\frac{q(s)}{p(s)}} ds \right. \\
&\quad + \int_t^{+\infty} e^{\xi} \left(\int_s^t p(\tau) (\varphi_k(\tau) - \varphi(\tau)) d\tau \right) p(s) \\
&\quad \left. \cdot \left(\varphi(s) + \sqrt{-\frac{q(s)}{p(s)}} \right) \sqrt{-\frac{q(s)}{p(s)}} ds \right|,
\end{aligned} \tag{123}$$

where ξ is between $e^{\int_s^t p(\tau)(\varphi_k(\tau)+\sqrt{-q(\tau)/p(\tau)})d\tau}$ and $e^{\int_s^t p(\tau)(\varphi(\tau)+\sqrt{-q(\tau)/p(\tau)})d\tau}$, thus ξ is between $2p_M\sqrt{-(q/p)_L}(t-s)$ and $2p_L\sqrt{-(q/p)_M}(t-s)$, hence we have

$$\begin{aligned}
&\left|(T\varphi_k)(t) - (T\varphi)(t)\right| \le \left| \int_t^{+\infty} e^{2p_L\sqrt{-(q/p)_M}(t-s)} \right| p(s) \\
&\quad \cdot \left| \sqrt{-\frac{q(s)}{p(s)}} ds + \int_t^{+\infty} e^{2p_L\sqrt{-(q/p)_M}(t-s)} \right. \\
&\quad \cdot \left(\int_s^t |p(\tau)| d\tau \right) \left| p(s) \right| \left(\varphi(s) + \sqrt{-\frac{q(s)}{p(s)}} \right) \\
&\quad \left. \cdot \sqrt{-\frac{q(s)}{p(s)}} ds \right| \rho(\varphi_k, \varphi) \\
&\le \left| \int_t^{+\infty} e^{-2p_L\sqrt{-(q/p)_M}(s-t)} p_M \sqrt{-\left(\frac{q}{p}\right)_L} ds \right. \\
&\quad + 2 \int_t^{+\infty} e^{-2p_L\sqrt{-(q/p)_M}(s-t)} (s-t) \\
&\quad \left. \cdot p_M^2 \left(-\left(\frac{q}{p}\right)_L \right) ds \right| \rho(\varphi_k, \varphi) \\
&= \left(\frac{p_M\sqrt{-(q/p)_L}}{2p_L\sqrt{-(q/p)_M}} + \frac{p_M^2(-(q/p)_L)}{2\left(p_L\sqrt{-(q/p)_M}\right)^2} \right) \\
&\quad \cdot \rho(\varphi_k, \varphi),
\end{aligned} \tag{124}$$

by (122), it follows that

$$(T\varphi_k)(t) \longrightarrow (T\varphi)(t), \quad (k \longrightarrow \infty) \tag{125}$$

and, therefore, T is continuous. By (118), it is easy to see that $T(\partial B) \subset B$, and according to Lemma 5, T has at least a fixed point on B; the fixed point is the periodic continuous solution $\gamma_1(t)$ of (2), and

$$\sqrt{-\left(\frac{q}{p}\right)_M} \le \gamma_1(t) \le \sqrt{-\left(\frac{q}{p}\right)_L}. \tag{126}$$

Let

$$y(t) = x(t) - \gamma_1(t), \tag{127}$$

where $x(t)$ is the unique solution of (2) with initial value $x(t_0) = x_0$, and $\gamma_1(t)$ is the periodic solution of (2); differentiating both sides of (127) along the solution of (2), we get

$$\begin{aligned}
\frac{dy}{dt} &= \frac{dx(t)}{dt} - \frac{d\gamma_1(t)}{dt} = p(t)\left(x^2(t) - \gamma_1^2(t)\right) \\
&= p(t)(x(t) + \gamma_1(t))(x(t) - \gamma_1(t)) \\
&= p(t)(x(t) - \gamma_1(t) + 2\gamma_1(t))(x(t) - \gamma_1(t))
\end{aligned}$$

$$\begin{aligned}&= 2p(t)\gamma_1(t)(x(t)-\gamma_1(t)) \\ &\quad + p(t)(x(t)-\gamma_1(t))^2 \\ &= 2p(t)\gamma_1(t)y + p(t)y^2.\end{aligned} \tag{128}$$

This is Bernoulli's equation. Let $u(t) = y^{-1}(t)$, and it can be turned into the following equation

$$\frac{du}{dt} = -2p(t)\gamma_1(t)u - p(t). \tag{129}$$

Note that

$$\begin{aligned}-2p_M\sqrt{-\left(\frac{q}{p}\right)_L} \le -2p(t)\gamma_1(t) \le -2p_L\sqrt{-\left(\frac{q}{p}\right)_M} \\ < 0,\end{aligned} \tag{130}$$

according to Lemma 3, (129) has a unique periodic continuous solution as follows

$$\zeta(t) = -\int_{-\infty}^{t} e^{-\int_s^t 2p(\tau)\gamma_1(\tau)d\tau} p(s)\,ds, \tag{131}$$

it is easy to know that

$$\begin{aligned}&\int_{-\infty}^{t}\left|e^{-\int_s^t 2p(\tau)\gamma_1(\tau)d\tau} p(s)\right| ds \\ &\quad = \int_{-\infty}^{t} e^{-\int_s^t 2p(\tau)\gamma_1(\tau)d\tau}\left|p(s)\right| ds \\ &\quad \le \int_{-\infty}^{t} e^{-2p_L\sqrt{-(q/p)_M}(t-s)} p_M ds = \frac{p_M}{2p_L\sqrt{-(q/p)_M}},\end{aligned} \tag{132}$$

and thus the infinite integral $\int_{-\infty}^{t}|e^{-\int_s^t 2p(\tau)\gamma_1(\tau)d\tau}p(s)|ds$ is convergent; thereby, the infinite integral $\zeta(t) = \int_{-\infty}^{t} e^{-\int_s^t 2p(\tau)\gamma_1(\tau)d\tau}p(s)ds$ is convergent.

In addition,

$$\begin{aligned}\zeta(t) &\ge -\int_{-\infty}^{t} e^{-\sqrt{-(q/p)_M}\int_s^t 2p(\tau)d\tau} p(s)\,ds \\ &= -\frac{1}{2\sqrt{-(p/q)_M}}\left[e^{-2\sqrt{-(q/p)_M}\int_s^t p(\tau)d\tau}\right]_{-\infty}^{t} \\ &= -\frac{1}{2\sqrt{-(p/q)_M}}\left[1 - e^{-2\sqrt{-(q/p)_M}\int_{-\infty}^t p(\tau)d\tau}\right] \\ &\ge -\frac{1}{2\sqrt{-(p/q)_M}}\left[1 - e^{-2\sqrt{-(q/p)_M}\int_{-\infty}^t p_M d\tau}\right] \\ &= -\frac{1}{2\sqrt{-(p/q)_M}},\end{aligned} \tag{133}$$

and

$$\begin{aligned}\zeta(t) &\le -\int_{-\infty}^{t} e^{-\sqrt{-(q/p)_L}\int_s^t 2p(\tau)d\tau} p(s)\,ds \\ &= -\frac{1}{2\sqrt{-(p/q)_L}}\left[e^{-2\sqrt{-(q/p)_L}\int_s^t p(\tau)d\tau}\right]_{-\infty}^{t} \\ &= -\frac{1}{2\sqrt{-(p/q)_L}}\left[1 - e^{-2\sqrt{-(q/p)_L}\int_{-\infty}^t p(\tau)d\tau}\right] \\ &\le -\frac{1}{2\sqrt{-(p/q)_L}}\left[1 - e^{-2\sqrt{-(q/p)_L}\int_{-\infty}^t p_L d\tau}\right] \\ &= -\frac{1}{2\sqrt{-(p/q)_L}},\end{aligned} \tag{134}$$

thus we have

$$-\frac{1}{2\sqrt{-(p/q)_M}} \le \zeta(t) \le -\frac{1}{2\sqrt{-(p/q)_L}}, \tag{135}$$

by (131), we know that (129) has a unique periodic continuous solution $\zeta(t)$, and by the transformations $u(t) = y^{-1}(t)$, $y(t) = x(t) - \gamma_1(t)$,we know that (2) has another periodic continuous solution $\gamma_2(t)$ as follows

$$\gamma_2(t) = \frac{1}{\zeta(t)} + \gamma_1(t). \tag{136}$$

Since $\gamma_1(t), \gamma_2(t)$ are periodic solution of (2), we have

$$\begin{aligned}&\frac{d\gamma_2(t)}{dt} \\ &\quad = p(t)\left(\gamma_2(t) - \sqrt{-\frac{q(t)}{p(t)}}\right)\left(\gamma_2(t) + \sqrt{-\frac{q(t)}{p(t)}}\right),\end{aligned} \tag{137}$$

$$\begin{aligned}&\frac{d\gamma_1(t)}{dt} \\ &\quad = p(t)\left(\gamma_1(t) - \sqrt{-\frac{q(t)}{p(t)}}\right)\left(\gamma_1(t) + \sqrt{-\frac{q(t)}{p(t)}}\right).\end{aligned} \tag{138}$$

Since $\gamma_1(t)$ is a periodic solution of (2), we only consider its maximum and minimum values in a cycle. Suppose $\gamma_1(t_1^*)$ is the minimum value of $\gamma_1(t)$, $\gamma_1(t_2^*)$ is the maximum value of $\gamma_1(t)$, t_1^* is the minimum value point of $\gamma_1(t)$, and t_2^* is the maximum value point of $\gamma_1(t)$, where $0 \le t_1^*, t_2^* \le \omega$; then we have

$$\begin{aligned}\frac{d\gamma_1(t_1^*)}{dt} &= 0, \\ \frac{d\gamma_1(t_2^*)}{dt} &= 0,\end{aligned} \tag{139}$$

thus it follows

$$\gamma_1(t_1^*) = \sqrt{-\frac{q(t_1^*)}{p(t_1^*)}}, \qquad \gamma_1(t_2^*) = \sqrt{-\frac{q(t_2^*)}{p(t_2^*)}}, \tag{140}$$

and it is easy to see that

$$\gamma_1(t_1^*) = \sqrt{-\frac{q(t_1^*)}{p(t_1^*)}} \geq \sqrt{-\left(\frac{q}{p}\right)_M}, \qquad \gamma_1(t_2^*) = \sqrt{-\frac{q(t_2^*)}{p(t_2^*)}} \leq \sqrt{-\left(\frac{q}{p}\right)_L}. \tag{141}$$

From (131), let

$$\begin{aligned}\frac{d(1/\zeta(t))}{dt} &= -\zeta^{-2}\frac{d\zeta(t)}{dt}\\ &= -\zeta^{-2}\left(-2p(t)\gamma_1(t)\zeta(t) - p(t)\right) = 0.\end{aligned} \tag{142}$$

Then we get that the possible extremums of $1/\zeta(t)$ satisfy

$$\frac{1}{\zeta(t^*)} = -2\gamma_1(t^*), \quad (0 \leq t^* \leq \omega) \tag{143}$$

so

$$\frac{1}{\zeta(t_1^*)} = -2\gamma_1(t_1^*), \qquad \frac{1}{\zeta(t_2^*)} = -2\gamma_1(t_2^*), \tag{144}$$

thus we have

$$\zeta(t_1^*) = -\frac{1}{2\gamma_1(t_1^*)}, \qquad \zeta(t_2^*) = -\frac{1}{2\gamma_1(t_2^*)}, \tag{145}$$

and it is easy to see that t_1^*, t_2^* are also extreme points of $\zeta(t)$, $\zeta(t_1^*)$ is the maximum value of $\zeta(t)$, and $\zeta(t_2^*)$ is the minimum value of $\zeta(t)$; thus $1/\zeta(t_1^*)$ is the minimum value of $1/\zeta(t)$ and $1/\zeta(t_2^*)$ is the maximum value of $1/\zeta(t)$. By (136), it follows that

$$\gamma_2(t_1^*) = \gamma_1(t_1^*) + \frac{1}{\zeta(t_1^*)} = -\gamma_1(t_1^*) \leq -\sqrt{-\left(\frac{q}{p}\right)_M}, \tag{146}$$

$$\gamma_2(t_2^*) = \gamma_1(t_2^*) + \frac{1}{\zeta(t_2^*)} = -\gamma_1(t_2^*) \geq -\sqrt{-\left(\frac{q}{p}\right)_L}, \tag{147}$$

$\gamma_2(t_1^*), \gamma_2(t_2^*)$ are two possible extremums of $\gamma_2(t)$. Moreover, let

$$\frac{d\gamma_2}{dt} = p(t)\gamma_2^2 + q(t) = 0, \tag{148}$$

and then we get from the following equation that all possible extreme points t^* of function $\gamma_2(t)$ satisfy

$$\gamma_2(t^*) = \pm\sqrt{\frac{q(t^*)}{p(t^*)}}, \quad (0 \leq t^* \leq \omega). \tag{149}$$

Take the negative sign of (149). Since they are the possible extremums of $\gamma_2(t)$, by (146), (147), and (149), we get

$$-\sqrt{-\left(\frac{q}{p}\right)_L} \leq \gamma_2(t) \leq -\sqrt{-\left(\frac{q}{p}\right)_M}. \tag{150}$$

(2) We prove the stability of two periodic solutions $\gamma_1(t)$ and $\gamma_2(t)$ of (2).

First, we prove the stability of the periodic solution $\gamma_2(t)$ of (2).

It is easy to know that the unique solution $u(t)$ of (115) with initial value $u(t_0) = u_0$ is

$$u(t) = e^{-\int_{t_0}^{t} 2p(s)\gamma_1(s)ds}u_0 - \int_{t_0}^{t} e^{-\int_s^t p(\tau)\gamma_1(\tau)d\tau}p(s)\,ds \tag{151}$$

$$\begin{aligned}&= e^{-\int_{t_0}^{t} 2p(s)\gamma_1(s)ds}u_0\\ &\quad + \int_{-\infty}^{t_0} e^{-\int_s^t 2p(\tau)\gamma_1(\tau)d\tau}p(s)\,ds\\ &\quad - \int_{-\infty}^{t} e^{-\int_s^t 2p(\tau)\gamma_1(\tau)d\tau}p(s)\,ds\\ &= e^{-\int_{t_0}^{t} 2p(s)\gamma_1(s)ds}u_0\\ &\quad + e^{-\int_{t_0}^{t} 2p(s)\gamma_1(s)ds}\int_{-\infty}^{t_0} e^{-\int_s^{t_0} 2p(\tau)\gamma_1(\tau)d\tau}p(s)\,ds\\ &\quad - \int_{-\infty}^{t} e^{-\int_s^t 2p(\tau)\gamma_1(\tau)d\tau}p(s)\,ds\end{aligned} \tag{152}$$

$$= e^{-\int_{t_0}^{t} 2p(s)\gamma_1(s)ds}\left[u_0 - \zeta(t_0)\right] + \zeta(t). \tag{153}$$

By (127) and $u(t) = y^{-1}(t)$, the unique solution $y(t)$ of (128) with initial value

$$y(t_0) = \frac{1}{u(t_0)} = x(t_0) - \gamma_1(t_0) \tag{154}$$

is

$$\begin{aligned}y(t) &= \frac{1}{e^{-\int_{t_0}^{t} 2p(s)\gamma_1(s)ds}\left[u_0 - \zeta(t_0)\right] + \zeta(t)}\\ &= \frac{1}{e^{-\int_{t_0}^{t} 2p(s)\gamma_1(s)ds}\left[1/(x(t_0) - \gamma_1(t_0)) - \zeta(t_0)\right] + \zeta(t)}.\end{aligned} \tag{155}$$

By (127), (136), and (151), we have

$$\begin{aligned}&\left|x(t)-\gamma_2(t)\right|\\&=\left|\frac{1}{e^{-\int_{t_0}^{t}2p(s)\gamma_1(s)ds}\left[1/\left(x(t_0)-\gamma_1(t_0)\right)-\zeta(t_0)\right]+\zeta(t)}\right.\\&\left.-\frac{1}{\zeta(t)}\right|.\end{aligned} \tag{156}$$

By (130), we have

$$e^{-\int_{t_0}^{t}2p(s)\gamma_1(s)ds}\longrightarrow 0 \quad (t\longrightarrow+\infty). \tag{157}$$

Following we discuss the sign of $1/(x(t_0)-\gamma_1(t_0))-\zeta(t_0)$ in following cases:

(i) If $1/(x(t_0)-\gamma_1(t_0))-\zeta(t_0)<0$, that is,

$$\frac{1}{\zeta(t_0)}+\gamma_1(t_0)<x(t_0)<\gamma_1(t_0), \tag{158}$$

by (135), (156), and (157), it follows that

$$\left|x(t)-\gamma_2(t)\right|\longrightarrow 0, \quad (t\longrightarrow+\infty), \tag{159}$$

and, therefore, the ω–periodic solution $\gamma_2(t)$ of (2) is attractive if given the initial value $1/\zeta(t_0)+\gamma_1(t_0)<x(t_0)<\gamma_1(t_0)$.

(ii) If $1/(x(t_0)-\gamma_1(t_0))-\zeta(t_0)=0$,that is, $x(t_0)=1/\zeta(t_0)+\gamma_1(t_0)$, at this time, the unique solution $x(t)$ of (2) with initial value $x(t_0)=1/\zeta(t_0)+\gamma_1(t_0)$ is just the periodic solution $\gamma_2(t)$, and $|x(t)-\gamma_2(t)|=0$.

(iii) If

$$\frac{1}{x(t_0)-\gamma_1(t_0)}-\zeta(t_0)>0, \tag{160}$$

from (156), (153), (157), and (154), we know

$$\left|x(t)-\gamma_2(t)\right|=\left|\frac{1}{u(t)}-\frac{1}{\zeta(t)}\right|, \tag{161}$$

$$u(t)-\zeta(t)\longrightarrow 0, \quad (t\longrightarrow+\infty) \tag{162}$$

$$u(t_0)=\frac{1}{x(t_0)-\gamma_1(t_0)}. \tag{163}$$

Now, we discuss $u(t_0)$ in two cases.

(I) If $1/(x(t_0)-\gamma_1(t_0))<0$, then $x(t_0)<\gamma_1(t_0)$. Thus we have

$$u(t_0)<0, \tag{164}$$

from (151), when $t>t_0$, it follows that

$$u(t)<u(t_0)<0, \quad (t>t_0) \tag{165}$$

therefore, we have

$$\left|x(t)-\gamma_2(t)\right|\longrightarrow 0, \quad (t\longrightarrow+\infty), \tag{166}$$

and thus the ω–periodic solution $\gamma_2(t)$ of (2) is attractive if given the initial value

$$x(t_0)<\gamma_1(t_0). \tag{167}$$

By (i), (ii), and (I) of (iii), the ω–periodic solution $\gamma_2(t)$ of (2) is attractive if given the initial value

$$x(t_0)\in D_1=\{x(t_0)\mid x(t_0)<\gamma_1(t_0)\}. \tag{168}$$

(II) If $1/(x(t_0)-\gamma_1(t_0))>0$, then

$$x(t_0)>\gamma_1(t_0), \tag{169}$$

thus

$$u(t_0)=\frac{1}{x(t_0)-\gamma_1(t_0)}>0, \tag{170}$$

by (165), there is a $t^{\star}>t_0$, such that

$$u(t^{\star})<0, \quad (t^{\star}>t_0) \tag{171}$$

according to zero point theorem, there exists a $t^{*}\in(t_0,t^{\star})$, such that

$$\begin{aligned}u(t^{*})={}&e^{-\int_{t_0}^{t^{*}}2p(s)\gamma_1(s)ds}\left[\frac{1}{x(t_0)-\gamma_1(t_0)}-\zeta(t_0)\right]\\&+\zeta(t^{*})=0,\end{aligned} \tag{172}$$

therefore, when $t\longrightarrow t^{*}$, we have

$$\begin{aligned}&e^{-\int_{t_0}^{t^{*}}2a(s)\gamma_1(s)+ds}\left[\frac{1}{x(t_0)-\gamma_1(t_0)}-\zeta(t_0)\right]+\zeta(t^{*})\\&\longrightarrow 0,\end{aligned} \tag{173}$$

and thus

$$\left|x(t)-\gamma_2(t)\right|\longrightarrow+\infty, \quad (t\longrightarrow t^{*}). \tag{174}$$

By (160) and (169), it follows that

$$x(t_0)>\gamma_1(t_0), \tag{175}$$

and thus the periodic solution $\gamma_2(t)$ of (2) is unstable if initial value $x(t_0)>\gamma_1(t_0)$.

(iiii)In addition, if $x(t_0)=\gamma_1(t_0)$, then $x(t)=\gamma_1(t)$, and we have

$$\left|x(t)-\gamma_2(t)\right|=\frac{1}{|\zeta(t)|}>0, \tag{176}$$

$\gamma_2(t)$ is unstable.

By (II) of (iii) and (iiii), we get that if given the initial value

$$x(t_0)\in D_2=\{x(t_0)\mid x(t_0)\geq\gamma_1(t_0)\}, \tag{177}$$

$\gamma_2(t)$ is unstable.

Next, we prove the stability of the periodic solutions $\gamma_1(t)$ of (2).

$$\begin{aligned}|x(t)-\gamma_1(t)| &= \left|x(t)-\gamma_1(t)-\frac{1}{\zeta(t)}+\frac{1}{\zeta(t)}\right| \\ &= \left|x(t)-\gamma_2(t)+\frac{1}{\zeta(t)}\right| \\ &\geq \frac{1}{|\zeta(t)|}-|x(t)-\gamma_2(t)|,\end{aligned} \tag{178}$$

when $x(t_0)\in D_1$,$|x(t)-\gamma_2(t)|\longrightarrow 0, (t\longrightarrow +\infty)$; that is to say, given any $\varepsilon>0$, there is a $T>0$, such that $|x(t)-\gamma_2(t)|<\varepsilon$ as $t\geq t_0+T$, so, when $t\geq t_0+T$, we have

$$|x(t)-\gamma_1(t)|>\frac{1}{|\zeta(t)|}-\varepsilon, \tag{179}$$

therefore, it follows that

$$|x(t)-\gamma_1(t)|\geq\frac{1}{|\zeta(t)|}, \tag{180}$$

and note that $|\zeta(t)|$ is bounded and positive on R, so $\gamma_1(t)$ is unstable if $x(t_0)\in D_1$.

When $x(t_0)\in D_2$, there are two cases:

(I) If $x(t_0)>\gamma_1(t_0)$, by (174), there exists a $t^*>t_0$, such that

$$|x(t)-\gamma_2(t)|\longrightarrow +\infty,\quad (t\longrightarrow t^*) \tag{181}$$

since $|\zeta(t)|$ is bounded and positive on R, by (136), we have

$$\begin{aligned}|x(t)-\gamma_1(t)| &= \left|x(t)-\gamma_2(t)+\frac{1}{\zeta(t)}\right| \\ &\geq |x(t)-\gamma_2(t)|-\frac{1}{|\zeta(t)|}\longrightarrow +\infty, \\ &\qquad (t\longrightarrow t^*)\end{aligned} \tag{182}$$

and thus $\gamma_1(t)$ is unstable.

(II) If $x(t_0)=\gamma_1(t_0)$, by (176),

$$|x(t)-\gamma_2(t)|=|\gamma_1(t)-\gamma_2(t)|=\frac{1}{|\zeta(t)|}>0, \tag{183}$$

$\gamma_1(t)$ is also unstable.

Thus $\gamma_1(t)$ is unstable if $x(t_0)\in D_2$.

Therefore, the ω–periodic solution $\gamma_1(t)$ of (2) is unstable on $D_1\bigcup D_2=R$.

This is the end of the proof of Theorem 10. □

Theorem 11. *Under the conditions of Theorem 10, (2) has exactly two ω–periodic continuous solutions, $\gamma_1(t)$ and $\gamma_2(t)$.*

Proof. The proof of the existence of $\gamma_1(t)$ and $\gamma_2(t)$ is seen in Theorem 10. Now, we prove that (2) has exactly two ω–periodic continuous solutions, $\gamma_1(t)$ and $\gamma_2(t)$.

We know that if $x(t_0)=\gamma_1(t_0)$, the unique solution of (2) is $\gamma_1(t)$, and if $x(t_0)=\gamma_2(t_0)=\zeta(t_0)+\gamma_1(t_0)$, the unique solution of (2) is $\gamma_2(t)$.

(I) If $x(t_0)>\gamma_1(t_0)$, by (174), the unique solution $x(t)$ of (2) satisfies

$$|x(t)|\longrightarrow +\infty,\quad (t\longrightarrow t^*) \tag{184}$$

and thus $x(t)$ cannot be periodic solution.

(II) If $x(t_0)<\gamma_1(t_0)$, we know that $\gamma_2(t)$ is attractive; thus the unique solution $x(t)$ of (2) is satisfied $|x(t)-\gamma_2(t)|\longrightarrow 0$ as $t\longrightarrow +\infty$, and hence $x(t)$ cannot be periodic solution; otherwise, there is a certain δ such that $|x(t)-\gamma_2(t)|\geq\delta>0$ for any $t\in R$.

Therefore, (2) has exactly two ω–periodic continuous solutions, $\gamma_1(t)$ and $\gamma_2(t)$.

This is the end of the proof of Theorem 11. □

4. Periodic Solutions on Riccati's Equation

From the proofs of Theorems 8–11, we can get two results about the existence of periodic solutions on (1).

Theorem 12. *Consider (1); $a(t), b(t), c(t)$ are ω–periodic continuous functions, and $a(t), b(t)$ are derivable on R; suppose that the following conditions hold:*

$$\begin{aligned}&(H_1)\ a(t)<0,\\ &(H_2)\ 4a^2(t)c(t)-a(t)b^2(t)+2a(t)b'(t)\\ &\qquad -2a'(t)b(t)>0,\end{aligned} \tag{185}$$

and then (1) has exactly two ω–periodic continuous solutions.

Proof. (1) can be turned into

$$\begin{aligned}\frac{dx}{dt} &= a(t)x^2+b(t)x+c(t) \\ &= a(t)\left(x+\frac{b(t)}{2a(t)}\right)^2+\frac{4a(t)c(t)-b^2(t)}{4a(t)}.\end{aligned} \tag{186}$$

(186) can be also turned into

$$\begin{aligned}\frac{d(x+b(t)/2a(t))}{dt} &= a(t)\left(x+\frac{b(t)}{2a(t)}\right)^2 \\ &\quad +\frac{4a(t)c(t)-b^2(t)}{4a(t)}+\frac{d(b(t)/2a(t))}{dt} \\ &= a(t)\left(x+\frac{b(t)}{2a(t)}\right)^2 \\ &\quad +\frac{4a^2(t)c(t)-a(t)b^2(t)+2a(t)b'(t)-2a'(t)b(t)}{4a^2(t)}.\end{aligned} \tag{187}$$

Let

$$u=x+\frac{b(t)}{2a(t)}, \tag{188}$$

then (187) is turned into

$$\begin{aligned}\frac{du}{dt} &= a(t)u^2 \\ &\quad +\frac{4a^2(t)c(t)-a(t)b^2(t)+2a(t)b'(t)-2a'(t)b(t)}{4a^2(t)}.\end{aligned} \tag{189}$$

By $(H_1), (H_2)$, (189) satisfies all the conditions of Theorems 8 and 9; according to Theorems 8 and 9, (189) has exactly two ω−periodic solutions $\gamma_1(t), \gamma_2(t)$; and by (188), (186) has exactly two ω−periodic solutions

$$\zeta_1(t) = \gamma_1(t) - \frac{b(t)}{2a(t)},$$
$$\zeta_2(t) = \gamma_2(t) - \frac{b(t)}{2a(t)}. \tag{190}$$

Similarly, we can get the following. □

Theorem 13. *Consider (1); $a(t), b(t), c(t)$ are ω−periodic continuous functions, and $a(t), b(t)$ are derivable on R; suppose that the following conditions hold:*

$$(H_1)\ a(t) > 0,$$
$$(H_2)\ 4a^2(t)c(t) - a(t)b^2(t) + 2a(t)b'(t) - 2a'(t)b(t) < 0, \tag{191}$$

and then (1) has exactly two ω−periodic continuous solutions.

Proof. The proof is similar to that of Theorem 12, so we omit it here. □

5. Example

The following example shows the feasibility of our main results.

Example 1. Consider the following equation:

$$\frac{dx}{dt} = (-2 + \sin t)x^2 + 3 - \cos t. \tag{192}$$

Here, $p(t) = -2 + \sin t, q(t) = 3 - \cos t$, and it is easy to calculate that

$$\sqrt{-\left(\frac{q}{p}\right)_M} = 0.9194,$$
$$\sqrt{-\left(\frac{q}{p}\right)_L} = 1.7761. \tag{193}$$

Clearly, conditions (H_1)-(H_2) of Theorems 8 and 9 are satisfied. It follows from Theorems 8 and 9 that (192) has exactly two 2π−periodic continuous solutions $\gamma_1(t)$ and $\gamma_2(t)$,

$$0.9194 = \sqrt{-\left(\frac{q}{p}\right)_M} \le \gamma_1(t) \le \sqrt{-\left(\frac{q}{p}\right)_L} = 1.7761, \tag{194}$$

and $\gamma_1(t)$ is attractive on $D_1 = \{x(t_0) \mid x(t_0) \le 1/\zeta(t_0) + \gamma_1(t_0)\}$, and unstable on $D_2 = \{x(t_0) \mid x(t_0) > 1/\zeta(t_0)+\gamma_1(t_0)\}$, where $x(t_0)$ is any given initial value of (192), and

$$\zeta(t) = \int_t^{+\infty} e^{-\int_s^t 2(-2+\sin t)\gamma_1(\tau)d\tau}(-2 + \sin s)\,ds. \tag{195}$$

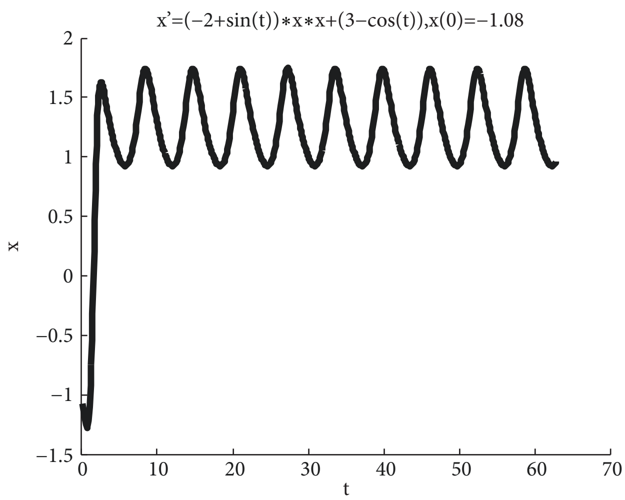

Figure 1: The curve of the solution of (192) with initial value $x(0) = -1.08$.

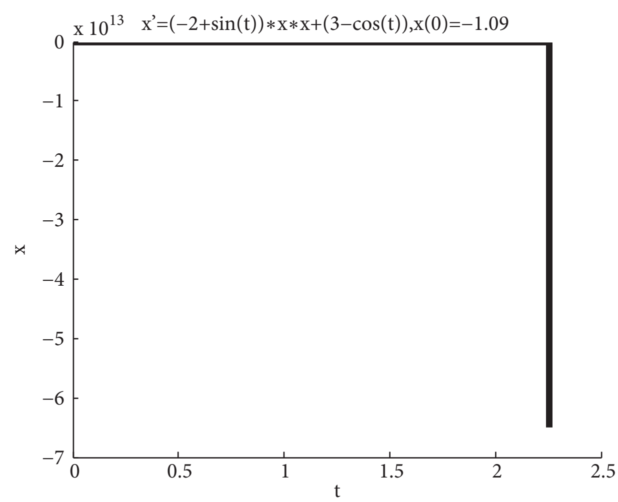

Figure 2: The curve of the solution of (192) with initial value $x(0) = -1.09$.

(2) Another ω−periodic continuous solution is $\gamma_2(t)$,

$$-1.7761 = -\sqrt{-\left(\frac{q}{p}\right)_L} \le \gamma_2(t) \le -\sqrt{-\left(\frac{q}{p}\right)_M} = -0.9194, \tag{196}$$

and $\gamma_2(t)$ is unstable on R.

From this example, using Matlab, we can deduce the value $-1.09 < \gamma_2(0) = 1/\zeta(0) + \gamma_1(0) < -1.08$; when initial value $x(0) \ge -1.08$, the solution curve of (192) tends to the curve of the periodic solution $\gamma_1(t)$ as t is achieved at a certain value (see Figure 1); when initial value $x(0) \le -1.09$, the solution curve of (192) arrives at $+\infty$ at some time t^* (see Figure 2).

6. Concluding Remarks

In this paper, when the coefficient functions of Riccati's type equation satisfy

$$p(t)\,q(t) < 0, \tag{197}$$

we obtain the existence and more accurate range of two periodic solutions of the equation by means of the fixed point theorem. This is a great improvement on the paper [1, 6] and provides a criterion for judging the existence and size range of periodic solutions of the equation, which has great application value in engineering technological and physical fields.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

The authors contributed to each part of this paper equally. The authors read and approved the final manuscript.

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