Hindawi
Mathematical Problems in Engineering
Volume 2018, Article ID 2940105, 9 pages
https://doi.org/10.1155/2018/2940105

Research Article

Robust Output Controllability Analysis and Control Design for Incomplete Boolean Networks with Disturbance Inputs

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Received 17 March 2018; Accepted 1 November 2018; Published 21 November 2018

Academic Editor: Sabri Arik

This paper addresses the problems of robust-output-controllability and robust optimal output control for incomplete Boolean control networks with disturbance inputs. First, by resorting to the semi-tensor product technique, the system is expressed as an algebraic form, based on which several necessary and sufficient conditions for the robust output controllability are presented. Second, the Mayer-type robust optimal output control issue is studied and an algorithm is established to find a control scheme which can minimize the cost functional regardless of the effect of disturbance inputs. Finally, a numerical example is given to demonstrate the effectiveness of the obtained new results.

1. Introduction

Boolean network (BN) was firstly introduced by Kauffman for modelling genetic regulatory networks (GRNs) [1]. Since then, it has been used to analyze and simulate cellular networks. A typical BN consists of N nodes, and each can take one of the following two values: 1 or 0, representing the gene is expressed or not, respectively. Furthermore, the state evolution of each node can be determined by Boolean functions. Although BN is a simplified model, it becomes a powerful tool in analysing GRNs. And some significant results were presented [2, 3].

Recently, D. Cheng and coauthors have developed an algebraic state space representation (ASSR) of Boolean control networks (BCNs) by using the semitensor product (STP) of matrices [4]. Using the ASSR, many fundamental and landmark results about BCNs, which include but are not limited to controllability and observability [5–8], stability and stabilization [9–11], disturbance decoupling [12, 13], network synchronization [14, 15], output tracking [16], output regulation [17, 18], and optimal control [19], have been obtained in the last few years. The STP is an interesting research topic, and we can refer to [20–23] for some other applications of it.

As is well known, output-controllability and optimal output control issues are fundamental concepts in a control theory field. The output controllability describes the ability of an external input to move the output from any initial condition to any final condition in a finite time interval. The optimal output deals with the problem of finding an external input for a given system such that a certain optimality output criterion is achieved. A controllable system is not necessarily output controllable, and an output controllable system is not necessarily controllable. Hence, they are also two important structural properties in modern control theory which reflect the dominant ability of the control inputs over outputs [24, 25]. As a suitable model of GRNs, the study of output controllability (reachability) and optimal control is also meaningful for BCNs, and there are some literatures studying on both of them. Reference [26] discussed the output reachability of BCNs, and several necessary and sufficient conditions were obtained for the verification of output reachability. The output-controllability and optimal output control problems of a state-dependent switched BCN were studied in [27], in which the output controllability criteria were presented and an optimal output control design algorithm for the Mayer-type optimal output problem was established.

Robust control is a branch of control theory that deals with uncertainty [28]. It aims to achieve robust performance, such as controllability, in the presence of uncertain parameters or disturbances. It is known that, in practical networks such as GRNs, the disturbances are ubiquitous, which primarily stem from environmental changes, biological uncertainties, and experimental noises [29, 30]. These disturbance inputs may lead to instability of networks or make it difficult to understand the network dynamics. To be consistent with the dynamical behaviours of the practical networks, disturbances should be a characteristic which has to be taken into account when modelling the GRNs by BNs [31]. Thus, two interesting problems with biological significance are robust-output-controllability and robust optimal output control for BCNs with disturbance inputs, which deserve investigation for theoretical developments as well as for practical applications.

In order to solve two classical intellectual problems: the wolf-sheep-cabbage puzzle and the missionaries-cannibals problem, the authors of [32] presented a new logical control system (LCS), called the incomplete LCS, in which certain controls can only be applied to certain states. Moreover, in [33], authors modeled a networked evolutionary game with bankruptcy mechanism as an LCS whose certain control-states should be forbidden. These cases show that the investigation of incomplete BCNs is of practical significance. Unfortunately, to the best of authors' knowledge, there is no result available for robust output controllability and robust optimal output control of incomplete BCNs with disturbance inputs, which motivates the study of this research.

This paper investigates the robust-output-controllability and robust optimal output control of incomplete BCNs with disturbance inputs. The main difficulties of extending the results obtained in [27, 34] contain the following two aspects: (1) the dynamics of a BCN with disturbance inputs is much more complicated than a BCN and, in this paper, the disturbance inputs are arbitrary; (2) the existence of control-state avoiding set makes the control law of this paper different from that of the existing results about BCNs since some control-states should be avoided when designing the control sequences. We present several necessary and sufficient conditions for the robust output controllability of incomplete BCNs with disturbance inputs. Moreover, a robust optimal output control design algorithm for the Mayer-type robust optimal output problem is also derived.

The remainder of this paper is organized as follows. Section 2 provides some preliminary results. In Section 3, we study the robust-output-controllability and robust optimal output control problems of incomplete BCNs with disturbance inputs, and present the main results of this paper. A numerical example is given in Section 4 to illustrate the validity of the obtained new results. Finally, conclusions are drawn in Section 5.

2. Preliminaries

Firstly, we list some notations which will be used throughout this paper.

(i) $\mathscr{D} := \{1, 0\}$ and $\mathscr{D}^n := \underbrace{\mathscr{D} \times \cdots \times \mathscr{D}}_{n}$.

(ii) δ_n^i: the i-th column of the identity matrix I_n.

(iii) $\Delta_n := \{\delta_n^i \mid i = 1, 2, \cdots, n\}$, and for compactness, $\Delta := \Delta_2$. By identifying $1 \sim \delta_2^1$ and $0 \sim \delta_2^2$, where "$\sim$" denotes two equivalent forms of the same object, the logical variable in $\mathscr{D}$ then takes value from Δ.

(iv) $Col(A)$ ($Row(A)$) is the set of columns (rows) of A. $Col_i(A)$ ($Row_i(A)$) is the i-th column (row) of A.

(v) $\mathscr{M}_{m\times n}$: the set of $m \times n$ real matrices. A matrix $A \in \mathscr{M}_{m\times n}$ is called a logical matrix, if the columns of A are elements of Δ_m. Denote by $\mathscr{L}_{m\times n}$ the set of $m \times n$ logical matrices.

(vi) If $A \in \mathscr{L}_{m\times n}$, by definition it can be expressed as $A = [\delta_m^{i_1}, \delta_m^{i_2}, \cdots, \delta_m^{i_n}]$, and its shorthand is $A = \delta_m [i_1 \ i_2 \ \cdots \ i_n]$.

(vii) A matrix $B \in \mathscr{M}_{m\times n}$ is called a Boolean matrix, if its entries $b_{ij} \in \mathscr{D}$, for every i, j. Denote by $\mathscr{B}_{m\times n}$ the set of $m \times n$ Boolean matrices.

(viii) A matrix $L \in \mathscr{M}_{m\times n}$ is called an incomplete logical matrix, if

$$Col(L) \subset \Delta_m \bigcup \{\mathbf{0}_m\}, \tag{1}$$

where $\mathbf{0}_m$ is an m dimensional zero vector. Identifying $\mathbf{0}_m$ with δ_m^0, an incomplete logical matrix L can briefly expressed as $L = \delta_m [i_1 \ i_2 \ \cdots \ i_n]$, where certain i_r could be 0. Denote by $\mathscr{L}_{m\times n}^I$ the set of $m \times n$ incomplete logical matrices.

(ix) For $A \in \mathscr{M}_{m\times r}, B \in \mathscr{M}_{n\times r}$,

$$\begin{aligned} A * B &\\ &= [Col_1(A) \ltimes Col_1(B), \cdots, Col_r(A) \ltimes Col_r(B)] \end{aligned} \tag{2}$$

is the Khatri-Rao product of A and B.

Some necessary definitions are given as follows.

Definition 1 (see [4]). For $A \in \mathscr{M}_{m\times n}$ and $B \in \mathscr{M}_{p\times q}$, the STP, denoted by $A \ltimes B$, is defined as follows:

$$A \ltimes B := (A \otimes I_{s/n})(B \otimes I_{s/p}), \tag{3}$$

where $s =$l.c.m.(n, p), denoting the least common multiple of n and p, and $\otimes$ denotes the Kronecker product.

It is noted that the STP is a generalization of the ordinary matrix product. In this paper, we simply call it "product" and omit the symbol "$\ltimes$" if no confusion arises.

Definition 2. The intersection of Boolean matrices is defined as

$$A \wedge B = (a_{ij} \wedge b_{ij}) \in \mathscr{B}_{m\times n}, \quad \forall A, B \in \mathscr{B}_{m\times n}. \tag{4}$$

Definition 3. Let $A = (a_{ij}) \in \mathscr{B}_{m\times n}$ be a matrix; then we call $A > 0$ if and only if $a_{ij} > 0, \forall i, j$.

Finally, we give some conclusions which will be used in the remainder of the paper.

Lemma 4 (see [4]). *Any logical function $f(x_1, \cdots, x_n)$ with logical arguments $x_1, \cdots, x_n \in \Delta$ can be expressed in a multilinear form as*

$$f(x_1, \cdots, x_n) = M_f x_1 \cdots x_n, \tag{5}$$

where $M_f \in \mathscr{L}_{2\times 2^n}$ is unique, called the structure matrix of logical function f.

Consider a disturbed BCN as follows:

$$\begin{aligned} x_1(t+1) &= f_1(\Xi(t), X(t), U(t)), \\ x_2(t+1) &= f_2(\Xi(t), X(t), U(t)), \\ &\vdots \\ x_n(t+1) &= f_n(\Xi(t), X(t), U(t)); \\ y_j(t) &= h_j(X(t)), \quad j = 1, \cdots, p, \end{aligned} \tag{6}$$

where $X(t) = (x_1(t), x_2(t), \cdots, x_n(t)) \in \mathscr{D}^n$ denotes the state variable, $U(t) = (u_1(t), \cdots, u_m(t)) \in \mathscr{D}^m$ denotes the control input, $\Xi(t) = (\xi_1(t), \cdots, \xi_r(t)) \in \mathscr{D}^r$ denotes the disturbance input, $Y(t) = (y_1(t), y_2(t), \cdots, y_p(t)) \in \mathscr{D}^p$ denotes the output variable, and $f_i : \mathscr{D}^{n+m+r} \longmapsto \mathscr{D}$, $i = 1, \cdots, n$, and $h_j : \mathscr{D}^n \longmapsto \mathscr{D}$, $j = 1, \cdots, p$ are Boolean functions. Denote the output trajectory of the system (6) by $Y(t; X(0), U^{t-1}, \Xi^{t-1})$, where $X(0) \in \mathscr{D}^n$, $U^{t-1} := \{U(0), \cdots, U(t-1)\} \subseteq \mathscr{D}^m$, and $\Xi^{t-1} := \{\Xi(0), \cdots, \Xi(t-1)\} \subseteq \mathscr{D}^r$ represent the initial state, the control sequence, and the sequence of disturbance inputs, respectively.

In order to convert the system (6) into an algebraic form, we define $x(t) = \ltimes_{i=1}^n x_i(t)$, $u(t) = \ltimes_{i=1}^m u_i(t)$, $\xi(t) = \ltimes_{i=1}^r \xi_i(t)$, and $y(t) = \ltimes_{i=1}^p y_i(t)$. Assume that the structure matrices for f_i, h_j are $F_i \in \mathscr{L}_{2\times 2^{n+m+r}}$, $H_j \in \mathscr{L}_{2\times 2^n}$, respectively, then the system (6) can be converted into the following algebraic form:

$$\begin{aligned} x(t+1) &= L\xi(t)u(t)x(t), \\ y(t) &= Hx(t), \end{aligned} \tag{7}$$

where $L = *_{i=1}^n F_i \in \mathscr{L}_{2^n\times 2^{n+m+r}}$ is called the state transition matrix, $H = *_{j=1}^n H_j \in \mathscr{L}_{2^p\times 2^n}$, and $*$ is the Khatri-Rao product.

3. Main Results

In this section, we study the robust-output-controllability and robust optimal output control of disturbed incomplete BCNs and present the main results of this paper.

3.1. An Algebraic Form for Disturbed Incomplete BCNs. First, we give the concept of disturbed incomplete BCNs.

Definition 5. Consider the disturbed BCN (7). Suppose certain controls are not applicable to certain states. Precisely, there exists a set of pairs

$$\Theta := \left\{ \left(\delta_{2^m}^{\alpha_i}, \delta_{2^n}^{\beta_i}\right) \mid i = 1, \cdots, s \right\} \subset \Delta_{2^m} \times \Delta_{2^n}, \tag{8}$$

such that the control $\delta_{2^m}^{\alpha_i}$ is not applicable to $\delta_{2^n}^{\beta_i}$. Θ is called the control-state avoiding set. Supposing that $\Theta \neq \emptyset$, then the disturbed BCN is called the disturbed incomplete BCN.

Consider the disturbed incomplete BCN (7) with control-state avoiding set Θ and split L into 2^r blocks as $L := [L_1, L_2, \cdots, L_{2^r}]$, where $L_k \in \mathscr{L}_{2^n\times 2^{n+m}}$, $k = 1, 2, \cdots, 2^r$. For $u(t) = \delta_{2^m}^{\alpha_i}$, $x(t) = \delta_{2^n}^{\beta_i}$, a straightforward computation shows that $u(t) \ltimes x(t) = \delta_{2^m}^{\alpha_i} \ltimes \delta_{2^n}^{\beta_i} = \delta_{2^{m+n}}^{a_i}$, where $a_i = (\alpha_i - 1)2^n + \beta_i$. Thus, when $\xi(t) = \delta_{2^r}^k$, $u(t) = \delta_{2^m}^{\alpha_i}$ and $x(t) = \delta_{2^n}^{\beta_i}$, we have

$$\begin{aligned} x(t+1) &= L\xi(t)u(t)x(t) = L_k u(t)x(t) = L_k \delta_{2^{m+n}}^{a_i} \\ &= Col_{a_i}(L_k), \end{aligned} \tag{9}$$

where $a_i = (\alpha_i - 1)2^n + \beta_i$ and $k = 1, 2, \cdots, 2^r$. If $(\delta_{2^m}^{\alpha_i}, \delta_{2^n}^{\beta_i}) \in \Theta$, we let $Col_{a_i}(L_k) = \delta_{2^n}^0$. In this case, the corresponding $x(t+1)$ does not exist. We also denote it briefly as $x(t+1) = \delta_{2^n}^0$. Then, it is easy to obtain the following result.

Proposition 6. *A disturbed incomplete BCN with control-state avoiding set Θ as in (8) has its algebraic expression as in (7) except that*

$$Col_{a_i}(L_k) = \delta_{2^n}^0, \tag{10}$$

where $a_i = (\alpha_i - 1)2^n + \beta_i$, $i = 1, 2, \cdots, s$, and $k = 1, 2, \cdots, 2^r$.

We denote the new blocks obtained from Proposition 6 by L_k^Θ, that is,

$$\begin{aligned} Col_{a_i}\left(L_k^\Theta\right) &= \delta_{2^n}^0, \quad a_i = (\alpha_i - 1)2^n + \beta_i; \\ Col_{a_i}\left(L_k^\Theta\right) &= Col_{a_i}(L_k), \quad \text{otherwise}. \end{aligned} \tag{11}$$

Let I^Θ be the matrix obtained from an identity matrix $I_{2^{n+m}\times 2^{n+m}}$ substituting the columns with indexes $a_1, \cdots, a_s$ by $\delta_{2^{n+m}}^0$; then L_k^Θ can be calculated by $L_k^\Theta = L_k I^\Theta$.

Construct a new matrix as follows:

$$L^\Theta = \left[L_1^\Theta, L_2^\Theta, \cdots, L_{2^r}^\Theta\right], \tag{12}$$

and then it is easy to see that L^Θ is the state transition matrix of a disturbed incomplete BCN with control-state avoiding set Θ. In fact, L^Θ can be further calculated as

$$\begin{aligned} L^\Theta &= \left[L_1^\Theta, L_2^\Theta, \cdots, L_{2^r}^\Theta\right] = \left[L_1 I^\Theta, L_2 I^\Theta, \cdots, L_{2^r} I^\Theta\right] \\ &= \left[L \ltimes \delta_{2^r}^1 \ltimes I^\Theta, L \ltimes \delta_{2^r}^2 \ltimes I^\Theta, \cdots, L \ltimes \delta_{2^r}^{2^r} \ltimes I^\Theta\right] \\ &= L \ltimes \left[\delta_{2^r}^1 \ltimes I^\Theta, \delta_{2^r}^2 \ltimes I^\Theta, \cdots, \delta_{2^r}^{2^r} \ltimes I^\Theta\right] \\ &= L\left(I_{2^r} \otimes I^\Theta\right). \end{aligned} \tag{13}$$

Based on (7) and above analysis, the following result is obvious.

Proposition 7. *Consider a disturbed incomplete BCN with control-state avoiding set* Θ. *It can be expressed as an algebraic form:*

$$x(t+1) = L^{\Theta}\xi(t)u(t)x(t), \quad y(t) = Hx(t), \tag{14}$$

where $L^{\Theta} \in \mathscr{L}^{I}_{2^n \times 2^{n+m+r}}$ *is given in (12).*

Proof. We only need to prove that the state trajectories of the disturbed incomplete logical system (14) are equivalent to that of the system (7) with control-state avoiding set Θ.

Denote $W = \{\delta^{a_i}_{2^{m+n}} = \delta^{\alpha_i}_{2^m} \ltimes \delta^{\beta_i}_{2^n} \mid (\delta^{\alpha_i}_{2^m}, \delta^{\beta_i}_{2^n}) \in \Theta\}$, where $a_i = (\alpha_i - 1)2^n + \beta_i$. It is obvious that W and Θ are equivalent. For any $\xi(t) = \delta^{k}_{2^r}$, $k = 1, \cdots, 2^r$, we first prove the case of $u(t)x(t) = \delta^{a_i}_{2^{m+n}} \in W$. A simple calculation shows that

$$\begin{aligned} x(t+1) &= L\xi(t)u(t)x(t) = L_k u(t)x(t) \\ &= Col_{a_i}(L_k) = \delta^{0}_{2^n} = Col_{a_i}(L^{\Theta}_k) \\ &= L^{\Theta}_k u(t)x(t) = L^{\Theta}\xi(t)u(t)x(t). \end{aligned} \tag{15}$$

When $u(t)x(t) = \delta^{a_i}_{2^{m+n}} \notin W$, we have

$$\begin{aligned} x(t+1) &= L\xi(t)u(t)x(t) = L_k u(t)x(t) \\ &= Col_{a_i}(L_k) = Col_{a_i}(L^{\Theta}_k) = L^{\Theta}_k u(t)x(t) \\ &= L^{\Theta}\xi(t)u(t)x(t). \end{aligned} \tag{16}$$

Hence, in both cases, the state trajectories of the disturbed incomplete logical system (14) are equivalent to that of the system (7) with control-state avoiding set W. Since W and Θ re equivalent, this completes the proof. □

3.2. Robust Output Controllability Analysis of Disturbed Incomplete BCNs. Now, we give definitions of the robust output controllability for disturbed incomplete BCNs.

Definition 8. Consider the disturbed incomplete BCN (14).

(i) $y_f \in \Delta_{2^p}$ is said to be s-robust-output-reachable from the initial state $x(0) \in \Delta_{2^n}$, if one can find a control sequence $u^{s-1} \subseteq \Delta_{2^m}$ such that for arbitrary disturbance inputs $\xi^{s-1} \subseteq \Delta_{2^r}$, the output at time $t = s$ satisfies $y(s; x(0), u^{s-1}, \xi^{s-1}) = y_f$, where $u^{s-1} := \{u(0), \cdots, u(s-1)\}$ and $\xi^{s-1} := \{\xi(0), \cdots, \xi(s-1)\}$. The s-robust-output-reachable set of the initial state $x(0)$ is denoted by $R_s(x(0))$.

(ii) The system (14) is said to be s-robust-output-controllable at the initial state $x(0) \in \Delta_{2^n}$, if for any output state $y_f \in \Delta_{2^p}$, one can find a control sequence $u^{s-1} \subseteq \Delta_{2^m}$ such that, for arbitrary disturbance inputs $\xi^{s-1} \subseteq \Delta_{2^r}$, the output at time $t = s$ satisfies $y(s; x(0), u^{s-1}, \xi^{s-1}) = y_f$. The system (14) is said to be s-robust-output-controllable, if the system is s-robust-output-controllable at any $x(0) \in \Delta_{2^n}$.

(iii) The system (14) is said to be robust-output-controllable at the initial state $x(0) \in \Delta_{2^n}$, if for any output state $y_f \in \Delta_{2^p}$, one can find a positive integer T and a control sequence $u^{T-1} \subseteq \Delta_{2^m}$ such that, for arbitrary disturbance inputs $\xi^{T-1} \subseteq \Delta_{2^r}$, the output at time $t = T$ is $y(T; x(0), u^{T-1}, \xi^{T-1}) = y_f$. The system (14) is said to be robust-output-controllable, if the system is robust-output-controllable at any $x(0) \in \Delta_{2^n}$.

To obtain necessary and sufficient conditions for the robust output controllability of disturbed incomplete BCNs, we first consider the robust reachability of disturbed incomplete BCNs.

Consider the disturbed incomplete BCN (14), and denote by

$$\widetilde{L}^{\Theta} = L^{\Theta}_1 \wedge L^{\Theta}_2 \wedge \cdots \wedge L^{\Theta}_{2^r}. \tag{17}$$

Split $\widetilde{L}^{\Theta}$ into 2^m square blocks as

$$\widetilde{L}^{\Theta} = \left[\widetilde{L}^{\Theta}_1, \widetilde{L}^{\Theta}_2, \cdots, \widetilde{L}^{\Theta}_{2^m}\right], \tag{18}$$

where $\widetilde{L}^{\Theta}_j \in \mathscr{L}^{I}_{2^n \times 2^n}$, $j = 1, 2, \cdots, 2^m$. Then we have the following result.

Theorem 9. *Consider the disturbed incomplete BCN (14). Let* $x_f = \delta^{q}_{2^n}$ *and* $x(0) = \delta^{r}_{2^n}$ *be given. Then,*

(i) x_f *is robust reachable from* $x(0)$ *at the s-th step, if and only if*

$$(M^s)_{q,r} > 0, \tag{19}$$

where $M = \sum_{j=1}^{2^m}(\widetilde{L}^{\Theta}_j)$ *and* $(M^s)_{q,r}$ *denotes the* (q,r)*th element of* M^s;

(ii) x_f *is robust reachable from* $x(0)$, *if and only if*

$$\sum_{s=1}^{2^{m+n+r}} (M^s)_{q,r} > 0. \tag{20}$$

Proof.

(i) (Necessity). We prove it by induction on s. Consider the case $s = 1$. It is assumed that $x(1) = \delta^{q}_{2^n}$ can be robust reachable from $x(0) = \delta^{r}_{2^n}$ at the first step; then one can find at least one control $u(0) = \delta^{j}_{2^m}$ such that, for any disturbance input $\xi(0) = \delta^{k}_{2^r}$, the following equation will hold:

$$\delta^{q}_{2^n} = L^{\Theta}_k \delta^{j}_{2^m} \delta^{r}_{2^n}, \quad \forall k = 1, 2, \cdots, 2^r. \tag{21}$$

Multiplying (21) from the left by $(\delta^{q}_{2^n})^T$ yields

$$1 = (\delta^{q}_{2^n})^T L^{\Theta}_k \delta^{j}_{2^m} \delta^{r}_{2^n}, \quad \forall k = 1, 2, \cdots, 2^r. \tag{22}$$

For each block $L^{\Theta}_k \in \mathscr{L}^{I}_{2^n \times 2^{n+m}}$, split it into 2^m equal blocks as

$$L^{\Theta}_k = \left[L^{\Theta}_{k1}, L^{\Theta}_{k2}, \cdots, L^{\Theta}_{k2^m}\right], \tag{23}$$

where L_k^{Θ} is given in (12). Then according to (22), we have

$$\left(L_{1j}^{\Theta}\right)_{q,r} = \left(L_{2j}^{\Theta}\right)_{q,r} = \cdots = \left(L_{2^r j}^{\Theta}\right)_{q,r} = 1. \tag{24}$$

Since $\widetilde{L}^{\Theta} = L_1^{\Theta} \wedge L_2^{\Theta} \wedge \cdots \wedge L_{2^r}^{\Theta}$, one knows that

$$\begin{aligned}\widetilde{L}^{\Theta}\delta_{2^m}^{j} &= \widetilde{L}_j^{\Theta} = \left(L_1^{\Theta} \wedge L_2^{\Theta} \wedge \cdots \wedge L_{2^r}^{\Theta}\right)\delta_{2^m}^{j} \\ &= \left(L_1^{\Theta}\delta_{2^m}^{j}\right) \wedge \left(L_2^{\Theta}\delta_{2^m}^{j}\right) \wedge \cdots \wedge \left(L_{2^r}^{\Theta}\delta_{2^m}^{j}\right) \\ &= L_{1j}^{\Theta} \wedge L_{2j}^{\Theta} \wedge \cdots \wedge L_{2^r j}^{\Theta}.\end{aligned} \tag{25}$$

Combining (24) and (25), we have

$$\left(\widetilde{L}_j^{\Theta}\right)_{q,r} = 1. \tag{26}$$

Thus,

$$(M)_{q,r} = \left(\widetilde{L}_1^{\Theta} + \widetilde{L}_2^{\Theta} + \cdots + \widetilde{L}_{2^m}^{\Theta}\right)_{q,r} \geq \left(\widetilde{L}_j^{\Theta}\right)_{q,r} = 1. \tag{27}$$

This proves the situation for $s = 1$.

For the induction step, we assume that, for any $l \leq s$, if $x(l) = \delta_{2^n}^{q}$ can be robust reachable from $x(0) = \delta_{2^n}^{r}$ at the lth step, then $(M^l)_{q,r} > 0$. In the following, we need to prove that (19) holds for $l = s + 1$. Assuming that $x(s+1) = \delta_{2^n}^{q}$ can be robust reachable from $x(0) = \delta_{2^n}^{r}$ at the $(s+1)$th step, then there must exist a path from $\delta_{2^n}^{r}$ to $\delta_{2^n}^{k_0}$ at the sth step and a path from $\delta_{2^n}^{k_0}$ to $\delta_{2^n}^{q}$ in one step. Applying the induction hypothesis, we have

$$\begin{aligned}\left(M^s\right)_{k_0,r} &> 0, \\ (M)_{q,k_0} &> 0.\end{aligned} \tag{28}$$

Therefore,

$$\begin{aligned}\left(M^{s+1}\right)_{q,r} &= \left(MM^s\right)_{q,r} = \sum_{k=1}^{2^n} (M)_{q,k}\left(M^s\right)_{k,r} \\ &\geq (M)_{q,k_0}\left(M^s\right)_{k_0,r} > 0.\end{aligned} \tag{29}$$

The proof of necessity is completed.

(Sufficiency). We still prove it by induction on s. From the proof of necessity, one can easily see that if $(M)_{q,r} > 0$, then (21) must hold, which implies that control $\delta_{2^m}^{j}$ can steer system (14) from $\delta_{2^m}^{r}$ to $\delta_{2^m}^{q}$ in one step under arbitrary disturbance inputs. This proves (19) for $s = 1$.

We assume that, for any $l \leq s$, if $(M^l)_{q,r} > 0$, then $\delta_{2^m}^{q}$ can be robust reachable from $\delta_{2^m}^{r}$ at the lth step. By (29), one can obviously see that if $(M^{s+1})_{q,r} > 0$, then there exists at least a positive integer k_0 such that $(M^s)_{k_0,r} > 0$ and $(M)_{q,k_0} > 0$. Based on the induction hypothesis, $\delta_{2^n}^{k_0}$ can be robust reachable from $\delta_{2^n}^{r}$ at the sth step and $\delta_{2^n}^{q}$ can be robust reachable from $\delta_{2^n}^{k_0}$ in one step. Thus, $\delta_{2^n}^{q}$ can be robust reachable from $\delta_{2^n}^{r}$ at the $(s+1)$th step.

The result (ii) can be easily obtained from (i). □

From Theorem 9 and Definition 8, one can obtain the following robust output controllability results.

Theorem 10. *Consider the disturbed incomplete BCN (14).*

(i) $y_f = \delta_{2^p}^{l}$ is s-robust-output-reachable from the initial state $x(0) = \delta_{2^n}^{r}$, if and only if

$$\left(HM^s\right)_{l,r} > 0; \tag{30}$$

(ii) The system (14) is s-robust-output-controllable at the initial state $x(0) = \delta_{2^n}^{r}$, if and only if

$$Col_r\left(HM^s\right) > 0; \tag{31}$$

(iii) The system (14) is s-robust-output-controllable, if and only if

$$HM^s > 0; \tag{32}$$

(iv) The system (14) is robust-output-controllable at the initial state $x(0) = \delta_{2^n}^{r}$, if and only if

$$\sum_{s=1}^{2^{m+n+r}} Col_r\left(HM^s\right) > 0; \tag{33}$$

(v) The system (14) is robust-output-controllable, if and only if

$$\sum_{s=1}^{2^{m+n+r}} \left(HM^s\right) > 0. \tag{34}$$

Proof.

(i) (Sufficiency). Assuming that (30) holds, and we need to prove that there exists a control sequence $u^{s-1} := \{u(t) : t = 0, \cdots, s-1\} \subseteq \Delta_{2^m}$ such that, for arbitrary disturbance inputs $\xi^{s-1} := \{\xi(t) : t = 0, \cdots, s-1\} \subseteq \Delta_{2^r}$, the output satisfies $y(s; \delta_{2^n}^{r}, u^{s-1}, \xi^{s-1}) = \delta_{2^p}^{l}$. Since

$$\left(HM^s\right)_{l,r} = \sum_{q=1}^{2^n} (H)_{l,q}\left(M^s\right)_{q,r} > 0, \tag{35}$$

there must exist an integer $1 \leq q_0 \leq 2^n$ such that $(H)_{l,q_0} = 1$ and $(M^s)_{q_0,r} > 0$.

Based on Theorem 9, $\delta_{2^n}^{q_0}$ is robust reachable from $\delta_{2^n}^{r}$ at the sth step. In other words, we can find a control sequence $u^{s-1} \subseteq \Delta_{2^m}$ such that

$$x\left(s; \delta_{2^n}^{r}, u^{s-1}, \xi^{s-1}\right) = \delta_{2^n}^{q_0}, \quad \forall \xi^{s-1} \subseteq \Delta_{2^p}. \tag{36}$$

Correspondingly,

$$\begin{aligned}y\left(s; \delta_{2^n}^{r}, u^{s-1}, \xi^{s-1}\right) = H\delta_{2^n}^{q_0} = Col_{q_0}(H) = \delta_{2^p}^{l}, \\ \forall \xi^{s-1} \subseteq \Delta_{2^p},\end{aligned} \tag{37}$$

which infers that $\delta_{2^p}^{l}$ is s-robust-output-reachable from $x(0) = \delta_{2^n}^{r}$.

(Necessity). Assuming that $\delta_{2^p}^{l}$ is s-robust-output-reachable from $\delta_{2^n}^{r}$, then there exists a sequence of control signals u^{s-1}

under which $y(s;\delta_{2^n}^r,u^{s-1},\xi^{s-1})=\delta_{2^p}^l$ regardless of the effect of disturbance inputs ξ^{s-1}. Since

$$y\left(s;\delta_{2^n}^r,u^{s-1},\xi^{s-1}\right)=Hx\left(s;\delta_{2^n}^r,u^{s-1},\xi^{s-1}\right), \tag{38}$$

letting $x(s;\delta_{2^n}^r,u^{s-1},\xi^{s-1})=\delta_{2^n}^{q_0}$, it is easy to check that $Col_{q_0}(H)=\delta_{2^p}^l$ and $\delta_{2^n}^{q_0}$ is robust reachable from $\delta_{2^n}^r$ at the sth step. In other words, $(H)_{l,q_0}=1$ and $(M^s)_{q_0,r}>0$. Hence,

$$\left(HM^s\right)_{l,r}=\sum_{i=1}^{2^n}(H)_{l,i}\left(M^s\right)_{i,r}\geq(H)_{l,q_0}\left(M^s\right)_{q_0,r}>0, \tag{39}$$

which means that (30) holds.

The results (ii) and (iii) can be derived from (i) and Definition 8 easily. Thus, we omit them.

Next, we prove result (iv).

(Sufficiency). Assuming that (33) holds, then for any $l\in\{1,2,\cdots,2^p\}$, there exists an integer $1\leq s\leq 2^{n+m+r}$ such that $(HM^s)_{l,r}>0$. Based on result (i), $y_f=\delta_{2^p}^l$ is s-robust-output-reachable from the initial state $x(0)$. For the arbitrariness of l, we know system (14) is robust-output-controllable at the initial state $x(0)=\delta_{2^n}^r$.

(Necessity). Suppose that the system (14) is robust-output-controllable at the initial state $x(0)=\delta_{2^n}^r$. Then for any output $y_f=\delta_{2^p}^l$, there exists an integer $1\leq s_l\leq 2^{m+n+r}$ such that, for arbitrary disturbance inputs, the inequality $(HM^{s_l})_{l,r}>0$ holds, $\forall l\in\{1,2,\cdots,2^p\}$. Thus,

$$\sum_{s=1}^{2^{m+n+r}}\left(HM^s\right)_{l,r}\geq\left(HM^{s_l}\right)_{l,r}>0, \qquad \forall l\in\left\{1,2,\cdots,2^p\right\}, \tag{40}$$

which means that (33) holds.

The result (v) follows from (i) and Definition 8, and we omit it. The proof is completed. □

Corollary 11. *By Theorem 10, for a given initial state $x(0)=\delta_{2^n}^r$ and an integer $s>0$, one can successively obtain that*

$$R_s\left(x\left(0\right)\right)=\left\{\delta_{2^p}^l:\left(HM^s\right)_{l,r}>0\right\}. \tag{41}$$

3.3. Robust Optimal Output Control of Disturbed Incomplete Boolean Control Networks. In this subsection, we investigate the Mayer-type robust optimal output control problem of disturbed incomplete Boolean control networks and present a new design procedure for the problem.

Consider a disturbed incomplete BCN with the algebraic state representation (14). The Mayer-type robust optimal output control problem can be described as follows: find a control sequence $u^{s-1}:=\{u(t):t=0,\cdots,s-1\}$ such that, for arbitrary disturbance inputs $\xi^{s-1}:=\{\xi(t):t=0,\cdots,s-1\}$, the cost functional

$$J\left(u^{s-1},\xi^{s-1};x\left(0\right)\right)=\lambda^Ty\left(s\right) \tag{42}$$

is minimized under the given initial state $x(0)$, where $\lambda=[\lambda_1\ \cdots\ \lambda_{2^p}]^T\in\mathscr{M}_{2^p\times 1}$ is a given constant vector and $s\geq 1$ is a fixed termination time.

Minimizing the cost functional J is equal to find out the minimum value of J under the following constraint:

$$y\left(s\right)\in R_s\left(x\left(0\right)\right)=\left\{\delta_{2^p}^{q_1},\cdots,\delta_{2^p}^{q_\alpha}\right\}, \tag{43}$$

And, meanwhile, we need to design a control sequence $\{u(0),u(1),\cdots,u(s-1)\}$ which steers $x(0)$ to the optimal terminal output $y^*(s)$ regardless of the effect of disturbance inputs, where $R_s(x(0))$ can be computed by (41).

It is noted that when $y(s)=\delta_{2^p}^{q_i}$, $J(u^{s-1},\xi^{s-1};x(0))=\lambda^Ty(s)=\lambda^T\delta_{2^p}^{q_i}=Col_{q_i}(\lambda^T)=\lambda_{q_i}$. Thus, to find out the minimum value of J, we just need to compute the minimum value of λ_q, $q=q_1,\cdots,q_\alpha$.

According to the above discussion, we obtain the following algorithm to find a sequence of controls $u(0),\cdots,u(s-1)$ such that, for any disturbance inputs $\xi(0),\cdots,\xi(s-1)$, the cost functional J is minimized at the fixed termination time s.

Algorithm 12.

Step 1. Compute the s-robust-output-reachable set with the initial state $x(0)=\delta_{2^n}^r$ by (41), denoted by $R_s(x(0))=\{\delta_{2^p}^{q_1},\cdots,\delta_{2^p}^{q_\alpha}\}$;

Step 2. Compute the optimal value $J^*=\min\{\lambda_q:q=q_1,\cdots,q_\alpha\}:=\lambda_{q_{w^*}}$;

Step 3. Let $y(s)=\delta_{2^p}^{q_{w^*}}$. Find an integer $1\leq q\leq 2^n$ such that $(H)_{q_{w^*},q}=1$ and $(M^s)_{q,r}>0$. Let $x(s)=\delta_{2^n}^q$;

Step 4. Find two integers $1\leq\beta\leq 2^n$ and $1\leq j\leq 2^m$ such that $(M)_{q,\beta}>0,(M^{s-1})_{\beta,r}>0$ and $(\widetilde{L}_j^\Theta)_{q,\beta}>0$. Let $x(s-1)=\delta_{2^n}^\beta$ and $u(s-1)=\delta_{2^m}^j$;

Step 5. If $s-1=1$, find j' such that $(\widetilde{L}_{j'}^\Theta)_{\beta,r}>0$ and let $u(0)=\delta_{2^m}^{j'}$, stop. Otherwise, replace s by $s-1$, and replace q by β. Go back to step 4.

Proposition 13. *The control sequence $\{u(0),\cdots,u(s-1)\}$ generated by Algorithm 12 can minimize the cost functional (42) at the fixed termination time s regardless of the effect of disturbance inputs.*

Proof. Since $y(s)=\delta_{2^p}^{q_{w^*}}$ can minimize the cost functional J, we just need to show that the control sequence $\{u(0),\cdots,u(s-1)\}$ can steer the state $x(0)$ to the objective output $y^*(s)$ under arbitrary disturbance inputs.

Notice that $y(s)\in R_s(x(0))$; by Theorem 10, we can find an integer q such that $(H)_{q_{w^*},q}=1$ and $(M^s)_{q,r}>0$. Let

$x(s) = \delta_{2^n}^q$. By (23), one knows that for arbitrary disturbance input $\xi(s-1) = \delta_{2^r}^k,\ k = 1, 2, \cdots, 2^r$, it follows the result

$$\begin{aligned} x(s) &= L^{\Theta}\xi(s-1)\,u(s-1)\,x(s-1) \\ &= L_k^{\Theta}u(s-1)\,x(s-1) = L_k^{\Theta}\delta_{2^m}^j\delta_{2^n}^{\beta} \\ &= Col_{\beta}\left(L_{kj}^{\Theta}\right) = \delta_{2^n}^q. \end{aligned} \tag{44}$$

To ensure that $\delta_{2^n}^q$ can be reached from $\delta_{2^n}^{\beta}$ in one step with the control $u(s-1) = \delta_{2^m}^j$ for arbitrary disturbance input, the following results should be kept:

$$\begin{aligned} \delta_{2^n}^q &= L_1^{\Theta}\delta_{2^m}^j\delta_{2^n}^{\beta} = Col_{\beta}\left(L_{1j}^{\Theta}\right), \\ \delta_{2^n}^q &= L_2^{\Theta}\delta_{2^m}^j\delta_{2^n}^{\beta} = Col_{\beta}\left(L_{2j}^{\Theta}\right), \\ &\vdots \\ \delta_{2^n}^q &= L_{2^r}^{\Theta}\delta_{2^m}^j\delta_{2^n}^{\beta} = Col_{\beta}\left(L_{2^r j}^{\Theta}\right). \end{aligned} \tag{45}$$

Combining (25) and (45), we obtain $Col_{\beta}(\widetilde{L}_j^{\Theta}) = \delta_{2^n}^q$.

In the same way, we can find integers j' and β' such that

$$\begin{aligned} &(M)_{\beta,\beta'} > 0, \\ &\left(M^{s-2}\right)_{\beta',r} > 0, \\ &\left(\widetilde{L}_{j'}^{\Theta}\right)_{\beta,\beta'} > 0, \end{aligned} \tag{46}$$

which implies that $\delta_{2^n}^{\beta}$ can be reached from $\delta_{2^n}^{\beta'}$ in one step with the control $u(s-2) = \delta_{2^m}^{j'}$ for arbitrary disturbance.

Do the same procedure, we can finally obtain the control sequence $\{u(0), \cdots, u(s-1)\}$ which steers the state $x(0)$ to the objective output $y^*(s)$ regardless of the effect of disturbance inputs. □

4. An Illustrative Example

In this section, we give an example to demonstrate the efficiency of the results obtained in this paper.

Consider an incomplete BCN with one disturbance input whose dynamic equations are given as follows:

$$\begin{aligned} x_1(t+1) &= \neg u_1(t) \wedge \left(x_2(t) \vee x_3(t)\right), \\ x_2(t+1) &= \neg u_1(t) \wedge u_2(t) \wedge x_1(t) \wedge \xi(t) \\ x_3(t+1) &= \neg u_1(t) \wedge \left(u_2(t) \vee \left(u_3(t) \wedge \left(x_1(t)\right)\right)\right) \end{aligned} \tag{47}$$

In this example, the measured outputs are assumed as

$$\begin{aligned} y_1(t) &= x_1(t) \wedge x_2(t) \vee x_3(t), \\ y_2(t) &= x_1(t) \wedge x_2(t) \vee \left(\neg x_1(t) \wedge \neg x_2(t)\right). \end{aligned} \tag{48}$$

Set $x(t) = \ltimes_{i=1}^3 x_i(t)$, $u(t) = \ltimes_{i=1}^3 u_i(t)$ and $y(t) = \ltimes_{i=1}^2 y_i(t)$. Assume the control-state avoiding set $\Theta := \{(\delta_8^1, \delta_8^2), (\delta_8^1, \delta_8^5), (\delta_8^5, \delta_8^6), (\delta_8^6, \delta_8^4), (\delta_8^7, \delta_8^4)\}$. Then we can express the disturbed incomplete BCN (47) in its algebraic form as

$$\begin{aligned} x(t+1) &= L^{\Theta}\xi(t)\,u(t)\,x(t), \\ y(t) &= Hx(t), \end{aligned} \tag{49}$$

where

$$\begin{aligned} L^{\Theta} = \delta_8[\ & 8\ 0\ 8\ 8\ 0\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8 \\ & 1\ 1\ 1\ 5\ 3\ 0\ 3\ 7\ 1\ 1\ 1\ 0\ 3\ 3\ 3\ 7\ 3\ 3\ 3\ 0\ 4\ 4\ 4\ 8\ 4\ 4\ 4\ 8\ 4\ 4\ 4\ 8 \\ & 8\ 0\ 8\ 8\ 0\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8 \\ & 3\ 3\ 3\ 7\ 3\ 0\ 3\ 7\ 3\ 3\ 3\ 0\ 3\ 3\ 3\ 7\ 3\ 3\ 3\ 0\ 4\ 4\ 4\ 8\ 4\ 4\ 4\ 8\ 4\ 4\ 4\ 8] \in \mathscr{L}_{2^3\times 2^{3+3+1}}^{I}, \end{aligned} \tag{50}$$

and $H = \delta_4[1\ 1\ 2\ 4\ 2\ 4\ 1\ 3]$.

Skipping a tedious calculation, we can obtain

$$HM^3 = \begin{bmatrix} 3 & 3 & 3 & 1 & 3 & 3 & 3 & 2 \\ 2 & 2 & 2 & 1 & 2 & 2 & 2 & 2 \\ 6 & 6 & 6 & 2 & 6 & 6 & 6 & 5 \\ 2 & 2 & 2 & 1 & 2 & 2 & 2 & 2 \end{bmatrix}, \tag{51}$$

where $M = \sum_{j=1}^{2^3} \widetilde{L}_j^{\Theta}$ and $\widetilde{L}^{\Theta} = L_1^{\Theta} \wedge L_2^{\Theta}$ can be calculated by (12) and (17). From Theorem 10, it is concluded that the system (49) is 3-robust-output-controllable. It is also robust-output-controllable.

(2) Consider the Mayer-type robust optimal output control problem of the disturbed incomplete Boolean control network (49). The constant vector $\lambda^T = [3\ 1\ 4\ 2]$, and we aim to determine a control strategy such that, for arbitrary disturbance input, the cost functional

$$J\left(u^{s-1}, \xi^{s-1}; x(0)\right) = \lambda^T y(s), \tag{52}$$

is minimized under the initial value $x(0) = \delta_8^4$ and $s = 3$.

Now, we follow Algorithm 12 step by step as follows:

Step 1. Compute the 3-robust-output-reachable set with the initial value $x(0) = \delta_8^4$, and obtain that $R_s(x(0)) = \delta_4[1, 2, 3, 4]$;

Step 2. $J^*=\min\{\lambda_q,\ q=1,2,3,4\}=1$, and $q_{w^*}=2$;

Step 3. Let $y(3)=\delta_4^2$, and find $q=3$ such that $(H)_{2,3}=1$ and $(M^3)_{3,4}>0$. So $x(3)=\delta_8^3$;

Step 4. We can find $\beta=7$, $j=6$ such that $(M)_{3,7}>0$, $(M^2)_{7,4}>0$, and $(\widetilde{L}_6^{\Theta})_{3,7}>0$. So $x(2)=\delta_8^7$ and $u(2)=\delta_8^6$;

Step 5. We can find $\beta'=8$, $j'=5$, $j''=2$ such that $(M)_{7,8}>0$, $(M)_{8,4}>0$, and $(\widetilde{L}_5^{\Theta})_{7,8}>0$, $(\widetilde{L}_2^{\Theta})_{8,4}>0$. So $x(1)=\delta_8^8$, $u(1)=\delta_8^5$ and $u(0)=\delta_8^2$.

At last, we obtain an optimal control $\{u(0),u(1),u(2)\}=\{\delta_8^2,\delta_8^5,\delta_8^6\}$ to minimize the cost functional J regardless of the effect of disturbance inputs.

From another aspect, for the initial value $x(0)=\delta_8^4$, under the control $\{u(0),u(1),u(2)\}=\{\delta_8^2,\delta_8^5,\delta_8^6\}$, the state trajectories of the system (49) are $x(1)=\delta_8^8$, $x(2)=\delta_8^7$, $x(3)=\delta_8^3$. At this time, $y(3)=\delta_4^2$, $J(u^{s-1},\xi^{s-1};x(0))=\lambda^T\delta_4^2=1$. Hence, the control $\{u(0),u(1),u(2)\}=\{\delta_8^2,\delta_8^5,\delta_8^6\}$ is an optimal control.

5. Conclusions

In this paper, we have investigated the robust-output-controllability and robust optimal output control problems of incomplete BCNs with disturbance inputs. We have proposed several necessary and sufficient conditions for the robust output controllability based on the algebraic representation of the system. Then, we have discussed the Mayer-type robust optimal output control issue and presented an algorithm to find a control scheme which can minimize the cost functional regardless of the effect of disturbance inputs. Finally, an illustrative example has been given to support the results. It should be pointed out that event-triggered control has attracted a great deal of attention from scholars in the last two decades. Hence, future works can study the following issues: (1) output-controllability of higher-order incomplete BCNs via event-triggered control and (2) event-triggered control for robust optimal output control of incomplete BCNs with disturbance inputs.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Authors' Contributions

All authors drafted the manuscript, and they read and approved the final version.

Acknowledgments

This work was supported by the National Natural Science Foundation of China (no. 11301247) and Science and Technology Project of Department of Education, Shandong Province (no. J15LI10).

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