

Research Article

On an Application of the Absolute Stability Theory to Sampled-Data Stabilization

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A nonlinear Lur'e-type plant with a sector bound nonlinearity is considered. The plant is stabilized by a discrete-time feedback signal with a nonperiodic uncertain sampling. The sampling control function is nonlinear and also obeys some sectoral constraints at discrete (sampling) times. The linear matrix inequality (LMI) conditions for the stability of the closed-loop system are obtained.

1. Introduction

Absolute stability theory of nonlinear systems with sectoral constraints goes back to works of A. I. Lur'e (see [1], some historical reviews can be found in [2, 3]). In [4] and subsequent papers the Lur'e problem was reduced to feasibility of a special system of Linear Matrix Inequalities (LMIs). Later, the advantage of the LMI approach to different problems of applied mathematics was comprehensively discussed in monograph [5] that launched a broad development of a specific computer software for exploring LMIs. In [6–8] the Lur'e theory was extended to multiple nonlinearities.

We will mention two basic concepts put forward by V. A. Yakubovich in 1960s–1970s. The first one is S-procedure [2, 9, 10] that is especially useful when we have to deal with several nonlinearities. The second one is integral-quadratic constraint (IQC), the concept that was initially introduced by V. A. Yakubovich in connection with the study of pulse-width modulated systems (a special class of sampled-data systems) [11]. Regrettably, the last paper was never translated and is almost unknown for a non-Russian reader. Notice that the impact of the early works of V. A. Yakubovich on the modern IQC theory was recognized in the review part of a widely known paper [12]. Sectoral constraints and IQCs proved to be instrumental for stability analysis of various classes of nonlinear control systems (see, e.g., [2]).

The third type of constraints, that can be named discrete-time constraints, was put forward by A. Kh. Gelig for nonlinear sampled-data systems [13]. Unlike the usual sectoral constraints, discrete-time constraints are valid not at all times, by only at some discrete-time instants lying in the sampling interval. The exact position of these instants depends on the type of a pulse modulation. In the further development of this approach it was proposed to exploit a Lyapunov–Krasovskii functional [14], but later it was found that for the stability analysis it is more convenient to combine discrete-time constraints with IQCs. Namely, in [15] it was proposed to employ the IQC based on Wirtinger integral inequality [16]. Since that time, the Wirtinger-based IQCs were used in a great number of publications for various types of sampled-data systems [17–26]. In particular, the problem of stabilization of a linear plant by means of a pulse-modulated signal was considered [21, 22]. The statements of the above works were formulated in terms of frequency-domain inequalities, but later some of them were restated in terms of LMIs [23–26].

The main idea of the Gelig's approach is a substitution of the initial train of pulses for a sequence of the average values of these pulses, with a supposition that these averages satisfy some instant constraints. The errors of such a substitution are estimated with the help of IQCs. Unlike other averaging theorems, the results of Gelig were not asymptotical, but

could be used for an estimation of the sampling frequency from below. For sufficiently high sampling frequencies the Gelig-type stability conditions reduce to the conventional absolute stability criteria (the circle criterion, the Popov criterion, and some others).

The problem of stabilization of a continuous-time plant by a sampled-data signal attracted much attention last decade; see a review paper [27] where the existing modern approaches are outlined. We can distinguish two main competing methods. The input delay approach was contributed by E. Fridman [28, 29] who considered a sampled-data signal as a special case of delayed signal. If $t_n, n \geq 0$, are sampling times, then a sampled signal $x(t_n)$ can be reformulated as a delayed signal $x(t - \tau(t))$ with $\tau(t) = t - t_n, t_n < t < t_{n+1}$, then Lyapunov–Krasovskii functionals can be applied.

An alternative is the IQCs approach that is more closely related to the absolute stability theory [30–33]. In [30] it was firstly proved that specially chosen IQCs give the same results as those previously obtained by the input delay method. In [31] it was demonstrated that by extending the IQC approach these results can be refined. Notice that the estimate for a L^2 -gain used in [30] can be considered as a reformulated Wirtinger inequality. In [32, 33] the stability problem was reduced to feasibility of an infinite number of LMIs with coefficients depending on time t . It was shown that under certain assumptions they can be checked only at a finite number of points. Though being more laborious, this approach leads to improvements of the previous results.

The most publications on sampled-data stabilization treat the case when the plant is linear and the discrete-time control implements the zero-order hold strategy. As for nonlinear plants (with single or multiple nonlinearities), their stabilization problem by a linear zero-order hold sampled-data control was considered in [34–37]. The technique used in these papers was based on the method of input delays and on constructing special Lyapunov–Krasovskii functionals. The paper [38] considered a multiple nonlinearity Lur'e system stabilized by a multirate discrete-data control. The consideration was based on the integral estimate from [39]; the result was formulated in the form of a frequency-domain inequality.

This paper aims to demonstrate how the classical absolute stability technique (Yakubovich's S-procedure, Lur'e sectoral constraints, IQCs, and Gelig's discrete-time constraints) can be applied to stability analysis of a nonlinear Lur'e-type plant under a sampled-data nonlinear stabilizing signal. The results thus obtained are parsimonious and easily verifiable, but they are compatible with those obtained by more sophisticated mathematical methods.

The paper is organized as follows. Firstly we describe a model that consists of a Lur'e-type nonlinear plant under a sampled-data nonlinear control. The sampling is supposed to be nonuniform, with a known upper bound of dwell-times. Further, we demonstrate how the sampled-data stabilization problem for a Lur'e system can be treated in the frame of the Gelig–Yakubovich approach to the absolute stability theory. The key role is played by the Yakubovich's S-procedure that in fact provides an alternative technique to using Lyapunov–Krasovski functionals. It is proved that for

a sufficiently high sampling frequency the stability criterion obtained is reduced to the circle criterion for absolute stability of continuous-time systems. Thus for high sampling frequencies the conservatism of the considered method is the same as that of the classical absolute stability criteria. Finally the main result is illustrated by an application to a simple first-order problem and to a sampled-data control of a mathematical pendulum.

2. System Model

Here we address a sampling-data counterpart of the circle stability criterion (see, e.g., [2, 40, 41]).

Consider a nonlinear Lur'e-type plant with a sectoral bound uncertainty

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bv(t), \\ \sigma(t) &= Cx(t), \\ v(t) &= \varphi(\sigma(t), t).\end{aligned}\tag{1}$$

Here the nonlinearity $\varphi(\cdot, \cdot)$ describes an intrinsic nonlinear feedback; it is continuous and obeys the sectoral bound

$$\mu_1 \leq \frac{\varphi(y, t)}{y} \leq \mu_2\tag{2}$$

for all real y, t . In other words, $v(t)$ satisfies a quadratic constraint [2]

$$(\mu_2\sigma(t) - v(t))(v(t) - \mu_1\sigma(t)) \geq 0\tag{3}$$

for all t . Here μ_1, μ_2 are scalars, and A, B, C are constant matrices of sizes $p \times p, p \times 1, 1 \times p$, respectively. Obviously, system (1) has a zero equilibrium $x(t) \equiv 0$, whose stability can be investigated with the help of the circle criterion of absolute stability [2]. However, we will be interested in the case when the zero equilibrium is unstable, so a sampled-data external feedback is used for its stabilization.

Let plant (1) be governed by a sampled-data external signal. Assume that we have a strictly increasing sequence of sampling times $t_0 < t_1 < t_2 < \dots$ with the lengths of the sampling intervals (dwell-times) $T_n = t_{n+1} - t_n$ estimated as

$$\delta_0 T \leq T_n \leq T, \quad \forall n \geq 0,\tag{4}$$

where δ_0, T are some positive constants. The sequence $\{t_n\}$ is uncertain, and it does not need to be periodic, and only estimates (4) matter. The ratio $1/T_n$ can be considered as an instant sampling frequency. Let plant (1) be controlled by a zero-order hold signal $u(t)$:

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bv(t) + B_u u(t), \\ \sigma_u(t) &= Kx(t),\end{aligned}\tag{5}$$

$$u(t) = u_n, \quad t_n \leq t < t_{n+1},\tag{6}$$

$$u_n = F(\sigma_u(t_n)).\tag{7}$$

Here B_u , K are vectors of sizes $p \times 1$, $1 \times p$, respectively, and $F(\cdot)$ is a nonlinear function (a modulation characteristic) that satisfies a sectoral constraint

$$\nu_1 \leq \frac{F(y)}{y} \leq \nu_2 \quad (8)$$

for all real y and some scalars ν_1, ν_2 , hence

$$(\nu_2 \sigma_u(t_n) - u_n)(u_n - \nu_1 \sigma_u(t_n)) \geq 0 \quad (9)$$

for all $n \geq 0$. Thus the quadratic constraint (9) holds not for all times t , but only for discrete times t_n (see [13]).

The zero-order hold signal $u(t)$ defined by (6), (7) can be considered as a special case of a square (rectangular) pulse

$$u(t) = \begin{cases} u_n, & t_n \leq t < t_n + \tau_n, \\ 0, & t_n + \tau_n \leq t < t_{n+1}, \end{cases} \quad (10)$$

where u_n , τ_n , and $1/T_n$ (with $T_n = t_{n+1} - t_n$) are the amplitude, the width and the instant frequency, respectively. The specific of the zero-order hold is that $\tau_n \equiv T_n$; thus all the three impulse parameters, the amplitude, the width, and the frequency, are modulated. The average of the pulse signal $u(t)$ considered on the n th sampling interval is

$$\bar{u}_n = \frac{1}{T_n} \int_{t_n}^{t_{n+1}} u(t) dt = u_n. \quad (11)$$

Hence $u(t) = \bar{u}_n$, $t_n < t < t_{n+1}$; i.e., the error of a substitution of a pulse for its average is equal to zero.

Formula (7) reads that the amplitude modulation is of the first kind [20, 42], so discrete constraints can be imposed at the points $t = t_n$. (In the theory of hybrid systems this type of modulation is termed as self-triggered control [43, 44].) Notice that for more elaborate types of pulse modulation the averaging method considers discrete constraints at some intermediate points $t = \tilde{t}_n$, $t_n < \tilde{t}_n < t_{n+1}$. Then the value $\sigma_u(\tilde{t}_n)$ can be interpreted as a signal with a deviating argument that can be not only delayed, but also advanced: $\sigma_u(t - \tau(t))$ with $\tau(t) = t - \tilde{t}_n$, $t_n < t < t_{n+1}$.

Here we will demonstrate how the approach developed in Theorem 3.3 [20] can be reformulated for this special case. We will use not the frequency-domain inequalities (as in [20]), but LMIs. The result will be augmented by an additional IQC taken from [31].

3. The Main Statement

Let us make an additional assumption on the nonlinearity $\varphi_0(\sigma, t)$. Suppose that there exists a scalar μ such that the function

$$\varphi_0(\sigma, t) = \varphi(\sigma, t) - \mu\sigma \quad (12)$$

is bounded for all σ, t . The following theorem presents LMI conditions for the zero asymptotic of the closed-loop system (5), (6), (7).

Theorem 1. Consider a nonlinear system (5), (6), (7) with a nonlinearity $v(t) = \varphi(\sigma(t), t)$ satisfying (3) and with a sampled-data control satisfying (4) and (9). Assume that there exist a symmetric $p \times p$ matrix H and scalars $\tau \geq 0$, $\varepsilon \geq 0$, $\vartheta \geq 0$ such that the following set of matrix inequalities is feasible:

$$H > 0, \quad (13)$$

$$\Pi = \begin{bmatrix} \Pi_{11} & \Pi_{12} & \Pi_{13} & \Pi_{14} \\ * & \Pi_{22} & \Pi_{23} & \Pi_{24} \\ * & * & \Pi_{33} & \Pi_{34} \\ * & * & * & \Pi_{44} \end{bmatrix} < 0 \quad (14)$$

with

$$\begin{aligned} \Pi_{11} &= HA + A^\top H - \varepsilon \mu_1 \mu_2 C^\top C - \tau \nu_1 \nu_2 K^\top K \\ &\quad + A^\top K^\top KA, \\ \Pi_{12} &= HB_u + \frac{1}{2} \tau (\nu_1 + \nu_2) K^\top + A^\top K^\top KB_u, \\ \Pi_{13} &= HB + \frac{1}{2} \varepsilon (\mu_1 + \mu_2) C^\top + A^\top K^\top KB, \\ \Pi_{14} &= \tau \nu_1 \nu_2 K^\top + \vartheta A^\top K^\top, \\ \Pi_{22} &= -\tau + (KB_u)^2, \\ \Pi_{23} &= KBKB_u, \\ \Pi_{24} &= -\frac{1}{2} \tau (\nu_1 + \nu_2) + \vartheta KB_u, \\ \Pi_{33} &= -\varepsilon + (KB)^2, \\ \Pi_{34} &= \vartheta KB, \\ \Pi_{44} &= -\tau \nu_1 \nu_2 - \gamma. \end{aligned} \quad (15)$$

Here $\gamma = \pi^2/(4T^2)$, \top denotes matrix transpose and asterisks stand for the matrix blocks symmetric with respect to the main diagonal. Inequalities (13), (14) are understood in the sense of positive and negative definiteness of quadratic forms. Then any solution of (5), (6), (7) is asymptotically zero: $x(t) \rightarrow 0$ as $t \rightarrow +\infty$ and $u_n \rightarrow 0$ as $n \rightarrow +\infty$.

Notice that if we abandon the above assumption on the function $\varphi_0(\sigma, t)$ we can assert only that any solution $x(t)$ is square integrable that is a weaker property than the zero asymptotic.

If the control gains vector K is given and fixed, inequalities (14), (15) present LMIs with respect to variables H , τ , ε , ϑ , γ . However, if K is considered as a design parameter and needs to be chosen, then we get a problem with nonlinear constraints.

4. Proof of the Main Statement

Let us introduce an auxiliary function

$$\xi(t) = \sigma_u(t) - \sigma_u(t_n) = K(x(t) - x(t_n)) \quad (16)$$

for $t_n \leq t < t_{n+1}$. As well as $u(t)$, the function $\xi(t)$ is discontinuous with jumps at the points $t = t_n, n \geq 0$.

The proof follows the mathematical technique conventional for the absolute stability theory (see, e.g., [2]). The three types of constraints will be used. Firstly, we will use a quadratic (Lur'e-type) constraint (3) that can be rewritten as

$$(\mu_2 Cx(t) - v(t))(v(t) - \mu_1 Cx(t)) \geq 0 \quad (17)$$

for all t . Secondly, consider a discrete-time (Gelig-type) constraint (9). With the help of (16) it can be rewritten as

$$\begin{aligned} & (\nu_2 \sigma_u(t_n) - u(t))(u(t) - \nu_1 \sigma_u(t_n)) \\ &= (\nu_2 \sigma_u(t) - \nu_2 \xi(t) - u(t)) \\ & \cdot (u(t) - \nu_1 \sigma_u(t) - \nu_1 \xi(t)) = (\nu_2 \sigma_u(t) - u(t)) \\ & \cdot (u(t) - \nu_1 \sigma_u(t)) - (\nu_1 + \nu_2) u(t) \xi(t) \\ & + 2\nu_1 \nu_2 \sigma_u(t) \xi(t) - \nu_1 \nu_2 \xi^2(t) \geq 0 \end{aligned} \quad (18)$$

for $t_n < t < t_{n+1}, n \geq 0$. Recall that $\sigma_u(t) = Kx(t)$.

Thirdly, let us take two integral-quadratic (Yakubovich type) constraints. The first one is based on the Wirtinger integral inequality (see [15, 17, 19, 20])

$$\int_{t_n}^{t_{n+1}} (\sigma_u(t) - \sigma_u(t_n))^2 dt \leq \frac{4T_n^2}{\pi^2} \int_{t_n}^{t_{n+1}} \dot{\sigma}_u^2(t) dt \quad (19)$$

for all $n \geq 0$. From (4) the last inequality implies

$$\int_{t_n}^{t_{n+1}} (\sigma_u(t) - \sigma_u(t_n))^2(t) dt \leq \Delta \int_{t_n}^{t_{n+1}} \dot{\sigma}_u^2(t) dt \quad (20)$$

with $\Delta = 4T^2/\pi^2$.

The second IQC can be extracted from [31]. Since $\dot{\sigma}_u(t) = \dot{\xi}(t), t_n < t < t_{n+1}$, and $\xi(t_n^+) = 0$, we get

$$\int_{t_n}^t \dot{\sigma}_u(s) \xi(s) ds = \int_{t_n}^t \dot{\xi}(s) \xi(s) ds = \frac{1}{2} \xi^2(t) \quad (21)$$

for $t_n < t < t_{n+1}$. Thus

$$\int_{t_n}^{t_{n+1}} \dot{\sigma}_u(s) \xi(s) ds = \frac{1}{2} \xi^2(t_{n+1}^-) \geq 0. \quad (22)$$

Following the S-procedure (see [2, 9]), let us define a quadratic form

$$\begin{aligned} G(x, u, v, \xi) &= \varepsilon (\mu_2 Cx - v)(v - \mu_1 Cx) \\ &+ \tau \left[(\nu_2 Kx - u)(u - \nu_1 Kx) - (\nu_1 + \nu_2) u\xi \right. \\ &+ 2\nu_1 \nu_2 Kx\xi - \nu_1 \nu_2 \xi^2 \left. \right] + \left[(KAx + KBv + KB_u u)^2 \right. \\ &\left. - \frac{\xi^2}{\Delta} \right] + 2\vartheta (KAx + KBv + KB_u u) \xi, \end{aligned} \quad (23)$$

where $\tau, \varepsilon, \vartheta$ are some nonnegative parameters. From (17), (18), (20), (22) it follows that

$$\int_{t_n}^{t_{n+1}} G(x(t), u(t), v(t), \xi(t)) dt \geq 0 \quad (24)$$

for any solution of (5), (6), (7) and any $n \geq 0$ (see also [11]).

Let us define a quadratic Lyapunov function $V(x) = x^T Hx$, where H is a symmetric matrix satisfying (14). Denote $X = \text{col}\{x, u, v, \xi\}$. Since

$$2x^T H(Ax + Bv + B_u u) + G(x, u, v, \xi) = X^T \Pi X, \quad (25)$$

linear matrix inequality (14) can be rewritten in terms of quadratic forms as

$$\begin{aligned} & 2x^T H(Ax + Bv + B_u u) + G(x, u, v, \xi) \\ & \leq -\varepsilon_0 (\|x\|^2 + u^2 + v^2 + \xi^2) \end{aligned} \quad (26)$$

for all vectors x and scalars u, v, ξ , where ε_0 is some (sufficiently small) positive number and $\|\cdot\|$ is the Euclidean vector norm. Along the solutions of (5), (6), (7) inequality (26) implies

$$\begin{aligned} & \dot{V}(x(t)) + G(x(t), u(t), v(t), \xi(t)) \\ & \leq -\varepsilon_0 (\|x(t)\|^2 + u(t)^2) \end{aligned} \quad (27)$$

for all $t_n < t < t_{n+1}, n \geq 0$. Notice the vector function $x(t)$ is continuous in t ; hence $V(x(t))$ is also continuous for all $t \geq t_0$. Integrating (27) and using (24) we get

$$\begin{aligned} V(x(t_n)) - V(x(t_0)) & \leq -\varepsilon_0 \int_{t_0}^{t_n} \|x(s)\|^2 ds \\ & - \varepsilon_0 \sum_{k=0}^{n-1} T_k u_k^2 \end{aligned} \quad (28)$$

for all $n \geq 1$. Since $H > 0$, (28) implies

$$\int_{t_0}^{t_n} \|x(s)\|^2 ds + \sum_{k=0}^{n-1} T_k u_k^2 \leq \frac{1}{\varepsilon_0} V(x(t_0)). \quad (29)$$

From (29) and (4) it follows that

$$\int_{t_0}^{+\infty} \|x(s)\|^2 ds < +\infty, \quad (30)$$

$$\sum_{k=0}^{\infty} u_k^2 < +\infty$$

that implies $u_n \rightarrow 0$ as $n \rightarrow \infty$.

Let us prove that $x(t)$ is asymptotically zero. From (28) we get $V(x(t_n)) \leq V(x(t_0))$ for $n \geq 0$. Since $H > 0$, we conclude that the sequence $\|x(t_n)\|$ is bounded for $n \geq 0$. The sequence u_n vanishes as $n \rightarrow \infty$, thus it is bounded, and the function $u(t)$ is also bounded for $t \geq t_0$. Introduce the matrix $A_\mu = A + \mu BC$. Then the first equation (5) can be rewritten as

$$\dot{x} = A_\mu x(t) + Bv_0(t) + B_u u(t) \quad (31)$$

where $A_\mu = A + \mu BC$, $v_0(t) = v(t) - \mu\sigma(t)$ and the function $v_0(t)$ is bounded for $t \geq t_0$. Integrating (31), we get

$$x(t) = e^{A_\mu(t-t_n)} x(t_n) + \int_{t_n}^t e^{A_\mu(t-s)} (Bv_0(s) + B_u u(s)) ds \quad (32)$$

for $t_n \leq t \leq t_{n+1}$. Hence

$$\|x(t)\| \leq e^{\|A_\mu\|T} \|x(t_n)\| + T e^{\|A_\mu\|T} \sup_{t \geq t_0} \|Bv_0(t) + B_u u(t)\| \quad (33)$$

for $t_n \leq t \leq t_{n+1}$. Because $\|x(t_n)\|$ is bounded, estimate (33) implies that $\|x(t)\|$ is bounded for $t \geq t_0$. Thus the right-hand side of (31) is also bounded, and so the function $\|x(t)\|^2$ is uniformly continuous. Applying Barbalat's lemma (see, e.g., [8]) we conclude that $x(t) \rightarrow 0$ as $t \rightarrow +\infty$.

5. Some Remarks to Theorem 1

Remark 2. Let us multiply formulas (15) elementwise by T and change the variables

$$\begin{aligned} H_0 &= TH, \\ \tau_0 &= T\tau, \\ \varepsilon_0 &= T\varepsilon, \\ \vartheta_0 &= T\vartheta, \\ \gamma_0 &= T\gamma. \end{aligned} \quad (34)$$

Using the Schur complement [5] inequality (14) can be rewritten as

$$\tilde{\Pi} + TL_1^T L_1 + \frac{1}{\tau_0 \nu_1 \nu_2 + \gamma_0} L_2^T L_2 < 0 \quad (35)$$

with

$$\begin{aligned} L_1 &= \begin{bmatrix} A^T K^T \\ KB_u \\ KB \end{bmatrix}, \\ L_2 &= \begin{bmatrix} \tau_0 \nu_1 \nu_2 K^T + \vartheta_0 A^T K^T \\ -\frac{1}{2} \tau_0 (\nu_1 + \nu_2) + \vartheta_0 KB_u \\ \vartheta_0 KB \end{bmatrix}, \\ \tilde{\Pi} &= \begin{bmatrix} \tilde{\Pi}_{11} & \tilde{\Pi}_{12} & \tilde{\Pi}_{13} \\ * & \tilde{\Pi}_{22} & \tilde{\Pi}_{23} \\ * & * & \tilde{\Pi}_{33} \end{bmatrix}, \end{aligned} \quad (36)$$

where

$$\begin{aligned} \tilde{\Pi}_{11} &= H_0 A + A^T H_0 - \varepsilon_0 \mu_1 \mu_2 C^T C - \tau_0 \nu_1 \nu_2 K^T K, \\ \tilde{\Pi}_{12} &= H_0 B_u + \frac{1}{2} \tau_0 (\nu_1 + \nu_2) K^T, \\ \tilde{\Pi}_{13} &= H_0 B + \frac{1}{2} \varepsilon_0 (\mu_1 + \mu_2) C^T, \\ \tilde{\Pi}_{22} &= -\tau_0, \\ \tilde{\Pi}_{23} &= 0, \\ \tilde{\Pi}_{33} &= -\varepsilon_0. \end{aligned} \quad (37)$$

Assume that the sampling frequency is sufficiently high; i.e.

$$\begin{aligned} T &\rightarrow +0, \\ \gamma_0 &= \frac{\pi^2}{4T} \rightarrow +\infty. \end{aligned} \quad (38)$$

Then (35) can be reduced to $\tilde{\Pi} < 0$. The last inequality ensures the fulfillment of the circle criterion of absolute stability for a continuous-time system

$$\begin{aligned} \dot{x} &= Ax(t) + Bv(t) + B_u u(t), \\ v(t) &= \varphi(Cx(t)), \\ u(t) &= F(Kx(t)) \end{aligned} \quad (39)$$

with quadratic constraints (17) and

$$(\nu_2 Kx(t) - u(t))(u(t) - \mu_1 Kx(t)) \geq 0 \quad (40)$$

for all t .

Remark 3. Let the conditions of Theorem 1 be satisfied. As it was shown above, this implies inequality (26). Let us set $v = \hat{\mu}Cx$, $u = \hat{\nu}Kx$, $\xi = 0$ in (26), where $\hat{\mu}$, $\hat{\nu}$ are some numbers such that

$$\begin{aligned} \mu_1 &\leq \hat{\mu} \leq \mu_2, \\ \nu_1 &\leq \hat{\nu} \leq \nu_2. \end{aligned} \quad (41)$$

Then from (26) we obtain $HA_L + A_L^T H < 0$ with

$$A_L = A + \hat{\mu}BC + \hat{\nu}B_u K. \quad (42)$$

Since $H > 0$, we conclude that the matrix A_L defined by (42) is Hurwitz stable for any numbers $\hat{\mu}$, $\hat{\nu}$ satisfying (41). This gives necessary conditions for the fulfillment of Theorem 1.

Remark 4. Let us discuss a relation of Theorem 1 of this paper to Theorem 3.2 [38]. The difference of the problem setting in Theorem 3.2 with that considered in this paper is threefold. Firstly, Theorem 3.2 studies not zero asymptotic, but a stronger property of exponential stability. Secondly, Theorem 3.2 considers a more general case of multiple nonlinearities and a multirate control. Thirdly, Theorem 3.2

is formulated not as an LMI, but as a frequency-domain inequality. However, a reformulation of Theorem 1 to this more general case presents no problem. (Frequency-domain counterparts of the LMI problem given here can be found in Chapter 3 [20].) The more interesting is to compare integral estimates used in both works. The proof of Theorem 3.2 is based on an inequality from Lemma 1 [39] that can be written as

$$(\sigma_u(t) - \sigma_u(t - \tau(t)))^2 \leq T \int_{t-\tau(t)}^t \dot{\sigma}_u^2(s) ds, \quad (43)$$

where $\tau(t) \leq T$. The advantage of estimate (43) is that it is valid for any function $\tau(t)$, $0 \leq \tau(t) \leq T$. However, for a special type of delay $\tau(t) = t - t_n$, $t_n \leq t < t_{n+1}$, the estimate based on (43) can be refined. Integrating (43) with this special delay we obtain

$$\int_{t_n}^{t_{n+1}} (\sigma_u(s) - \sigma_u(t_n))^2 \leq T^2 \int_{t_n}^{t_{n+1}} \dot{\sigma}_u^2(s) ds. \quad (44)$$

Obviously, (44) is more conservative than the Wirtinger inequality (20) with the multiplier $4T^2/\pi^2$ in the right-hand side.

6. Example: First-Order System

Consider a simplest first-order model

$$\dot{x}(t) = -F(x(t_n)), \quad t_n \leq t < t_{n+1}, \quad (45)$$

where $x(t)$ is a scalar function and the nonlinearity $F(\cdot)$ satisfies the sectoral bounds (8) with some scalars $\nu_1, \nu_2, 0 < \nu_1 < \nu_2$.

Let us apply Theorem 1. Equation (45) can be rewritten in the form (5) with $A = 0, B = 0, C = 0, B_u = -1, K = 1$. Let us take $\vartheta = 0$. Then inequality (14) takes the form

$$\begin{bmatrix} -\tau\nu_1\nu_2 & -H + \frac{1}{2}\tau(\nu_1 + \nu_2) & 0 & \tau\nu_1\nu_2 \\ * & 1 - \tau & 0 & -\frac{1}{2}\tau(\nu_1 + \nu_2) \\ 0 & 0 & -\varepsilon & 0 \\ * & * & 0 & -\tau\nu_1\nu_2 - \gamma \end{bmatrix} < 0, \quad (46)$$

where the asterisks stand for the symmetric entries with respect to the main diagonal. Take any positive number for ε and

$$\begin{aligned} H &= \nu_2, \\ \tau &= \frac{2\nu_2}{\nu_2 - \nu_1}. \end{aligned} \quad (47)$$

Then $H - (1/2)\tau(\nu_1 + \nu_2) = -\tau\nu_1$, and inequality (46) is reduced to

$$\begin{bmatrix} \nu_1\nu_2 & -\nu_1 & -\nu_1\nu_2 \\ * & 1 - \alpha & \frac{1}{2}(\nu_1 + \nu_2) \\ * & * & \nu_1\nu_2 + \gamma\alpha \end{bmatrix} > 0, \quad (48)$$

where $\alpha = 1/\tau$. By applying Sylvester's criterion we conclude that (48) is satisfied provided that $\gamma > \nu_2^2$. The latter inequality can be rewritten as

$$T < \frac{\pi}{2\nu_2} \approx \frac{1.57}{\nu_2}. \quad (49)$$

Let us estimate the conservatism of estimate (49). Introduce notation $x_n = x(t_n)$. Integrating (45) we come to a discrete-time map

$$x_{n+1} = \left(1 - \frac{F(x_n)}{x_n} T_n\right) x_n. \quad (50)$$

Map (50) is contracting if

$$\left|1 - \frac{F(x_n)}{x_n} T_n\right| < 1 \quad (51)$$

that is guaranteed if $T < 2/\nu_2$.

The case when the function $F(\cdot)$ is linear was considered previously in [29, 30]. Assume $F(x) = x$, then we can apply Theorem 1 with $\nu_2 = 1, \nu_1 \rightarrow 1 - 0$, then we come to the estimate $T < \pi/2$. This inequality is consistent with the result obtained in [30] (with the help of IQCs), but Proposition 1 of [29] gives the more accurate estimate $T < 1.99$ (with the help of Lyapunov–Krasovskii functionals).

Notice that unlike the linear case, the model considered in Theorem 1 is much more general. Observe also that from the practical standpoint the systems' stability is not the only issue that should be taken into account. If a sampling frequency is close to the boundary of the stability region, the decay rate of solutions may decrease dramatically. On the other hand, (50) reduces to

$$x_{n+1} = (1 - T_n) x_n \quad (52)$$

for the linear system considered. Thus if the sampling period is chosen $T_n = 1$, the zero equilibrium is attained in one iteration.

7. Numerical Example: Mathematical Pendulum

Following [34], consider an equation of a computer controlled pendulum

$$\ddot{\theta}(t) = -\frac{g}{l} \sin \theta(t) + \frac{1}{ml^2} u(t), \quad (53)$$

$$u(t) = F(k_1\theta(t_n) + k_2\dot{\theta}(t_n)), \quad t_n \leq t < t_{n+1},$$

for $n \geq 0$. Let the sampling times satisfy the bound $|t_{n+1} - t_n| \leq T, n \geq 0$. The parameters of (53) are $g = 9.8 \text{ m/s}^2, l = 1 \text{ m}, m = 2 \text{ kg}$. The function $\varphi(\sigma) = \sin \sigma$ is bounded and satisfies a sectoral constraint (2) with $\mu_1 = -0.2173, \mu_2 = 1$ (see [34]). Assume that the function $F(\cdot)$ obeys (8) with $\nu_1 = 0.95, \nu_2 = 1.05$ (i.e., its slope can deviate as $1 \pm 5\%$).

Define $x_1 = \theta, x_2 = \dot{\theta}$. Then we have

$$\begin{aligned} A &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \\ B &= \begin{bmatrix} 0 \\ -b_1 \end{bmatrix}, \\ B_u &= \begin{bmatrix} 0 \\ b_2 \end{bmatrix}, \\ C &= [1 \ 0], \\ K &= [k_1 \ k_2], \end{aligned} \tag{54}$$

where $b_1 = g/l, b_2 = 1/(ml^2)$. With the help of Remark 3 let us obtain upper bounds for feasible values of feedback gains k_1, k_2 . Suppose that numbers $\hat{\mu}, \hat{\nu}$ satisfy (41). Then the matrix A_L defined by (42) must be Hurwitz stable. Since

$$A_L = \begin{bmatrix} 0 & 1 \\ -\hat{\mu}b_1 + \hat{\nu}b_2k_1 & \hat{\nu}b_2k_2 \end{bmatrix}. \tag{55}$$

The conditions for Hurwitz stability of the matrix A_L are

$$\begin{aligned} k_1 &< \mu_1 \frac{gml}{\nu_2} \approx -4.056, \\ k_2 &< 0. \end{aligned} \tag{56}$$

Let us apply MATLAB software with YALMIP package for interface and SeDuMi solver for semidefinite programming [45, 46]. Feasible values of k_1, k_2 were found by a manual search within region (56), then k_1, k_2 were fixed while γ was minimized. It was discovered that the conditions of Theorem 1 are fulfilled for

$$\begin{aligned} T &= 0.2336, \\ k_1 &= -23.6, \\ k_2 &= -11.4 \end{aligned} \tag{57}$$

with

$$\begin{aligned} H &= \begin{bmatrix} 12731 & 2034 \\ 2034 & 554 \end{bmatrix}, \\ \varepsilon &= 78496, \\ \tau &= 1002.7, \\ \vartheta &= 0, \\ \gamma &= 45.1981. \end{aligned} \tag{58}$$

Thus the lower bound for the sampling frequency is $1/T = 4.28$ Hz.

Let us compare the above result with Example 4.1 of [34], where the linear discrete-time control of (53) was treated (in our notation $F(\sigma_u) = \sigma_u$, so $\nu_1 = \nu_2 = 1$). We are primarily

interested in the case of a nonuniform sampling which was considered in Theorem 1 [34]. From that theorem it was found that the pendulum system is stable with $T = 0.191$ and $k_1 = -23.6, k_2 = -6$. It is seen that even for this special case the lower bound for the sampling frequency provided in [34] is $1/T = 5.23$ Hz, which is greater than ours. However, for the case of a periodic sampling and a linear discrete-time control the estimate given in [34] is better than ours: $1/0.302 = 3.31$ Hz. Notice that the simulation for the case of a periodic sampling and a linear discrete-time control gave the minimal sampling rate $1/0.432 = 2.31$ Hz (see Table 1 [34]).

8. Conclusion

The paper discusses an application of the absolute stability theory to a sampled-data stabilization of a nonlinear Lur'e-type system. The design of the stabilizing feedback is reduced to optimization problem for some system of matrix inequalities. The mathematical considerations are based on the Gelig–Yakubovich approach to the stability of sampled-data systems, including S-procedure and specific integral-quadratic constraints. When the sampling frequency is sufficiently high, the main statement of this paper reduces to the circle criterion of absolute stability with two nonlinearities. Illustrative examples demonstrate sufficiently good agreement with the previously known results on linear sampled-data control.

Data Availability

All data generated or analysed during this study are included in the present article.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this paper.

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