Research Article

Homogenization Problem in a Domain with Double Oscillating Boundary

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In this paper, we study the convergence of solutions for homogenization problems about the Poisson equation in a domain with double oscillating locally periodic boundary. Such a problem arises in the processing of devices with very small features. We utilize second-order Taylor expansion of boundary data in combination with boundary correctors to obtain the convergence rate in H^1 -norm. This work explores the domain with double oscillating boundary and also shows the influence of the amplitudes and periods of the oscillations to convergence rates of solutions.

1. Introduction

Several important problems arising in physics and engineering lead to considering boundary value problems in domains with oscillating boundaries. Such problems arise in the context of fluid flows over a rough surface [1, 2], of reinforcement by a thin layer [3], or of electromagnetic scattering by an obstacle with a periodic coating [4, 5]. Indeed, the oscillating boundary results can be applied to the homogenization of neutronic diffusion or transport equation [6–8]. Studying the oscillating boundary is also the key for determining interface transmission conditions in various mechanical problems [9–11].

In this paper, we are interested in the following boundary value problem for the Poisson equation in a domain with double oscillating locally periodic boundary:

$$\begin{aligned} -\Delta u_\varepsilon &= f(x) \quad \text{in } \Omega_\varepsilon, \\ \frac{\partial u_\varepsilon}{\partial \nu_\varepsilon} + p\left(x', \frac{x'}{\varepsilon^\alpha}\right) u_\varepsilon &= g\left(x', \frac{x'}{\varepsilon^\alpha}\right) \quad \text{on } \Gamma_1^\varepsilon, \\ \frac{\partial u_\varepsilon}{\partial \nu_\varepsilon} + q\left(x', \frac{x'}{\varepsilon^\beta}\right) u_\varepsilon &= h\left(x', \frac{x'}{\varepsilon^\beta}\right) \quad \text{on } \Gamma_2^\varepsilon, \\ u_\varepsilon &= 0 \quad \text{on } \Gamma_3. \end{aligned} \quad (1)$$

where $x' = (x_1, x_2, \dots, x_{n-1}) \in (0, 1)^{n-1}$ and ν_ε is the outward unit normal. The domain Ω_ε is a bounded domain in $\mathbb{R}^n$ and $\partial\Omega_\varepsilon = \Gamma_1^\varepsilon \cup \Gamma_2^\varepsilon \cup \Gamma_3$, where hypersurfaces

$$\Gamma_1^\varepsilon : x_n = \varepsilon^\alpha S\left(x', \frac{x'}{\varepsilon^\alpha}\right), \quad \alpha > 0, \quad (2)$$

and

$$\Gamma_2^\varepsilon : x_n = 1 - \varepsilon^\beta T\left(x', \frac{x'}{\varepsilon^\beta}\right), \quad \beta > 0. \quad (3)$$

Without loss of generality, we assume that Ω_ε is a domain bounded by hyperplanes $\{x_i = 0\}$ and $\{x_i = 1\}$ ($i = 1, 2, \dots, n-1$) and hypersurfaces Γ_1^ε and Γ_2^ε .

We suppose that

$$\begin{aligned} p(x', \xi') &\geq 0, \\ q(x', \eta') &\geq 0, \\ S(x', \xi') &\geq 0, \\ T(x', \eta') &\geq 0, \end{aligned} \quad (4)$$

$$\text{where } \xi' = \frac{x'}{\varepsilon^\alpha} \text{ and } \eta' = \frac{x'}{\varepsilon^\beta}.$$

The conditions $p(x', \xi') \geq 0$ and $q(x', \eta') \geq 0$ ensure the existence and uniqueness of solution for problem (1). This can be guaranteed by Lax-Milgram Theorem. The assumptions $S(x', \xi') \geq 0$ and $T(x', \eta') \geq 0$ ensure that $0 \leq x_n \leq 1$. This makes the proof a bit more simpler. In fact, all the results of this paper remain valid even without the nonnegativity of $S(x', \xi')$ and $T(x', \eta')$.

We also impose the smoothness condition and periodicity condition

$$\begin{aligned} & p(x', \xi'), g(x', \xi') \text{ and } S(x', \xi') \\ & \text{are smooth functions in } (x', \xi') \text{ and } 1 \\ & - \text{periodic in } \xi', \\ & q(x', \eta'), h(x', \eta') \text{ and } T(x', \eta') \\ & \text{are smooth functions in } (x', \eta') \text{ and } 1 \\ & - \text{periodic in } \eta'. \end{aligned} \quad (5)$$

Here we do not aim to obtain the optimal smoothness but rather focus on the method itself.

In order to avoid technical difficulties, we assume that

$$\begin{aligned} g(x', \xi') &\equiv 0, \\ h(x', \eta') &\equiv 0, \\ S(x', \xi') &\equiv 0, \\ T(x', \eta') &\equiv 0, \end{aligned} \quad (6)$$

on $x_i = 0$ and $x_i = 1$,

where $i = 1, 2, \dots, n-1$. The conditions on g and h are compatibility conditions. The assumptions about S and T are needed for dealing with the situation when the oscillating boundary intersects the flat boundary.

As the authors are aware, there are many papers about results of convergence rates for elliptic homogenization problems with oscillating boundary data. In 1997, A. Friedman, B. Hu, and Y. Liu [12] studied two-dimensional domain, whose boundary is oscillating according to three scales. They extended the results of Belyaev [13, 14] to the three-scale oscillating boundary. In 1999, G. A. Chechkin, A. Friedman, and A. L. Piatnitski [15] considered such problem including some parameters and obtained some error estimates in H^1 -norm. Their [15] method is following a general procedure in homogenization but without using correctors.

Recently, there has been a surge of activity in the theory of homogenization in domain with oscillating boundary data. In 2012, D. Gérard and N. Masmoudi [16] studied the homogenization of elliptic system with Dirichlet boundary conditions, when the coefficients of the system and the boundary data are ε -periodic. They obtained the solutions convergence in L^2 with a power rate in ε . In 2013, H. Aleksanyan, H. Shahgholian, and P. Sjölin [17] studied the the boundary value homogenization for Dirichlet problem. In particular,

they proved pointwise and L^p convergence results. Their method is based on analysis of oscillatory integrals. The papers [18, 19] were devoted to the investigations of the homogenization problem for the Poisson equation in a thin domain with an oscillating boundary.

For such domain with rapidly oscillating boundary, there are many directions enabling us to consider. Boundary-value problems involving rapidly oscillating boundaries or interfaces appear in many fields of physics and engineering sciences, and it will be interesting to investigate the asymptotic behavior of the solutions of spectral problems. Y. Amirat, G. A. Chechkin, and R. R. Gadyl'shin were devoted to prove the H^1 convergence of the eigenvalues and eigenfunctions to the eigenvalues and eigenfunctions of the homogenization problem, via the traditional method of asymptotic expansions. Moreover, it is worth noticing that many mathematical works have been contributed to the asymptotic analysis of problems in domains with random microstructure. For instance, in 2011, Y. Amirat and other scholars, after adding some extra assumptions on the random variables, were able to obtain the convergence results of solutions in H^1 . This is also an interesting problem.

One may consult several outstanding sources [20–26] for background and overview of the homogenization theory.

The main difference of the present work in relation to previous existing work is that this work explores the domain with double oscillating boundary and also shows the influence of the amplitudes and periods of the oscillations to convergence rates of solutions. Meanwhile, with different traditional asymptotic expansions method, this paper improves the convergence rate results in H^1 -norm by virtue of boundary correctors that can be used to obtain effective approximation.

We now describe the outline of this paper. Section 2 contains some basic formulas and estimates which are important to obtain error estimates. In Section 3, we show that the solution u_ε of problem (1) converges to the solution u_0 of the corresponding homogenized problem in the H^1 -norm with error estimate up to order of $O(\varepsilon^{\alpha/2} + \varepsilon^{\beta/2})$. In Section 4, we construct correctors that play important role in improving the power in ε . Next, we improve the error estimate up to order of $O(\varepsilon^\alpha + \varepsilon^\beta)$ in Section 5. This can be obtained via the correctors. Following the same line of research, in Section 6, we obtain an approximation to u_ε with error estimate up to order of $(\varepsilon^{3\alpha/2} + \varepsilon^{3\beta/2})$ in H^1 -norm by using correctors and second-order Taylor's expansion.

2. Preliminaries

We consider the homogenized problem associated with problem (1) in the form

$$-\Delta u_0 = f(x) \quad \text{in } \Omega_0,$$

$$\frac{\partial u_0}{\partial \nu_0} + P(x') u_0 = G(x') \quad \text{on } \Gamma_1,$$

$$\begin{aligned} \frac{\partial u_0}{\partial \nu_0} + Q(x') u_0 &= H(x') \quad \text{on } \Gamma_2, \\ u_0 &= 0 \quad \text{on } \Gamma_3, \end{aligned} \quad (7)$$

where $\Omega_0 \subset (0, 1)^n$ and ν_0 is the outward unit normal. Here $\Gamma_1 = \{x' \in [0, 1]^{n-1}, x_n = 0\}$ and $\Gamma_2 = \{x' \in [0, 1]^{n-1}, x_n = 1\}$. Functions $P(x')$, $Q(x')$, $G(x')$, and $H(x')$ are defined as follows:

$$\begin{aligned} P(x') &= \int_0^1 \cdots \int_0^1 \left[1 + |\nabla_{\xi'} S(x', \xi')|^2 \right]^{1/2} p(x', \xi') d\xi', \\ G(x') &= \int_0^1 \cdots \int_0^1 \left[1 + |\nabla_{\xi'} S(x', \xi')|^2 \right]^{1/2} g(x', \xi') d\xi', \\ Q(x') &= \int_0^1 \cdots \int_0^1 \left[1 + |\nabla_{\eta'} T(x', \eta')|^2 \right]^{1/2} q(x', \eta') d\eta', \\ H(x') &= \int_0^1 \cdots \int_0^1 \left[1 + |\nabla_{\eta'} T(x', \eta')|^2 \right]^{1/2} h(x', \eta') d\eta'. \end{aligned} \quad (8)$$

As a preliminary step, we shall prove some propositions.

Proposition 1. *There exists a constant C independent of ε such that, for any $v \in H^1(\Omega_0)$, the following inequalities*

$$\begin{aligned} \left\| v \left(x', \varepsilon^\alpha S \left(x', \frac{x'}{\varepsilon^\alpha} \right) \right) - v(x', 0) \right\|_{L^2(\Gamma_1)} \\ \leq C \varepsilon^{\alpha/2} \|v\|_{H^1(\Omega_0)} \end{aligned} \quad (9)$$

and

$$\begin{aligned} \left\| v \left(x', 1 - \varepsilon^\beta T \left(x', \frac{x'}{\varepsilon^\beta} \right) \right) - v(x', 1) \right\|_{L^2(\Gamma_2)} \\ \leq C \varepsilon^{\beta/2} \|v\|_{H^1(\Omega_0)} \end{aligned} \quad (10)$$

hold.

Proof. Without loss of generality, we may assume $v \in C^\infty(\Omega_0)$. A simple calculation then gives

$$\begin{aligned} v \left(x', \varepsilon^\alpha S \left(x', \frac{x'}{\varepsilon^\alpha} \right) \right) - v(x', 0) \\ = \int_0^{\varepsilon^\alpha S(x', x'/\varepsilon^\alpha)} \frac{\partial v(x', t)}{\partial t} dt. \end{aligned} \quad (11)$$

It follows from Hölder's inequality that

$$\begin{aligned} \left| v \left(x', \varepsilon^\alpha S \left(x', \frac{x'}{\varepsilon^\alpha} \right) \right) - v(x', 0) \right|^2 \\ \leq C \varepsilon^\alpha \int_0^{\varepsilon^\alpha S(x', x'/\varepsilon^\alpha)} \left| \frac{\partial v(x', t)}{\partial t} \right|^2 dt. \end{aligned} \quad (12)$$

Integrating over Γ_1 , we obtain (9). The proof of (10) is similar to (9). This completes the proof. $\square$

Proposition 2. *There exists a constant C independent of ε such that, for any $v \in H^1(\Omega_0)$, the following estimate*

$$\|v\|_{L^2(\Omega_0 \setminus \Omega_\varepsilon)} \leq C \left(\varepsilon^{\alpha/2} + \varepsilon^{\beta/2} \right) \|v\|_{H^1(\Omega_0)} \quad (13)$$

is valid.

Proof. Note that

$$v(x', 1) - v(x', \tau) = \int_\tau^1 \frac{\partial v(x', t)}{\partial t} dt, \quad (14)$$

where $\tau \in [1 - \varepsilon^\beta T(x', x'/\varepsilon^\beta), 1]$.

It is easy to see that

$$v^2(x', \tau) \leq 2v^2(x', 1) + 2(1 - \tau) \int_\tau^1 \left| \frac{\partial v(x', t)}{\partial t} \right|^2 dt. \quad (15)$$

Let $\tau = x_{n,2} = 1 - \varepsilon^\beta T(x', x'/\varepsilon^\beta)$, and, integrating over $(0, 1)^{n-1}$, we obtain

$$\begin{aligned} \int_0^1 \cdots \int_0^1 \int_{x_{n,2}}^1 v^2(x', x_{n,2}) dx \\ \leq 2\varepsilon^\beta \max T \|v\|_{L^2(0,1)^{n-1}} \end{aligned} \quad (16)$$

$$+ 2\varepsilon^{2\beta} \max T^2 \int_0^1 \cdots \int_0^1 \int_{x_{n,2}}^1 \left| \frac{\partial v(x', t)}{\partial t} \right|^2 dt.$$

A similar computation yields the following result:

$$\begin{aligned} \int_0^1 \cdots \int_0^1 \int_0^{x_{n,1}} v^2(x', x_{n,1}) dx \\ \leq 2\varepsilon^\alpha \max S \|v\|_{L^2(0,1)^{n-1}} \\ + 2\varepsilon^{2\alpha} \max S^2 \int_0^1 \cdots \int_0^1 \int_0^{x_{n,1}} \left| \frac{\partial v(x', t)}{\partial t} \right|^2 dt, \end{aligned} \quad (17)$$

where $x_{n,1} = \varepsilon^\alpha S(x', x'/\varepsilon^\alpha)$.

Combining these two terms, we obtain (13). This completes the proof. $\square$

Proposition 3. *Let $m(x', \xi')$, $n(x', \eta')$ be 1-periodic in ξ' , η' and satisfy*

$$\begin{aligned} \int_0^1 \cdots \int_0^1 m(x', \xi') d\xi' &= 0 \quad \text{and} \\ \int_0^1 \cdots \int_0^1 n(x', \eta') d\eta' &= 0 \end{aligned} \quad (18)$$

respectively. Then, these inequalities

$$\begin{aligned} \left| \int_{\Gamma_1} m \left(x', \frac{x'}{\varepsilon^\alpha} \right) v(x') dx' \right| &\leq C \varepsilon^{\alpha/2} \|v\|_{H^{1/2}(\Gamma_1)} \quad \text{and} \\ \left| \int_{\Gamma_2} n \left(x', \frac{x'}{\varepsilon^\beta} \right) v(x') dx' \right| &\leq C \varepsilon^{\beta/2} \|v\|_{H^{1/2}(\Gamma_2)} \end{aligned} \quad (19)$$

are satisfied.

Proof. This proposition has been proved by G. A. Chechkin, A. Friedman, and A. L. Piatnitski in [15]. $\square$

Proposition 4. *There exists a constant C independent of ε such that, for all $v \in H^1(\Omega_0)$, the following estimates*

$$\left| \int_{\Gamma_1^\varepsilon} g\left(x', \frac{x'}{\varepsilon^\alpha}\right) v\left(x', \varepsilon^\alpha S\left(x', \frac{x'}{\varepsilon^\alpha}\right)\right) d\sigma_1 - \int_{\Gamma_1} G(x') v(x', 0) dx' \right| \leq C\varepsilon^{\alpha/2} \|v\|_{H^1(\Omega_0)} \quad (20)$$

and

$$\left| \int_{\Gamma_2^\varepsilon} h\left(x', \frac{x'}{\varepsilon^\beta}\right) v\left(x', 1 - \varepsilon^\beta T\left(x', \frac{x'}{\varepsilon^\beta}\right)\right) d\sigma_2 - \int_{\Gamma_2} H(x') v(x', 1) dx' \right| \leq C\varepsilon^{\beta/2} \|v\|_{H^1(\Omega_0)}. \quad (21)$$

take place.

Proof. A direct computation shows that

$$\begin{aligned} & \left| \int_{\Gamma_1^\varepsilon} g\left(x', \frac{x'}{\varepsilon^\alpha}\right) v\left(x', \varepsilon^\alpha S\left(x', \frac{x'}{\varepsilon^\alpha}\right)\right) d\sigma_1 - \int_{\Gamma_1} G(x') v(x', 0) dx' \right| \\ &= \left| \int_{\Gamma_1} g\left(x', \frac{x'}{\varepsilon^\alpha}\right) v\left(x', \varepsilon^\alpha S\left(x', \frac{x'}{\varepsilon^\alpha}\right)\right) [1 + |\nabla_{\xi'} S|^2]^{1/2} dx' - \int_{\Gamma_1} G(x') v(x', 0) dx' \right| + O(\varepsilon^\alpha) \|v\|_{H^1(\Omega_0)} \\ &\leq \left| \int_{\Gamma_1} g\left(x', \frac{x'}{\varepsilon^\alpha}\right) \left(v\left(x', \varepsilon^\alpha S\left(x', \frac{x'}{\varepsilon^\alpha}\right)\right) - v(x', 0) \right) [1 + |\nabla_{\xi'} S|^2]^{1/2} dx' \right| \\ &+ \left| \int_{\Gamma_1} g\left(x', \frac{x'}{\varepsilon^\alpha}\right) v(x', 0) [1 + |\nabla_{\xi'} S|^2]^{1/2} dx' - \int_{\Gamma_1} G(x') v(x', 0) dx' \right| + C\varepsilon^\alpha \|v\|_{H^1(\Omega_0)}. \end{aligned} \quad (22)$$

This, together with Propositions 1 and 3, yields estimate (20). Similarly, one can prove (21). $\square$

Proposition 5. *There exists a constant C independent of ε such that, for all $v \in H^1(\Omega_0)$, the following inequalities*

$$\left| \int_{\Gamma_1^\varepsilon} p\left(x', \frac{x'}{\varepsilon^\alpha}\right) u_0\left(x', \frac{x'}{\varepsilon^\alpha}\right) v\left(x', \varepsilon^\alpha S\left(x', \frac{x'}{\varepsilon^\alpha}\right)\right) d\sigma_1 - \int_{\Gamma_1} P(x') u_0(x', 0) v(x', 0) dx' \right| \leq C\varepsilon^{\alpha/2} \|u_0\|_{H^1(\Omega_0)} \|v\|_{H^1(\Omega_0)} \quad (23)$$

and

$$\left| \int_{\Gamma_2^\varepsilon} q\left(x', \frac{x'}{\varepsilon^\beta}\right) u_0\left(x', \frac{x'}{\varepsilon^\beta}\right) v\left(x', 1 - \varepsilon^\beta T\left(x', \frac{x'}{\varepsilon^\beta}\right)\right) d\sigma_2 - \int_{\Gamma_2} Q(x') u_0(x', 1) v(x', 1) dx' \right| \leq C\varepsilon^{\beta/2} \|u_0\|_{H^1(\Omega_0)} \|v\|_{H^1(\Omega_0)}. \quad (24)$$

hold true.

Proof. This proposition can be proved in the same way as Proposition 4. $\square$

3. Error Estimate up to $O(\varepsilon^{\alpha/2} + \varepsilon^{\beta/2})$

In this section, we will prove the error estimate up to the order of $O(\varepsilon^{\alpha/2} + \varepsilon^{\beta/2})$. Our main result is the following theorem.

Theorem 6 (let $f \in L^2(\Omega_\varepsilon)$). *Assume that u_ε is a solution of problem (1)-(3). Suppose that $p(x', \xi')$, $g(x', \xi')$, $q(x', \eta')$, $h(x', \eta')$, $S(x', \xi')$, and $T(x', \eta')$ satisfy (4)-(6). Then there exists a constant C which does not depend on ε , such that*

$$\|u_\varepsilon - u_0\|_{H^1(\Omega_\varepsilon)} \leq C(\varepsilon^{\alpha/2} + \varepsilon^{\beta/2}). \quad (25)$$

Remark 7. Following general procedure in homogenization, we introduce the associated bilinear form

$$B(u, v) \doteq \int_{\Omega_\varepsilon} \nabla u \cdot \nabla v dx + \int_{\Gamma_1^\varepsilon} p u v d\sigma_1 + \int_{\Gamma_2^\varepsilon} q u v d\sigma_2, \quad (26)$$

where

$$\begin{aligned} d\sigma_1 &= \left[1 + \sum_{i=1}^{n-1} \left(\varepsilon^\alpha \frac{\partial S}{\partial x_i} + \frac{\partial S}{\partial \xi_i} \right)^2 \right]^{1/2} dx' \\ &= \left[1 + |\nabla_{\xi'} S|^2 \right]^{1/2} dx' + O(\varepsilon^\alpha) dx', \\ d\sigma_2 &= \left[1 + \sum_{i=1}^{n-1} \left(\varepsilon^\beta \frac{\partial T}{\partial x_i} + \frac{\partial T}{\partial \xi_i} \right)^2 \right]^{1/2} dx' \\ &= \left[1 + |\nabla_{\eta'} T|^2 \right]^{1/2} dx' + O(\varepsilon^\beta) dx'. \end{aligned} \quad (27)$$

We shall establish

$$B(u_\varepsilon - u_0, v) \leq C(\varepsilon^{\alpha/2} + \varepsilon^{\beta/2}) \|v\|_{H^1(\Omega_0)}. \quad (28)$$

Then, we extend $u_\varepsilon - u_0$ into Ω_0 without increasing the H^1 -norm by more than a multiplicative constant. What remains is just to take $v = u_\varepsilon - u_0$ and use Poincaré's inequality. Then this theorem will be proved.

Proof. Since $\Omega_\varepsilon \subset \Omega_0$, every function $f \in L^2(\Omega_\varepsilon)$ can be easily extended by 0 in $\Omega_0 \setminus \Omega_\varepsilon$ to become a function of $L^2(\Omega_0)$. For any $v \in H^1(\Omega_0)$ and $v = 0$ on Γ_3 , it is easy to see that

$$\begin{aligned} B(u_\varepsilon - u_0, v) &= \int_{\Omega_\varepsilon} \nabla(u_\varepsilon - u_0) \cdot \nabla v dx \\ &\quad + \int_{\Gamma_1^\varepsilon} p(u_\varepsilon - u_0) v d\sigma_1 \\ &\quad + \int_{\Gamma_2^\varepsilon} q(u_\varepsilon - u_0) v d\sigma_2 \\ &= \int_{\Omega_\varepsilon} f v dx + \int_{\Gamma_1^\varepsilon} g v d\sigma_1 - \int_{\Gamma_1^\varepsilon} p u_0 v d\sigma_1 \\ &\quad - \int_{\Omega_\varepsilon} \nabla u_0 \cdot \nabla v dx + \int_{\Gamma_2^\varepsilon} h v d\sigma_2 \\ &\quad - \int_{\Gamma_2^\varepsilon} q u_0 v d\sigma_2 \\ &= - \int_{\Omega_0} \nabla u_0 \cdot \nabla v dx - \int_{\Gamma_1} p u_0 v dx' \\ &\quad - \int_{\Gamma_2} q u_0 v dx' + \int_{\Gamma_1} p u_0 v dx' \\ &\quad - \int_{\Gamma_1^\varepsilon} p u_0 v d\sigma_1 + \int_{\Gamma_2} q u_0 v dx' \\ &\quad - \int_{\Gamma_2^\varepsilon} q u_0 v d\sigma_2 + \int_{\Omega_\varepsilon} f v dx \end{aligned}$$

$$\begin{aligned} &+ \int_{\Gamma_1^\varepsilon} g v d\sigma_1 + \int_{\Gamma_2^\varepsilon} h v d\sigma_2 \\ &+ \int_{\Omega_0 \setminus \Omega_\varepsilon} \nabla u_0 \cdot \nabla v dx \\ &= - \int_{\Omega_0 \setminus \Omega_\varepsilon} f v dx + \int_{\Omega_0 \setminus \Omega_\varepsilon} \nabla u_0 \cdot \nabla v dx \\ &+ \int_{\Gamma_1^\varepsilon} g v d\sigma_1 - \int_{\Gamma_1} G v dx' + \int_{\Gamma_2^\varepsilon} h v d\sigma_2 \\ &- \int_{\Gamma_2} H v dx' + \int_{\Gamma_1} P u_0 v dx' \\ &- \int_{\Gamma_1^\varepsilon} p u_0 v d\sigma_1 + \int_{\Gamma_2} Q u_0 v dx' \\ &- \int_{\Gamma_2^\varepsilon} q u_0 v d\sigma_2. \end{aligned} \quad (29)$$

It follows from Proposition 2 that

$$\left| \int_{\Omega_0 \setminus \Omega_\varepsilon} f v dx \right| \leq C(\varepsilon^{\alpha/2} + \varepsilon^{\beta/2}) \|f\|_{L^2(\Omega_0)} \|v\|_{H^1(\Omega_0)} \quad (30)$$

and

$$\left| \int_{\Omega_0 \setminus \Omega_\varepsilon} \nabla u_0 \cdot \nabla v dx \right| \leq C(\varepsilon^{\alpha/2} + \varepsilon^{\beta/2}) \|v\|_{H^1(\Omega_0)}, \quad (31)$$

where we have used the uniform boundedness of $\|u_0\|_{H^2(\Omega_0)}$. Then, according to Propositions 4 and 5, we conclude that

$$\begin{aligned} &\left| \int_{\Omega_\varepsilon} \nabla(u_\varepsilon - u_0) \cdot \nabla v dx + \int_{\Gamma_1^\varepsilon} p(u_\varepsilon - u_0) v d\sigma_1 \right. \\ &\quad \left. + \int_{\Gamma_2^\varepsilon} q(u_\varepsilon - u_0) v d\sigma_2 \right| \leq C(\varepsilon^{\alpha/2} + \varepsilon^{\beta/2}) \|v\|_{H^1(\Omega_0)}. \end{aligned} \quad (32)$$

Choosing $v = u_\varepsilon - u_0$ and using condition (4) ($p \geq 0$, $q \geq 0$) and Poincaré's inequality, we obtain the desired result. This completes the proof. $\square$

4. Construction of Correctors u_1 and u_2

In order to improve the power of ε , in this section, we shall construct the correctors u_1 and u_2 .

Firstly, we introduce the harmonic functions $M_i(x', \xi)$, and x' is treated as a parameter, as solutions of

$$\begin{aligned}
\Delta M_i &= 0 \quad \text{in } \widetilde{D}, \\
\frac{\partial M_0}{\partial \nu_\xi} &= g - \frac{p}{P} G \quad \text{on } \widetilde{\Gamma}, \\
\frac{\partial M_j}{\partial \nu_\xi} &= \frac{-1}{\left[1 + |\nabla_{\xi'} S(x', \xi')|^2\right]^{1/2}} \frac{\partial S(x', \xi')}{\partial \xi_j} \\
&\quad j = 1, 2, \dots, n-1 \quad \text{on } \widetilde{\Gamma}, \quad (33) \\
\frac{\partial M_n}{\partial \nu_\xi} &= \frac{1}{\left[1 + |\nabla_{\xi'} S(x', \xi')|^2\right]^{1/2}} - \frac{p}{P} \quad \text{on } \widetilde{\Gamma}, \\
|M_i| &\leq C \exp(-\lambda \xi_i) \quad i = 0, 1, \dots, n, \quad \lambda > 0, \\
M_i(x', \xi) &\text{ is 1-periodic in } \xi',
\end{aligned}$$

where

$$\widetilde{D} = \{(\xi', \xi_n) : \xi' \in (0, 1)^{n-1}, S(x', \xi') < \xi_n < +\infty\}, \quad (34)$$

and

$$\widetilde{\Gamma} : \xi_n = S(x', \xi'). \quad (35)$$

This system was first introduced by A. G. Belyaev [13, 14].

To ensure these solutions M_i exist, we need to verify the compatibility condition holds true. A simple calculation then gives

$$\begin{aligned}
&\int_{\partial \widetilde{D}} \frac{\partial M_0}{\partial \nu_\xi} d\sigma_\xi \\
&= \int_0^1 \cdots \int_0^1 \left(g - \frac{p}{P} G\right) \left[1 + |\nabla_{\xi'} S(x', \xi')|^2\right]^{1/2} d\xi' \\
&= 0,
\end{aligned} \quad (36)$$

$$\int_{\partial \widetilde{D}} \frac{\partial M_j}{\partial \nu_\xi} d\sigma_\xi = \int_0^1 \cdots \int_0^1 -\frac{\partial S(x', \xi')}{\partial \xi_j} d\xi' = 0,$$

where we have used (5) and

$$\begin{aligned}
&\int_{\partial \widetilde{D}} \frac{\partial M_n}{\partial \nu_\xi} d\sigma_\xi \\
&= \int_0^1 \cdots \int_0^1 \left(1 - \frac{p}{P} \left[1 + |\nabla_{\xi'} S(x', \xi')|^2\right]^{1/2}\right) d\xi' \\
&= 0.
\end{aligned} \quad (37)$$

Therefore, these harmonic functions M_i exist.

Then we define u_1 as follows:

$$u_1(x, \xi) = M_0(x', \xi) + \sum_{k=1}^n \frac{\partial u_0}{\partial x_k}(x) M_k(x', \xi). \quad (38)$$

Using the same technique, we construct corrector u_2 as follows.

Assume harmonic functions $N_i(x', \eta)$, x' is treated as a parameter, as solutions of

$$\begin{aligned}
\Delta N_i &= 0 \quad \text{in } \widehat{D}, \\
\frac{\partial N_0}{\partial \nu_\eta} &= h - \frac{q}{Q} H \quad \text{on } \widehat{\Gamma}, \\
\frac{\partial N_j}{\partial \nu_\eta} &= \frac{-1}{\left[1 + |\nabla_{\eta'} T(x', \eta')|^2\right]^{1/2}} \frac{\partial T(x', \eta')}{\partial \eta_j} \\
&\quad j = 1, 2, \dots, n-1 \quad \text{on } \widehat{\Gamma}, \quad (39) \\
\frac{\partial N_n}{\partial \nu_\eta} &= \frac{-1}{\left[1 + |\nabla_{\eta'} T(x', \eta')|^2\right]^{1/2}} + \frac{q}{Q} \quad \text{on } \widehat{\Gamma}, \\
|N_i| &\leq C \exp(-\theta \eta_i) \quad i = 0, 1, \dots, n, \quad \theta > 0, \\
N_i(x', \eta') &\text{ is 1-periodic in } \eta',
\end{aligned}$$

where

$$\begin{aligned}
\widehat{D} &= \{(\eta', \eta_n) : \eta' \in (0, 1)^{n-1}, -\infty < \eta_n < 1 \\
&\quad - T(x', \eta')\}, \quad (40)
\end{aligned}$$

$$\nu_\eta = (\nu_\eta^1, \nu_\eta^2, \dots, \nu_\eta^n) \text{ is the outward normal,}$$

and the hypersurface

$$\widehat{\Gamma} : \eta_n = T(x', \eta'). \quad (41)$$

It is easy to verify the compatibility condition holds true. We define u_2 as follows:

$$u_2(x, \eta) = N_0(x', \eta) + \sum_{k=1}^n \frac{\partial u_0}{\partial x_k}(x) N_k(x', \eta). \quad (42)$$

5. Error Estimate up to $O(\varepsilon^\alpha + \varepsilon^\beta)$

In this section, we will prove the error estimate up to the order of $O(\varepsilon^\alpha + \varepsilon^\beta)$. The main technique is using the correctors u_1 and u_2 . Our main result is the following theorem.

Theorem 8 (let $f \in L^2(\Omega_\varepsilon)$). Assume that u_ε is a solution of problem (1)-(3). Suppose that $p(x', \xi')$, $g(x', \xi')$, $q(x', \eta')$, $h(x', \eta')$, $S(x', \xi')$, and $T(x', \eta')$ satisfy (4)-(6). Then there exists a constant C which does not depend on ε , such that

$$\|u_\varepsilon - u_0 - \varepsilon^\alpha u_1 - \varepsilon^\beta u_2\|_{H^1(\Omega_\varepsilon)} \leq C(\varepsilon^\alpha + \varepsilon^\beta), \quad (43)$$

where correctors u_1 and u_2 are defined in (38) and (42), respectively.

Proof. Analogously to Section 3, for any $v \in H^1(\Omega_0)$ and $v = 0$ on Γ_3 , we consider

$$\begin{aligned}
&B(u_\varepsilon - u_0 - \varepsilon^\alpha u_1 - \varepsilon^\beta u_2, v) \\
&= B(u_\varepsilon - u_0, v) - \int_{\Omega_\varepsilon} \nabla_\xi u_1 \cdot \nabla v dx
\end{aligned} \quad (44)$$

$$- \int_{\Omega_\varepsilon} \nabla_\eta u_2 \cdot \nabla v dx + O(\varepsilon^\alpha + \varepsilon^\beta) \|v\|_{H^1(\Omega_0)}.$$

Firstly, let us estimate the term $B(u_\varepsilon - u_0, v)$. Clearly,

$$\begin{aligned}
B(u_\varepsilon - u_0, v) &= \int_{\Omega_\varepsilon} \nabla(u_\varepsilon - u_0) \cdot \nabla v dx \\
&+ \int_{\Gamma_1^\varepsilon} p(u_\varepsilon - u_0) v d\sigma_1 \\
&+ \int_{\Gamma_2^\varepsilon} q(u_\varepsilon - u_0) v d\sigma_2 \\
&= \int_{\Gamma_1^\varepsilon} \left(-pu_0 - \frac{\partial u_0}{\partial \nu_\varepsilon} + g \right) v d\sigma_1 \\
&+ \int_{\Gamma_2^\varepsilon} \left(-qu_0 - \frac{\partial u_0}{\partial \nu_\varepsilon} + h \right) v d\sigma_2
\end{aligned} \tag{45}$$

It follows from (7) that

$$\begin{aligned}
\frac{\partial u_0}{\partial \nu_0} + P(x') u_0 &= G(x') \text{ on } \Gamma_1, \\
\frac{\partial u_0}{\partial \nu_0} + Q(x') u_0 &= H(x') \text{ on } \Gamma_2.
\end{aligned} \tag{46}$$

Hence

$$\begin{aligned}
\left| pu_0 + \frac{p}{P} \left(\frac{\partial u_0}{\partial \nu_0} - G \right) \right|_{\Gamma_1^\varepsilon} &\leq C\varepsilon^\alpha, \\
\left| qu_0 + \frac{q}{Q} \left(\frac{\partial u_0}{\partial \nu_0} - H \right) \right|_{\Gamma_2^\varepsilon} &\leq C\varepsilon^\beta.
\end{aligned} \tag{47}$$

This gives

$$\begin{aligned}
B(u_\varepsilon - u_0, v) &= \int_{\Gamma_1^\varepsilon} \left(g - \frac{p}{P} G \right) v d\sigma_1 \\
&+ \int_{\Gamma_1^\varepsilon} \left(\frac{p}{P} \frac{\partial u_0}{\partial \nu_0} - \frac{\partial u_0}{\partial \nu_\varepsilon} \right) v d\sigma_1
\end{aligned}$$

$$\begin{aligned}
&+ O(\varepsilon^\alpha) \|v\|_{H^1(\Omega_0)} \\
&+ \int_{\Gamma_2^\varepsilon} \left(h - \frac{q}{Q} H \right) v d\sigma_2 \\
&+ \int_{\Gamma_2^\varepsilon} \left(\frac{q}{Q} \frac{\partial u_0}{\partial \nu_0} - \frac{\partial u_0}{\partial \nu_\varepsilon} \right) v d\sigma_2 \\
&+ O(\varepsilon^\beta) \|v\|_{H^1(\Omega_0)}.
\end{aligned} \tag{48}$$

In view of (44), we obtain

$$\begin{aligned}
B(u_\varepsilon - u_0 - \varepsilon^\alpha u_1 - \varepsilon^\beta u_2, v) &= \int_{\Gamma_1^\varepsilon} \left(g - \frac{p}{P} G \right) v d\sigma_1 + \int_{\Gamma_1^\varepsilon} \left(\frac{p}{P} \frac{\partial u_0}{\partial \nu_0} - \frac{\partial u_0}{\partial \nu_\varepsilon} \right) v d\sigma_1 \\
&- \int_{\Omega_\varepsilon} \nabla_\xi u_1 \cdot \nabla v dx + O(\varepsilon^\alpha) \|v\|_{H^1(\Omega_0)} \\
&+ \int_{\Gamma_2^\varepsilon} \left(h - \frac{q}{Q} H \right) v d\sigma_2 \\
&+ \int_{\Gamma_2^\varepsilon} \left(\frac{q}{Q} \frac{\partial u_0}{\partial \nu_0} - \frac{\partial u_0}{\partial \nu_\varepsilon} \right) v d\sigma_2 - \int_{\Omega_\varepsilon} \nabla_\eta u_2 \cdot \nabla v dx \\
&+ O(\varepsilon^\beta) \|v\|_{H^1(\Omega_0)}.
\end{aligned} \tag{49}$$

We shall next deal with the term $\int_{\Omega_\varepsilon} \nabla_\xi u_1 \cdot \nabla v dx$. Using the change of variables, we find that

$$\int_{\Omega_\varepsilon} \nabla_\xi u_1 \cdot \nabla v dx = \varepsilon^{(n-1)\alpha} \sum_i \int_{\widetilde{\Omega}_i} \nabla_\xi u_1 \cdot \nabla_\xi v d\xi, \tag{50}$$

where

$$\widetilde{\Omega}_i = \left\{ (\xi', \xi_n) \mid \xi' \in \left(0, \frac{1}{\varepsilon^\alpha} \right)^{n-1}, S < \xi_n < \frac{1}{\varepsilon^\alpha} - \varepsilon^{\beta-\alpha} T \right\} \cap \left\{ \xi' \in \left(\frac{A_i}{\varepsilon^\alpha}, \frac{A_{i+1}}{\varepsilon^\alpha} \right)^{n-1} \right\}, \tag{51}$$

where $A_{i+1} - A_i = \varepsilon^\alpha$, $i = 0, 1, 2, \dots$, and we assume that $A_0 = 0$.

In view of integration by parts and the fact that $\Delta_\xi u_1 = 0$, we get

$$\int_{\Omega_\varepsilon} \nabla_\xi u_1 \cdot \nabla v dx = \varepsilon^{(n-1)\alpha} \sum_i \int_{\widetilde{\Gamma}_i} \frac{\partial u_1}{\partial \nu_\xi} v d\sigma_\xi, \tag{52}$$

where

$$\widetilde{\Gamma}_i : \left\{ \xi_n = S(x', \xi'), \xi_i \in \left(\frac{A_i}{\varepsilon^\alpha}, \frac{A_{i+1}}{\varepsilon^\alpha} \right)^{n-1} \right\}. \tag{53}$$

It follows from (38) that

$$\begin{aligned}
\int_{\Omega_\varepsilon} \nabla_\xi u_1 \cdot \nabla v dx &= \varepsilon^{(n-1)\alpha} \sum_i \left(\int_{\widetilde{\Gamma}_i} \frac{\partial M_0}{\partial \nu_\xi} v d\sigma_\xi \right. \\
&+ \sum_{j=1}^{n-1} \int_{\widetilde{\Gamma}_i} \frac{\partial M_j}{\partial \nu_\xi} \frac{\partial u_0}{\partial x_j} v d\sigma_\xi + \int_{\widetilde{\Gamma}_i} \frac{\partial M_n}{\partial \nu_\xi} \frac{\partial u_0}{\partial x_n} v d\sigma_\xi \Big)
\end{aligned} \tag{54}$$

$$\div A_1 + A_2 + A_3,$$

where

$$\begin{aligned}
A_1 &= \varepsilon^{(n-1)\alpha} \sum_i \int_{\bar{\Gamma}_i} \frac{\partial M_0}{\partial \nu_\xi} v d\sigma_\xi = \varepsilon^{(n-1)\alpha} \sum_i \int_{\bar{\Gamma}_i} \left(g - \frac{p}{P} G \right) v d\sigma_\xi \\
&= \varepsilon^{(n-1)\alpha} \sum_i \int_{A_i/\varepsilon^\alpha}^{A_{i+1}/\varepsilon^\alpha} \cdots \int_{A_i/\varepsilon^\alpha}^{A_{i+1}/\varepsilon^\alpha} \left(g - \frac{p}{P} G \right) v \left[1 + |\nabla_{\xi'} S(x', \xi')|^2 \right]^{1/2} d\xi', \\
A_2 &= \varepsilon^{(n-1)\alpha} \sum_{j=1}^{n-1} \sum_i \int_{\bar{\Gamma}_i} \frac{\partial M_j}{\partial \nu_\xi} \frac{\partial u_0}{\partial x_j} v d\sigma_\xi = \varepsilon^{(n-1)\alpha} \sum_{j=1}^{n-1} \sum_i \int_{\bar{\Gamma}_i} \frac{-1}{\left[1 + |\nabla_{\xi'} S(x', \xi')|^2 \right]^{1/2}} \frac{\partial S(x', \xi')}{\partial \xi_j} \frac{\partial u_0}{\partial x_j} v d\sigma_\xi \\
&= \varepsilon^{(n-1)\alpha} \sum_{j=1}^{n-1} \sum_i \int_{A_i/\varepsilon^\alpha}^{A_{i+1}/\varepsilon^\alpha} \cdots \int_{A_i/\varepsilon^\alpha}^{A_{i+1}/\varepsilon^\alpha} -\frac{\partial S(x', \xi')}{\partial \xi_j} \frac{\partial u_0}{\partial x_j} v d\xi', \text{ and} \\
A_3 &= \varepsilon^{(n-1)\alpha} \sum_i \int_{\bar{\Gamma}_i} \frac{\partial M_n}{\partial \nu_\xi} \frac{\partial u_0}{\partial x_n} v d\sigma_\xi = \varepsilon^{(n-1)\alpha} \sum_i \int_{\bar{\Gamma}_i} \left(\frac{1}{\left[1 + |\nabla_{\xi'} S(x', \xi')|^2 \right]^{1/2}} - \frac{p}{P} \right) \frac{\partial u_0}{\partial x_n} v d\sigma_\xi \\
&= \varepsilon^{(n-1)\alpha} \sum_i \int_{A_i/\varepsilon^\alpha}^{A_{i+1}/\varepsilon^\alpha} \cdots \int_{A_i/\varepsilon^\alpha}^{A_{i+1}/\varepsilon^\alpha} \left(1 - \frac{p}{P} \left[1 + |\nabla_{\xi'} S(x', \xi')|^2 \right]^{1/2} \right) \frac{\partial u_0}{\partial x_n} v d\xi'.
\end{aligned} \tag{55}$$

Also, note that

where we have used the change of variables $x' = \varepsilon^\alpha \xi'$. Hence

$$\begin{aligned}
&\int_{\Gamma_1^\varepsilon} \left(g - \frac{p}{P} G \right) v d\sigma_1 = \int_{\Gamma_1} \left(g - \frac{p}{P} G \right) v \left[1 + |\nabla_{\xi'} S(x', \xi')|^2 \right]^{1/2} dx' + O(\varepsilon^\alpha) \|v\|_{H^1(\Omega_0)} \\
&= \varepsilon^{(n-1)\alpha} \sum_i \int_{A_i/\varepsilon^\alpha}^{A_{i+1}/\varepsilon^\alpha} \cdots \int_{A_i/\varepsilon^\alpha}^{A_{i+1}/\varepsilon^\alpha} \left(g - \frac{p}{P} G \right) v \left[1 + |\nabla_{\xi'} S(x', \xi')|^2 \right]^{1/2} d\xi' + O(\varepsilon^\alpha) \\
&\quad \cdot \|v\|_{H^1(\Omega_0)},
\end{aligned} \tag{56}$$

Next, we shall evaluate the term $\int_{\Gamma_1^\varepsilon} ((p/P)(\partial u_0/\partial \nu_0) - \partial u_0/\partial \nu_\varepsilon) v d\sigma_1$.

Also, note that since $\nu_0 = (0, \dots, 0, -1)$ and $\nu_\varepsilon = (\nabla_{\xi'} S, -1)(1/[1 + |\nabla_{\xi'} S(x', \xi')|^2]^{1/2}) + O(\varepsilon^\alpha)$,

$$\begin{aligned}
\int_{\Gamma_1^\varepsilon} \left(\frac{p}{P} \frac{\partial u_0}{\partial \nu_0} - \frac{\partial u_0}{\partial \nu_\varepsilon} \right) v d\sigma_1 &= \int_{\Gamma_1^\varepsilon} \left(\frac{p}{P} \nabla u_0 \cdot \nu_0 - \nabla u_0 \cdot \nu_\varepsilon \right) v d\sigma_1 \\
&= \int_{\Gamma_1^\varepsilon} \frac{\partial u_0}{\partial x_n} \left(\frac{1}{\left[1 + |\nabla_{\xi'} S(x', \xi')|^2 \right]^{1/2}} - \frac{p}{P} \right) v d\sigma_1 \\
&\quad - \sum_{j=1}^{n-1} \int_{\Gamma_1^\varepsilon} \frac{\partial u_0}{\partial x_j} \frac{1}{\left[1 + |\nabla_{\xi'} S(x', \xi')|^2 \right]^{1/2}} \frac{\partial S}{\partial \xi_j} v d\sigma_1 + O(\varepsilon^\alpha) \|v\|_{H^1(\Omega_0)} \\
&= \int_{\Gamma_1} \frac{\partial u_0}{\partial x_n} \left(1 - \frac{p}{P} \left[1 + |\nabla_{\xi'} S(x', \xi')|^2 \right]^{1/2} \right) v dx' - \sum_{j=1}^{n-1} \int_{\Gamma_1} \frac{\partial u_0}{\partial x_j} \frac{\partial S}{\partial \xi_j} v dx' + O(\varepsilon^\alpha) \|v\|_{H^1(\Omega_0)} \\
&= \varepsilon^{(n-1)\alpha} \sum_i \int_{A_i/\varepsilon^\alpha}^{A_{i+1}/\varepsilon^\alpha} \cdots \int_{A_i/\varepsilon^\alpha}^{A_{i+1}/\varepsilon^\alpha} \frac{\partial u_0}{\partial x_n} \left(1 - \frac{p}{P} \left[1 + |\nabla_{\xi'} S(x', \xi')|^2 \right]^{1/2} \right) v d\xi' \\
&\quad - \varepsilon^{(n-1)\alpha} \sum_{j=1}^{n-1} \sum_i \int_{A_i/\varepsilon^\alpha}^{A_{i+1}/\varepsilon^\alpha} \cdots \int_{A_i/\varepsilon^\alpha}^{A_{i+1}/\varepsilon^\alpha} \frac{\partial u_0}{\partial x_j} \frac{\partial S}{\partial \xi_j} v d\xi' + O(\varepsilon^\alpha) \|v\|_{H^1(\Omega_0)}.
\end{aligned} \tag{58}$$

By the definition of A_2 and A_3 in (54), this implies that

$$\int_{\Gamma_1^\varepsilon} \left(\frac{p}{P} \frac{\partial u_0}{\partial \nu_0} - \frac{\partial u_0}{\partial \nu_\varepsilon} \right) \nu d\sigma_1 - A_2 - A_3 = O(\varepsilon^\alpha) \|v\|_{H^1(\Omega_0)}. \quad (59)$$

In a similar way, we obtain that

$$\int_{\Omega_\varepsilon} \nabla_{\eta'} u_2 \cdot \nabla \nu dx \doteq B_1 + B_2 + B_3, \quad (60)$$

where

$$\begin{aligned} B_1 &= \varepsilon^{(n-1)\alpha} \sum_i \int_{B_i/\varepsilon^\beta}^{B_{i+1}/\varepsilon^\beta} \cdots \int_{B_i/\varepsilon^\beta}^{B_{i+1}/\varepsilon^\beta} \left(h - \frac{q}{Q} H \right) \nu \left[1 + |\nabla_{\eta'} T(x', \eta')|^2 \right]^{1/2} d\eta', \\ B_2 &= \varepsilon^{(n-1)\alpha} \sum_{j=1}^{n-1} \sum_i \int_{B_i/\varepsilon^\beta}^{B_{i+1}/\varepsilon^\beta} \cdots \int_{B_i/\varepsilon^\beta}^{B_{i+1}/\varepsilon^\beta} -\frac{\partial T(x', \eta')}{\partial \eta_j} \frac{\partial u_0}{\partial x_j} \nu d\eta', \\ B_3 &= \varepsilon^{(n-1)\alpha} \sum_i \int_{B_i/\varepsilon^\beta}^{B_{i+1}/\varepsilon^\beta} \cdots \int_{B_i/\varepsilon^\beta}^{B_{i+1}/\varepsilon^\beta} \left(-1 + \frac{q}{Q} \left[1 + |\nabla_{\eta'} T(x', \eta')|^2 \right]^{1/2} \right) \frac{\partial u_0}{\partial x_n} \nu d\eta'. \end{aligned} \quad (61)$$

Using the same technique, we also have

$$\begin{aligned} \int_{\Gamma_2^\varepsilon} \left(h - \frac{q}{Q} H \right) \nu d\sigma_2 - B_1 &= O(\varepsilon^\beta) \|v\|_{H^1(\Omega_0)} \quad \text{and} \\ \int_{\Gamma_2^\varepsilon} \left(\frac{q}{Q} \frac{\partial u_0}{\partial \nu_0} - \frac{\partial u_0}{\partial \nu_\varepsilon} \right) \nu d\sigma_2 - B_2 - B_3 &= O(\varepsilon^\beta) \|v\|_{H^1(\Omega_0)}. \end{aligned} \quad (62)$$

Combining all these terms and choosing $v = u_\varepsilon - u_0 - \varepsilon^\alpha(u_1 - u_1|_{x_i=1}) - \varepsilon^\beta(u_2 - u_2|_{x_i=1})$, we conclude that

$$\|u_\varepsilon - u_0 - \varepsilon^\alpha u_1 - \varepsilon^\beta u_2\|_{H^1(\Omega_\varepsilon)} \leq C(\varepsilon^\alpha + \varepsilon^\beta), \quad (63)$$

where we have used Poincaré's inequality and the fact

$$\begin{aligned} \varepsilon^\alpha u_1|_{x_i=1} + \varepsilon^\beta u_2|_{x_i=1} &\leq C(\varepsilon^\alpha + \varepsilon^\beta) \left[\exp\left(-\frac{\lambda}{\varepsilon^\alpha}\right) + \exp\left(-\frac{\theta}{\varepsilon^\beta}\right) \right], \end{aligned} \quad (64)$$

where $i = 1, 2, \dots, n-1$.

This completes the proof. $\square$

6. Error Estimate up to $O(\varepsilon^{3\alpha/2} + \varepsilon^{3\beta/2})$

In this section, we will improve the error estimate up to the order of $O(\varepsilon^{3\alpha/2} + \varepsilon^{3\beta/2})$. The main technique is using the correctors and second-order Taylor's expansion. Our main result is the following theorem.

Theorem 9 (let $f \in L^2(\Omega_\varepsilon)$). Assume that u_ε is a solution of problem (1)-(3). Suppose that $p(x', \xi')$, $g(x', \xi')$, $q(x', \eta')$, $h(x', \eta')$, $S(x', \xi')$, and $T(x', \eta')$ satisfy (4)-(6). Then there exists a constant C that does not depend on ε , such that

$$\begin{aligned} \|u_\varepsilon - u_0 - \varepsilon^\alpha u_1 - \varepsilon^\beta u_2 + \varepsilon^\alpha \phi_1 + \varepsilon^\beta \phi_2\|_{H^1(\Omega_\varepsilon)} &\leq C(\varepsilon^{3\alpha/2} + \varepsilon^{3\beta/2}), \end{aligned} \quad (65)$$

where correctors ϕ_1 and ϕ_2 are given by (88) and (93).

Proof. Following the same line of research, we consider

$$\begin{aligned} B(u_\varepsilon - u_0, v) &= \int_{\Gamma_1^\varepsilon} \left(-p u_0 - \frac{\partial u_0}{\partial \nu_\varepsilon} + g \right) \nu d\sigma_1 \\ &\quad + \int_{\Gamma_2^\varepsilon} \left(-q u_0 - \frac{\partial u_0}{\partial \nu_\varepsilon} + h \right) \nu d\sigma_2. \end{aligned} \quad (66)$$

This, together with (7) and the second-order Taylor's expansion, gives

$$\begin{aligned} &\left(\frac{\partial u_0}{\partial \nu_0} + P(x') u_0 - G(x') \right)_{\Gamma_1^\varepsilon} \\ &= \varepsilon^\alpha S(x', \xi') \frac{\partial}{\partial x_n} \left(\frac{\partial u_0}{\partial \nu_0} + P(x') u_0 - G(x') \right)_{\Gamma_1} \\ &\quad + O(\varepsilon^{2\alpha}), \text{ and} \\ &\left(\frac{\partial u_0}{\partial \nu_0} + Q(x') u_0 - H(x') \right)_{\Gamma_2^\varepsilon} \\ &= -\varepsilon^\beta T(x', \eta') \frac{\partial}{\partial x_n} \left(\frac{\partial u_0}{\partial \nu_0} + Q(x') u_0 - H(x') \right)_{\Gamma_2} \\ &\quad + O(\varepsilon^{2\beta}). \end{aligned} \quad (67)$$

It follows that

$$\begin{aligned}
& B(u_\varepsilon - u_0, v) \\
&= \int_{\Gamma_1^\varepsilon} \left(g - \frac{p}{P} G \right) v d\sigma_1 + \int_{\Gamma_1^\varepsilon} \left(\frac{p}{P} \frac{\partial u_0}{\partial \nu_0} - \frac{\partial u_0}{\partial \nu_\varepsilon} \right) v d\sigma_1 \\
&\quad - \int_{\Gamma_1^\varepsilon} \varepsilon^\alpha \frac{p}{P} S(x', \xi') \frac{\partial}{\partial x_n} \left(\frac{\partial u_0}{\partial \nu_0} + P(x') u_0 \right) v d\sigma_1 \\
&\quad + O(\varepsilon^{2\alpha}) \|v\|_{H^1(\Omega_0)} + \int_{\Gamma_2^\varepsilon} \left(h - \frac{q}{Q} H \right) v d\sigma_2 \quad (68) \\
&\quad + \int_{\Gamma_2^\varepsilon} \left(\frac{q}{Q} \frac{\partial u_0}{\partial \nu_0} - \frac{\partial u_0}{\partial \nu_\varepsilon} \right) v d\sigma_2 \\
&\quad + \int_{\Gamma_2^\varepsilon} \varepsilon^\beta \frac{q}{Q} T(x', \eta') \frac{\partial}{\partial x_n} \left(\frac{\partial u_0}{\partial \nu_0} + Q(x') u_0 \right) v d\sigma_2 \\
&\quad + O(\varepsilon^{2\beta}) \|v\|_{H^1(\Omega_0)}.
\end{aligned}$$

Note that

$$\begin{aligned}
B(-\varepsilon^\alpha u_1 - \varepsilon^\beta u_2, v) &= \int_{\Omega_\varepsilon} \varepsilon^\alpha \Delta u_1 v dx \\
&\quad - \int_{\Gamma_1^\varepsilon} \varepsilon^\alpha \frac{\partial u_1}{\partial \nu_\varepsilon} v d\sigma_1 \\
&\quad - \int_{\Gamma_1^\varepsilon} \varepsilon^\alpha p u_1 v d\sigma_1 \\
&\quad + \int_{\Omega_\varepsilon} \varepsilon^\beta \Delta u_2 v dx \\
&\quad - \int_{\Gamma_2^\varepsilon} \varepsilon^\beta \frac{\partial u_2}{\partial \nu_\varepsilon} v d\sigma_2 \\
&\quad - \int_{\Gamma_2^\varepsilon} \varepsilon^\beta q u_2 v d\sigma_2 \\
&= \int_{\Omega_\varepsilon} \varepsilon^\alpha \Delta_x u_1 v dx \quad (69) \\
&\quad + \int_{\Omega_\varepsilon} 2\nabla_x \cdot \nabla_{\xi'} u_1 v dx \\
&\quad - \int_{\Gamma_1^\varepsilon} \varepsilon^\alpha \frac{\partial u_1}{\partial \nu_\varepsilon} v d\sigma_1 \\
&\quad + \int_{\Omega_\varepsilon} \varepsilon^\beta \Delta_x u_2 v dx \\
&\quad + \int_{\Omega_\varepsilon} 2\nabla_x \cdot \nabla_{\eta'} u_2 v dx \\
&\quad - \int_{\Gamma_1^\varepsilon} \varepsilon^\alpha p u_1 v d\sigma_1 \\
&\quad - \int_{\Gamma_2^\varepsilon} \varepsilon^\beta \frac{\partial u_2}{\partial \nu_\varepsilon} v d\sigma_2 \\
&\quad - \int_{\Gamma_2^\varepsilon} \varepsilon^\beta q u_2 v d\sigma_2,
\end{aligned}$$

where we have used $\Delta_\xi u_1 = 0$ and $\Delta_\eta u_2 = 0$.

Thus we obtain

$$\begin{aligned}
& B(u_\varepsilon - u_0 - \varepsilon^\alpha u_1 - \varepsilon^\beta u_2, v) \\
&= \int_{\Gamma_1^\varepsilon} \left(g - \frac{p}{P} G \right) v d\sigma_1 \\
&\quad + \int_{\Gamma_1^\varepsilon} \left(\frac{p}{P} \frac{\partial u_0}{\partial \nu_0} - \frac{\partial u_0}{\partial \nu_\varepsilon} \right) v d\sigma_1 \\
&\quad + \int_{\Omega_\varepsilon} 2\nabla_x \cdot \nabla_{\xi'} u_1 v dx \\
&\quad - \int_{\Gamma_1^\varepsilon} \varepsilon^\alpha \frac{p}{P} S(x', \xi') \frac{\partial}{\partial x_n} \left(\frac{\partial u_0}{\partial \nu_0} + P(x') u_0 \right) \big|_{\Gamma_1} v d\sigma_1 \\
&\quad + O(\varepsilon^{2\alpha}) \|v\|_{H^1(\Omega_0)}, \\
&\quad + \int_{\Omega_\varepsilon} \varepsilon^\alpha \Delta_x u_1 v dx \\
&\quad - \int_{\Gamma_1^\varepsilon} \varepsilon^\alpha p u_1 v d\sigma_1 \\
&\quad - \int_{\Gamma_1^\varepsilon} \varepsilon^\alpha \frac{\partial u_1}{\partial \nu_\varepsilon} v d\sigma_1 \quad (70) \\
&\quad + \int_{\Gamma_2^\varepsilon} \left(h - \frac{q}{Q} H \right) v d\sigma_2 \\
&\quad + \int_{\Gamma_2^\varepsilon} \left(\frac{q}{Q} \frac{\partial u_0}{\partial \nu_0} - \frac{\partial u_0}{\partial \nu_\varepsilon} \right) v d\sigma_2 \\
&\quad + \int_{\Omega_\varepsilon} 2\nabla_x \cdot \nabla_{\eta'} u_2 v dx \\
&\quad + \int_{\Gamma_2^\varepsilon} \varepsilon^\beta \frac{q}{Q} T(x', \eta') \frac{\partial}{\partial x_n} \left(\frac{\partial u_0}{\partial \nu_0} + Q(x') u_0 \right) \big|_{\Gamma_2} v d\sigma_2 \\
&\quad + O(\varepsilon^{2\beta}) \|v\|_{H^1(\Omega_0)}, \\
&\quad + \int_{\Omega_\varepsilon} \varepsilon^\beta \Delta_x u_2 v dx \\
&\quad - \int_{\Gamma_2^\varepsilon} \varepsilon^\beta q u_2 v d\sigma_2 \\
&\quad - \int_{\Gamma_2^\varepsilon} \varepsilon^\beta \frac{\partial u_2}{\partial \nu_\varepsilon} v d\sigma_2.
\end{aligned}$$

Note that on Γ_1^ε

$$\begin{aligned}
& \nu_0 = (0, \dots, 0, -1), \\
& \nu_\varepsilon = (\nabla_{\xi'} S, -1) \frac{1}{\left[1 + |\nabla_{\xi'} S|^2\right]^{1/2}} \\
& \quad + \varepsilon^\alpha (\nabla_{x'} S, \nabla_{x'} S \cdot \nabla_{\xi'} S) \frac{1}{\left[1 + |\nabla_{\xi'} S|^2\right]^{3/2}} \quad (71) \\
& \quad + O(\varepsilon^{2\alpha})
\end{aligned}$$

and on Γ_2^ε

$$\nu_0 = (0, \dots, 0, 1), \quad (72)$$

$$\begin{aligned} \nu_\varepsilon &= (\nabla_{\eta'} T, 1) \frac{1}{[1 + |\nabla_{\eta'} T|^2]^{1/2}} \\ &+ \varepsilon^\beta (\nabla_{x'} T, -\nabla_{x'} T \cdot \nabla_{\eta'} T) \frac{1}{[1 + |\nabla_{\eta'} T|^2]^{3/2}} \\ &+ O(\varepsilon^{2\beta}). \end{aligned} \quad (73)$$

In view of (33)-(42), we obtain

$$\begin{aligned} &\left| \int_{\Omega_\varepsilon} (\varepsilon^\alpha \Delta_x u_1 + \varepsilon^\beta \Delta_x u_2) v dx \right| \\ &\leq C \varepsilon^\alpha \int_{\Omega_\varepsilon} \exp\left(-\frac{\lambda x_1}{\varepsilon^\alpha}\right) |v| dx \\ &+ C \varepsilon^\beta \int_{\Omega_\varepsilon} \exp\left(-\frac{\theta x_1}{\varepsilon^\beta}\right) |v| dx \leq C (\varepsilon^{3\alpha/2} \\ &+ \varepsilon^{3\beta/2}) \|v\|_{H^1(\Omega_0)}, \\ &2 \int_{\Omega_\varepsilon} \nabla_x \cdot \nabla_\xi u_1 v dx = 2 \varepsilon^\alpha \int_{\Gamma_1^\varepsilon} \nabla_x u_1 \cdot \nu_\varepsilon v d\sigma_1 \\ &- 2 \varepsilon^\alpha \int_{\Omega_\varepsilon} \nabla_x u_1 \cdot \nabla_x v dx \leq 2 \varepsilon^\alpha \int_{\Gamma_1^\varepsilon} \nabla_x u_1 \cdot \nu_\varepsilon v d\sigma_1 \\ &+ C \varepsilon^\alpha \int_{\Omega_\varepsilon} \exp(-\lambda \xi_1) |\nabla_x v| dx \leq 2 \varepsilon^\alpha \int_{\Gamma_1^\varepsilon} \nabla_x u_1 \\ &\cdot (\nabla_{\xi'} S, -1) \frac{1}{[1 + |\nabla_{\xi'} S|^2]^{1/2}} v d\sigma_1 \\ &+ C \varepsilon^{3\alpha/2} \|v\|_{H^1(\Omega_0)} + 2 \varepsilon^{2\alpha} \int_{\Gamma_1^\varepsilon} \nabla_x u_1 \\ &\cdot (\nabla_{x'} S, \nabla_{x'} S \cdot \nabla_{\xi'} S) \frac{1}{[1 + |\nabla_{\xi'} S|^2]^{3/2}} v d\sigma_1 \end{aligned} \quad (74)$$

and

$$\begin{aligned} &2 \int_{\Omega_\varepsilon} \nabla_x \cdot \nabla_{\eta'} u_2 v dx \leq 2 \varepsilon^\beta \int_{\Gamma_2^\varepsilon} \nabla_x u_2 \\ &\cdot (\nabla_{\eta'} T, 1) \frac{1}{[1 + |\nabla_{\eta'} T|^2]^{1/2}} v d\sigma_2 \\ &+ C \varepsilon^{3\beta/2} \|v\|_{H^1(\Omega_0)} + 2 \varepsilon^{2\beta} \int_{\Gamma_2^\varepsilon} \nabla_x u_2 \\ &\cdot (\nabla_{x'} T, -\nabla_{x'} T \cdot \nabla_{\eta'} T) \frac{1}{[1 + |\nabla_{\eta'} T|^2]^{3/2}} v d\sigma_2. \end{aligned} \quad (76)$$

It follows from (38) that

$$\begin{aligned} \varepsilon^\alpha \frac{\partial u_1}{\partial \nu_\varepsilon} \Big|_{\Gamma_1^\varepsilon} &= (\nabla_{\xi'} u_1 \cdot \nu_\varepsilon + \varepsilon^\alpha \nabla_x u_1 \cdot \nu_\varepsilon) \Big|_{\Gamma_1^\varepsilon} \\ &= \bar{A}_1 + \bar{A}_2 + O(\varepsilon^{2\alpha}), \end{aligned} \quad (77)$$

where

$$\begin{aligned} \bar{A}_1 &= \left(g - \frac{p}{P} G\right) - \nabla_{x'} u_0 \cdot \nabla_{\xi'} S \frac{1}{[1 + |\nabla_{\xi'} S|^2]^{1/2}} \\ &+ \frac{\partial u_0}{\partial x_n} \left(\frac{1}{[1 + |\nabla_{\xi'} S|^2]^{1/2}} - \frac{p}{P} \right) \text{ and} \\ \bar{A}_2 &= \varepsilon^\alpha \nabla_x u_1 \cdot (\nabla_{\xi'} S, -1) \frac{1}{[1 + |\nabla_{\xi'} S|^2]^{1/2}}. \end{aligned} \quad (78)$$

Combining all these terms in (77), we have proved that

$$\begin{aligned} \int_{\Gamma_1^\varepsilon} \varepsilon^\alpha \frac{\partial u_1}{\partial \nu_\varepsilon} v d\sigma_1 &= \int_{\Gamma_1^\varepsilon} \bar{A}_1 v d\sigma_1 + \int_{\Gamma_1^\varepsilon} \bar{A}_2 v d\sigma_1 \\ &+ O(\varepsilon^{2\alpha}) \|v\|_{H^1(\Omega_0)}. \end{aligned} \quad (79)$$

A similar calculation yields the following result:

$$\begin{aligned} &\int_{\Gamma_1^\varepsilon} \left(\frac{p}{P} \frac{\partial u_0}{\partial \nu_\varepsilon} - \frac{\partial u_0}{\partial \nu_\varepsilon} \right) v d\sigma_1 \\ &= \int_{\Gamma_1^\varepsilon} -\frac{p}{P} \frac{\partial u_0}{\partial x_n} v d\sigma_1 - \bar{A} + O(\varepsilon^{2\alpha}) \|v\|_{H^1(\Omega_0)}, \end{aligned} \quad (80)$$

where

$$\begin{aligned} \bar{A} &= \int_{\Gamma_1^\varepsilon} \frac{1}{[1 + |\nabla_{\xi'} S|^2]^{1/2}} \left(\nabla_{x'} u_0 \cdot \nabla_{\xi'} S - \frac{\partial u_0}{\partial x_n} \right) \\ &+ \varepsilon^\alpha \nabla u_0 \\ &\cdot (\nabla_{x'} S, \nabla_{x'} S \cdot \nabla_{\xi'} S) \frac{1}{[1 + |\nabla_{\xi'} S|^2]^{3/2}} v d\sigma_1. \end{aligned} \quad (81)$$

In the same way, it follows from (42) that

$$\begin{aligned} \varepsilon^\beta \frac{\partial u_2}{\partial \nu_\varepsilon} \Big|_{\Gamma_2^\varepsilon} &= (\nabla_{\eta'} u_2 \cdot \nu_\varepsilon + \varepsilon^\beta \nabla_x u_2 \cdot \nu_\varepsilon) \Big|_{\Gamma_2^\varepsilon} \\ &= \bar{B}_1 + \bar{B}_2 + O(\varepsilon^{2\beta}), \end{aligned} \quad (82)$$

where

$$\begin{aligned} \tilde{B}_1 = & \left(h - \frac{q}{Q} H \right) - \nabla_{x'} u_0 \cdot \nabla_{\eta'} T \frac{1}{[1 + |\nabla_{\xi'} T|^2]^{1/2}} \\ & + \frac{\partial u_0}{\partial x_n} \left(\frac{-1}{[1 + |\nabla_{\eta'} T|^2]^{1/2}} + \frac{q}{Q} \right) \text{ and} \end{aligned} \quad (83)$$

$$\tilde{B}_2 = \varepsilon^\beta \nabla_x u_2 \cdot (\nabla_{\eta'} T, 1) \frac{1}{[1 + |\nabla_{\eta'} T|^2]^{1/2}}.$$

Combining all these terms in (82), we have proved that

$$\begin{aligned} \int_{\Gamma_2^\varepsilon} \varepsilon^\beta \frac{\partial u_2}{\partial \nu_\varepsilon} \nu d\sigma_2 &= \int_{\Gamma_2^\varepsilon} \tilde{B}_1 \nu d\sigma_2 + \int_{\Gamma_2^\varepsilon} \tilde{B}_2 \nu d\sigma_2 \\ &+ O(\varepsilon^{2\beta}) \|v\|_{H^1(\Omega_0)}. \end{aligned} \quad (84)$$

Similarly, we obtain the following result:

$$\begin{aligned} \int_{\Gamma_2^\varepsilon} \left(\frac{q}{Q} \frac{\partial u_0}{\partial \nu_0} - \frac{\partial u_0}{\partial \nu_\varepsilon} \right) \nu d\sigma_2 \\ = \int_{\Gamma_2^\varepsilon} \frac{q}{Q} \frac{\partial u_0}{\partial x_n} \nu d\sigma_2 - \tilde{B} + O(\varepsilon^{2\beta}) \|v\|_{H^1(\Omega_0)}, \end{aligned} \quad (85)$$

where

$$\begin{aligned} \tilde{B} = & \int_{\Gamma_2^\varepsilon} \frac{1}{[1 + |\nabla_{\eta'} T|^2]^{1/2}} \left(\nabla_{x'} u_0 \cdot \nabla_{\eta'} T + \frac{\partial u_0}{\partial x_n} \right) \\ & + \varepsilon^\beta \nabla u_0 \\ & \cdot (\nabla_{x'} T, -\nabla_{x'} T \cdot \nabla_{\eta'} T) \frac{1}{[1 + |\nabla_{\eta'} T|^2]^{3/2}} \nu d\sigma_2. \end{aligned} \quad (86)$$

Next, we shall introduce two new correctors so as to cancel lower power of ε^α and ε^β .

For any given function of the form $\tilde{g}$, suppose that $\tilde{u}_\varepsilon$ is the solution of the following problem:

$$\begin{aligned} -\Delta \tilde{u}_\varepsilon &= 0 \quad \text{in } \Omega_\varepsilon, \\ \frac{\partial \tilde{u}_\varepsilon}{\partial \nu_\varepsilon} + p \left(x', \frac{x'}{\varepsilon^\alpha} \right) \tilde{u}_\varepsilon &= \tilde{g} \left(x', \frac{x'}{\varepsilon^\alpha} \right) \quad \text{on } \Gamma_1^\varepsilon, \\ \tilde{u}_\varepsilon &= 0 \quad \text{on } \partial\Omega_\varepsilon \setminus \Gamma_1^\varepsilon. \end{aligned} \quad (87)$$

Meanwhile, we assume that ϕ_1 is the solution of the homogenized problem:

$$\begin{aligned} -\Delta \phi_1 &= 0 \quad \text{in } \Omega_0, \\ \frac{\partial \phi_1}{\partial \nu_0} + P(x') \phi_1 &= \tilde{G}(x') \quad \text{on } \Gamma_1, \\ \phi_1 &= 0 \quad \text{on } \partial\Omega_0 \setminus \Gamma_1, \end{aligned} \quad (88)$$

where $P(x')$ and Γ_1^ε are defined in (8) and (2),

$$\begin{aligned} \tilde{G}(x') \\ = \int_0^1 \cdots \int_0^1 [1 + |\nabla_{\xi'} S(x', \xi')|^2]^{1/2} \tilde{g}(x', \xi') d\xi'. \end{aligned} \quad (89)$$

Clearly, it is a special case of problem (1).

We may invoke Theorem 8 to conclude that

$$\|\tilde{u}_\varepsilon - \phi_1 - \varepsilon^\alpha \tilde{u}_1\|_{H^1(\Omega_\varepsilon)} \leq C\varepsilon^\alpha, \quad (90)$$

where $\tilde{u}_1$ follows essentially the same steps as Section 4, with the obvious modifications.

It follows that

$$\begin{aligned} B(\varepsilon^\alpha \phi_1, \nu) &= B(\varepsilon^\alpha (\phi_1 - \tilde{u}_\varepsilon + \varepsilon^\alpha \tilde{u}_1), \nu) \\ &+ B(\varepsilon^\alpha \tilde{u}_\varepsilon - \varepsilon^{2\alpha} \tilde{u}_1, \nu) \\ &= B(\varepsilon^\alpha \tilde{u}_\varepsilon, \nu) + O(\varepsilon^{2\alpha}) \|v\|_{H^1(\Omega_0)}. \end{aligned} \quad (91)$$

In a similar way, for any given function of the form $\tilde{h}$, suppose that $\hat{u}_\varepsilon$ is the solution of the following problem:

$$\begin{aligned} -\Delta \hat{u}_\varepsilon &= 0 \quad \text{in } \Omega_\varepsilon, \\ \frac{\partial \hat{u}_\varepsilon}{\partial \nu_\varepsilon} + q \left(x', \frac{x'}{\varepsilon^\beta} \right) \hat{u}_\varepsilon &= \hat{h} \left(x', \frac{x'}{\varepsilon^\beta} \right) \quad \text{on } \Gamma_2^\varepsilon, \\ \hat{u}_\varepsilon &= 0 \quad \text{on } \partial\Omega_\varepsilon \setminus \Gamma_2^\varepsilon. \end{aligned} \quad (92)$$

Analogously to ϕ_1 , let ϕ_2 be the solution of the homogenized problem

$$\begin{aligned} -\Delta \phi_2 &= 0 \quad \text{in } \Omega_0, \\ \frac{\partial \phi_2}{\partial \nu_0} + Q(x') \phi_2 &= \hat{H}(x') \quad \text{on } \Gamma_2, \\ \phi_2 &= 0 \quad \text{on } \partial\Omega_0 \setminus \Gamma_2, \end{aligned} \quad (93)$$

where $Q(x')$ and Γ_2^ε are defined in (8) and (3):

$$\begin{aligned} \hat{H}(x') \\ = \int_0^1 \cdots \int_0^1 [1 + |\nabla_{\eta'} T(x', \eta')|^2]^{1/2} \hat{h}(x', \eta') d\eta'. \end{aligned} \quad (94)$$

In view of Theorem 8,

$$\|\tilde{u}_\varepsilon - \phi_2 - \varepsilon^\beta \hat{u}_2\|_{H^1(\Omega_\varepsilon)} \leq C\varepsilon^\beta, \quad (95)$$

where $\hat{u}_2$ follows essentially the same steps as Section 4, with the obvious modifications.

It follows that

$$B(\varepsilon^\beta \phi_2, \nu) = B(\varepsilon^\beta \hat{u}_\varepsilon, \nu) + O(\varepsilon^{2\beta}) \|v\|_{H^1(\Omega_0)}. \quad (96)$$

We now write

$$\begin{aligned} B(u_\varepsilon - u_0 - \varepsilon^\alpha u_1 - \varepsilon^\beta u_2 + \varepsilon^\alpha \phi_1 + \varepsilon^\beta \phi_2, \nu) \\ = \varepsilon^\alpha \int_{\Gamma_1} \tilde{g} \nu dx' + \varepsilon^\beta \int_{\Gamma_2} \hat{h} \nu dx' - \varepsilon^\alpha \sum_{i=1}^3 I_i - \varepsilon^\beta \sum_{i=1}^3 L_i \\ + O(\varepsilon^{2\alpha} + \varepsilon^{2\beta}) \|v\|_{H^1(\Omega_0)}, \end{aligned} \quad (97)$$

where

$$\begin{aligned}
I_1 &= \int_{\Gamma_1} \nabla u_0 \cdot (\nabla_{x'} S, \nabla_{x'} S \cdot \nabla_{\xi'} S) v dx', \\
I_2 &= \int_{\Gamma_1^c} \frac{p}{P} S \frac{\partial}{\partial x_n} \left(\frac{\partial u_0}{\partial v_0} + P u_0 \right) \Big|_{\Gamma_1} v d\sigma_1, \\
I_3 &= - \int_{\Gamma_1} \nabla_x u_1 \cdot (\nabla_{x'} S, -1) v dx', \text{ and} \\
L_1 &= \int_{\Gamma_2} \nabla u_0 \cdot (\nabla_{x'} T, -\nabla_{x'} T \cdot \nabla_{\eta'} T) v dx', \\
L_2 &= - \int_{\Gamma_2^c} \frac{q}{Q} T \frac{\partial}{\partial x_n} \left(\frac{\partial u_0}{\partial v_0} + Q u_0 \right) \Big|_{\Gamma_2} v d\sigma_2, \\
L_3 &= - \int_{\Gamma_2} \nabla_x u_2 \cdot (\nabla_{x'} T, 1) v dx'.
\end{aligned} \tag{98}$$

Then we just need to take that

$$\begin{aligned}
\tilde{g} &= \sum_{i=1}^3 \tilde{I}_i \text{ and} \\
\hat{h} &= \sum_{i=1}^3 \hat{L}_i,
\end{aligned} \tag{99}$$

where

$$\begin{aligned}
\tilde{I}_1 &= \nabla u_0 \cdot (\nabla_{x'} S, \nabla_{x'} S \cdot \nabla_{\xi'} S), \\
\tilde{I}_2 &= \frac{p}{P} S \frac{\partial}{\partial x_n} \left(\frac{\partial u_0}{\partial v_0} + P u_0 \right) \Big|_{\Gamma_1} \left[1 + |\nabla_{\xi'} S(x', \xi')|^2 \right]^{1/2}, \\
\tilde{I}_3 &= -\nabla_x u_1 \cdot (\nabla_{x'} S, -1), \\
\hat{L}_1 &= \nabla u_0 \cdot (\nabla_{x'} T, -\nabla_{x'} T \cdot \nabla_{\eta'} T), \\
\hat{L}_2 &= -\frac{q}{Q} T \frac{\partial}{\partial x_n} \left(\frac{\partial u_0}{\partial v_0} + Q u_0 \right) \Big|_{\Gamma_2} \left[1 + |\nabla_{\eta'} T(x', \eta')|^2 \right]^{1/2}, \\
\hat{L}_3 &= -\nabla_x u_2 \cdot (\nabla_{x'} T, 1).
\end{aligned} \tag{100}$$

Combining all these terms (70)-(97) and choosing $v = u_\varepsilon - u_0 - \varepsilon^\alpha (u_1 - u_1|_{x_i=1}) - \varepsilon^\beta (u_2 - u_2|_{x_i=1}) + \varepsilon^\alpha \phi_1 + \varepsilon^\beta \phi_2$, the proof is completed. $\square$

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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