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Research Article

Filter Trust Region Method for Nonlinear Semi-Infinite Programming Problem

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We present a filter trust region method for nonlinear semi-infinite programming. Based on the discretization technique and motivated by the multiobjective programming, we transform the semi-infinite problem into a finite one. Together with the filter technique, we propose a modified method that avoids the merit function. Compared with the existing methods, our method is more flexible and easier to implement. Under some mild conditions, the convergent properties are proved. Moreover, the numerical results are reported in the end.

1. Introduction

Consider nonlinear semi-infinite programming problem (SIP) as follows:

$$\text{SIP}: \min_{x \in D \subseteq R^n} f(x) \quad \text{s.t. } g(x, w) \leq 0, \ w \in \Omega, \tag{1}$$

where D is a compact set, Ω is a closed set, $f : R^n \longrightarrow R$ is a twice continuously differentiable function, and $g : R^n \times \Omega \longrightarrow R$ is continuous and differentiable about variables x and w. For convenience, consider only the case of $\Omega = [a, b]$; a and b are real numbers.

The SIP problem arises in various applications such as approximation theory, optimal control, eigenvalue computation, mechanical stress of materials, pollution control, and statistical design. The research of SIP can be traced back to linear semi-infinite programming [1] and it was also proposed in researching Fritz-John conditions [2]. Then the SIP was formally proposed by Charnes, Coopper, and Kortanek in 1962 [3]. It is difficult to solve SIP because the constraints in SIP are infinite. Numerical methods for solving SIP may be divided into discretization methods and continuous methods. Discretization methods are based on nonlinear programming problems which are obtained by discretization of the original problem and incorporate some grid-refinement strategy [4, 5]. Our new method might be useful in the discretization context for semi-infinite programming since it drastically decreases the number of constraints.

For continuous methods, Hettich and Honstede present an iterative method which attempts to find a solution satisfying optimality conditions of the original problem. However, this method is only locally convergent and is very restrictive for practical use [6]. Jian et al. present a Norm-Relaxed Method of Feasible Directions algorithm, in order to obtain global convergence [7]. Coope and Watson have proposed a Sequential Quadratic Programming (SQP) method that utilizes an exact L_1 penalty function and global convergence obtained [8]. Then, a globally convergent SQP method for SIP is proposed which utilizes an exact L_∞ penalty function and trust region methods [9]. For optimization problems with smooth inequality constraints, the penalty function method is, in general, recognized as an efficient method. In [10, 11], in order to solve SIP, the smooth approximate functions in integral form are appended to the objective function by using the concept of the penalty function, but the error caused by taking the smooth approximation of the continuous inequality constraints cannot be avoided. Moreover, it is difficult to find a suitable penalty parameter. If the penalty parameter is too large, then any monotonic method would be forced to follow the nonlinear constraint manifold very closely, resulting in much shortened Newton steps and slow convergence.

To avoid using the above drawbacks, Fletcher and Leyyfer proposed a filter method in 2002 [12]. In their method, instead of combining the objective and constraint violation into a single function, they view the original problem as a biobjective optimization problem that minimizes objective function and constraint violation function. Consequently, filter technique has been employed in many approaches, for instance, SQP methods [13], interior point approaches [14], bundle techniques [15], and SIP [16]. But, in [16], the search direction is obtained in a linear equation. The structure of filter is complex.

Motivated by the above ideas, we propose a filter trust region algorithm. We transform the SIP into a finite problem based on a modified discretion method. Then, the search direction is obtained by the modified trust region quadratic subproblem. Moreover, without the merit function, we adopt the filter technique to decide whether a new iteration point is acceptable or not. Compared with the existing methods, there are two main advantages in our presented algorithm: (1) the modified quadratic subproblem is always feasible; (2) unlike the continuous method, the penalty parameter is avoided in our presented method, and actually the algorithm has a certain nonmonotonicity property. Under some wild conditions, the convergent properties of the proposed method are proved. The numerical results show the effectiveness of our approach.

The remainder of this paper is organized as follows. The algorithm is described in Section 2. In Section 3, we analyze the convergent properties of proposed algorithm. Numerical results are reported in the final section.

2. Description of Algorithm

For solving SIP, discretization of the original problem SIP, consider the following problem:

$$SIP_q : \min_{x\in R^n} f(x) \quad \text{s.t. } g(x,w) \le 0, w \in \Omega_q, \tag{2}$$

where $\Omega_q = \{w \in \Omega \mid w = a + i\cdot(b-a)/q,\ i = 0,1,2,\dots,q\}$ is the discrete subset of Ω that appeared in SIP problem (1). q is a positive integer (in general, $q \ge 100$) that represents the discrete level. In practice, depending on the length of [a, b], a and b are real numbers, suppose $q \ge 100|b-a|$.

For any $x \in R^n$, denote $\varphi(x) = \max\{0, g(x,w), w \in \Omega_q\}$,

$\Omega^-(x) = \{w \in \Omega_q \mid g(x,w) \le 0\}$, $\Omega^+(x) = \{w \in \Omega_q \mid g(x,w) > 0\}$,

$\Omega_0^-(x) = \{w \in \Omega^-(x) \mid g(x,w) = 0\}$, $\Omega_0^+(x) = \{w \in \Omega^+(x) \mid g(x,w) = \varphi(x)\}$, $\Omega_0(x) = \Omega_0^-(x) \cup \Omega_0^+(x)$, $X = \{x_k \mid g(x_k,w) \le 0, w \in \Omega_q\}$.

Suppose the current iteration point is $x_k \in R^n$, $\Omega_k^-(x) = \{w \in \Omega^q \mid g(x_k,w) \le 0\}$, $\Omega_k^+(x) = \{w \in \Omega^q \mid g(x_k,w) > 0\}$, $\varphi_k = \max\{0, g(x_k,w), w \in \Omega_q\}$. $\Omega_k = \Omega_k^-(x) \cup \Omega_k^+$.

In order to improve the degree of accuracy of optimal solution of SIP, we observed that trust region method is exceedingly important for ensuring global convergence while retaining fast local convergence in optimization algorithms, so the trust region condition is added to the quadratic subproblem of (2).

For the current k-th iteration, d_k is obtained by the following modified trust region quadratic subproblem (QP):

$$\begin{aligned}
\text{(QP)}\ \min\quad & \gamma_0 z + \frac{1}{2} d^T H_k d \\
\text{s.t.}\quad & \nabla f(x_k)^T d \le \gamma_0 z, \\
& g(x_k,w) + \nabla_x g(x_k,w)^T d \le \gamma_w z, \\
& \qquad\qquad \forall w \in \Omega_k^-, \\
& g(x_k,w) + \nabla_x g(x_k,w)^T d \le \gamma_w z + \varphi_k, \\
& \qquad\qquad \forall w \in \Omega_k^+, \\
& \|d\| \le \Delta_k,
\end{aligned} \tag{3}$$

where $\gamma_0, \gamma_w > 0$ are positive constants, H_k is a symmetric positive definite matrix, and $\Delta_k \ge 0$ is the k-th trust region radius.

Denote

$$q^{(k)}(z,d) = \gamma_0 z + \frac{1}{2} d^T H_k d. \tag{4}$$

Define the actual reduction of $f(x)$ as

$$f(x_k) - f(x_k + d_k). \tag{5}$$

The predict reduction of $f(x)$ as

$$q^{(k)}(0,0) - q^{(k)}(z_k, d_k). \tag{6}$$

Here $q^{(k)}(z_k,d_k) = \gamma_0 z_k + (1/2) d_k^T H_k d_k$.

Define

$$\rho_k = \frac{Ared_k}{Pred_k} = \frac{f(x_k) - f(x_k + d_k)}{q^{(k)}(0,0) - q^{(k)}(z_k,d_k)}. \tag{7}$$

The parameter ρ_k is used to decide whether the predicted reduction of $f(x)$ is close to the actual reduction of $f(x)$ or not.

In the proposed algorithm, the filter technique is used to decide whether a trial point is acceptable or not.

Definition 1. A pair (h_k, f_k) is dominated by (h_j, f_j) if and only if $h_k \le h_j$ and $f_k \le f_j$ for each $j \ne k$.

Definition 2. A filter set $\mathcal{F}$ is a set of pairs (h, f) such that no pair dominates any other.

To ensure the convergence, some additional conditions are required to decide whether to accept a trial point to the filter or not. The traditional acceptable criterion is as follows.

Definition 3. A trial point x is called acceptable to the filter $\mathcal{F}$ if and only if either

$$h(x) \le h_j$$
$$or\ \ f(x) \le f_j, \tag{8}$$
$$\forall (h_j, f_j) \in \mathcal{F}.$$

To avoid convergence to infeasible limit points, we add an envelope around the current filter. So criterion (8) can be changed to the following criterion.

Definition 4. A trial point x is called acceptable to the filter $\mathcal{F}$ if and only if either

$$h(x) \le \beta h_j$$
$$or\ \ f(x) \le f_j - \gamma h(x), \tag{9}$$
$$\forall (h_j, f_j) \in \mathcal{F}$$

where $0 < \gamma < \beta < 1$ are constants. In practice, β is close to 1 and γ is close to 0.

In the current k-th iteration, let $h_k = \max\{0, g(x_k, w), w \in \Omega_q\}$, $f_k = f(x_k)$. Then our algorithm (FTR) is presented as follows.

FTR Algorithm

Step 1 (initialization). Given $x_0 \in R^n$, Δ_0, and $\varepsilon > 0$ set k=0, $\mathcal{F} = \Phi$, $q \ge 100|b-a|$, $\gamma > 0$, $\beta > 0$, $0 \le \varepsilon < 1$, $\rho_0 > 0$, $\gamma_0 > 0$, and $\gamma_w > 0$.

Step 2. Solve (3) to obtain d_k, z_k. If $|z_k| \le \varepsilon$, stop; otherwise, set $x_k^+ = x_k + d_k$, and go to Step 2.

Step 3. Calculate $\rho_k = Ared_k / Pred_k = (f(x_k) - f(x_k + d_k))/(q^{(k)}(0,0) - q^{(k)}(z_k, d_k))$, where $q^{(k)}(z_k, d_k) = \gamma_0 z_k + (1/2) d_k^T H_k d_k$.

Step 4. If $\rho_k \ge \rho_0$, set k:=k+1, $\Delta_k = 2\Delta_k$ and update parameter H and go to Step 1; otherwise go to Step 4.

Step 5. If (h_k, f_k) is acceptable for current filter, add the (h_k, f_k) to filter, set k:=k+1, $\Delta_k = 2\Delta_k$, update parameter H, and go to Step 1; otherwise, set k:=k, $\Delta_k = (1/2)\Delta_k$ and go to Step 1.

Suppose that QP has local optimal solution (d_k, z_k); i.e., (d_k, z_k) is a Karush-Kuhn-Tucker (KKT) point of QP. So there exists multiplier $\mu_k \in R^n$ and $\mu_w^k \in R^{q+1}$, $w \in \Omega_q$, satisfy:

$$H_k d_k + \mu_k \nabla f(x_k) + \sum_{w \in \Omega_q} \mu_w^k \nabla_x g(x_k, w) + \mu_d d = 0, \tag{10}$$

$$\mu_k \gamma_0 + \sum_{w \in \Omega_q} \mu_w^k \gamma_w = \gamma_0, \tag{11}$$

$$\nabla f(x_k)^T d - \gamma_0 z \le 0, \tag{12}$$

$$g(x_k, w) + \nabla_x g(x_k, w)^T d - \gamma_w z \le 0, \quad \forall w \in \Omega_k^-, \tag{13}$$

$$g(x_k, w) + \nabla_x g(x_k, w)^T d - \gamma_w z - \varphi_k \le 0, \quad \forall w \in \Omega_k^+, \tag{14}$$

$$\|d\| \le \triangle_k, \tag{15}$$

$$\mu_k \left[\nabla f(x_k)^T d - \gamma_0 z\right] = 0, \tag{16}$$

$$\mu_w^k \left[g(x_k, w) + \nabla_x g(x_k, w)^T d - \gamma_w z\right] = 0, \quad \forall w \in \Omega_k^-, \tag{17}$$

$$\mu_w^k \left[g(x_k, w) + \nabla_x g(x_k, w)^T d - \gamma_w z - \varphi_k\right] = 0, \quad \forall w \in \Omega_k^+, \tag{18}$$

$$\mu_d (\|d\| - \triangle_k) = 0, \tag{19}$$

$$\mu_k, \mu_w^k, \mu_d \ge 0. \tag{20}$$

Similarly, if x_k is a KKT point for problem SIP_q, then there exists a multiplier vector $\mu_k \in R^n$ and $\lambda_k \in R^{q+1}$ such that the following formulas hold:

$$\mu_k \nabla f(x_k) + \sum_{w \in \Omega} \lambda_k \nabla_x g(x_k, w) = 0, \tag{21}$$

$$g(x_k, w) \le 0, \quad w \in \Omega, \tag{22}$$

$$\lambda_k g(x_k, w) = 0, \tag{23}$$

$$\lambda_k \ge 0. \tag{24}$$

The following lemma shows that our algorithm is well defined.

Assumption A1. $f : R^n \longrightarrow R$ and $g : R^n \times \Omega_q \longrightarrow R$ are continuous and differentiable.

Lemma 5. *Assume Assumption A1 holds and $x_k \in R^n$; then (i) subproblem (QP) has a unique solution; (ii) (z_k, d_k) is an optimal solution of subproblem (QP) if and only if (z_k, d_k) is a Karush-Kuhn-Tucker (KKT) point of it.*

Proof. (i) Obviously, $(z_k, d_k) = (0,0)$ is a feasible solution of subproblem (QP) since $g(x_k, w) \le 0$ for all $w \in \Omega_k^-$ as well as $g(x_k, w) \le \varphi_k$ for all $w \in \Omega_k^+$; beside $(z_k, d_k) = (0,0)$ also satisfy $\nabla f(x_k)^T d \le \gamma_0 z$ and $\|d\| \le \Delta_k$, so the feasible set of subproblem (QP) is not empty. On the other hand, for each feasible solution (z, d) of subproblem (QP), in view of the first constraint $\nabla f(x_k)^T d \le \gamma_0 z$ of subproblem (QP), we know that the value of objective function of subproblem (QP) satisfies the following inequality:

$$\gamma_0 z_k + \frac{1}{2} d_k^T H_k d_k \ge \nabla f(x_k)^T d_k + \frac{1}{2} d_k^T H_k d_k. \tag{25}$$

Therefore, the value of objective function of subproblem (QP) is bounded from below because of the symmetric positive definite property of B_k; that is, there exists a constant a such that

$$\inf\left\{\gamma_0 z_k + \frac{1}{2}d_k^T H_k d_k : (z_k, d_k) \in X_k\right\} = a \in (-\infty, +\infty). \tag{26}$$

Here the feasible set

$$X_k = \left\{(z_k, d_k) : \nabla f(x_k)^T d_k \le \gamma_0 z_k;\ g(x_k, w) + \nabla_x g(x_k, w)^T d_k \le \gamma_w z_k,\ w \in \Omega_k^-;\ g(x_k, w) + \nabla_x g(x_k, w)^T d_k \le \gamma_w z_k + \varphi_k,\ w \in \Omega_k^+;\ \|d_k\| \le \Delta_k\right\} \tag{27}$$

According to Bolzano-Weierstrass theorem, bounded series must have a convergent subsequence. Therefore, there exists a sequence $\{z_{k_j}, d_{k_j}\} \subseteq X_k$ such that

$$\gamma_0 z_{k_j} + \frac{1}{2}d_{k_j}^T H_k d_{k_j} \longrightarrow a, \quad j \longrightarrow \infty. \tag{28}$$

Moreover, for j large enough, we have

$$\begin{aligned} a + \frac{1}{2} &\ge \gamma_0 z_{k_j} + \frac{1}{2}d_{k_j}^T H_k d_{k_j} \\ &\ge \nabla f(x_k)^T d_k + \frac{1}{2}d_{k_j}^T H_k d_{k_j}. \end{aligned} \tag{29}$$

Note that the matrix B_k is positive definite, as well as the boundedness of $\{d_{k_j}\}_j$, so the sequence $\{z_{k_j}\}_j$ is bounded. Therefore, there exists a subset K such that

$$\begin{aligned} \lim_{j\in K}\left(z_{k_j}, d_{k_j}\right) &= (z_k, d_k) \in X_k, \\ a &= \gamma_0 z_k + \frac{1}{2}d_k^T H_k d_k. \end{aligned} \tag{30}$$

Hence (z_k, d_k) is an optimal solution of subproblem (QP). In addition, the subproblem (QP) is equal to the following constrained optimization:

$$\begin{aligned} \min_{d_k\in R^n} \quad & \left\{\frac{1}{2}d_k^T H_k d_k + \max\left\{\nabla f(x_k)^T d_k; \frac{\gamma_0}{\gamma_w}\left(g(x_k, w) + \nabla_x g(x_k, w)^T d_k\right),\ w \in \Omega_k^-; \frac{\gamma_0}{\gamma_w}\left(g(x_k, w) + \nabla_x g(x_k, w)^T d_k - \varphi_k\right),\ w \in \Omega_k^+\right\}\right\} \\ \text{s.t.} \quad & \|d_k\| \le \Delta_k. \end{aligned} \tag{31}$$

Obviously, the first term of the objective function is strictly convex about variable d_k and the second term is convex about variable d_k; thus the objective function of the problem above is strictly convex [18]. Besides, the constrained function of (31) is a convex function. Combining the convexity of R^n, we obtain problem (31) which is a convex programming and its optimal solution d_k is unique. Therefore, the optimal solution of subproblem (QP) is unique.

(ii) If (z_k, d_k) is a KKT point for subproblem (QP), then it is an optimal solution of subproblem (QP) since subproblem (QP) is a convex programming. Conversely, if (z_k, d_k) is an optimal solution of subproblem (QP), note that Abadie's Constraint Qualification [19] holds automatically since the constraints of subproblem (QP) are linear; then (z_k, d_k) is a KKT point for subproblem (QP). The proof is completed. □

From Lemma 5, we know QP has only optimal solution (z_k, d_k); i.e., QP has only KKT point. So there exists multiplier $\mu_k \in R^n$ and $\mu_w^k \in R^{q+1}, w \in \Omega_q$, satisfy (10)-(20).

Similarly, if x_k is a KKT point for problem SIP_q, then there exists a multiplier vector $\mu_k \in R^n$ and $\lambda_k \in R^{q+1}$ satisfy (21)-(24).

Assumption A2. Assume the Mangasarian-Fromovitz Constraint Qualification (MFCQ) holds at any $x \in R^n$; i.e., there exists a vector d such that $\{d \mid \nabla_x g(x, w)^T d < 0,\ \forall w \in \Omega_0(x)\} \ne \emptyset$.

Lemma 6. *Assume Assumptions A1 and A2 hold, (z_k, d_k) is an optimal solution of subproblem (QP); then (i) $\gamma_0 z_k + (1/2)d_k^T H_k d_k \le 0, z_k \le 0$; (ii) $z_k = 0$ if and only if $d_k = 0$ if and only if x_k is a KKT point for subproblem (QP).*

Proof. (i) From the fact that $(z_k, d_k) = (0, 0)$ is a feasible solution of subproblem (QP) and (z_k, d_k) is an optimal solution of subproblem, then $\gamma_0 z_k + (1/2)d_k^T H_k d_k \le 0$. Besides B_k is positive definite, and $\gamma_0 > 0$; one has $z_k \le -(1/2\gamma_0)d_k^T H_k d_k \le 0$.

(ii) From (i), we have $\gamma_0 z_k + (1/2)d_k^T H_k d_k \le 0$. If $z_k = 0$, then $(1/2)d_k^T H_k d_k \le 0$; taking into account the positive definite property of B_k, one has $d_k = 0$. Conversely, if $d_k = 0$, we denote the feasible set of problem SIP_q:

$$X = \left\{x_k \mid g(x_k, w) \le 0,\ w \in \Omega_q\right\}, \tag{32}$$

and two cases $x_k \in X$ and $x_k \notin X$ are considered, respectively. Firstly, suppose that $x_k \in X$, i.e., $\varphi_k = 0$, from the first constraint of subproblem $\nabla f(x_k)^T d \le \gamma_0 z$, we have $0 = \nabla f(x_k)^T d_k \le \gamma_0 z_k$. Combining with the result that $z_k \le 0$ and $\gamma_0 > 0$, we know that the conclusion $z_k = 0$ holds. Now suppose that $x_k \notin X$, i.e., $\varphi_k > 0$, then the set $\Omega_k^+(x_k) \ne \emptyset$ and let $w_0 \in \Omega_q$ such that $g(x, w_0) = \varphi_k$. So from the constraints $g(x_k, w) + \nabla_x g(x_k, w)^T d_k \le \gamma_w z_k + \varphi_k$ of subproblem (QP) one gets

$$z_k \geq \frac{g(x_k, w) - \varphi_k + \nabla_x g(x_k, w)^T d_k}{\gamma_w} = \frac{\nabla_x g(x_k, w)^T d_k}{\gamma_w} = 0. \tag{33}$$

Together with $z_k \leq 0$, one gets $z_k = 0$. Conclusion (ii) is obtained. □

Lemma 7. *Assume $z_k = 0$ and $x_k \notin X$, and X is defined as (32); then the set $\{d \mid \nabla_x g(x,w)^T d < 0, \forall w \in \Omega_0(x)\} = \emptyset$.*

Proof. In view of the definition of X and $x_k \notin X$, it follows that $\Omega_k^+ \neq \emptyset$. Now we prove by contradiction that the first conclusion of Lemma 7 holds. Suppose that there exists a vector $\overline{d} \in R^n$ such that

$$\nabla_x g(x, w)^T \overline{d} < 0, \quad \forall w \in \Omega_0(x). \tag{34}$$

the following result holds if the positive parameter λ is small enough

$$\overline{z}(\lambda) = \max\left\{\frac{1}{\gamma_0}(\nabla f(x_k))^T \cdot (\lambda\overline{d}); \frac{1}{\gamma_w}\left(g(x_k, w) + \nabla_x g(x_k, w)^T(\lambda\overline{d})\right), w \in \Omega_k^-; \frac{1}{\gamma_w}\left(g(x_k, w) + \nabla_x g(x_k, w)^T(\lambda\overline{d}) - \varphi_k\right), w \in \Omega_k^+\right\} \tag{35}$$

Obviously, for $\lambda > 0$ small enough, $(\overline{z}, \lambda\overline{d})$ is a feasible solution of (3); furthermore, $\gamma_0\overline{z} + (1/2)(\lambda\overline{d})^T B_K(\lambda\overline{d}) < 0$. Therefore

$$z_k \leq z_k + \frac{1}{2}(d_k)^T H_k(d_k) \leq \overline{z} + \frac{1}{2\gamma_0}\left(\lambda\overline{d}\right)^T B_K\left(\lambda\overline{d}\right) < 0. \tag{36}$$

This is a contradiction to $z_k = 0$. Hence Lemma 7 follows. The proof is completed. □

Lemma 8. *Assume Assumptions A1 and A2 hold; if $x_k \notin X$, X is defined as (32), then $z_k < 0$.*

Proof. By contradiction, suppose that $z_k < 0$ does not hold; then $z_k = 0$, and we have $\{d \mid \nabla_x g(x,w)^T d < 0, \forall w \in \Omega_0(x)\} = \emptyset$ from Lemma 7; this conclusion contradicts to Assumption A2. The proof is completed. □

Lemma 9. *Assume Assumptions A1 and A2 hold. Then the multiplier sequence $\{\mu_k\}$ and $\mu_w^k (w \in \Omega_q)$ are both bounded.*

Proof. In view of the formula $\gamma_0 = \mu_k\gamma_0 + \sum_{w\in\Omega_q}\mu_w^k\gamma_w$ and the nonnegative properties of multipliers μ_w^k, γ_w and parameter $\gamma_0 > 0$, we can obtain the boundedness of sequences $\{\mu_k\}$ and $\{\mu_w^k\}$. □

Lemma 10. *Assume Assumptions A1 and A2 hold and (z_k, d_k) is an optimal solution of subproblem (QP); then $z_k = 0$ if and only if x_k is a KKT point of SIP_q.*

Proof. Suppose that $z_k = 0$, we can get $d_k = 0, x_k \in X, \varphi_k = 0$ by Lemma 6, so $\Omega_k^+(x_k) = \emptyset$, $\Omega_k^-(x_k) = \Omega_q$, and the KKT conditions (10)-(20) can be rewritten as

$$\mu_k \nabla f(x_k) + \sum_{w\in\Omega_q}\mu_w^k \nabla_x g(x_k, w) = 0, \tag{37}$$

$$\mu_k\gamma_0 + \sum_{w\in\Omega_q}\mu_w^k\gamma_w = \gamma_0, \tag{38}$$

$$g(x_k, w) \leq 0, \quad \forall w \in \Omega_q, \tag{39}$$

$$\mu_w^k g(x_k, w) = 0, \quad \forall w \in \Omega_q, \tag{40}$$

$$\mu_k, \mu_w^k \geq 0. \tag{41}$$

In view of the formula $\gamma_0 = \mu_k\gamma_0 + \sum_{w\in\Omega_q}\mu_w^k\gamma_w$ and the nonnegative properties of multipliers μ_w^k, γ_w and parameter $\gamma_0 > 0$, we can obtain the boundedness of sequences $\{\mu_k\}$ and $\{\mu_w^k\}$. We prove that $\mu_k \neq 0$. By contradiction, suppose that $\mu_k = 0$, then in view of the equation $\gamma_0 = \mu_k\gamma_0 + \sum_{w\in\Omega_q}\mu_w^k\gamma_w$ one can get $\mu_w^k \neq 0$; i.e., $g(x_k, w) < 0$, one knows that $\mu_w^k = 0$ from $\mu_w^k g(x_k, w) = 0$, $\forall w \in \Omega_q$. Therefore, $\mu_k\nabla f(x_k) + \sum_{w\in\Omega_q}\mu_w^k\nabla_x g(x_k, w) = 0$ becomes

$$\sum_{w\in\Omega_q}\mu_w^k \nabla_x g(x_k, w) = 0. \tag{42}$$

Choosing a vector d^* satisfying MFCQ at the point x_k and multiplying the equality above and the constraints of subproblem for $w \in \Omega_0(x)$ by d^*, we obtain

$$\sum_{w\in\Omega_q}\mu_w^k \nabla_x g(x_k, w) d^* = 0, \quad \nabla_x g(x_k, w) d^* < 0. \tag{43}$$

This is a contradiction and the conclusion that $\mu_k \neq 0$ is true.

Let

$$\lambda_k = \frac{\mu_w^k}{\mu_k}, \quad w \in \Omega_q(x_k), \tag{44}$$

and then the KKT condition of SIP_q holds at x_k with the multiplier vector λ_k, so x_k is a KKT point for SIP_q.

Now, to prove the necessity of theorem, note that if x_k is a KKT point for SIP_q, namely, (37)-(41) holds, then $(0,0)$ satisfies KKT conditions (37)-(41) of SIP_q with multipliers

$$\mu_k = \frac{\gamma_0 - \sum_{w\in\Omega_q}\mu_w^k\gamma_w}{\gamma_0}, \tag{45}$$

$$\mu_w^k = \lambda_k\mu_k. \tag{46}$$

From Lemma 5 we obtain the uniqueness of the KKT point. So the uniqueness of the KKT point shows that $z_k = 0$. The whole proof is completed. □

From Lemma 10, $z_k = 0$ can be used as terminating condition.

Lemma 11. *FTR algorithm cannot cycle infinitely many times among the inner cycle (Step 1⟶ Step 2⟶ Step 4⟶ Step 1).*

Proof. Suppose the contrary, then it follows from the rules of FTR algorithm that one has $\Delta_k \longrightarrow 0$, $d_k \longrightarrow 0$, while $\rho_k \leq \rho_0$ is maintained. Consequently, we have

$$\begin{aligned}\left|\rho_k - 1\right| &= \frac{\left|Ared_k - Pred_k\right|}{\left|Pred_k\right|} \\ &\leq \frac{o\left\|d_k\right\|^2}{\left|\nabla f\left(x_k\right)^T d_k + \frac{1}{2} d_k^T H_k d_k\right|}.\end{aligned} \tag{47}$$

so

$$\begin{aligned}0 \leq \left|\rho_k - 1\right| &= \frac{\left|Ared_k - Pred_k\right|}{\left|Pred_k\right|} \\ &\leq \frac{o\left\|d_k\right\|^2}{\left|\nabla f\left(x_k\right)^T d_k + \frac{1}{2} d_k^T H_k d_k\right|} \longrightarrow 0, \quad as\ \Delta_k \longrightarrow 0.\end{aligned} \tag{48}$$

This indicates that we eventually have $\rho_k > \rho_0$. This is a contradiction and the desired conclusion follows. This proof is completed. □

3. Convergent Properties

In this section, under appropriate conditions, we establish the convergence of FTR algorithm. For this purpose, the following basic assumption is necessary.

Assumptions A3. Assume the sequence $\{x_k\}$ generated by FTR algorithm is bounded.

Assumptions A4. Assume there exists $a, b > 0$, and $a, b \in N_+$, such that $a\|d\|^2 \leq d^T H_k d \leq b\|d\|^2$.

To analyze the convergence property of FTR algorithm, we need the following lemma.

Lemma 12. *Assume Assumptions A3 and A4 hold; then, sequences $\{d_k\}$ and $\{z_k\}$ generated by FTR algorithm are bounded.*

Proof. Firstly, prove $\{d_k\}$ is bounded. By the contradiction, suppose $\{d_k\}$ is unbounded, there exists infinite index set K, such that $\|d_k\| \longrightarrow +\infty$ as $k \in K, k \longrightarrow +\infty$. Together with $z_k \leq 0$, this is a contradiction with $\nabla f(x_k)^T d_k \leq \gamma_0 z$, so $\{d_k\}$ is bounded.

Now, because of $\{d_k\}$ is bounded and $f(x)$ is continuous and differentiable, thus $\{\nabla f(x_k)^T d_k\}$ is bounded. According to $\nabla f(x_k)^T d_k \leq \gamma_0 z$ and $\gamma_0 > 0$ one has

$$\frac{\nabla f\left(x_k\right)^T d_k}{\gamma_0} \leq z_k; \tag{49}$$

furthermore, together with $z_k \leq 0$, $\{z_k\}$ is bounded. This proof is completed. □

The following result is obtained.

Theorem 13. *Assume Assumptions A3 and A4 hold, K is an infinite iterative index set; if $d_k \longrightarrow 0$, $k \in K$, then any accumulation point of $\{x_k\}$, $k \in K$, generated by FTR algorithm is the KKT point of SIP_q.*

Proof. Suppose x^* is the accumulation point of $\{x_k\}, k \in K$. Denote

$$\begin{aligned}L_0^-\left(x_k\right) &= \left\{w \in \Omega_q^- \mid g\left(x_k, w\right) + \nabla g\left(x_k, w\right)^T d_k \right.\\ &\left.= \gamma_w z_k\right\} \\ L_0^+\left(x_k\right) &= \left\{w \in \Omega_q^+ \mid g\left(x_k, w\right) + \nabla g\left(x_k, w\right)^T d_k \right.\\ &\left.= \gamma_w z_k + \varphi_k\right\} \\ L_0\left(x_k\right) &= L_0^-\left(x_k\right) \cup L_0^+\left(x_k\right)\end{aligned} \tag{50}$$

$L_0(x_k)$ is the subset of Ω_q.

First, we prove $z_k \longrightarrow 0, k \in K$ as $d_k \longrightarrow 0, k \in K$.

If $\varphi_k = 0$, then $x_k \in X_q$; besides, $\nabla f(x_k)^T d_k \leq \gamma_0 z_k$, $z_k \leq 0$, so $\nabla f(x_k)^T d_k/\gamma_0 \leq z_k \leq 0$; thus $z_k \longrightarrow 0$ as $d_k \longrightarrow 0, k \in K$.

If $\varphi_k > 0$, then $\Omega_q^+ \neq \emptyset$; there exists $w_0 \in \Omega_q^+$ such that $\nabla g(x_k, w_0)^T d^k \leq \gamma_{w_0} z_k \leq 0, k \in K$. Thus $z_k \longrightarrow 0, k \in K$ as $d_k \longrightarrow 0, k \in K$.

Then, from Lemmas 9 and 10, the conclusion is obtained. □

Because we use the filter technique in FTR algorithm, the following lemma is necessary.

Lemma 14. *Assume there are infinitely many points added to the filter. Then*

$$\lim_{i \longrightarrow \infty} h\left(x_i\right) = 0. \tag{51}$$

Proof. If the theorem is not true, there would have infinite components in K_1, which is defined as follows:

$$K_1 = \left\{k \mid h_k > \varepsilon\right\}. \tag{52}$$

We assume that $|f_k| \leq M$ for all k without loss of generality, where M is a positive constant. The following two cases are considered, respectively.

(i) If $\min_{i \in K_1}\{f_i(x)\}$ exists, let $f_{min} = \min_{i \in K_1}\{f_i(x)\}$, $h_{min} = \min_{i \in K_1} h(x) = \min_{i \in K_1}\|g^+\|$, and $g^+ = \max\{0, g(x_k, w)\}$. Then, according to the definition of the filter, the other components, which lie behind x_{min} in the filter, satisfy

$$h_k \leq \beta h_{min}, f_k \geq f_{min}. \tag{53}$$

Then, all the filter points, which enter the filter behind x_{min} can be covered with a square, whose area is no more than $2Mh_{min}$. We consider the area lies to the southwest of the filter in this square. When a new point x_{np} enters the filter, the next point x_{np+1} should lie to the southwest of the points in the filter F_{np} and the area which lies to the southwest of F_{np} in the square is smaller than that of F_{np+1}. Therefore, we think that the area is reduced if a new point enters the filter. When a point is added to the filter, its h is less than every point, which lies to the left of this point, to more than $(1-\beta)\varepsilon$. Its f is less than every point, which lies to the right of this point, to more than $\gamma\varepsilon$. Therefore, the area of this square, more than $(1-\beta)\gamma\varepsilon^2$, will be reduced. Thus, the area will be reduced to 0 after finite times. When the area is zero, it means that a point does not enter K_1, which is contradicted with the infiniteness of K_1.

(ii) If $\min_{i\in K_1}\{f_i(x)\}$ does not exist, because $f(x)$ and $g(x,w)$ are continuous and differentiable, let

$$f_{min} = \inf_{i\in K_1} \{f_i(x)\}. \tag{54}$$

From the definition of $\inf f(x)$, there exists f_{np} such that $f_{np} \geq f_{min}$ and $f_{np} \leq f_{min} + \gamma\varepsilon$. Then, according to the definition of the filter, the other components, which lie behind x_{np} in the filter, satisfy

$$\begin{aligned} h_k &\leq \beta h_{np}, \\ f_k &\geq f_{np} - \gamma\varepsilon. \end{aligned} \tag{55}$$

Using the same techniques as that in (i), the conclusion can also obtained which is contradicted with the infiniteness of K_1. So K_1 is infinity; the result is gotten.

Thus, the conclusion is obtained. □

As for the case of finitely many points added to the filter, it is apparent that the following result is true.

Lemma 15. *Assume there are finitely many points added to the filter. Then $h(x_k) = 0$, where k is the number of points added to the filter.*

Proof. From the terminating condition of Step 2 in FTR algorithm, one has $z_k = 0$ and also $d_k = 0$; furthermore $h(x_k) = 0$. Thus, if there are finitely many points added to the filter, the index of the last point is the number of points added to the filter. The proof is completed. □

The following result is obtained.

Theorem 16. *Assume Assumptions A3 and A4 hold; if $\{x_k\}$ generated by FTR algorithm is a bounded sequence, then $\{x_k\}$ has an accumulation point satisfying the KKT conditions of SIP_q.*

Proof. Suppose x^* is an accumulation point of sequence $\{x_k\}$, by Lemma 14, $\lim_{i\longrightarrow\infty} h(x_i) = 0$, so there exists K such that $h(x_i) \longrightarrow 0, i \longrightarrow \infty,\ i \in K$. Taking the limits about KKT conditions (10)-(20) of subproblem and $h(x_i)$, together with Lemma 9, we have $\varphi(x^*) = 0$, $h(x^*) = 0$ so $x^* \in X_q$; thus x^* is the KKT point of SIP_q. The proof is completed. □

4. Numerical Results

In this section, we report our preliminary numerical test results. FTR algorithm described in Section 2 was then implemented in Matlab R2016a. We compared the performance of the FTR algorithm with [17]. The following four problems were tested. Throughout the computational experiments, the parameters used in algorithm were $q = 100$, $\gamma = 0.95$, $\beta = 0.05$, $\varepsilon = 10^{-4}$, $\rho_0 = 0.75$, $\gamma_0 = 1$, and $\gamma_w = 1$.

Problem 1.

$$\begin{aligned} \min_{x\in R^n} \quad & f(x) = 1.21 e^{x_1} + e^{x_2} \\ \text{s.t.} \quad & g(x,w) = w - e^{x_1+x_2} \leq 0, \quad w \in \Omega = [0,1]. \\ & x_0 = (0.8, 0.9)^T. \end{aligned} \tag{56}$$

Problem 2.

$$\begin{aligned} \min_{x\in R^n} \quad & f(x) \\ & = \left(x_1 - 2x_2 + 5x_2^2 - x_2^3 - 13\right)^2 \\ & \quad + \left(x_1 - 14x_2 + x_2^2 + x_2^3 - 29\right)^2 \\ \text{s.t.} \quad & g(x,w) = x_1^2 + 2x_2 w^2 + e^{x_1+x_2} - e^{w} \leq 0, \\ & \qquad w \in \Omega = [0,50]. \\ & x_0 = (1,1)^T. \end{aligned} \tag{57}$$

Problem 3.

$$\begin{aligned} \min_{x\in R^n} \quad & f(x) = \frac{1}{3}x_1^2 + \frac{1}{2}x_1 + x_2^2 \\ \text{s.t.} \quad & g(x,w) = \left(1 - x_1^2 w^2\right)^2 - x_1 w^2 - x_2^2 + x_2 \leq 0, \\ & \qquad w \in \Omega = [0,1]. \\ & x_0 = (-2,-2)^T. \end{aligned} \tag{58}$$

Problem 4.

$$\begin{aligned} \min_{x\in R^n} \quad & f(x) = \sum_{i=1}^{n} e^{x_i} \\ \text{s.t.} \quad & g(x,w) = \frac{1}{1+w^2} - \sum_{i=1}^{n} x_i w^{i-1} \leq 0, \\ & \qquad w \in \Omega = [-1,1]. \\ & x_0 = (1, 0 \cdots 0, \cdots, 0)^T. \end{aligned} \tag{59}$$

Table 1: The comparison between FTR algorithm and algorithm in [17] with same conditions.

	FTR Algorithm				Algorithm in [17]	
	n	Iter	x^*	$f(x^*)$	Iter	$f(x^*)$
Pro1	2	2	$(-0.0952, 0.0953)^T$	2.200132	3	2.200142
Pro2	2	4	$(0.7199, -1.1504)^T$	97.159372	9	97.16956
Pro3	2	2	$(0.7500, -0.6181)^T$	0.194599	17	0.194726
Pro4	15	2	$(1.0001, -0.0193, -0.0366, -0.0370, \cdots - 0.0370)^T$	16.22755	9	16.22757
Pro4	20	2	$(1.0001, -0.0193, -0.0266, -0.0267, \cdots - .00267)^T$	21.22514	7	21.22519

In Table 1, the problem is denoted by Pro; e.g., Problem 1 is denoted by Pro1, the dimension of x is denoted by n, the number of iterations is denoted by Iter, the optimal solution is denoted by x^*, and the optimal value is denoted by $f(x^*)$.

In Table 1, the limited numerical results show that the proposed algorithm can give an accurate solution quickly (the iterate number is less than that in [17]), which indicates that the algorithm herein is effective. Moreover, it has been observed that the computational results are not very sensitive to the choice of the initial point and the parameter n, showing that the proposed algorithm is stable.

5. Conclusion

In this paper, we present a filter trust region method. Based on the discrete technique, we transform nonlinear semi-infinite programming into a finite problem. And the search direction is obtained by a modified trust region quadratic subproblem which is always feasible. Besides, we adopt the filter technique to decide whether a new iteration point is acceptable or not, so the penalty parameter which is always used in most of the existing methods is avoided. Under some wild conditions, the global convergence of the proposed method was obtained. The main advantage of the presented method is that our method is more flexible and easier to implement, the modified trust region quadratic subproblem is always feasible, and penalty parameter is avoided. The numerical results show that our presented method is effective and stable.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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