Research Article

Existence and Stability of Periodic Solution Related to Valveless Pumping

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Valveless pumping, also known as Liebau effect, can be described as the unidirectional flow of liquid in a system without valves that is caused by the asymmetry of placing of the periodically working pump. Recently, the research in this field has been reevoked, partially due to its possible application in nanotechnologies. In this paper, a configuration of one pipe and one tank is considered from the mathematical point of view. Qualitative properties of a class of nonlinear differential equations that model the assumed system configuration are investigated. New sufficient conditions for the existence of positive T -periodic solutions are given. Correspondingly, exponential stability of periodic solution is treated. Presented results are new. They extend and complement earlier ones in the literature.

1. Introduction

Valveless pumping represents a mechanism of fluid propagation in one direction in a system where valves are not presented. This type of mechanism was described by German cardiologist Gerhard Liebau for the first time in 1954. Working with patients suffering from severe aortic insufficiency led him to the idea that unidirectional blood propagation could be achieved without valves. To check his assumptions, he demonstrated a valveless pumping in a system consisting of two tanks connected by a rubber tube. Via periodic compression of the tube, located asymmetrically along the length of it, he pumped water from the lower tank to the upper one without the necessity of a valve to ensure a preferential direction of the flow [1]. As Liebau had assumed, the valveless circulation has been later observed in early stages of human embryonic life. In this stage, the heart is only tubular with complete absence of valves; however, the blood circulates in one direction through the cardiovascular system [2]. Many experimental and simulation works have been published on the subject of Liebau phenomenon in order to explain its physical nature, as well as the conditions of its occurrence (for example, [3–6]). Better understanding of the valveless pumping allowed transferring the knowledge to the

technical sphere where, for example, valveless micropumps have been designed [7].

In recent years efforts to investigate the analytical solution to the mathematical model of Liebau phenomenon have arisen. In [8], the existence of periodic solution for configurations of two tanks connected with rigid pipe and of three tanks connected with rigid pipes, respectively, is shown. On the one hand, the model with one tank and one rigid pipe is the simplest in configuration. On the other hand, it appeared to be the most difficult when the existence of solution is considered. The one pipe–one tank problem is more closely examined in [5, 6] where some significant results on the existence of positive periodic solutions are obtained. In [5], the mathematical model of this configuration is derived, resulting in the differential equation with singularities. Applying a suitable substitution, the aforementioned differential equation is transformed into regular one of the form

$$x''(t) + ax'(t) = \frac{e(t)}{\mu} x^{1-2\mu}(t) - \frac{c}{\mu} x^{1-\mu}(t), \quad (1)$$

$$t \in (0, T),$$

where $a \geq 0$, $c > 0$, $0 < \mu < 1/2$, and $e(t)$ is continuous and T -periodic on R . Likewise the authors in [6], we consider the generalization of this equation in the form

$$x''(t) + ax'(t) + q(t)x^\beta(t) - r(t)x^\alpha(t) = 0, \quad t \geq t_0, \quad (2)$$

where $a \geq 0$, $q, r \in C([t_0, \infty), R)$. Whereas the authors in [5, 6] consider only the case $0 < \alpha < \beta < 1$, we consider also more general case $\alpha, \beta \in (0, \infty)$.

Qualitative properties of solutions of differential equations are studied, for example, in [9–15]. In [13], the authors investigate Lasota and the Wazewska-Czyzewska model for the survival of red blood cells in an animal. Model is represented by the first order nonlinear delay differential equation. Another interesting model is treated in [9] where the authors study the periodicity of the Nicholson's blowflies differential equations.

The purpose of this paper is primarily mathematical. We focus on the existence and exponential stability of a positive T -periodic solution of nonlinear differential equation (2) where $a \geq 0$, $\alpha, \beta \in (0, \infty)$, and $q, r \in C([t_0, \infty), R)$. In Section 2 there are given sufficient conditions for the existence of a positive T -periodic solution. Their application is illustrated on the example where the existence of 2π -periodic solution of (2) is shown for given functions $q(t)$ and $r(t)$. The exponential stability of a positive T -periodic solution is treated in Section 3. The obtained results are, consequently, applied on the problem of valveless pumping in one pipe–one tank configuration (Section 4). Sufficient conditions for the existence and exponential stability are reformulated for (1). Furthermore, the comparison of our main results and main results from [5, 6] for this equation is given in the Example 10. The results for the existence of positive T -periodic solution and its exponential stability, presented in this paper, are new, extending and complementing some earlier ones in the literature.

2. Existence of a Positive Periodic Solution

We study the existence of a positive T -periodic solution of (2) in this section. In the sequel, the following fixed point theorem will be used to prove some of the main results in the paper.

Theorem 1 (Schauder's fixed point theorem [14, 16]). *Let Ω be a closed, convex, and nonempty subset of a Banach space X . Let $S : \Omega \rightarrow \Omega$ be a continuous mapping such that $S\Omega$ is a relatively compact subset of X . Then S has at least one fixed point in Ω . That is, there exists an $x \in \Omega$ such that $Sx = x$.*

Theorem 2 states sufficient conditions for the existence of the periodic solution of equation (2). Conditions (3)–(5) guarantee that the operator $S : \Omega \rightarrow \Omega$. In addition, conditions (4), (5) guarantee that $(Sx)(t)$ is T -periodic function.

Theorem 2. *Suppose that there exist function $k \in C([t_0, \infty), R)$ and constants m, M such that*

$$0 < m \leq \exp \left(\int_{t_0}^t [-a + k(s)] ds \right) \leq M, \quad t \geq t_0, \quad (3)$$

$$\int_t^{t+T} [-a + k(s)] ds = 0, \quad t \geq t_0, \quad (4)$$

$$\begin{aligned} & k(t) \exp \left(\int_{t_0}^t [-a + k(s)] ds \right) \\ &= \int_{t_0}^t \left[r(s) \exp \left(\alpha \int_{t_0}^s [-a + k(v)] dv \right) \right. \\ &\quad \left. - q(s) \exp \left(\beta \int_{t_0}^s [-a + k(v)] dv \right) \right] ds, \\ &\quad t \geq t_0. \end{aligned} \quad (5)$$

Then (2) has a positive T -periodic solution.

Proof. Let $X = \{x \in C([t_0, \infty), R)\}$ be a Banach space with the norm $\|x\| = \sup_{t \geq t_0} |x(t)|$. We define a closed, bounded, and convex subset Ω of X as

$$\begin{aligned} \Omega &= \left\{ x \in X : x(t+T) = x(t), \quad t \geq t_0, \quad m \leq x(t) \right. \\ &\quad \leq M, \quad t \geq t_0, \quad \frac{1}{x(t)} \int_{t_0}^t [r(s)x^\alpha(s) - q(s)x^\beta(s)] ds \\ &\quad \left. = k(t), \quad t \geq t_0 \right\}. \end{aligned} \quad (6)$$

and the operator $S : \Omega \rightarrow X$ as

$$\begin{aligned} (Sx)(t) &= \exp \left(\int_{t_0}^t (-a \right. \\ &\quad \left. + \frac{1}{x(s)} \int_{t_0}^s [r(v)x^\alpha(v) - q(v)x^\beta(v)] dv \right) ds, \\ &\quad t \geq t_0. \end{aligned} \quad (7)$$

We need to show that for any $x \in \Omega$, $Sx \in \Omega$. According to (3), for every $x \in \Omega$ and $t \geq t_0$ we obtain

$$\begin{aligned} (Sx)(t) &= \exp \left(\int_{t_0}^t \left(-a \right. \right. \\ &\quad \left. \left. + \frac{1}{x(s)} \int_{t_0}^s [r(v)x^\alpha(v) - q(v)x^\beta(v)] dv \right) ds \right) \\ &= \exp \left(\int_{t_0}^t [-a + k(s)] ds \right) \leq M \end{aligned} \quad (8)$$

as well as

$$\begin{aligned}
(Sx)(t) &= \exp \left(\int_{t_0}^t \left(-a \right. \right. \\
&\quad \left. \left. + \frac{1}{x(s)} \int_{t_0}^s [r(v) x^\alpha(v) - q(v) x^\beta(v)] dv \right) ds \right) \quad (9) \\
&= \exp \left(\int_{t_0}^t [-a + k(s)] ds \right) \geq m.
\end{aligned}$$

With regard to (5), for every $x \in \Omega$ and $t \geq t_0$

$$\begin{aligned}
k(t)(Sx)(t) &= k(t) \\
&\cdot \exp \left(\int_{t_0}^t \left(-a + \frac{1}{x(s)} \int_{t_0}^s [r(v) x^\alpha(v) - q(v) x^\beta(v)] dv \right) ds \right) \\
&= k(t) \exp \left(\int_{t_0}^t [-a + k(s)] ds \right) \\
&= \int_{t_0}^t \left[r(s) \exp \left(\alpha \int_{t_0}^s [-a + k(v)] dv \right) \right. \\
&\quad \left. - q(s) \exp \left(\beta \int_{t_0}^s [-a + k(v)] dv \right) \right] ds \\
&= \int_{t_0}^t [r(s) (Sx)^\alpha(s) - q(s) (Sx)^\beta(s)] ds. \quad (10)
\end{aligned}$$

Finally, we show that for $x \in \Omega$, $t \geq t_0$, the function $(Sx)(t)$ is T -periodic. For $x \in \Omega$, $t \geq t_0$ and with regard to (4)

$$\begin{aligned}
(Sx)(t+T) &= \exp \left(\int_{t_0}^{t+T} \left(-a \right. \right. \\
&\quad \left. \left. + \frac{1}{x(s)} \int_{t_0}^s [r(v) x^\alpha(v) - q(v) x^\beta(v)] dv \right) ds \right) \\
&= \exp \left(\int_{t_0}^t \left(-a \right. \right. \\
&\quad \left. \left. + \frac{1}{x(s)} \int_{t_0}^s [r(v) x^\alpha(v) - q(v) x^\beta(v)] dv \right) ds \right) \\
&\quad \times \exp \left(\int_t^{t+T} \left(-a \right. \right. \\
&\quad \left. \left. + \frac{1}{x(s)} \int_{t_0}^s [r(v) x^\alpha(v) - q(v) x^\beta(v)] dv \right) ds \right) \\
&= \exp \left(\int_{t_0}^t \left(-a \right. \right. \\
&\quad \left. \left. + \frac{1}{x(s)} \int_{t_0}^s [r(v) x^\alpha(v) - q(v) x^\beta(v)] dv \right) ds \right) \\
&\quad \times \exp \left(\int_t^{t+T} [-a + k(s)] ds \right) = (Sx)(t). \quad (11)
\end{aligned}$$

This implies that $(Sx)(t)$ is T -periodic on $[t_0, \infty)$. Thus, we have proved that $Sx \in \Omega$ for any $x \in \Omega$.

Now we need to prove that S is completely continuous. First, we show that S is continuous. Let $x_k = x_k(t) \in \Omega$ be such that $x_k(t) \rightarrow x(t) \in \Omega$ as $k \rightarrow \infty$. For $t \geq t_0$ we obtain

$$\begin{aligned}
|(Sx_k)(t) - (Sx)(t)| &= \left| \exp \left(\int_{t_0}^t \left(-a + \frac{1}{x_k(s)} \int_{t_0}^s [r(v) x_k^\alpha(v) - q(v) x_k^\beta(v)] dv \right) ds \right) \right. \\
&\quad \left. - \exp \left(\int_{t_0}^t \left(-a + \frac{1}{x(s)} \int_{t_0}^s [r(v) x^\alpha(v) - q(v) x^\beta(v)] dv \right) ds \right) \right|. \quad (12)
\end{aligned}$$

With respect to the Lebesgue dominated convergence theorem

$$\lim_{k \rightarrow \infty} \|(Sx_k)(t) - (Sx)(t)\| = 0. \quad (13)$$

This means that S is continuous.

Further, we prove that $S\Omega$ is relatively compact. It is sufficient to show by the Arzela-Ascoli theorem that the family of functions $\{Sx : x \in \Omega\}$ is uniformly bounded and equicontinuous on every finite subinterval of $[t_0, \infty)$. The uniform boundedness follows from the definition of Ω . For $t \geq t_0, x \in \Omega$ we get

$$\left| \frac{d}{dt} (Sx)(t) \right| = \left| -a + \frac{1}{x(t)} \int_{t_0}^t [r(v) x^\alpha(v) - q(v) x^\beta(v)] dv \right|$$

$$\begin{aligned}
&\cdot x^\beta(v)] dv \times \exp \left(\int_{t_0}^t \left(-a + \frac{1}{x(s)} \right. \right. \\
&\quad \left. \left. \cdot \int_{t_0}^s [r(v) x^\alpha(v) - q(v) x^\beta(v)] dv \right) ds \right) \\
&= |-a + k(t)| \exp \left(\int_{t_0}^t [-a + k(s)] ds \right) \leq M_1,
\end{aligned}$$

$$M_1 > 0. \quad (14)$$

This shows the equicontinuity of the family $S\Omega$, (cf. [14], p. 265). Hence, $S\Omega$ is relatively compact and therefore S is completely continuous. With respect to Theorem 1, there is an $x_0 \in \Omega$ such that $Sx_0 = x_0$.

Consequently, $x_0(t)$ is a positive T -periodic solution of (2). The proof is complete. $\square$

Example 3. Consider the nonlinear differential equation (2) where $a \geq 0$, $\alpha, \beta \in (0, \infty)$, and

$$\begin{aligned} q(t) &= -\frac{1}{\alpha^2} (\cos^2 t) \exp\left(\frac{1-\beta}{\alpha} (\sin t - \sin t_0)\right), \\ r(t) &= \frac{1}{\alpha} (a \cos t - \sin t) \exp\left(\frac{1-\alpha}{\alpha} (\sin t - \sin t_0)\right). \end{aligned} \quad (15)$$

Here t_0 is such that $\cos t_0 = -a\alpha$, $0 \leq a\alpha \leq 1$.

We set

$$k(t) = a + \frac{1}{\alpha} \cos t. \quad (16)$$

Condition (4) for $T = 2\pi$ is

$$\int_t^{t+T} [-a + k(s)] ds = \frac{1}{\alpha} \int_t^{t+2\pi} \cos s ds = 0, \quad t \geq t_0. \quad (17)$$

For condition (5), we obtain

$$\begin{aligned} k(t) \exp\left(\int_{t_0}^t [-a + k(s)] ds\right) \\ = \left(a + \frac{1}{\alpha} \cos t\right) \exp\left(\frac{1}{\alpha} \int_{t_0}^t \cos s ds\right) \\ = \left(a + \frac{1}{\alpha} \cos t\right) \exp\left(\frac{1}{\alpha} (\sin t - \sin t_0)\right), \quad t \geq t_0. \end{aligned} \quad (18)$$

Also

$$\begin{aligned} \int_{t_0}^t \left[r(s) \exp\left(\alpha \int_{t_0}^s [-a + k(v)] dv\right) - q(s) \right. \\ \cdot \exp\left(\beta \int_{t_0}^s [-a + k(v)] dv\right) \Big] ds \\ = \int_{t_0}^t \left[\frac{1}{\alpha} (a \cos s - \sin s) \exp\left(\frac{1}{\alpha} (\sin s - \sin t_0)\right) \right. \\ \cdot \exp\left(\frac{1-\alpha}{\alpha} (\sin s - \sin t_0)\right) \exp(\sin s - \sin t_0) \\ + \frac{1}{\alpha^2} (\cos^2 s) \exp\left(\frac{1-\beta}{\alpha} (\sin s - \sin t_0)\right) \\ \cdot \exp\left(\frac{\beta}{\alpha} (\sin s - \sin t_0)\right) \Big] ds \\ = \int_{t_0}^t \left[\frac{1}{\alpha} (a \cos s - \sin s) \exp\left(\frac{1}{\alpha} (\sin s - \sin t_0)\right) \right. \\ + \frac{1}{\alpha^2} (\cos^2 s) \exp\left(\frac{1}{\alpha} (\sin s - \sin t_0)\right) \Big] ds \\ = \int_{t_0}^t \frac{1}{\alpha} \left(a \cos s - \sin s + \frac{1}{\alpha} \cos^2 s \right) \\ \cdot \exp\left(\frac{1}{\alpha} (\sin s - \sin t_0)\right) ds \end{aligned}$$

$$\begin{aligned} &= \int_{t_0}^t \frac{d}{ds} \left[\left(a + \frac{1}{\alpha} \cos s\right) \right. \\ &\cdot \exp\left(\frac{1}{\alpha} (\sin s - \sin t_0)\right) \Big] ds = \left(a + \frac{1}{\alpha} \cos t\right) \\ &\cdot \exp\left(\frac{1}{\alpha} (\sin t - \sin t_0)\right) - \left(a + \frac{1}{\alpha} \cos t_0\right) = \left(a \right. \\ &+ \frac{1}{\alpha} \cos t\right) \exp\left(\frac{1}{\alpha} (\sin t - \sin t_0)\right), \quad t \geq t_0, \end{aligned} \quad (19)$$

since $\cos t_0 = -a\alpha$. It is easy to see that condition (3) also holds. Thus, the conditions of Theorem 2 are satisfied and (2) has a positive $T = 2\pi$ -periodic solution.

3. Stability of a Positive Periodic Solution

Here we consider the exponential stability of a positive periodic solution of (2). Let $x_1(t)$ denote the positive T -periodic solution of (2) with the initial condition $x_1(t_0) = 1$. Let $x(t)$ denote another solution of (2) with initial condition $x(t_0) = c_1 > 0$, $c_1 \neq 1$. Let $y(t) = x(t) - x_1(t)$, $t \in [t_0, \infty)$, and $x'(t_0) + ax(t_0) - x_1'(t_0) - ax_1(t_0) = 0$.

After integration of (2), we get

$$\begin{aligned} \int_{t_0}^t x''(s) ds + a \int_{t_0}^t x'(s) ds \\ + \int_{t_0}^t [q(s) x^\beta(s) - r(s) x^\alpha(s)] ds = 0, \\ [x'(s)]_{t_0}^t \\ + a [x(s)]_{t_0}^t = \int_{t_0}^t [r(s) x^\alpha(s) - q(s) x^\beta(s)] ds \\ x'(t) - x'(t_0) + ax(t) \\ - ax(t_0) = \int_{t_0}^t [r(s) x^\alpha(s) - q(s) x^\beta(s)] ds. \end{aligned} \quad (20)$$

Similarly, integrating (2) for $x_1(t)$ leads to

$$\begin{aligned} x_1'(t) - x_1'(t_0) + ax_1(t) - ax_1(t_0) \\ = \int_{t_0}^t [r(s) x_1^\alpha(s) - q(s) x_1^\beta(s)] ds. \end{aligned} \quad (21)$$

Consequently,

$$\begin{aligned} y'(t) &= x'(t) - x_1'(t) = x'(t_0) + ax(t_0) - x_1'(t_0) \\ &- ax_1(t_0) - a[x(t) - x_1(t)] \\ &+ \int_{t_0}^t [r(s) (x^\alpha(s) - x_1^\alpha(s)) \\ &- q(s) (x^\beta(s) - x_1^\beta(s))] ds = -ay(t) \\ &+ \int_{t_0}^t [r(s) (x^\alpha(s) - x_1^\alpha(s)) \\ &- q(s) (x^\beta(s) - x_1^\beta(s))] ds. \end{aligned} \quad (22)$$

By the mean value theorem, we obtain

$$\begin{aligned} y'(t) = & -ay(t) + \int_{t_0}^t [r(s) \alpha x_*(s)^{\alpha-1} (x(s) - x_1(s)) \\ & - q(s) \beta x_0^{\beta-1}(s) (x(s) - x_1(s))] ds = -ay(t) \\ & + \int_{t_0}^t [r(s) \alpha x_*^{\alpha-1}(s) - q(s) \beta x_0^{\beta-1}(s)] y(s) ds, \end{aligned} \quad (23)$$

$$t \geq t_0.$$

$x_*, x_0 \in [x, x_1]$ or $x_*, x_0 \in [x_1, x]$.

Let us assume that the function

$$f(t, x) = -ax(t) + \int_{t_0}^t [r(s) x^\alpha(s) - q(s) x^\beta(s)] ds, \quad (24)$$

$$t \geq t_0$$

satisfies Lipschitz-type condition with respect to $m \leq x \leq M$.

Definition 4. Let $x_1(t)$ be a positive solution of (2). Let there exist constants $K_{x_1}, \lambda > 0$ for every solution $x(t)$ of (2) such that $0 < m_* \leq x(t) \leq M_*, m_* \leq m, M_* \geq M, x'(t_0) + ax(t_0) - x'_1(t_0) - ax_1(t_0) = 0$ and $|x(t) - x_1(t)| < K_{x_1} e^{-\lambda t}$ for all $t > t_0$.

Then $x_1(t)$ is said to be exponentially stable.

In the next theorem, we establish sufficient conditions for the exponential stability of the positive solution $x_1(t)$ of (2).

Theorem 5. Suppose that $q, r \in C([t_0, \infty), (0, \infty))$ and there exist function $k \in C([t_0, \infty), \mathbb{R})$ and constants m, M such that (3)–(5) hold. Let $a > 0, 0 < \alpha < \beta < 1$ and there exist constants $m_*, M_* \in (0, \infty)$ such that $m_* \leq m, M_* \geq M$ and

$$\alpha m_*^{\alpha-1} r(t) - \beta M_*^{\beta-1} q(t) \leq 0 \quad \text{for } t \geq t_0. \quad (25)$$

Then (2) has a positive T -periodic solution which is exponentially stable.

Proof. Conditions (3)–(5) imply that (2) has a positive T -periodic solution $x_1(t)$. Let $x(t)$ be a solution of (2) such that $m_* \leq x(t) \leq M_*, x'(t_0) + ax(t_0) - x'_1(t_0) - ax_1(t_0) = 0$. We show that there exists $\lambda \in (0, \infty)$ such that

$$|x(t) - x_1(t)| < K_{x_1} e^{-\lambda t}, \quad t > t_0, \quad (26)$$

where $K_{x_1} = e^{\lambda t_0} |y(t_0)| + 1$.

We consider the Lyapunov function

$$L(t) = |y(t)| e^{\lambda t}, \quad t \geq t_0, \quad \lambda \in (0, a). \quad (27)$$

Let us claim that $L(t) < K_{x_1}$ for $t > t_0$. Furthermore, let there exists $t_* > t_0$ such that $L(t_*) = K_{x_1}$ and $L(t) < K_{x_1}$ for $t \in [t_0, t_*)$. Calculating the upper left derivative of $L(t)$ along the solution of (23), we obtain

$$\begin{aligned} D^-(L(t)) \leq & -a |y(t)| e^{\lambda t} \\ & + e^{\lambda t} \int_{t_0}^t [\alpha x_*^{\alpha-1}(s) r(s) - \beta x_0^{\beta-1}(s) q(s)] |y(s)| ds \\ & + \lambda |y(t)| e^{\lambda t}, \quad t \geq t_0. \end{aligned} \quad (28)$$

For $t = t_*$ we get

$$\begin{aligned} 0 \leq D^-(L(t_*)) \leq & (\lambda - a) |y(t_*)| e^{\lambda t_*} \\ & + e^{\lambda t_*} \int_{t_0}^{t_*} [\alpha x_*^{\alpha-1}(s) r(s) - \beta x_0^{\beta-1}(s) q(s)] \\ & \cdot |y(s)| ds \leq (\lambda - a) |y(t_*)| e^{\lambda t_*} \\ & + e^{\lambda t_*} \int_{t_0}^{t_*} [\alpha m_*^{\alpha-1} r(s) - \beta M_*^{\beta-1} q(s)] |y(s)| ds \\ \leq & (\lambda - a) |y(t_*)| e^{\lambda t_*} = (\lambda - a) K_{x_1} < 0, \end{aligned} \quad (29)$$

which is a contradiction. Thus, we have

$$|y(t)| e^{\lambda t} < K_{x_1} \quad \text{for } t > t_0 \text{ and some } \lambda \in (0, a). \quad (30)$$

The proof is complete. $\square$

4. Application in a Pipe-Tank Configuration

In [5], authors J. Cid, G. Propst and M. Tvrdý established sufficient conditions for the existence and the asymptotic stability of a positive periodic solution for a pipe-tank flow configuration. Such flow configuration is a special case of valveless systems of moving fluid [5, 6].

According to authors Cid et al., the problem of fluid motion in the pipe in [5] can be reformulated as a periodic boundary value problem

$$u''(t) + au'(t) = \frac{1}{u(t)} (e(t) - b(u'(t))^2) - c, \quad (31)$$

$$t \in [0, T],$$

$$u(0) = u(T), \quad u'(0) = u'(T). \quad (32)$$

With regard to the physical meaning of the involved parameters, we may assume

$$a \geq 0, \quad b > 1, \quad c > 0 \quad (33)$$

and $e(t)$ is continuous and T -periodic on $\mathbb{R}$.

The change of variables

$$u = x^\mu, \quad \text{where } \mu = \frac{1}{b+1} \quad (34)$$

transforms the singular problem (31) to the regular one

$$\begin{aligned} x''(t) + ax'(t) + q(t) x^\beta(t) - r(t) x^\alpha(t) &= 0, \\ t \in [0, T], \\ x(0) &= x(T), \\ x'(0) &= x'(T), \end{aligned} \quad (35)$$

where

$$\begin{aligned} r(t) &= \frac{e(t)}{\mu}, \\ q(t) &= \frac{c}{\mu}, \\ \alpha &= 1 - 2\mu, \\ \beta &= 1 - \mu. \end{aligned} \quad (36)$$

From previous text, it follows that $0 < \alpha < \beta < 1$.

The main results of the paper [5] are summarized in Theorems 6 and 7.

Theorem 6 (see [5]). Assume (33) and let $a > 0$ and $e_* > 0$. Then problem (31) has a positive solution provided that the following inequality holds:

$$\frac{(b+1)c^2}{4e_*} < \left(\frac{\pi}{T}\right)^2 + \frac{a^2}{4}. \quad (37)$$

Theorem 7 (see [5]). Assume (33) and let $a > 0$ and $e_* > 0$. Then problem (31) has at least one asymptotically stable positive solution provided that the following inequalities hold:

$$\frac{c^2 [b(e_*)^2 - (b-1)(e_*)^2]}{e_* (e_*)^2} < \left(\frac{\pi}{T}\right)^2 + \frac{a^2}{4} \quad (38)$$

and

$$(b-1)e^* < be_*, \quad (39)$$

where $e_* = \min\{e(t) : t \in [0, T]\}$, $e^* = \max\{e(t) : t \in [0, T]\}$.

The proofs of Theorems 6 and 7 rely on the method of lower and upper solution. For more details about the model and main results, we refer readers to [5, 6] and the references cited therein.

Our aim is to establish new sufficient conditions for the existence and the exponential stability of positive T -periodic solution of the equation

$$x''(t) + ax'(t) + \frac{1}{\mu} [cx^\beta(t) - e(t)x^\alpha(t)] = 0, \quad (40)$$

$$t \geq t_0.$$

With respect to Theorems 2 and 5, we obtain the following result.

Theorem 8. Suppose that $a \geq 0$, $0 < \alpha < \beta < 1$ and there exist function $k \in C([t_0, \infty), \mathbb{R})$ and constants m, M such that (3) and (4) hold and

$$\begin{aligned} & k(t) \exp \left(\int_{t_0}^t [-a + k(s)] ds \right) \\ &= \int_{t_0}^t \frac{1}{\mu} \left[e(s) \exp \left(\alpha \int_{t_0}^s (-a + k(v)) dv \right) \right. \\ &\quad \left. - c \exp \left(\beta \int_{t_0}^s (-a + k(v)) dv \right) \right] ds, \quad t \geq t_0. \end{aligned} \quad (41)$$

Then (40) has a positive T -periodic solution.

Theorem 9. Suppose that $e \in C([t_0, \infty), (0, \infty))$, $a > 0$, $0 < \alpha < \beta < 1$, and $c > 0$ and there exist function $k \in C([t_0, \infty), \mathbb{R})$ and constants m, M such that (3), (4), and (41) hold. Let, in addition, there exist constants $m_*, M_* \in (0, \infty)$ such that $m_* \leq m, M_* \geq M$ and

$$\alpha m_*^{\alpha-1} e(t) - \beta M_*^{\beta-1} c \leq 0 \quad \text{for } t \geq t_0. \quad (42)$$

Then (40) has a positive T -periodic solution which is exponentially stable.

The results of Theorems 8 and 9 are illustrated by the example.

Example 10. Let us consider the nonlinear differential equation

$$x''(t) + ax'(t) + q(t)x^\beta(t) - r(t)x^\alpha(t) = 0, \quad (43)$$

$$t \geq t_0,$$

where $a \in (0, \infty)$, $0 < \alpha < \beta < 1$ and

$$\begin{aligned} q(t) &= c(b+1) = \frac{c}{\mu}, \quad b > 1, \quad c > 0, \\ r(t) &= \frac{a \cos t - \sin t}{d + \sin t_0} \left(\frac{d + \sin t_0}{d + \sin t} \right)^\alpha \\ &\quad + c(b+1) \left(\frac{d + \sin t}{d + \sin t_0} \right)^{\beta-\alpha} = \frac{e(t)}{\mu}, \quad d > 1, \end{aligned} \quad (44)$$

where t_0 is such that $a(d + \sin t_0) + \cos t_0 = 0$.

We set

$$k(t) = a + \frac{\cos t}{d + \sin t}. \quad (45)$$

Then, for condition (4) and $T = 2\pi$, we get

$$\begin{aligned} \int_t^{t+T} [-a + k(s)] ds &= \int_t^{t+2\pi} \frac{\cos s}{d + \sin s} ds \\ &= \ln(d + \sin(t + 2\pi)) \\ &\quad - \ln(d + \sin t) = 0, \quad t \geq t_0. \end{aligned} \quad (46)$$

For condition (41), we obtain

$$\begin{aligned} & k(t) \exp \left(\int_{t_0}^t [-a + k(s)] ds \right) \\ &= \left(a + \frac{\cos t}{d + \sin t} \right) \exp \left(\int_{t_0}^t \frac{\cos s}{d + \sin s} ds \right) \\ &= \left(a + \frac{\cos t}{d + \sin t} \right) \exp \left(\ln \frac{d + \sin t}{d + \sin t_0} \right) \\ &= \left(a + \frac{\cos t}{d + \sin t} \right) \frac{d + \sin t}{d + \sin t_0} \\ &= \frac{a(d + \sin t) + \cos t}{d + \sin t_0}. \end{aligned} \quad (47)$$

We also get

$$\begin{aligned} & \int_{t_0}^t \left[r(s) \exp \left(\alpha \int_{t_0}^s [-a + k(v)] dv \right) - q(s) \right. \\ &\quad \cdot \exp \left(\beta \int_{t_0}^s [-a + k(v)] dv \right) \Big] ds \\ &= \int_{t_0}^t \left[\left(\frac{a \cos s - \sin s}{d + \sin t_0} \left(\frac{d + \sin t_0}{d + \sin s} \right)^\alpha \right. \right. \\ &\quad \left. \left. + c(b+1) \left(\frac{d + \sin s}{d + \sin t_0} \right)^{\beta-\alpha} \right) \right] \\ &\quad \times \exp \left(\alpha \int_{t_0}^s \frac{\cos v}{d + \sin v} dv \right) - c(b+1) \end{aligned}$$

$$\begin{aligned}
& \cdot \exp \left(\beta \int_{t_0}^s \frac{\cos v}{d + \sin v} dv \right) ds \\
& = \int_{t_0}^t \left(\left[\frac{a \cos s - \sin s}{d + \sin t_0} \left(\frac{d + \sin t_0}{d + \sin s} \right)^\alpha \right. \right. \\
& \quad \left. \left. + c(b+1) \left(\frac{d + \sin s}{d + \sin t_0} \right)^{\beta-\alpha} \right] \times \left(\frac{d + \sin s}{d + \sin t_0} \right)^\alpha \right. \\
& \quad \left. - c(b+1) \left(\frac{d + \sin s}{d + \sin t_0} \right)^\beta \right) ds \\
& = \int_{t_0}^t \frac{a \cos s - \sin s}{d + \sin t_0} ds = \frac{1}{d + \sin t_0} (a \sin t + \cos t \\
& \quad - a \sin t_0 - \cos t_0) = \frac{a(d + \sin t) + \cos t}{d + \sin t_0},
\end{aligned} \tag{48}$$

since $a(d + \sin t_0) + \cos t_0 = 0$. The condition (3) is also satisfied.

Thus, conditions (3), (4), and (41) of Theorem 8 are satisfied and (43) has a positive $T = 2\pi$ -periodic solution

$$\begin{aligned}
x(t) &= \exp \left(\int_{t_0}^t [-a + k(s)] ds \right) \\
&= \exp \left(\int_{t_0}^t \frac{\cos s}{d + \sin s} ds \right) = \exp \left(\ln \frac{d + \sin t}{d + \sin t_0} \right) \\
&= \frac{d + \sin t}{d + \sin t_0}, \quad t \geq t_0,
\end{aligned} \tag{49}$$

$$a(d + \sin t_0) + \cos t_0 = 0.$$

For

$$\begin{aligned}
t_0 &= \pi, \\
a &= \frac{1}{4}, \\
d &= 4, \\
b &= 3, \\
c &= 2,
\end{aligned} \tag{50}$$

we get

$$\begin{aligned}
\mu &= \frac{1}{b+1} = \frac{1}{4}, \\
\alpha &= 1 - 2\mu = \frac{1}{2}, \\
\beta &= 1 - \mu = \frac{3}{4}, \\
e(t) &= \mu r(t) = \frac{1}{4} \left[\frac{1}{4} \left(\frac{1}{4} \cos t - \sin t \right) \left(\frac{4}{4 + \sin t} \right)^{0.5} \right. \\
& \quad \left. + 8 \left(\frac{4 + \sin t}{4} \right)^{0.25} \right]
\end{aligned} \tag{51}$$

and $e_* \approx 1.931$, $e^* \approx 2.061$. When we set constants $m_* = 0.6 < m = 0.75$, $M_* = 1.4 > M = 1.25$, condition (42) has a form

$$\frac{1}{2} 0.6^{-0.5} e(t) - \frac{3}{2} 1.4^{-0.25} < 0, \quad t \geq \pi. \tag{52}$$

FIGURE 1: Numerical simulations of problem (43).

According to Theorem 9, solution $x(t)$ is exponentially stable. The numerical simulation in Figure 1 supports the conclusion.

Let us check the existence and the asymptotic stability of the solution of (43) according to Theorems 6 and 7, respectively, with regard to considered values of parameters. We can see that conditions (37) and (38) are not satisfied. Also condition

$$q^* < \min \left\{ \left(\frac{\pi}{T} \right)^2 + \left(\frac{a}{2} \right)^2, r_* \right\}, \tag{53}$$

where $q^* = \max\{q(t) : t \in [0, T]\}$, $r_* = \min\{r(t) : t \in [0, T]\}$, from Corollary 3.5 [6] is not satisfied.

As is illustrated on this example, our results provide extension to previously obtained results in [5, 6].

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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