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Research Article

Solutions and Painlevé Property for the KdV Equation with Self-Consistent Source

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In this paper, the polynomial solutions in terms of Jacobi's elliptic functions of the KdV equation with a self-consistent source (KdV-SCS) are presented. The extended (G'/G)-expansion method is utilized to obtain exact traveling wave solutions of the KdV-SCS, which finally are expressed in terms of the hyperbolic function, the trigonometric function, and the rational function. Meanwhile we find the Lie point symmetry and Lie symmetry group and give several group-invariant solutions for the KdV-SCS. Finally, we supplement the results of the Painlevé property in our previous work and get the Bäcklund transformations of the KdV-SCS.

1. Introduction

It is well known that the soliton equations with self-consistent sources (SESCSs) can exhibit abundant nonlinear dynamics compared to soliton equations themselves and have important physical applications [1]. These SESCSs are usually used to describe interactions between different solitary waves and are relevant in some problems related with hydrodynamics, solid state physics, or plasma physics [2–4]. The sources appear in solitary waves with nonconstant velocity and lead to a variety of dynamics of physical models [2]. For example, the KdV equation with a self-consistent source (KdV-SCS) describes the interaction of long and short capillary-gravity waves [5, 6]. During the past four decades or so searching for explicit solutions of nonlinear evolution equations by using various different methods is the main goal for many researchers. Many powerful methods to construct exact solutions of nonlinear evolution equations have been established and developed, which can be used to deal with the SESCSs as well. For instance, some equations with self-consistent sources have been studied by the inverse scattering method [5, 7], Darboux transformation method [8, 9], and Hirota method [10–12].

In recent years, many effectively straightforward methods have been proposed such as the Jacobi elliptic function expansion method [13, 14], the tanh-function expansion method [15, 16], the F-expansion method [17, 18], and the (G'/G)-expansion method [19–21]. Motivated by the previous works, we focus our attention on the following nonlinear partial differential equations (PDEs):

$$u_t + u_{xxx} + 12uu_x = \left(v^2\right)_x, \tag{1a}$$

$$v_{xx} + 2uv = \lambda v, \tag{1b}$$

where λ is an arbitrary constant. In fact, (1a)-(1b) is a reduced form of the KdV equation with source [22, 23].

The general KdV-SCS has been discussed from various aspects, such that, with Wronskian technique, the mixed rational-soliton solutions for the KdV-SCS are obtained [22]. The complexion solutions of the KdV-SCS are presented by the Darboux transformation [23]. The soliton solutions for the KdV hierarchy with self-consistent sources are obtained by the inverse scattering method [5]. The integration of KdV-SCS and higher KdV-SCS in the class of periodic functions are, respectively, studied in [24, 25]. However, the Jacobi's elliptic function solutions, the group-invariant solutions by the Lie group approach [26], and the extended (G'/G)-expansion method for the KdV-SCS have not been presented.

This paper is organized as follows. In Section 2, we construct the polynomial solutions in terms of Jacobi's elliptic functions for (1a)-(1b). In Section 3, using the extended (G'/G)-expansion method, we obtain the exact traveling wave solutions of (1a)-(1b). In Section 4, we give the group-invariant solutions of (1a)-(1b) by the Lie group approach. In Section 5, the results of the Painlevé property for (1a)-(1b) are supplemented. Section 6 is a brief conclusion.

2. Jacobi's Elliptic Function Solutions

In this section we mainly construct the polynomial solutions of (1a)-(1b) in terms of Jacobi's elliptic functions [27].

Consider the following tripled Riccati equations:

$$\begin{aligned} f' &= gh, \\ g' &= -fh, \\ h' &= -M^2 fg, \end{aligned} \tag{2}$$

where $' = d/d\xi$ and modulus $0 < M < 1$, which have three solutions as listed

$$\begin{aligned} f &= \operatorname{sn}(\xi; M), \\ g &= \operatorname{cn}(\xi; M), \\ h &= \operatorname{dn}(\xi; M). \end{aligned} \tag{3}$$

Solutions (3) satisfy

$$\begin{aligned} \operatorname{sn}^2(\xi; M) &= 1 - \operatorname{cn}^2(\xi; M), \\ \operatorname{dn}^2(\xi; M) &= 1 - M^2 \operatorname{sn}^2(\xi; M). \end{aligned} \tag{4}$$

Step J1. Transform (1a)-(1b) to ODEs.

To begin with, by using the travelling wave transformation,

$$\xi = kx + \omega t, \tag{5}$$

where ξ is referred to the traveling wave variable and k and ω represent the amplitude and velocity of the traveling wave, respectively, and setting $U(\xi) = u(x,t)$, $V(\xi) = v(x,t)$, system (1a)-(1b) is transformed into the following nonlinear ordinary differential equations (ODEs):

$$\omega U + 6kU^2 - kV^2 + k^3 U'' = 0, \tag{6a}$$

$$k^2 V'' + 2UV - \lambda V = 0, \tag{6b}$$

while $' = d/d\xi$.

Step J2. Determine the expressions of the polynomial solutions.

Suppose that the solution of (6a)-(6b) can be expressed as the following finite series:

$$\begin{aligned} U(\xi) &= \sum_{i=0}^{m} a_i f^i + \sum_{i=1}^{m} b_i g f^{i-1} + \sum_{i=1}^{m} c_i h f^{i-1}, \\ V(\xi) &= \sum_{i=0}^{n} A_i f^i + \sum_{i=1}^{n} B_i g f^{i-1} + \sum_{i=1}^{n} C_i h f^{i-1}, \end{aligned} \tag{7}$$

where a_i, b_i, c_i, A_i, B_i, and C_i are constants to be determined later and the positive integers m,n are determined by balancing the highest nonlinear terms and the highest-order partial derivative terms in (6a)-(6b) [28]. Then we get $m = n = 2$, and

$$\begin{aligned} U(\xi) &= a_0 + a_1 f + a_2 f^2 + b_1 g + b_2 g f + c_1 h + c_2 h f, \\ V(\xi) &= A_0 + A_1 f + A_2 f^2 + B_1 g + B_2 g f + C_1 h \\ &\quad + C_2 h f. \end{aligned} \tag{8}$$

Step J3. Derive the algebraic system for the coefficients a_i, b_i, c_i, A_i, B_i, and C_i.

Substituting (8) into (6a)-(6b) and repeatedly applying (4) and (2), and collecting all terms of same power of f, g, h together and setting each coefficient of the polynomials to zero, we get a system of algebraic equations for the unknowns $a_i, b_i, c_i, A_i, B_i, C_i, M, k$, and ω, which on solving gives five sets of solutions for an algebraic system omitted here.

Step J4. Build and test the Jacobi's elliptic function solutions.

Substituting the above five sets of solutions separately into (8) and replacing f, g, h with (3), we get the Jacobi's elliptic function solutions for (1a)-(1b) as follows.

Solution 1:

$$\begin{aligned} u(x,t) &= \frac{1}{2} \frac{3k^2\lambda - \lambda A_0 + 12k^4 - 3k^2 A_0}{3k^2 - A_0} \\ &\quad - \frac{3}{4} \frac{A_0 (4k^2 - A_0)}{3k^2 - A_0} \operatorname{sn}^2(\xi; M), \\ v(x,t) &= A_0 - \frac{3}{2} \frac{A_0 (4k^2 - A_0)}{3k^2 - A_0} \operatorname{sn}^2(\xi; M), \\ M &= \pm \frac{1}{k} \sqrt{\frac{4k^2 A_0 - A_0^2}{12k^2 - 4A_0}}, \\ A_0 &= \frac{30k^3 + 6k\lambda + \omega \pm \sqrt{-300k^6 + 36k^2\lambda^2 + 12k\lambda\omega + \omega^2}}{10k}. \end{aligned} \tag{9}$$

Solution 2:

$$\begin{aligned} u(x,t) &= \frac{1}{2} \frac{3k^2\lambda + \lambda A_0 + 12k^4 + 3k^2 A_0}{3k^2 + A_0} \\ &\quad + \frac{3}{4} \frac{A_0 (4k^2 + A_0)}{3k^2 + A_0} \operatorname{sn}^2(\xi; M), \\ v(x,t) &= A_0 - \frac{3}{2} \frac{A_0 (4k^2 + A_0)}{3k^2 + A_0} \operatorname{sn}^2(\xi; M), \\ M &= \pm \frac{1}{k} \sqrt{-\frac{4k^2 A_0 + A_0^2}{12k^2 + 4A_0}}, \end{aligned}$$

$$A_0 = -\frac{30k^3 + 6k\lambda + \omega \pm \sqrt{-300k^6 + 36k^2\lambda^2 + 12k\lambda\omega + \omega^2}}{10k}. \quad (10)$$

Solution 3:

$$u(x,t) = \frac{k^2M^2}{2} + \frac{\lambda}{2} - k^2M^2\mathrm{sn}^2(\xi; M),$$
$$v(x,t) = \pm\sqrt{\omega k + 2k^4M^2 + 6k^2\lambda - 4k^4}\,\mathrm{dn}(\xi; M). \quad (11)$$

Solution 4:

$$u(x,t) = \frac{k^2}{2} + \frac{\lambda}{2} - k^2M^2\mathrm{sn}^2(\xi; M),$$
$$v(x,t) = \pm\sqrt{2k^4 - 4k^4M^2 + \omega k + 6k^2\lambda}\,M\mathrm{cn}(\xi; M). \quad (12)$$

Solution 5:

$$u(x,t) = \frac{1}{2}k^2 + \frac{1}{2}k^2M^2 + \frac{\lambda}{2} - k^2M^2\mathrm{sn}^2(\xi; M),$$
$$v(x,t) = \pm\sqrt{-k(2k^3 + 2k^3M^2 + \omega + 6k\lambda)}\,M\mathrm{sn}(\xi; M), \quad (13)$$

where $\xi = kx+\omega t$, k, ω are arbitrary constants and $0 < M < 1$.

According to $\mathrm{sn}(\xi; 0) = \sin(\xi)$, $\mathrm{sn}(\xi; 1) = \tanh(\xi)$, $\mathrm{cn}(\xi; 0) = \cos(\xi)$, $\mathrm{cn}(\xi; 1) = \mathrm{sech}(\xi)$, and $\mathrm{cn}(\sqrt{m}\xi; 1/m) = \mathrm{dn}(\xi; m)$, the above solutions can be expressed in terms of hyperbolic functions and the trigonometric functions. When setting $M = 1$ in (9), we get $A_0 = 2k^2$ and $A_0 = 6k^2$. Substituting $A_0 = 2k^2$ into (9) and applying $\tanh^2\xi + \mathrm{sech}^2\xi = 1$, we get the following typical travelling wave solution from (9):

$$u(x,t) = \frac{\lambda}{2} + 3k^2\mathrm{sech}^2(kx + \omega t),$$
$$v(x,t) = -4k^2 + 6k^2\mathrm{sech}^2(kx + \omega t). \quad (14)$$

Likewise, when making $M = 1$ in (10)-(13), we can also get solutions in terms of hyperbolic functions. It shows that the Jacobi's elliptic function expansion method is more general than that of the hyperbolic function expansion method [29].

3. Extended (G'/G)-Expansion Method to KdV-SCS

3.1. Description of the Extended (G'/G)-Expansion Method. Let us have a look at the extended (G'/G)-expansion method briefly. For a given nonlinear PDE,

$$F(u, u_t, u_x, u_{xt}, u_{xx}, u_{tt}, \ldots) = 0, \quad (15)$$

where $u = u(x,t)$ and F is a polynomial about $u(x,t)$ and its various partial derivatives.

To begin with, using the following traveling wave transformation,

$$u(x,t) = U(\xi), \quad \xi = kx + \omega t, \quad (16)$$

Equation (15) reduces to the following ODE:

$$P(U, U', U'', \ldots, U^{(n)}) = 0, \quad (17)$$

where $' = d/d\xi$.

Suppose that the solution of (17) can be expressed as a finite series in (G'/G)

$$U(\xi) = \sum_{i=0}^{m} a_i\left(\frac{G'}{G}\right)^i, \quad a_m \neq 0, \; m \in \mathbb{N}, \quad (18)$$

where we let $G = G(\xi)$ satisfy the following ODE instead of that form in [30]

$$GG'' = \alpha G'^2 + \beta GG' + \gamma G^2, \quad (19)$$

where $a_i (i = 0, 1, 2, \ldots, m)$, α, β, γ are constants to be determined later, and the positive integer m can be determined the same as before.

Substituting (18) along with (19) into (17) and equating the coefficients of each power of G'/G to zeros, we obtain a system of algebraic equations for the unknowns $a_i (i = 1, 2, \ldots, m)$, k, and ω. Then we can determine the unknowns.

Equation (19) possesses solutions listed below.

When $\beta^2 - 4(\alpha - 1)\gamma > 0$, $\alpha \neq 1$,

$$\frac{G'}{G} = \frac{\beta}{2-2\alpha} + \frac{\sqrt{\beta^2 - 4\alpha\gamma + 4\gamma}\left(C_1\sinh\left((1/2)\sqrt{\beta^2 - 4\alpha\gamma + 4\gamma}\xi\right) + C_2\cosh\left((1/2)\sqrt{\beta^2 - 4\alpha\gamma + 4\gamma}\xi\right)\right)}{(2-2\alpha)\left(C_1\cosh\left((1/2)\sqrt{\beta^2 - 4\alpha\gamma + 4\gamma}\xi\right) + C_2\sinh\left((1/2)\sqrt{\beta^2 - 4\alpha\gamma + 4\gamma}\xi\right)\right)}. \quad (20)$$

When $\beta^2 - 4(\alpha - 1)\gamma < 0$, $\alpha \neq 1$,

$$\frac{G'}{G} = \frac{\beta}{2-2\alpha} + \frac{\sqrt{-\beta^2 + 4\alpha\gamma - 4\gamma}\left(-C_1\sin\left((1/2)\sqrt{-\beta^2 + 4\alpha\gamma - 4\gamma}\xi\right) + C_2\cos\left((1/2)\sqrt{-\beta^2 + 4\alpha\gamma - 4\gamma}\xi\right)\right)}{(2-2\alpha)\left(C_1\cos\left((1/2)\sqrt{-\beta^2 + 4\alpha\gamma - 4\gamma}\xi\right) + C_2\sin\left((1/2)\sqrt{-\beta^2 + 4\alpha\gamma - 4\gamma}\xi\right)\right)}. \quad (21)$$

When $\beta^2 - 4(\alpha - 1)\gamma = 0, \alpha \neq 1$,

$$\frac{G'}{G} = -\frac{\xi\beta + 2 + C_1\beta}{2(\xi\alpha - \xi + C_1\alpha - C_1)}. \quad (22)$$

In (20)-(22), C_1, C_2 are arbitrary constants.

Therefore, by the sign of the discriminant $\beta^2 - 4(\alpha - 1)\gamma$, we can obtain the exact solutions of (15).

3.2. Application of the KdV-SCS. In this section, we demonstrate the extended G'/G-expansion method on the KdV-SCS (1a)-(1b).

Introducing the travelling wave transformations

$$\begin{aligned} u(x,t) &= U(\xi), \\ v(x,t) &= V(\xi), \end{aligned} \quad (23)$$

where $\xi = kx + \omega t$, k, ω will be determined later. Similar to the strategy in Step J1, substituting $U(\xi) = U$, $V(\xi) = V$ into ((1a)-(1b)), one can transform (1a)-(1b) into nonlinear ODEs as (6a)-(6b).

Now, we make the ansatz $U(\xi) = \sum_{i=0}^{m} a_i(G'/G)^i$, $V(\xi) = \sum_{i=0}^{n} b_i(G'/G)^i$ for the solutions of (6a)-(6b). Likewise, using the homogeneous balance method we obtain $m = n = 2$. Therefore, the solutions of (6a)-(6b) have the following extended forms:

$$\begin{aligned} U(\xi) &= a_0 + a_1\left(\frac{G'}{G}\right) + a_2\left(\frac{G'}{G}\right)^2, \\ V(\xi) &= b_0 + b_1\left(\frac{G'}{G}\right) + b_2\left(\frac{G'}{G}\right)^2. \end{aligned} \quad (24)$$

Substituting (24) and (19) into (6a)-(6b) and collecting all terms with the same power of G'/G together, the left-hand sides of (6a)-(6b) are converted into other polynomials in G'/G. Equating each coefficient of the polynomials to zero yields a set of simultaneous algebraic equations for k, ω, a_0, a_1, a_2, b_0, b_1, and b_2 omitted here. Solving the algebraic system leads to several types of traveling wave solutions under various parameter constraints for the KdV-SCS ((1a)-(1b)).

Case 1. $\beta^2 - 4\alpha\gamma + 4\gamma > 0, \alpha \neq 1$.

In this case, we get five sets traveling wave solutions of the KdV-SCS (1a)-(1b).

Solution 1. The first set of the unknowns under this condition reads

$$\begin{aligned} k &= k, \\ \omega &= k\left(4k^2\gamma\alpha - k^2\beta^2 - 4k^2\gamma - 6\lambda \right. \\ &\quad \left. + \frac{8k^2\lambda\gamma\alpha - 2k^2\beta^2\lambda - 8k^2\lambda\gamma - 6\lambda^2}{4k^2\gamma\alpha - k^2\beta^2 - 4k^2\gamma - 2\lambda}\right), \\ a_0 &= -k^2\gamma\alpha + k^2\gamma + \frac{\lambda}{2}, \\ a_1 &= -\beta(\alpha - 1)k^2, \\ a_2 &= -(1-\alpha)^2 k^2, \\ b_0 &= \pm\frac{1}{2}\beta k\sqrt{-\frac{8k^2\lambda\gamma\alpha - 2k^2\beta^2\lambda - 8k^2\lambda\gamma - 6\lambda^2}{4k^2\gamma\alpha - k^2\beta^2 - 4k^2\gamma - 2\lambda}}, \\ b_1 &= \pm\sqrt{-\frac{8k^2\lambda\gamma\alpha - 2k^2\beta^2\lambda - 8k^2\lambda\gamma - 6\lambda^2}{4k^2\gamma\alpha - k^2\beta^2 - 4k^2\gamma - 2\lambda}}(\alpha - 1)k, \\ b_2 &= 0. \end{aligned} \quad (25)$$

Substituting (25) into (24), we get the following solution expression equations:

$$\begin{aligned} U(\xi) &= -k^2\gamma\alpha + k^2\gamma + \frac{\lambda}{2} - \beta(\alpha - 1)k^2\left(\frac{G'}{G}\right) \\ &\quad - (1-\alpha)^2 k^2\left(\frac{G'}{G}\right)^2, \end{aligned} \quad (26a)$$

$$\begin{aligned} V(\xi) &= \pm\frac{1}{2}\beta k\sqrt{-\frac{8k^2\lambda\gamma\alpha - 2k^2\beta^2\lambda - 8k^2\lambda\gamma - 6\lambda^2}{4k^2\gamma\alpha - k^2\beta^2 - 4k^2\gamma - 2\lambda}} \\ &\quad \pm\sqrt{-\frac{8k^2\lambda\gamma\alpha - 2k^2\beta^2\lambda - 8k^2\lambda\gamma - 6\lambda^2}{4k^2\gamma\alpha - k^2\beta^2 - 4k^2\gamma - 2\lambda}}(\alpha - 1) \\ &\quad \cdot k\left(\frac{G'}{G}\right). \end{aligned} \quad (26b)$$

Whence, from the ansatz (26a)-(26b) together with (20), the first set of traveling wave solutions in terms of hyperbolic functions to (1a)-(1b) reads

$$\begin{aligned} u(x,t) &= -k^2\gamma\alpha + k^2\gamma + \frac{\lambda}{2} - \beta(\alpha - 1) \\ &\quad \cdot k^2\left(\frac{\sqrt{\beta^2 - 4\alpha\gamma + 4\gamma}\left(C_1\sinh\left((1/2)\sqrt{\beta^2 - 4\alpha\gamma + 4\gamma}\xi\right) + C_2\cosh\left((1/2)\sqrt{\beta^2 - 4\alpha\gamma + 4\gamma}\xi\right)\right)}{(2 - 2\alpha)\left(C_1\cosh\left((1/2)\sqrt{\beta^2 - 4\alpha\gamma + 4\gamma}\xi\right) + C_2\sinh\left((1/2)\sqrt{\beta^2 - 4\alpha\gamma + 4\gamma}\xi\right)\right)} + \frac{\beta}{2 - 2\alpha}\right) \\ &\quad - (1-\alpha)^2 \end{aligned}$$

$$\cdot k^2 \left(\frac{\sqrt{\beta^2 - 4\alpha\gamma + 4\gamma}\left(C_1 \sinh\left((1/2)\sqrt{\beta^2 - 4\alpha\gamma + 4\gamma}\xi\right) + C_2 \cosh\left((1/2)\sqrt{\beta^2 - 4\alpha\gamma + 4\gamma}\xi\right)\right)}{(2-2\alpha)\left(C_1 \cosh\left((1/2)\sqrt{\beta^2 - 4\alpha\gamma + 4\gamma}\xi\right) + C_2 \sinh\left((1/2)\sqrt{\beta^2 - 4\alpha\gamma + 4\gamma}\xi\right)\right)} + \frac{\beta}{2-2\alpha} \right)^2, \tag{27a}$$

$$v(x,t) = \pm\frac{1}{2}\beta k \sqrt{-\frac{8k^2\lambda\gamma\alpha - 2k^2\beta^2\lambda - 8k^2\lambda\gamma - 6\lambda^2}{4k^2\gamma\alpha - k^2\beta^2 - 4k^2\gamma - 2\lambda}} \pm \sqrt{-\frac{8k^2\lambda\gamma\alpha - 2k^2\beta^2\lambda - 8k^2\lambda\gamma - 6\lambda^2}{4k^2\gamma\alpha - k^2\beta^2 - 4k^2\gamma - 2\lambda}}(\alpha - 1)$$
$$\cdot k \left(\frac{\sqrt{\beta^2 - 4\alpha\gamma + 4\gamma}\left(C_1 \sinh\left((1/2)\sqrt{\beta^2 - 4\alpha\gamma + 4\gamma}\xi\right) + C_2 \cosh\left((1/2)\sqrt{\beta^2 - 4\alpha\gamma + 4\gamma}\xi\right)\right)}{(2-2\alpha)\left(C_1 \cosh\left((1/2)\sqrt{\beta^2 - 4\alpha\gamma + 4\gamma}\xi\right) + C_2 \sinh\left((1/2)\sqrt{\beta^2 - 4\alpha\gamma + 4\gamma}\xi\right)\right)} + \frac{\beta}{2-2\alpha} \right), \tag{27b}$$

where C_1 and C_2 are arbitrary constants and the traveling wave variable is

$$\xi = kx + k\left(4k^2\gamma\alpha - k^2\beta^2 - 4k^2\gamma - 6\lambda + \frac{8k^2\lambda\gamma\alpha - 2k^2\beta^2\lambda - 8k^2\lambda\gamma - 6\lambda^2}{4k^2\gamma\alpha - k^2\beta^2 - 4k^2\gamma - 2\lambda} + 8k^2\lambda\gamma\alpha - 2k^2\beta^2\lambda - 8k^2\lambda\gamma - 6\lambda^2\right)^2 t. \tag{28}$$

Solution 2. The second set of the unknowns under this condition reads

$$k = \pm\sqrt{\frac{\lambda}{4\alpha\gamma - \beta^2 - 4\gamma}},$$
$$\omega = \mp\sqrt{\frac{\lambda}{4\alpha\gamma - \beta^2 - 4\gamma}}\lambda,$$
$$a_0 = -\frac{\lambda\left(2\gamma\alpha + \beta^2 - 2\gamma\right)}{2\left(4\alpha\gamma - \beta^2 - 4\gamma\right)},$$
$$a_1 = -\frac{3\beta\lambda(\alpha - 1)}{4\alpha\gamma - \beta^2 - 4\gamma},$$
$$a_2 = -\frac{3\lambda(\alpha - 1)^2}{4\alpha\gamma - \beta^2 - 4\gamma},$$
$$b_0 = \pm\frac{\lambda\left(2\gamma\alpha + \beta^2 - 2\gamma\right)}{4\alpha\gamma - \beta^2 - 4\gamma},$$
$$b_1 = \pm\frac{6\beta\lambda(\alpha - 1)}{4\alpha\gamma - \beta^2 - 4\gamma},$$
$$b_2 = \pm\frac{6\lambda(\alpha - 1)^2}{4\alpha\gamma - \beta^2 - 4\gamma}. \tag{29}$$

Substituting (29) into (24), we get the following solution expression equations:

$$U(\xi) = -\frac{\lambda\left(2\gamma\alpha + \beta^2 - 2\gamma\right)}{2\left(4\alpha\gamma - \beta^2 - 4\gamma\right)} - \frac{3\beta\lambda(\alpha - 1)}{4\alpha\gamma - \beta^2 - 4\gamma}\left(\frac{G'}{G}\right) - \frac{3\lambda(\alpha - 1)^2}{4\alpha\gamma - \beta^2 - 4\gamma}\left(\frac{G'}{G}\right)^2, \tag{30a}$$

$$V(\xi) = \pm\frac{\lambda\left(2\gamma\alpha + \beta^2 - 2\gamma\right)}{4\alpha\gamma - \beta^2 - 4\gamma} \pm \frac{6\beta\lambda(\alpha - 1)}{4\alpha\gamma - \beta^2 - 4\gamma}\left(\frac{G'}{G}\right) \pm \frac{6\lambda(\alpha - 1)^2}{4\alpha\gamma - \beta^2 - 4\gamma}\left(\frac{G'}{G}\right)^2. \tag{30b}$$

Whence, from the ansatz (30a)-(30b) together with (20), the second set of traveling wave solutions in terms of hyperbolic functions to (1a)-(1b) reads

$$u(x,t) = -\frac{\lambda\left(2\gamma\alpha + \beta^2 - 2\gamma\right)}{2\left(4\alpha\gamma - \beta^2 - 4\gamma\right)}$$
$$- \frac{3\beta\lambda(\alpha - 1)}{4\alpha\gamma - \beta^2 - 4\gamma}\left(\frac{\sqrt{\beta^2 - 4\alpha\gamma + 4\gamma}\left(C_1 \sinh\left((1/2)\sqrt{\beta^2 - 4\alpha\gamma + 4\gamma}\xi\right) + C_2 \cosh\left((1/2)\sqrt{\beta^2 - 4\alpha\gamma + 4\gamma}\xi\right)\right)}{(2-2\alpha)\left(C_1 \cosh\left((1/2)\sqrt{\beta^2 - 4\alpha\gamma + 4\gamma}\xi\right) + C_2 \sinh\left((1/2)\sqrt{\beta^2 - 4\alpha\gamma + 4\gamma}\xi\right)\right)}\right.$$

$$+\frac{\beta}{2-2\alpha}\Bigg)$$

$$-\frac{3\lambda(\alpha-1)^2}{4\alpha\gamma-\beta^2-4\gamma}\left(\frac{\sqrt{\beta^2-4\alpha\gamma+4\gamma}\left(C_1\sinh\left((1/2)\sqrt{\beta^2-4\alpha\gamma+4\gamma}\xi\right)+C_2\cosh\left((1/2)\sqrt{\beta^2-4\alpha\gamma+4\gamma}\xi\right)\right)}{(2-2\alpha)\left(C_1\cosh\left((1/2)\sqrt{\beta^2-4\alpha\gamma+4\gamma}\xi\right)+C_2\sinh\left((1/2)\sqrt{\beta^2-4\alpha\gamma+4\gamma}\xi\right)\right)}\right.$$

$$\left.+\frac{\beta}{2-2\alpha}\right)^2, \tag{31a}$$

$$v(x,t)=\pm\frac{\lambda\left(2\gamma\alpha+\beta^2-2\gamma\right)}{4\alpha\gamma-\beta^2-4\gamma}$$

$$\pm\frac{6\beta\lambda(\alpha-1)}{4\alpha\gamma-\beta^2-4\gamma}\left(\frac{\sqrt{\beta^2-4\alpha\gamma+4\gamma}\left(C_1\sinh\left((1/2)\sqrt{\beta^2-4\alpha\gamma+4\gamma}\xi\right)+C_2\cosh\left((1/2)\sqrt{\beta^2-4\alpha\gamma+4\gamma}\xi\right)\right)}{(2-2\alpha)\left(C_1\cosh\left((1/2)\sqrt{\beta^2-4\alpha\gamma+4\gamma}\xi\right)+C_2\sinh\left((1/2)\sqrt{\beta^2-4\alpha\gamma+4\gamma}\xi\right)\right)}\right.$$

$$\left.+\frac{\beta}{2-2\alpha}\right) \tag{31b}$$

$$\pm\frac{6\lambda(\alpha-1)^2}{4\alpha\gamma-\beta^2-4\gamma}\left(\frac{\sqrt{\beta^2-4\alpha\gamma+4\gamma}\left(C_1\sinh\left((1/2)\sqrt{\beta^2-4\alpha\gamma+4\gamma}\xi\right)+C_2\cosh\left((1/2)\sqrt{\beta^2-4\alpha\gamma+4\gamma}\xi\right)\right)}{(2-2\alpha)\left(C_1\cosh\left((1/2)\sqrt{\beta^2-4\alpha\gamma+4\gamma}\xi\right)+C_2\sinh\left((1/2)\sqrt{\beta^2-4\alpha\gamma+4\gamma}\xi\right)\right)}\right.$$

$$\left.+\frac{\beta}{2-2\alpha}\right)^2,$$

where C_1 and C_2 are arbitrary constants and the traveling wave variable is

$$\xi=\pm\sqrt{\frac{\lambda}{4\alpha\gamma-\beta^2-4\gamma}}x\mp\sqrt{\frac{\lambda}{4\alpha\gamma-\beta^2-4\gamma}}\lambda t \quad \text{with } \lambda<0. \tag{32}$$

Solution 3. The third set of the unknowns under this condition reads

$$k=\pm\sqrt{-\frac{\lambda}{4\alpha\gamma-\beta^2-4\gamma}},$$

$$\omega=\mp\sqrt{-\frac{\lambda}{4\alpha\gamma-\beta^2-4\gamma}}\lambda,$$

$$a_0=\frac{3\gamma\lambda(\alpha-1)}{4\alpha\gamma-\beta^2-4\gamma},$$

$$a_1=\frac{3\beta\lambda(\alpha-1)}{4\alpha\gamma-\beta^2-4\gamma},$$

$$a_2=\frac{3\lambda(\alpha-1)^2}{4\alpha\gamma-\beta^2-4\gamma},$$

$$b_0=\pm\frac{6\gamma\lambda(\alpha-1)}{4\alpha\gamma-\beta^2-4\gamma},$$

$$b_1=\pm\frac{6\beta\lambda(\alpha-1)}{4\alpha\gamma-\beta^2-4\gamma},$$

$$b_2=\pm\frac{6\lambda(\alpha-1)^2}{4\alpha\gamma-\beta^2-4\gamma}. \tag{33}$$

Substituting (33) into (24), we get the following solution expression equations:

$$U(\xi)=\frac{3\gamma\lambda(\alpha-1)}{4\alpha\gamma-\beta^2-4\gamma}+\frac{3\beta\lambda(\alpha-1)}{4\alpha\gamma-\beta^2-4\gamma}\left(\frac{G'}{G}\right)$$

$$+\frac{3\lambda(\alpha-1)^2}{4\alpha\gamma-\beta^2-4\gamma}\left(\frac{G'}{G}\right)^2, \tag{34a}$$

$$V(\xi)=\pm\frac{6\gamma\lambda(\alpha-1)}{4\alpha\gamma-\beta^2-4\gamma}\pm\frac{6\beta\lambda(\alpha-1)}{4\alpha\gamma-\beta^2-4\gamma}\left(\frac{G'}{G}\right)$$

$$\pm\frac{6\lambda(\alpha-1)^2}{4\alpha\gamma-\beta^2-4\gamma}\left(\frac{G'}{G}\right)^2. \tag{34b}$$

Whence, from the ansatz (34a)-(34b) together with (20), the third set of traveling wave solutions in terms of hyperbolic functions to (1a)-(1b) reads

$$\begin{aligned} u(x,t) &= \frac{3\gamma\lambda(\alpha-1)}{4\alpha\gamma-\beta^2-4\gamma} \\ &+ \frac{3\beta\lambda(\alpha-1)}{4\alpha\gamma-\beta^2-4\gamma}\left(\frac{\sqrt{\beta^2-4\alpha\gamma+4\gamma}\left(C_1\sinh\left((1/2)\sqrt{\beta^2-4\alpha\gamma+4\gamma}\xi\right)+C_2\cosh\left((1/2)\sqrt{\beta^2-4\alpha\gamma+4\gamma}\xi\right)\right)}{(2-2\alpha)\left(C_1\cosh\left((1/2)\sqrt{\beta^2-4\alpha\gamma+4\gamma}\xi\right)+C_2\sinh\left((1/2)\sqrt{\beta^2-4\alpha\gamma+4\gamma}\xi\right)\right)}\right. \\ &\left.+\frac{\beta}{2-2\alpha}\right) \\ &+ \frac{3\lambda(\alpha-1)^2}{4\alpha\gamma-\beta^2-4\gamma}\left(\frac{\sqrt{\beta^2-4\alpha\gamma+4\gamma}\left(C_1\sinh\left((1/2)\sqrt{\beta^2-4\alpha\gamma+4\gamma}\xi\right)+C_2\cosh\left((1/2)\sqrt{\beta^2-4\alpha\gamma+4\gamma}\xi\right)\right)}{(2-2\alpha)\left(C_1\cosh\left((1/2)\sqrt{\beta^2-4\alpha\gamma+4\gamma}\xi\right)+C_2\sinh\left((1/2)\sqrt{\beta^2-4\alpha\gamma+4\gamma}\xi\right)\right)}\right. \\ &\left.+\frac{\beta}{2-2\alpha}\right)^2, \end{aligned} \tag{35a}$$

$$\begin{aligned} v(x,t) &= \pm\frac{6\gamma\lambda(\alpha-1)}{4\alpha\gamma-\beta^2-4\gamma} \\ &\pm \frac{6\beta\lambda(\alpha-1)}{4\alpha\gamma-\beta^2-4\gamma}\left(\frac{\sqrt{\beta^2-4\alpha\gamma+4\gamma}\left(C_1\sinh\left((1/2)\sqrt{\beta^2-4\alpha\gamma+4\gamma}\xi\right)+C_2\cosh\left((1/2)\sqrt{\beta^2-4\alpha\gamma+4\gamma}\xi\right)\right)}{(2-2\alpha)\left(C_1\cosh\left((1/2)\sqrt{\beta^2-4\alpha\gamma+4\gamma}\xi\right)+C_2\sinh\left((1/2)\sqrt{\beta^2-4\alpha\gamma+4\gamma}\xi\right)\right)}\right. \\ &\left.+\frac{\beta}{2-2\alpha}\right) \\ &\pm \frac{6\lambda(\alpha-1)^2}{4\alpha\gamma-\beta^2-4\gamma}\left(\frac{\sqrt{\beta^2-4\alpha\gamma+4\gamma}\left(C_1\sinh\left((1/2)\sqrt{\beta^2-4\alpha\gamma+4\gamma}\xi\right)+C_2\cosh\left((1/2)\sqrt{\beta^2-4\alpha\gamma+4\gamma}\xi\right)\right)}{(2-2\alpha)\left(C_1\cosh\left((1/2)\sqrt{\beta^2-4\alpha\gamma+4\gamma}\xi\right)+C_2\sinh\left((1/2)\sqrt{\beta^2-4\alpha\gamma+4\gamma}\xi\right)\right)}\right. \\ &\left.+\frac{\beta}{2-2\alpha}\right)^2, \end{aligned} \tag{35b}$$

where C_1 and C_2 are arbitrary constants and the traveling wave variable is

$$\xi = \pm\sqrt{-\frac{\lambda}{4\alpha\gamma-\beta^2-4\gamma}}x \mp \sqrt{-\frac{\lambda}{4\alpha\gamma-\beta^2-4\gamma}}\lambda t \quad \text{with } \lambda > 0. \tag{36}$$

Solution 4. The fourth set of the unknowns under this condition reads

$$\begin{aligned}
k &= \pm\sqrt{\frac{3\lambda}{8\alpha\gamma-2\beta^2-8\gamma}},\\
\omega &= \pm\frac{3}{2}\sqrt{\frac{3\lambda}{8\alpha\gamma-2\beta^2-8\gamma}}\lambda,\\
a_0 &= -\frac{\lambda\left(5\gamma\alpha+\beta^2-5\gamma\right)}{2\left(4\alpha\gamma-\beta^2-4\gamma\right)},\\
a_1 &= -\frac{9\beta\lambda(\alpha-1)}{2\left(4\alpha\gamma-\beta^2-4\gamma\right)},\\
a_2 &= -\frac{9\lambda(\alpha-1)^2}{2\left(4\alpha\gamma-\beta^2-4\gamma\right)},\\
b_0 &= \pm\frac{3\lambda\left(2\gamma\alpha+\beta^2-2\gamma\right)}{2\left(4\alpha\gamma-\beta^2-4\gamma\right)},\\
b_1 &= \pm\frac{9\beta\lambda(\alpha-1)}{4\alpha\gamma-\beta^2-4\gamma},\\
b_2 &= \pm\frac{9\lambda(\alpha-1)^2}{4\alpha\gamma-\beta^2-4\gamma}.
\end{aligned} \tag{37}$$

Substituting (37) into (24), we get the following solution expression equations:

$$U(\xi) = -\frac{\lambda\left(5\gamma\alpha+\beta^2-5\gamma\right)}{2\left(4\alpha\gamma-\beta^2-4\gamma\right)} - \frac{9\beta\lambda(\alpha-1)}{2\left(4\alpha\gamma-\beta^2-4\gamma\right)}\left(\frac{G'}{G}\right) - \frac{9\lambda(\alpha-1)^2}{2\left(4\alpha\gamma-\beta^2-4\gamma\right)}\left(\frac{G'}{G}\right)^2, \tag{38a}$$

$$V(\xi) = \pm\frac{3\lambda\left(2\gamma\alpha+\beta^2-2\gamma\right)}{2\left(4\alpha\gamma-\beta^2-4\gamma\right)} \pm \frac{9\beta\lambda(\alpha-1)}{4\alpha\gamma-\beta^2-4\gamma}\left(\frac{G'}{G}\right) \pm \frac{9\lambda(\alpha-1)^2}{4\alpha\gamma-\beta^2-4\gamma}\left(\frac{G'}{G}\right)^2. \tag{38b}$$

Whence, from the ansatz (38a)-(38b) together with (20), the fourth set of traveling wave solutions in terms of hyperbolic functions to (1a)-(1b) reads

$$\begin{aligned}
u(x,t) &= -\frac{\lambda\left(5\gamma\alpha+\beta^2-5\gamma\right)}{2\left(4\alpha\gamma-\beta^2-4\gamma\right)}\\
&- \frac{9\beta\lambda(\alpha-1)}{2\left(4\alpha\gamma-\beta^2-4\gamma\right)}\left(\frac{\sqrt{\beta^2-4\alpha\gamma+4\gamma}\left(C_1\sinh\left((1/2)\sqrt{\beta^2-4\alpha\gamma+4\gamma}\xi\right)+C_2\cosh\left((1/2)\sqrt{\beta^2-4\alpha\gamma+4\gamma}\xi\right)\right)}{(2-2\alpha)\left(C_1\cosh\left((1/2)\sqrt{\beta^2-4\alpha\gamma+4\gamma}\xi\right)+C_2\sinh\left((1/2)\sqrt{\beta^2-4\alpha\gamma+4\gamma}\xi\right)\right)}\right.\\
&\left.+\frac{\beta}{2-2\alpha}\right)\\
&- \frac{9\lambda(\alpha-1)^2}{2\left(4\alpha\gamma-\beta^2-4\gamma\right)}\left(\frac{\sqrt{\beta^2-4\alpha\gamma+4\gamma}\left(C_1\sinh\left((1/2)\sqrt{\beta^2-4\alpha\gamma+4\gamma}\xi\right)+C_2\cosh\left((1/2)\sqrt{\beta^2-4\alpha\gamma+4\gamma}\xi\right)\right)}{(2-2\alpha)\left(C_1\cosh\left((1/2)\sqrt{\beta^2-4\alpha\gamma+4\gamma}\xi\right)+C_2\sinh\left((1/2)\sqrt{\beta^2-4\alpha\gamma+4\gamma}\xi\right)\right)}\right.\\
&\left.+\frac{\beta}{2-2\alpha}\right)^2,
\end{aligned} \tag{39a}$$

$$v(x,t) = \pm\frac{3\lambda\left(2\gamma\alpha+\beta^2-2\gamma\right)}{2\left(4\alpha\gamma-\beta^2-4\gamma\right)} \pm \frac{9\beta\lambda(\alpha-1)}{4\alpha\gamma-\beta^2-4\gamma}\left(\frac{\sqrt{\beta^2-4\alpha\gamma+4\gamma}\left(C_1\sinh\left((1/2)\sqrt{\beta^2-4\alpha\gamma+4\gamma}\xi\right)+C_2\cosh\left((1/2)\sqrt{\beta^2-4\alpha\gamma+4\gamma}\xi\right)\right)}{(2-2\alpha)\left(C_1\cosh\left((1/2)\sqrt{\beta^2-4\alpha\gamma+4\gamma}\xi\right)+C_2\sinh\left((1/2)\sqrt{\beta^2-4\alpha\gamma+4\gamma}\xi\right)\right)} + \frac{\beta}{2-2\alpha}\right)$$
$$\pm \frac{9\lambda(\alpha-1)^2}{4\alpha\gamma-\beta^2-4\gamma}\left(\frac{\sqrt{\beta^2-4\alpha\gamma+4\gamma}\left(C_1\sinh\left((1/2)\sqrt{\beta^2-4\alpha\gamma+4\gamma}\xi\right)+C_2\cosh\left((1/2)\sqrt{\beta^2-4\alpha\gamma+4\gamma}\xi\right)\right)}{(2-2\alpha)\left(C_1\cosh\left((1/2)\sqrt{\beta^2-4\alpha\gamma+4\gamma}\xi\right)+C_2\sinh\left((1/2)\sqrt{\beta^2-4\alpha\gamma+4\gamma}\xi\right)\right)} + \frac{\beta}{2-2\alpha}\right)^2, \tag{39b}$$

where C_1 and C_2 are arbitrary constants and the traveling wave variable is

$$\xi = \pm\sqrt{\frac{3\lambda}{8\alpha\gamma-2\beta^2-8\gamma}}x \pm \frac{3}{2}\sqrt{\frac{3\lambda}{8\alpha\gamma-2\beta^2-8\gamma}}\lambda t \quad \textit{with } \lambda < 0. \tag{40}$$

Solution 5. The fifth set of the unknowns under this condition reads

$$k = \pm\sqrt{\frac{-3\lambda}{8\alpha\gamma-2\beta^2-8\gamma}},$$
$$\omega = \pm\frac{3}{2}\sqrt{\frac{-3\lambda}{8\alpha\gamma-2\beta^2-8\gamma}}\lambda,$$
$$a_0 = \frac{\lambda\left(14\gamma\alpha+\beta^2-14\gamma\right)}{4\left(4\alpha\gamma-\beta^2-4\gamma\right)},$$
$$a_1 = \frac{9\beta\lambda(\alpha-1)}{2\left(4\alpha\gamma-\beta^2-4\gamma\right)},$$
$$a_2 = \frac{9\lambda(\alpha-1)^2}{2\left(4\alpha\gamma-\beta^2-4\gamma\right)},$$
$$b_0 = \pm\frac{9\gamma\lambda(\alpha-1)}{4\alpha\gamma-\beta^2-4\gamma},$$
$$b_1 = \pm\frac{9\beta\lambda(\alpha-1)}{4\alpha\gamma-\beta^2-4\gamma},$$
$$b_2 = \pm\frac{9\lambda(\alpha-1)^2}{4\alpha\gamma-\beta^2-4\gamma}. \tag{41}$$

Substituting (41) into (24), we get the following solution expression equations:

$$U(\xi) = \frac{\lambda\left(14\gamma\alpha+\beta^2-14\gamma\right)}{4\left(4\alpha\gamma-\beta^2-4\gamma\right)} + \frac{9\beta\lambda(\alpha-1)}{2\left(4\alpha\gamma-\beta^2-4\gamma\right)}\left(\frac{G'}{G}\right) + \frac{9\lambda(\alpha-1)^2}{2\left(4\alpha\gamma-\beta^2-4\gamma\right)}\left(\frac{G'}{G}\right)^2, \tag{42a}$$

$$V(\xi) = \pm\frac{9\gamma\lambda(\alpha-1)}{4\alpha\gamma-\beta^2-4\gamma} \pm \frac{9\beta\lambda(\alpha-1)}{4\alpha\gamma-\beta^2-4\gamma}\left(\frac{G'}{G}\right) \pm \frac{9\lambda(\alpha-1)^2}{4\alpha\gamma-\beta^2-4\gamma}\left(\frac{G'}{G}\right)^2. \tag{42b}$$

Whence, from the ansatz (42a)-(42b) together with (20), the fifth set of traveling wave solutions in terms of hyperbolic functions to (1a)-(1b) reads

$$u(x,t) = \frac{\lambda\left(14\gamma\alpha+\beta^2-14\gamma\right)}{4\left(4\alpha\gamma-\beta^2-4\gamma\right)}$$

$$+\frac{9\beta\lambda(\alpha-1)}{2(4\alpha\gamma-\beta^2-4\gamma)}\left(\frac{\sqrt{\beta^2-4\alpha\gamma+4\gamma}\left(C_1\sinh\left((1/2)\sqrt{\beta^2-4\alpha\gamma+4\gamma}\xi\right)+C_2\cosh\left((1/2)\sqrt{\beta^2-4\alpha\gamma+4\gamma}\xi\right)\right)}{(2-2\alpha)\left(C_1\cosh\left((1/2)\sqrt{\beta^2-4\alpha\gamma+4\gamma}\xi\right)+C_2\sinh\left((1/2)\sqrt{\beta^2-4\alpha\gamma+4\gamma}\xi\right)\right)}+\frac{\beta}{2-2\alpha}\right)$$

$$+\frac{9\lambda(\alpha-1)^2}{2(4\alpha\gamma-\beta^2-4\gamma)}\left(\frac{\sqrt{\beta^2-4\alpha\gamma+4\gamma}\left(C_1\sinh\left((1/2)\sqrt{\beta^2-4\alpha\gamma+4\gamma}\xi\right)+C_2\cosh\left((1/2)\sqrt{\beta^2-4\alpha\gamma+4\gamma}\xi\right)\right)}{(2-2\alpha)\left(C_1\cosh\left((1/2)\sqrt{\beta^2-4\alpha\gamma+4\gamma}\xi\right)+C_2\sinh\left((1/2)\sqrt{\beta^2-4\alpha\gamma+4\gamma}\xi\right)\right)}+\frac{\beta}{2-2\alpha}\right)^2, \tag{43a}$$

$$v(x,t)=\pm\frac{9\gamma\lambda(\alpha-1)}{4\alpha\gamma-\beta^2-4\gamma}$$

$$\pm\frac{9\beta\lambda(\alpha-1)}{4\alpha\gamma-\beta^2-4\gamma}\left(\frac{\sqrt{\beta^2-4\alpha\gamma+4\gamma}\left(C_1\sinh\left((1/2)\sqrt{\beta^2-4\alpha\gamma+4\gamma}\xi\right)+C_2\cosh\left((1/2)\sqrt{\beta^2-4\alpha\gamma+4\gamma}\xi\right)\right)}{(2-2\alpha)\left(C_1\cosh\left((1/2)\sqrt{\beta^2-4\alpha\gamma+4\gamma}\xi\right)+C_2\sinh\left((1/2)\sqrt{\beta^2-4\alpha\gamma+4\gamma}\xi\right)\right)}+\frac{\beta}{2-2\alpha}\right) \tag{43b}$$

$$\pm\frac{9\lambda(\alpha-1)^2}{4\alpha\gamma-\beta^2-4\gamma}\left(\frac{\sqrt{\beta^2-4\alpha\gamma+4\gamma}\left(C_1\sinh\left((1/2)\sqrt{\beta^2-4\alpha\gamma+4\gamma}\xi\right)+C_2\cosh\left((1/2)\sqrt{\beta^2-4\alpha\gamma+4\gamma}\xi\right)\right)}{(2-2\alpha)\left(C_1\cosh\left((1/2)\sqrt{\beta^2-4\alpha\gamma+4\gamma}\xi\right)+C_2\sinh\left((1/2)\sqrt{\beta^2-4\alpha\gamma+4\gamma}\xi\right)\right)}+\frac{\beta}{2-2\alpha}\right)^2,$$

where C_1 and C_2 are arbitrary constants and the traveling wave variable is

$$\xi=\pm\sqrt{\frac{-3\lambda}{8\alpha\gamma-2\beta^2-8\gamma}}x\pm\frac{3}{2}\sqrt{\frac{-3\lambda}{8\alpha\gamma-2\beta^2-8\gamma}}\lambda t \quad \text{with } \lambda>0. \tag{44}$$

Case 2. $\beta^2-4\alpha\gamma+4\gamma<0, \alpha\neq 1$.

In this case, each set of the unknowns and the corresponding $U(\xi)$ and $V(\xi)$ on this occasion consists with that in the case of $\beta^2-4\alpha\gamma+4\gamma>0$. We only need to replace G'/G of each group of solutions in Case 1 to (21). So the other five sets of traveling wave solutions are obtained and denoted by solutions 6-10.

Solution 11. The set of the unknowns under this condition reads

$$k=\pm\frac{\sqrt{-a_2}}{\alpha-1},$$

$$\omega=\mp3\frac{\sqrt{-a_2}\lambda}{\alpha-1},$$

$$a_0=\frac{1}{4}\frac{2\alpha^2\lambda-4\alpha\lambda+2\lambda+\beta^2a_2}{(\alpha-1)^2},$$

$$a_1=\frac{\beta a_2}{\alpha-1},$$

$$b_0=\pm\frac{1}{2}\frac{\sqrt{3\lambda a_2}\beta}{\alpha-1},$$

$$b_1 = \pm\sqrt{3\lambda a_2},$$
$$b_2 = 0. \quad (45)$$

Substituting (45) into (24) together with (22), we get the eleventh set of travelling wave solutions in terms of rational functions to (1a)-(1b) as follows:

$$u(x,t) = \frac{1}{4}\frac{2\alpha^2\lambda - 4\alpha\lambda + 2\lambda + \beta^2 a_2}{(\alpha-1)^2} - \frac{1}{2}\frac{\beta a_2(\xi\beta + 2 + C_1\beta)}{(\alpha-1)(\xi\alpha - \xi + C_1\alpha - C_1)} + \frac{1}{4}\frac{a_2(\xi\beta + 2 + C_1\beta)^2}{(\xi\alpha - \xi + C_1\alpha - C_1)^2}, \quad (46a)$$

$$v(x,t) = \pm\frac{1}{2}\frac{\sqrt{3\lambda a_2}\beta}{\alpha-1} \mp \frac{1}{2}\frac{\sqrt{3\lambda a_2}(\xi\beta + 2 + C_1\beta)}{\xi\alpha - \xi + C_1\alpha - C_1}, \quad (46b)$$

where C_1 is arbitrary constant and $a_2 < 0$ and the traveling wave variable is

$$\xi = \pm\frac{\sqrt{-a_2}}{\alpha-1}x \mp 3\frac{\sqrt{-a_2}\lambda}{\alpha-1}t \quad with\ \lambda < 0. \quad (47)$$

4. Group-Invariant Solution

As that in [31], one can suppose that the system of (1a)-(1b) admits a set of one-parameter (ε) Lie group of point transformations:

$$\begin{aligned} x^* &= x, \\ t^* &= t, \\ u^* &= u + \varepsilon\sigma_1(x,t,u,v) + O(\epsilon^2), \\ v^* &= v + \varepsilon\sigma_2(x,t,u,v) + O(\epsilon^2). \end{aligned} \quad (48)$$

As known, σ_1 and σ_2 are called the classical Lie point symmetry of (1a)-(1b) which satisfy

$$\begin{aligned} V_1 &= \sigma_1\frac{\partial}{\partial u}, \\ \sigma_1 &= U(x,t,u,v) - X(x,t,u,v)\frac{\partial}{\partial x}u - T(x,t,u,v)\frac{\partial}{\partial t}u, \\ V_2 &= \sigma_2\frac{\partial}{\partial v}, \\ \sigma_2 &= V(x,t,u,v) - X(x,t,u,v)\frac{\partial}{\partial x}v - T(x,t,u,v)\frac{\partial}{\partial t}v, \end{aligned} \quad (49)$$

where $X(x,t,u,v)$, $T(x,t,u,v)$, $U(x,t,u,v)$, and $V(x,t,u,v)$ are infinitesimals, which can be obtained by solving a determining equations, and V_1, V_2 are the corresponding Lie point symmetry generators of group (48). Obviously, σ_1 and σ_2 satisfy

$$\begin{aligned} \frac{\partial}{\partial\varepsilon}F(u + \varepsilon\sigma_1)\Big|_{\varepsilon=0} &= 0, \\ \frac{\partial}{\partial\varepsilon}F(v + \varepsilon\sigma_2)\Big|_{\varepsilon=0} &= 0. \end{aligned} \quad (50)$$

Theorem 1. *The determining equations possess the following properties:*

(i) $U(x,t,u,v)$ and $V(x,t,u,v)$ are linear functions about u and v, respectively, $T(x,t,u,v)$ is a function only about t, and $X(x,t,u,v)$ is independent of u and v.

(ii) $X(x,t,u,v)$, $T(x,t,u,v)$, $U(x,t,u,v)$, and $V(x,t,u,v)$ satisfy the following equations:

$$\begin{aligned} \Big\{ &-3\frac{\partial}{\partial x}X + \frac{\partial}{\partial t}T, \frac{\partial^2}{\partial x^2}X - 2\frac{\partial^2}{\partial x\partial v}V, -3\frac{\partial^2}{\partial x\partial u}U \\ &+ 3\frac{\partial^2}{\partial x^2}X, 4v\frac{\partial}{\partial x}X + 2v\frac{\partial}{\partial v}V - 2\left(\frac{\partial}{\partial u}U\right)v + 2V, \\ &-\frac{\partial^3}{\partial x^3}U - \frac{\partial}{\partial t}U + 2v\frac{\partial}{\partial x}V - 12u\frac{\partial}{\partial x}U, \\ &- 24\left(\frac{\partial}{\partial x}X\right)u + \frac{\partial}{\partial t}X + \frac{\partial^3}{\partial x^3}X - 3\frac{\partial^3}{\partial x^2\partial u}U \\ &- 12U, -4\left(\frac{\partial}{\partial x}X\right)uv + 2\left(\frac{\partial}{\partial x}X\right)\lambda v + 2\left(\frac{\partial}{\partial v}V\right) \\ &\cdot uv - \left(\frac{\partial}{\partial v}V\right)\lambda v - 2vU - 2uV + \lambda V - \frac{\partial^2}{\partial x^2}V, \frac{\partial}{\partial u} \\ &\cdot T, \frac{\partial}{\partial v}T, \frac{\partial}{\partial x}T, \frac{\partial}{\partial v}U, \frac{\partial}{\partial u}V, \frac{\partial}{\partial u}X, \frac{\partial}{\partial v}X, \frac{\partial^2}{\partial v^2}V, \frac{\partial^3}{\partial x\partial u^2} \\ &\cdot U\Big\}, \end{aligned} \quad (51)$$

which on solving yields

$$\begin{aligned} X(x,t,u,v) &= \frac{(x + 12\lambda t)\theta_1}{3} + \theta_3, \\ T(x,t,u,v) &= \theta_1 t + \theta_2, \\ U(x,t,u,v) &= -\frac{(2u - \lambda)\theta_1}{3}, \\ V(x,t,u,v) &= -\frac{2}{3}v\theta_1. \end{aligned} \quad (52)$$

Once the infinitesimals $X(x,t,u,v)$, $T(x,t,u,v)$, $U(x,t,u,v)$, and $V(x,t,u,v)$ are determined, we can obtain the following Lie symmetry groups.

In Lie symmetry group if $\theta_1 \neq 0$,

$$\begin{aligned} x1(\epsilon) &= \left(x - 6\lambda t + \frac{3\theta_3 - 18\lambda\theta_2}{\theta_1}\right) e^{(\theta_1/3)\epsilon} \\ &\quad + \left(6\lambda t + \frac{6\lambda\theta_2}{\theta_1}\right) e^{\theta_1 \epsilon} + \frac{12\lambda\theta_2 - 3\theta_3}{\theta_1}, \\ t1(\epsilon) &= \left(t + \frac{\theta_2}{\theta_1}\right) e^{\theta_1\epsilon} - \frac{\theta_2}{\theta_1}, \\ u1(\epsilon) &= \left(u - \frac{\lambda}{2}\right) e^{-(2\theta_1/3)\epsilon} + \frac{\lambda}{2}, \\ v1(\epsilon) &= v e^{-(2\theta_1/3)\epsilon}. \end{aligned} \tag{53}$$

Hence, solving the well-known characteristic equations gives birth to the first group-invariant solution (the so-called similarity solution)

$$\begin{aligned} u(x,t) &= \frac{1}{2}\lambda + \frac{U(\xi)}{(\theta_1 t + \theta_2)^{2/3}}, \\ v(x,t) &= \frac{V(\xi)}{(\theta_1 t + \theta_2)^{2/3}}, \end{aligned} \tag{54}$$

where the similarity variable is

$$\xi = \frac{x}{\sqrt[3]{\theta_1 t + \theta_2}} - \frac{6\lambda t}{\sqrt[3]{\theta_1 t + \theta_2}} + \frac{3\theta_3 - 18\lambda\theta_2}{\sqrt[3]{\theta_1 t + \theta_2}\theta_1} \tag{55}$$

and $U(\xi)$, $V(\xi)$ satisfy

$$\begin{aligned} U''' &= -12UU' + 2VV' + \frac{2}{3}\theta_1 U \\ &\quad - \frac{1}{3}\frac{-(\theta_1 t + \theta_2)^{2/3} x\theta_1 + 6(\theta_1 t + \theta_2)^{2/3}\theta_1\lambda t - 3(\theta_1 t + \theta_2)^{2/3}\theta_3 + 18\lambda(\theta_1 t + \theta_2)^{2/3}\theta_2}{\theta_1 t + \theta_2} U', \end{aligned} \tag{56a}$$

$$V'' = -2UV, \tag{56b}$$

When $\theta_1 = 0$, we obtain another Lie symmetry group:

$$\begin{aligned} x1(\epsilon) &= x + \theta_3\epsilon, \\ t1(\epsilon) &= t + \theta_2\epsilon, \\ u1(\epsilon) &= u, \\ v1(\epsilon) &= v. \end{aligned} \tag{57}$$

Meanwhile the second group-invariant solution is characterized by $u(x,t) = U(\xi)$, $v(x,t) = V(\xi)$, and $U(\xi)$, $V(\xi)$ satisfy

$$U''' = \frac{\theta_3}{\theta_2}U' - 12UU' + 2VV', \tag{58a}$$

$$V'' = -2UV + \lambda V, \tag{58b}$$

where the similarity variable

$$\xi = x - \frac{\theta_3}{\theta_2}t. \tag{59}$$

5. The Painlevé Property of the KdV-SCS

We have explored the Painlevé property of (1a)-(1b) in [32], where we only gave two principal branches. Now we obtain other principal branches, and on the basis of that we give more exact expansions of (1a)-(1b).

The expansions of (1a)-(1b) about the singular manifold have the forms,

$$\begin{aligned} u(x,t) &= \sum_{j=0}^{\infty} u_j(x,t)\,\phi(x,t)^{j+\mu}, \\ v(x,t) &= \sum_{j=0}^{\infty} v_j(x,t)\,\phi(x,t)^{j+\nu}. \end{aligned} \tag{60}$$

By using the WTC Painlevé test [33], we first obtain the following three principal branches:

$$\begin{aligned} &(i): u \sim -\phi_x^2\phi^{-2},\ v \sim v_0\phi^{-1},\ v_0 \text{ arbitrary}; \\ &(ii): u \sim -3\phi_x^2\phi^{-2},\ v \sim 6\phi_x^2\phi^{-2}; \\ &(iii): u \sim -3\phi_x^2\phi^{-2},\ v \sim -6\phi_x^2\phi^{-2}. \end{aligned} \tag{61}$$

Principal branch (i): it turns out that the branch (i) is the only principal branch, with resonances

$$r = -1, 0, 3, 4, 6. \tag{62}$$

The resonance −1 is always present, since it corresponds to the arbitrariness of ϕ, while $r = 0$ comes from the arbitrary constant v_0 in the leading order term of the expansion for v; the other three values arise from arbitrary coefficients higher up in the series for u, v, so that altogether there should be five arbitrary constants appearing in these Laurent series.

Next referring to the procedure in [34], we give the truncated expansion forms for (i) as follows:

$$u = (\ln \phi)_{xx} + \tilde{u}, \quad \tilde{u} \equiv u_2, \tag{63a}$$

$$v = \frac{v_0}{\phi} + \tilde{v}, \quad \tilde{v} \equiv v_1, \tag{63b}$$

where u, v, ϕ, u_2, v_0, and v_1 all are functions of x, t.

We can take the coefficients in the expansions (60) to be functions of t only, as this is referred to the 'reduced ansatz' of Kruskal [35]. So we get the coefficients of expansions (60) for (i) as follows:

$$\begin{aligned}
u_0(t) &= -1,\\
v_0(t) &= v_0(t);\\
u_1(t) &= 0,\\
v_1(t) &= 0;\\
u_2(t) &= \frac{1}{12}\psi'(t) - \frac{1}{12}v_0(t)^2,\\
v_2(t) &= -\frac{\lambda}{2}v_0(t) + \frac{1}{12}v_0(t)\psi'(t) - \frac{1}{12}v_0(t)^3;\\
u_3(t) &= 0,\\
v_3(t) &= v_3(t);\\
u_4(t) &= u_4(t),\\
v_4(t) &= -\frac{\lambda^2}{8}v_0(t) + \frac{\lambda}{24}v_0(t)\psi'(t) - \frac{\lambda}{24}v_0(t)^3\\
&\quad - \frac{1}{288}v_0(t)\psi'(t)^2 + \frac{1}{144}\psi'(t)v_0(t)^3\\
&\quad - \frac{1}{288}v_0(t)^5 - \frac{1}{2}v_0(t)u_4(t);\\
u_5(t) &= -\frac{1}{3}v_0(t)v_3(t) + \frac{1}{72}\psi''(t) - \frac{1}{36}v_0(t)v_0'(t),\\
v_5(t) &= -\frac{1}{60}v_3(t)\psi'(t) + \frac{1}{12}v_3(t)v_0(t)^2\\
&\quad - \frac{1}{360}v_0(t)\psi''(t) + \frac{1}{180}v_0(t)^2v_0'(t)\\
&\quad + \frac{\lambda}{10}v_3(t);\\
u_6(t) &= u_6(t),\\
v_6(t) &= \frac{\lambda^2}{288}v_0(t)\psi'(t) - \frac{\lambda^2}{288}v_0(t)^3\\
&\quad - \frac{\lambda}{1728}v_0(t)\psi'(t)^2 + \frac{\lambda}{864}v_0(t)^3\psi'(t)\\
&\quad - \frac{\lambda}{1728}v_0(t)^5 + \frac{1}{31104}v_0(t)\psi'(t)^3\\
&\quad - \frac{1}{10368}v_0(t)^3\psi'(t)^2\\
&\quad + \frac{1}{10368}v_0(t)^5\psi'(t) - \frac{1}{31104}v_0(t)^7\\
&\quad - \frac{1}{216}v_0(t)u_4(t)\psi'(t)\\
&\quad + \frac{1}{216}u_4(t)v_0(t)^3 + \frac{\lambda}{36}u_4(t)v_0(t)\\
&\quad - \frac{\lambda^3}{144}v_0(t) - \frac{1}{9}u_6(t)v_0(t),
\end{aligned} \tag{64}$$

where $\psi(t)$ is an arbitrary function. As can be seen from the above expressions, the resonance conditions at $r = 3, 4, 6$ corresponding to $v_3(t)$, $u_4(t)$, and $u_6(t)$ are satisfied.

Nonprincipal branch (ii): the second branch (ii) has resonances

$$r = -1, -3, 4, 6, 8. \tag{65}$$

The presence of $r = -3$ means that this is a nonprincipal branch. Then there should be four arbitrary constants appearing in these Laurent series.

We give the truncated expansion forms for (ii) as follows:

$$u = 3(\ln \phi)_{xx} + \tilde{u}, \quad \tilde{u} \equiv u_2, \tag{66a}$$

$$v = -6(\ln \phi)_{xx} + \tilde{v}, \quad \tilde{v} \equiv v_2, \tag{66b}$$

where u, v, ϕ, u_2, and v_2 all are functions of x, t.

Referring to the 'reduced ansatz' of Kruskal, we get the coefficients of expansions (60) for (ii) as follows:

$$\begin{aligned}
u_0(t) &= -3,\\
v_0(t) &= 6;\\
u_1(t) &= 0,\\
v_1(t) &= 0;\\
u_2(t) &= \frac{1}{20}\psi'(t) + \frac{\lambda}{5},\\
v_2(t) &= -\frac{3\lambda}{5} + \frac{1}{10}\psi'(t);\\
u_3(t) &= 0,\\
v_3(t) &= 0;\\
u_4(t) &= u_4(t),\\
v_4(t) &= -\frac{3\lambda}{100}\psi'(t) + \frac{1}{400}\psi'(t)^2 + \frac{9\lambda^2}{100} + 3u_4(t);\\
u_5(t) &= 0,\\
v_5(t) &= \frac{1}{240}\psi''(t);\\
u_6(t) &= -\frac{1}{24}u_4(t)\psi'(t) + \frac{\lambda}{4}u_4(t) - \frac{1}{2}v_6(t)\\
&\quad - \frac{9\lambda^2}{4000}\psi'(t) + \frac{3\lambda}{8000}\psi'(t)^2
\end{aligned}$$

$$
\begin{aligned}
& -\frac{1}{48000}\psi'(t)^3+\frac{9\lambda^3}{2000}, \\
& v_6(t)=v_6(t); \\
& u_7(t)=\frac{\lambda}{2000}\psi''(t)+\frac{7}{120}u_4'(t) \\
& \quad -\frac{1}{12000}\psi''(t)\psi'(t), \\
& v_7(t)=-\frac{1}{20}u_4'(t)-\frac{\lambda}{4000}\psi''(t) \\
& \quad +\frac{1}{24000}\psi''(t)\psi'(t); \\
& u_8(t)=-\frac{1}{2}u_4(t)^2-2v_8(t)-\frac{\lambda}{300}\psi'(t)u_4(t) \\
& \quad +\frac{1}{3600}u_4(t)\psi'(t)^2+\frac{3\lambda^2}{40000}\psi'(t)^2 \\
& \quad -\frac{\lambda}{120000}\psi'(t)^3-\frac{3\lambda^3}{10000}\psi'(t) \\
& \quad +\frac{\lambda^2}{100}u_4(t)+\frac{1}{2880000}\psi'(t)^4+\frac{9\lambda^4}{20000}, \\
& v_8(t)=v_8(t).
\end{aligned} \tag{67}
$$

As can be seen from the above expressions, the resonance conditions at $r=4,6,8$ corresponding to $u_4(t)$, $v_6(t)$, and $v_8(t)$ are satisfied.

Nonprincipal branch (iii): the third branch (iii) has resonances

$$r=-1,-3,4,6,8. \tag{68}$$

We give the truncated expansion forms for (iii) as follows:

$$u=3(\ln\phi)_{xx}+\tilde{u},\quad \tilde{u}\equiv u_2, \tag{69a}$$

$$v=6(\ln\phi)_{xx}+\tilde{v},\quad \tilde{v}\equiv v_2, \tag{69b}$$

where u, v, ϕ, u_2, and v_2 all are functions of x,t.

Referring to the 'reduced ansatz' of Kruskal, we get the coefficients of expansions (60) for (iii) as follows:

$$
\begin{aligned}
& u_0(t)=-3, \\
& v_0(t)=-6; \\
& u_1(t)=0, \\
& v_1(t)=0; \\
& u_2(t)=\frac{1}{20}\psi'(t)+\frac{\lambda}{5}, \\
& v_2(t)=\frac{3\lambda}{5}-\frac{1}{10}\psi'(t); \\
& u_3(t)=0, \\
& v_3(t)=0; \\
& u_4(t)=u_4(t), \\
& v_4(t)=\frac{3\lambda}{100}\psi'(t)-\frac{1}{400}\psi'(t)^2-\frac{9\lambda^2}{100}-3u_4(t); \\
& u_5(t)=0, \\
& v_5(t)=-\frac{1}{240}\psi''(t); \\
& u_6(t)=u_6(t), \\
& v_6(t)=\frac{1}{12}u_4(t)\psi'(t)-\frac{\lambda}{2}u_4(t)+2u_6(t) \\
& \quad +\frac{9\lambda^2}{2000}\psi'(t)-\frac{3\lambda}{4000}\psi'(t)^2 \\
& \quad +\frac{1}{24000}\psi'(t)^3-\frac{9\lambda^3}{1000}; \\
& u_7(t)=\frac{\lambda}{2000}\psi''(t)+\frac{7}{120}u_4'(t) \\
& \quad -\frac{1}{12000}\psi''(t)\psi'(t), \\
& v_7(t)=\frac{1}{20}u_4'(t)+\frac{\lambda}{4000}\psi''(t)-\frac{1}{24000}\psi''(t)\psi'(t); \\
& u_8(t)=-\frac{1}{2}u_4(t)^2+2v_8(t)-\frac{\lambda}{300}\psi'(t)u_4(t) \\
& \quad +\frac{1}{3600}u_4(t)\psi'(t)^2+\frac{3\lambda^2}{40000}\psi'(t)^2 \\
& \quad -\frac{\lambda}{120000}\psi'(t)^3-\frac{3\lambda^3}{10000}\psi'(t) \\
& \quad +\frac{\lambda^2}{100}u_4(t)+\frac{1}{2880000}\psi'(t)^4+\frac{9\lambda^4}{20000}, \\
& v_8(t)=v_8(t).
\end{aligned} \tag{70}
$$

As seen from the above expression, the resonance conditions at $r=4,6,8$ corresponding to $u_4(t)$, $u_6(t)$, and $v_8(t)$ are satisfied.

6. Conclusion

As demonstrated above, we get the polynomial solutions of the KdV equation with a self-consistent source (KdV-SCS) which are expressed in terms of Jacobi's elliptic functions. The extended (G'/G)-expansion method has been successfully applied in this paper to deal with the new exact traveling wave solutions of the KdV-SCS. As a result, the hyperbolic function solutions, the trigonometric function solutions, and the rational function solutions with arbitrary parameters are obtained. The arbitrary parameters imply that those corresponding solutions have abundant local structures. Meanwhile, we give the reduction forms and the group-invariant solution of the KdV-SCS. By the WTC Painlevé test method, we show that the KdV-SCS passes the Painlevé test. And the truncated expansion form gives the Bäcklund transformations of (1a)-(1b). In fact, these methods are also readily applicable to a large variety of nonlinear partial differential equations; we indeed can obtain some new analytical solutions for many nonlinear differential equations. And it is very satisfying to see that more analytical solutions for the physically interesting equation (1a)-(1b) can be obtained by these most fundamental but widely applicable approaches. As far as we know, our solutions have not been reported in previous literature.

Data Availability

The *Maple* data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally to the writing of this paper and approved the final manuscript.

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