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Research Article

Heronian Mean Operator of Linguistic Neutrosophic Cubic Numbers and Their Multiple Attribute Decision-Making Methods

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Many aggregation operators in multiattribute decisions assume that attributes are independent of each other; this leads to an unreasonable situation in information aggregation and decision-making. Heronian mean is the aggregation operator that can embody the interaction between attributes. In this paper, we merge the linguistic neutrosophic cubic number (LNCN) and the Heronian mean operator together to develop a LNCN generalized weighted Heronian mean (LNCNGWHM) operator and a LNCN three-parameter weighted Heronian mean (LNCNTPWHM) operator and then discuss their properties. Further, two multiattribute decision methods based on the proposed LNCNGWHM or LNCNTPWHM operator are introduced under LNCN environment. Finally, an example is used to indicate the effectiveness of the developed methods.

1. Introduction

In multiple attribute decision-making problems, the general attribute values are usually difficult to be given as the form of real numbers and instead of the form of uncertain variables, such as fuzzy numbers, intuitionistic fuzzy numbers, and linguistic variables (LVs). Fuzzy set (FS) was proposed by Zadeh to measure things that cannot be described with accurate information or accurate probability distribution [1]. The intuitionistic fuzzy set (IFS) proposed by Atanassov consists of the membership and nonmembership information, which is an effective extension of fuzzy set theory [2]. LVs are used to evaluate attributes in multiattribute decision-making while attributes cannot be evaluated by numerical values. Linguistic set (LS) $L = \{L_0, L_1, L_2, L_3, \ldots L_g\}$ (g is an even number) was proposed by Zadeh to deal with the approximate reasoning problems [3, 4]. Since FS, IFS, and LS are effective tools to deal with the approximate reasoning problems, many researches have studied them [5–11]. However, in the literature [7–11], only incomplete information can be effectively expressed, and then indeterminate and inconsistent information cannot be described effectively. In order to make up for the deficiency of literature [7–11], Smarandache put forward the theory of neutrosophic set (NS) consisting of three parts: truth $T(x)$, falsity $F(x)$, and indeterminacy $I(x)$ [12]. Wang and Smarandache also proposed a single value neutrosophic set (SVNS), which satisfies $T(x), I(x), F(x) \in [0,1]$, and $0 \leq T(x)+ I(x)+F(x) \leq 3$ [13]. IF the three components $T(x), I(x)$, and $F(x)$ in NS are interval numbers, Wang and Zhang introduced an interval neutrosophic set (INS) [14]. Next, some scholars put NS and LS together to develop their new set of concepts. First, Ye defined the interval neutrosophic uncertain linguistic set (INULS) and its multiple attribute decision-making (MADM) method [15]. Then, Ye proposed single valued neutrosophic linguistic numbers (SVNLNs) for multiple attribute group decision-making (MAGDM) [16]. Further, Fang and Ye proposed the linguistic neutrosophic number (LNN) consisting of the truth, falsity, and indeterminacy linguistic degrees [17]. Recently, Ye also proposed a new concept of a linguistic neutrosophic cubic number (LNCN) to extend neutrosophic cubic sets to linguistic neutrosophic arguments [18]. So far, there is little study on LNCN. However, many aggregation operators in multiattribute decisions assume that attributes are independent of each other; this leads to an unreasonable situation in information aggregation and decision-making. Then, Heronian mean is the aggregation operator that can embody the interaction between attributes. So, in this paper, we propose the LNCN generalized weighted Heronian mean (LNCNGWHM) operator and the LNCN three-parameter weighted Heronian mean (LNCNTPWHM) operator and investigate their properties.

The remaining organizations of this paper are listed as follows. Section 2 describes some concepts of LNN, LNCN, GWHM, and TPWHM. Section 3 proposes the LNCNGWHM and LNCNTPWHM operators and investigates their properties. Section 4 establishes MADM methods by using the LNCNGWHM and LNCNTPWHM operators. Section 5 provides an illustrative example to demonstrate the application and effectiveness of the proposed methods. Section 6 gives conclusions of this paper.

2. Some Basic Concepts

2.1. Linguistic Neutrosophic Numbers, Linguistic Neutrosophic Cubic Numbers, and Their Operational Laws

Definition 1 ([17]). Set $L = \{l_j \mid l_0 \le l_j \le l_g, j \in [0, g]\}$ as a language term set; g is an even number. Then a LNN can be defined as follows:

$$m = \left\langle l_\alpha, l_\beta, l_\gamma \right\rangle, \quad (1)$$

in which $l_\alpha, l_\beta, l_\gamma \in L$ and $\alpha, \beta, \gamma \in [0, g]$, l_α, l_β, and l_γ represent the truth, indeterminacy, and falsity variables, respectively, in linguistic terms.

Definition 2 ([17]). Set $m = \langle l_\alpha, l_\beta, l_\gamma \rangle$, $m_1 = \langle l_{\alpha1}, l_{\beta1}, l_{\gamma1} \rangle$, and $m_2 = \langle l_{\alpha2}, l_{\beta2}, l_{\gamma2} \rangle$ as three LNNs in L and a real number $\delta \ge 0$; then the operational laws of LNNs are as follows:

$$\begin{aligned} m_1 \oplus m_2 &= \left\langle l_{\alpha1}, l_{\beta1}, l_{\gamma1} \right\rangle \oplus \left\langle l_{\alpha2}, l_{\beta2}, l_{\gamma2} \right\rangle \\ &= \left\langle l_{\alpha1+\alpha2-\alpha1\alpha2/g}, l_{\beta1\beta2/g}, l_{\gamma1\gamma2/g} \right\rangle; \end{aligned} \quad (2)$$

$$\begin{aligned} m_1 \otimes m_2 &= \left\langle l_{\alpha1}, l_{\beta1}, l_{\gamma1} \right\rangle \otimes \left\langle l_{\alpha2}, l_{\beta2}, l_{\gamma2} \right\rangle \\ &= \left\langle l_{\alpha1\alpha2/g}, l_{\beta1+\beta2-\beta1\beta2/g}, l_{\gamma1+\gamma2-\gamma1\gamma2/g} \right\rangle; \end{aligned} \quad (3)$$

$$\begin{aligned} \delta m &= \delta \left\langle l_\alpha, l_\beta, l_\gamma \right\rangle \\ &= \left\langle l_{g-g(1-\alpha/g)^\delta}, l_{g(\beta/g)^\delta}, l_{g(\gamma/g)^\delta} \right\rangle; \end{aligned} \quad (4)$$

$$\begin{aligned} m^\delta &= \left\langle l_\alpha, l_\beta, l_\gamma \right\rangle^\delta \\ &= \left\langle l_{g(\alpha/g)^\delta}, l_{g-g(1-\beta/g)^\delta}, l_{g-g(1-\gamma/g)^\delta} \right\rangle. \end{aligned} \quad (5)$$

Definition 3 ([18]). Set $L = \{l_j \mid l_0 \le l_j \le l_g, j \in [0, g]\}$ as a language term set; g is an even number. Then a LNCN can be defined as follows:

$$m = (u, c), \quad (6)$$

where $u = \langle [l_{\alpha_b}, l_{\alpha_t}], [l_{\beta_b}, l_{\beta_t}], [l_{\gamma_b}, l_{\gamma_t}] \rangle$ expresses an uncertain LNN and $[l_{\alpha_b}, l_{\alpha_t}], [l_{\beta_b}, l_{\beta_t}], [l_{\gamma_b}, l_{\gamma_t}]$ represent, respectively, the truth, the indeterminacy, and the falsity uncertain linguistic variables for $l_{\alpha_b}, l_{\alpha_t}, l_{\beta_b}, l_{\beta_t}, l_{\gamma_b}, l_{\gamma_t} \in L$ and $\alpha_b, \alpha_t, \beta_b, \beta_t, \gamma_b, \gamma_t \in [0, g]$, $\alpha_b \le \alpha_t, \beta_b \le \beta_t, \gamma_b \le \gamma_t$; $c = \langle l_\alpha, l_\beta, l_\gamma \rangle$ expresses a LNN, in which $l_\alpha, l_\beta, l_\gamma \in L$ and $\alpha, \beta, \gamma \in [0, g]$, and l_α, l_β *and* l_γ represent the truth, the indeterminacy, and the falsity linguistic variables, respectively, in linguistic terms.

Definition 4 ([18]). Set $m = (\langle [l_{\alpha_b}, l_{\alpha_t}], [l_{\beta_b}, l_{\beta_t}], [l_{\gamma_b}, l_{\gamma_t}] \rangle, \langle l_\alpha, l_\beta, l_\gamma \rangle)$ as a LNCN in L, then, one calls m

① an internal LNCN if $\alpha_b \le \alpha \le \alpha_t, \beta_b \le \beta \le \beta_t, \gamma_b \le \gamma \le \gamma_t$;

② an external LNCN if $\alpha \notin [\alpha_b, \alpha_t]$, $\beta \notin [\beta_b, \beta_t]$ and $\gamma \notin [\gamma_b, \gamma_t]$.

Definition 5 ([18]). Set $m_1 = (\langle [l_{\alpha_{b1}}, l_{\alpha_{t1}}], [l_{\beta_{b1}}, l_{\beta_{t1}}], [l_{\gamma_{b1}}, l_{\gamma_{t1}}] \rangle, \langle l_{\alpha1}, l_{\beta1}, l_{\gamma1} \rangle)$, and $m_2 = (\langle [l_{\alpha_{b2}}, l_{\alpha_{t2}}], [l_{\beta_{b2}}, l_{\beta_{t2}}], [l_{\gamma_{b2}}, l_{\gamma_{t2}}] \rangle, \langle l_{\alpha2}, l_{\beta2}, l_{\gamma2} \rangle)$ as two LNCNs in L and a real number $\delta \ge 0$; then the operational laws of LNCNs are as follows:

$$\begin{aligned} m_1 \oplus m_2 &= \left(\left\langle [l_{\alpha_{b1}}, l_{\alpha_{t1}}], [l_{\beta_{b1}}, l_{\beta_{t1}}], [l_{\gamma_{b1}}, l_{\gamma_{t1}}] \right\rangle, \left\langle l_{\alpha1}, l_{\beta1}, l_{\gamma1} \right\rangle \right) \\ &\quad \oplus \left(\left\langle [l_{\alpha_{b2}}, l_{\alpha_{t2}}], [l_{\beta_{b2}}, l_{\beta_{t2}}], [l_{\gamma_{b2}}, l_{\gamma_{t2}}] \right\rangle, \left\langle l_{\alpha2}, l_{\beta2}, l_{\gamma2} \right\rangle \right) \\ &= \left(\left\langle [l_{\alpha_{b1}+\alpha_{b2}-\alpha_{b1}\alpha_{b2}/g}, l_{\alpha_{t1}+\alpha_{t2}-\alpha_{t1}\alpha_{t2}/g}], [l_{\beta_{b1}\beta_{b2}/g}, l_{\beta_{t1}\beta_{t2}/g}], \right. \right. \\ &\quad \left. \left. [l_{\gamma_{b1}\gamma_{b2}/g}, l_{\gamma_{t1}\gamma_{t2}/g}] \right\rangle, \left\langle l_{\alpha1+\alpha2-\alpha1\alpha2/g}, l_{\beta1\beta2/g}, l_{\gamma_1\gamma_2/g} \right\rangle \right); \\ m_1 \otimes m_2 &= \left(\left\langle [l_{\alpha_{b1}}, l_{\alpha_{t1}}], [l_{\beta_{b1}}, l_{\beta_{t1}}], [l_{\gamma_{b1}}, l_{\gamma_{t1}}] \right\rangle, \left\langle l_{\alpha1}, l_{\beta1}, l_{\gamma1} \right\rangle \right) \end{aligned} \quad (7)$$

$$\otimes\left(\left\langle\left[l_{\alpha_{b2}},l_{\alpha_{t2}}\right],\left[l_{\beta_{b2}},l_{\beta_{t2}}\right],\left[l_{\gamma_{b2}},l_{\gamma_{t2}}\right]\right\rangle,\left\langle l_{\alpha2},l_{\beta2},l_{\gamma2}\right\rangle\right)$$
$$=\left(\left\langle\left[l_{\alpha_{b1}\alpha_{b2}/g},l_{\alpha_{t1}\alpha_{t2}/g}\right],\left[l_{\beta_{b1}+\beta_{b2}-\beta_{b1}\beta_{b2}/g},l_{\beta_{t1}+\beta_{t2}-\beta_{t1}\beta_{t2}/g}\right],\left[l_{\gamma_{b1}+\gamma_{b2}-\gamma_{b1}\gamma_{b2}/g},l_{\gamma_{t1}+\gamma_{t2}-\gamma_{t1}\gamma_{t2}/g}\right]\right\rangle,\left\langle l_{\alpha1\alpha2/g},l_{\beta1+\beta2-\beta1\beta2/g},l_{\gamma1+\gamma2-\gamma1\gamma2/g}\right\rangle\right); \quad (8)$$

$$\delta m_1=\delta\left(\left\langle\left[l_{\alpha_{b1}},l_{\alpha_{t1}}\right],\left[l_{\beta_{b1}},l_{\beta_{t1}}\right],\left[l_{\gamma_{b1}},l_{\gamma_{t1}}\right]\right\rangle,\left\langle l_{\alpha1},l_{\beta1},l_{\gamma1}\right\rangle\right)$$
$$=\left(\left\langle\left[l_{g-g(1-\alpha_{b1}/g)^{\delta}},l_{g-g(1-\alpha_{t1}/g)^{\delta}}\right],\left[l_{g(\beta_{b1}/g)^{\delta}},l_{g(\beta_{t1}/g)^{\delta}}\right],\left[l_{g(\gamma_{b1}/g)^{\delta}},l_{g(\gamma_{t1}/g)^{\delta}}\right]\right\rangle,\left\langle l_{g-g(1-\alpha1/g)^{\delta}},l_{g(\beta1/g)^{\delta}},l_{g(\gamma1/g)^{\delta}}\right\rangle\right); \quad (9)$$

$$m_1^{\delta}=\left(\left\langle\left[l_{\alpha_{b1}},l_{\alpha_{t1}}\right],\left[l_{\beta_{b1}},l_{\beta_{t1}}\right],\left[l_{\gamma_{b1}},l_{\gamma_{t1}}\right]\right\rangle,\left\langle l_{\alpha1},l_{\beta1},l_{\gamma1}\right\rangle\right)^{\delta}$$
$$=\left(\left\langle\left[l_{g(\alpha_{b1}/g)^{\delta}},l_{g(\alpha_{t1}/g)^{\delta}}\right],\left[l_{g-g(1-\beta_{b1}/g)^{\delta}},l_{g-g(1-\beta_{t1}/g)^{\delta}}\right],\left[l_{g-g(1-\gamma_{b1}/g)^{\delta}},l_{g-g(1-\gamma_{t1}/g)^{\delta}}\right]\right\rangle,\left\langle l_{g(\alpha1/g)^{\delta}},l_{g-g(1-\beta1/g)^{\delta}},l_{g-g(1-\gamma1/g)^{\delta}}\right\rangle\right). \quad (10)$$

Definition 6 ([18]). Set $m = (\langle[l_{\alpha_b},l_{\alpha_t}],[l_{\beta_b},l_{\beta_t}],[l_{\gamma_b},l_{\gamma_t}]\rangle, \langle l_\alpha,l_\beta,l_\gamma\rangle)$ as a LNCN in L; then the score $S(m)$, accuracy $A(m)$, and certain $C(m)$ functions can be defined, respectively, as follows:

$$S(m)=\frac{1}{9g}\left[\left(4g+\alpha_b+\alpha_t-\beta_b-\beta_t-\gamma_b-\gamma_t\right)+\left(2g+\alpha-\beta-\gamma\right)\right]; \quad (11)$$

$$A(m)=\frac{1}{3g}\left[\left(\alpha_b+\alpha_t-\beta_b-\beta_t\right)+\left(\alpha-\beta\right)\right]; \quad (12)$$

$$C(m)=\frac{1}{3g}\left(\alpha_b+\alpha_t+\alpha\right). \quad (13)$$

Definition 7 ([18]). Set $m_1 = (\langle[l_{\alpha_{b1}},l_{\alpha_{t1}}],[l_{\beta_{b1}},l_{\beta_{t1}}],[l_{\gamma_{b1}},l_{\gamma_{t1}}]\rangle, \langle l_{\alpha1},l_{\beta1},l_{\gamma1}\rangle)$, $m_2 = (\langle[l_{\alpha_{b2}},l_{\alpha_{t2}}],[l_{\beta_{b2}},l_{\beta_{t2}}],[l_{\gamma_{b2}},l_{\gamma_{t2}}]\rangle, \langle l_{\alpha2}, l_{\beta2},l_{\gamma2}\rangle)$ as two LNCNs in L; then

if $S(m_1) > S(m_2)$, then $m_1 > m_2$;

if $S(m_1) = S(m_2)$ then

if $A(m_1) > A(m_2)$, then $m_1 > m_2$;

if $A(m_1) > A(m_2)$ and $C(m_1) > C(m_2)$, then $m_1 > m_2$;

if $A(m_1) = A(m_2)$ and $C(m_1) = C(m_2)$, then $m_1 = m_2$.

2.2. Generalized Weighted Heronian Mean and Three-Parameter Weighted Heronian Mean Operators

Definition 8 ([19]). Let $w = (w_1, w_2, \ldots, w_n)$ be the relative weight of m_i for $m_i \in R$ $(i = 1,2,\ldots,n)$, $w_i \in [0,1]$, and $\sum_{i=1}^{n} w_i = 1$ and $p,q \geq 0$. Then

$$GWHM^{p,q}(m_1,m_2,\ldots m_n)=\left(\frac{1}{\lambda}\bigoplus_{i=1}^{n}\bigoplus_{j=i}^{n}\left(w_i{}^{p}w_j{}^{q}m_i{}^{p}\otimes m_j{}^{q}\right)\right)^{1/(p+q)}, \quad (14)$$

where $\lambda = \sum_{i=1}^{n}\sum_{j=i}^{n} w_i{}^{p}w_j{}^{q}$ is called a generalized weighted Heronian mean (GWHM) operator.

Definition 9 ([19]). Let $w=(w_1, w_2, \ldots, w_n)$ be the relative weight of m_i *for* $m_i \in R$ $(i = 1,2,\ldots,n)$, $w_i \in [0,1]$, and $\sum_{i=1}^{n} w_i = 1$ and $p,q,r \geq 0$. Then

$$TPWHM^{p,q,r}(m_1,m_2,\ldots m_n)=\left(\frac{1}{\lambda}\bigoplus_{i=1}^{n}\bigoplus_{j=i}^{n}\bigoplus_{k=j}^{n}\left(w_i{}^{p}w_j{}^{q}w_k{}^{r}m_i{}^{p}\otimes m_j{}^{q}\otimes m_k{}^{r}\right)\right)^{1/(p+q+r)}, \quad (15)$$

where $\lambda = \sum_{i=1}^{n}\sum_{j=i}^{n}\sum_{k=j}^{n} w_i^{p} w_j^{q} w_k^{r}$ is called a three-parameter weighted Heronian mean (TPWHM) operator.

3. Two Aggregation Operators of LNCNs

3.1. GWHM Operator of LNCNs

Definition 10. Set $m_i = (\langle[l_{\alpha_{bi}}, l_{\alpha_{ti}}], [l_{\beta_{bi}}, l_{\beta_{ti}}], [l_{\gamma_{bi}}, l_{\gamma_{ti}}]\rangle, \langle l_{\alpha i}, l_{\beta i}, l_{\gamma i}\rangle)$ $(i = 1, 2, \ldots, n)$ as a collection of LNCNs in L; then the LNCNGWHM operator can be defined as

$$LNCNGWHM^{p,q}(m_1, m_2, \ldots m_n) = \left(\frac{1}{\lambda}\bigoplus_{i=1}^{n}\bigoplus_{j=i}^{n}\left(w_i^{p} w_j^{q} m_i^{p} \otimes m_j^{q}\right)\right)^{1/(p+q)}, \quad (16)$$

where $\lambda = \sum_{i=1}^{n}\sum_{j=i}^{n} w_i^{p} w_j^{q}$, $w_i \in [0,1]$ and $\sum_{i=1}^{n} w_i = 1$.

Then we can use Definitions 5 and 10 to get the following theorem.

Theorem 11. *Set $m_i = (\langle[l_{\alpha_{bi}}, l_{\alpha_{ti}}], [l_{\beta_{bi}}, l_{\beta_{ti}}], [l_{\gamma_{bi}}, l_{\gamma_{ti}}]\rangle, \langle l_{\alpha i}, l_{\beta i}, l_{\gamma i}\rangle)$ $(i = 1, 2, \ldots, n)$ as a collection of LNCNs in L; then by LNCNGWHM operator, the aggregation result of m_i is still a LNCN, which is given by the following form:*

$$\begin{aligned}
&LNCNGWHM^{p,q}(m_1, m_2, \ldots m_n) = \left(\frac{1}{\lambda}\bigoplus_{i=1}^{n}\bigoplus_{j=i}^{n}\left(w_i^{p} w_j^{q} m_i^{p} \otimes m_j^{q}\right)\right)^{1/(p+q)} \\
&= \Bigg(\Bigg\langle\Bigg[l_{g(1-(\prod_{i=1}^{n}\prod_{j=i}^{n}(1-(\alpha_{bi}/g)^{p}(\alpha_{bj}/g)^{q})^{w_i^{p}w_j^{q}})^{1/\lambda})^{1/(p+q)}}, l_{g(1-(\prod_{i=1}^{n}\prod_{j=i}^{n}(1-(\alpha_{ti}/g)^{p}(\alpha_{tj}/g)^{q})^{w_i^{p}w_j^{q}})^{1/\lambda})^{1/(p+q)}}\Bigg], \\
&\Bigg[l_{g-g(1-(\prod_{i=1}^{n}\prod_{j=i}^{n}(1-(1-\beta_{bi}/g)^{p}(1-\beta_{bj}/g)^{q})^{w_i^{p}w_j^{q}})^{1/\lambda})^{1/(p+q)}}, l_{g-g(1-(\prod_{i=1}^{n}\prod_{j=i}^{n}(1-(1-\beta_{ti}/g)^{p}(1-\beta_{tj}/g)^{q})^{w_i^{p}w_j^{q}})^{1/\lambda})^{1/(p+q)}}\Bigg], \\
&\Bigg[l_{g-g(1-(\prod_{i=1}^{n}\prod_{j=i}^{n}(1-(1-\gamma_{bi}/g)^{p}(1-\gamma_{bj}/g)^{q})^{w_i^{p}w_j^{q}})^{1/\lambda})^{1/(p+q)}}, l_{g-g(1-(\prod_{i=1}^{n}\prod_{j=i}^{n}(1-(1-\gamma_{ti}/g)^{p}(1-\gamma_{tj}/g)^{q})^{w_i^{p}w_j^{q}})^{1/\lambda})^{1/(p+q)}}\Bigg]\Bigg\rangle, \\
&\Bigg\langle l_{g(1-(\prod_{i=1}^{n}\prod_{j=i}^{n}(1-(\alpha i/g)^{p}(\alpha j/g)^{q})^{w_i^{p}w_j^{q}})^{1/\lambda})^{1/(p+q)}}, l_{g-g(1-(\prod_{i=1}^{n}\prod_{j=i}^{n}(1-(1-\beta i/g)^{p}(1-\beta j/g)^{q})^{w_i^{p}w_j^{q}})^{1/\lambda})^{1/(p+q)}}, \\
&l_{g-g(1-(\prod_{i=1}^{n}\prod_{j=i}^{n}(1-(1-\gamma i/g)^{p}(1-\gamma j/g)^{q})^{w_i^{p}w_j^{q}})^{1/\lambda})^{1/(p+q)}}\Bigg\rangle\Bigg),
\end{aligned} \quad (17)$$

where $\lambda = \sum_{i=1}^{n}\sum_{j=i}^{n} w_i^{p} w_j^{q}$, $w_i \in [0,1]$, and $\sum_{i=1}^{n} w_i = 1$.

Proof.

(1)

$$\begin{aligned}
m_i^{p} = &\left(\left\langle\left[l_{g(\alpha_{bi}/g)^{p}}, l_{g(\alpha_{ti}/g)^{p}}\right], \left[l_{g-g(1-\beta_{bi}/g)^{p}}, l_{g-g(1-\beta_{ti}/g)^{p}}\right],\right.\right. \\
&\left.\left[l_{g-g(1-\gamma_{bi}/g)^{p}}, l_{g-g(1-\gamma_{ti}/g)^{p}}\right]\right\rangle, \\
&\left.\left\langle l_{g(\alpha i/g)^{p}}, l_{g-g(1-\beta i/g)^{p}}, l_{g-g(1-\gamma i/g)^{p}}\right\rangle\right);
\end{aligned} \quad (18)$$

(2)

$$\begin{aligned}
m_j^{q} = &\left(\left\langle\left[l_{g(\alpha_{bj}/g)^{q}}, l_{g(\alpha_{tj}/g)^{q}}\right],\right.\right. \\
&\left[l_{g-g(1-\beta_{bj}/g)^{q}}, l_{g-g(1-\beta_{tj}/g)^{q}}\right], \\
&\left.\left[l_{g-g(1-\gamma_{bj}/g)^{q}}, l_{g-g(1-\gamma_{tj}/g)^{q}}\right]\right\rangle, \left\langle l_{g(\alpha j/g)^{q}}, l_{g-g(1-\beta j/g)^{q}},\right. \\
&\left.\left. l_{g-g(1-\gamma j/g)^{q}}\right\rangle\right);
\end{aligned} \quad (19)$$

(3)

$$\begin{aligned}
&m_i^{p} \otimes m_j^{q} \\
&= \left(\left\langle\left[l_{g(\alpha_{bi}/g)^{p} g(\alpha_{bj}/g)^{q}/g}, l_{g(\alpha_{ti}/g)^{p} g(\alpha_{tj}/g)^{q}/g}\right],\right.\right. \\
&\left[l_{g-g(1-\beta_{bi}/g)^{p}+g-g(1-\beta_{bj}/g)^{q}-(g-g(1-\beta_{bi}/g)^{p})(g-g(1-\beta_{bj}/g)^{q})/g}, l_{g-g(1-\beta_{ti}/g)^{p}+g-g(1-\beta_{tj}/g)^{q}-(g-g(1-\beta_{ti}/g)^{p})(g-g(1-\beta_{tj}/g)^{q})/g}\right], \\
&\left.\left[l_{g-g(1-\gamma_{bi}/g)^{p}+g-g(1-\gamma_{bj}/g)^{q}-(g-g(1-\gamma_{bi}/g)^{p})(g-g(1-\gamma_{bj}/g)^{q})/g}, l_{g-g(1-\gamma_{ti}/g)^{p}+g-g(1-\gamma_{tj}/g)^{q}-(g-g(1-\gamma_{ti}/g)^{p})(g-g(1-\gamma_{tj}/g)^{q})/g}\right]\right\rangle, \\
&\left.\left\langle l_{(g(\alpha i/g)^{p} g(\alpha j/g)^{q})/g}, l_{g-g(1-\beta i/g)^{p}+g-g(1-\beta j/g)^{q}-(g-g(1-\beta i/g)^{p})(g-g(1-\beta j/g)^{q})/g}, l_{g-g(1-\gamma i/g)^{p}+g-g(1-\gamma j/g)^{q}-(g-g(1-\gamma i/g)^{p})(g-g(1-\gamma j/g)^{q})/g}\right\rangle\right)
\end{aligned}$$

$$= \Big(\Big\langle \Big[l_{g(\alpha_{bi}/g)^p(\alpha_{bj}/g)^q}, l_{g(\alpha_{ti}/g)^p(\alpha_{tj}/g)^q} \Big], \Big[l_{g-g(1-\beta_{bi}/g)^p(1-\beta_{bj}/g)^q}, l_{g-g(1-\beta_{ti}/g)^p(1-\beta_{tj}/g)^q} \Big], \Big[l_{g-g(1-\gamma_{bi}/g)^p(1-\gamma_{bj}/g)^q}, l_{g-g(1-\gamma_{ti}/g)^p(1-\gamma_{tj}/g)^q} \Big] \Big\rangle, \Big\langle l_{g(\alpha i/g)^p(\alpha j/g)^q}, l_{g-g(1-\beta i/g)^p(1-\beta j/g)^q}, l_{g-g(1-\gamma i/g)^p(1-\gamma j/g)^q} \Big\rangle \Big); \quad (20)$$

(4)

$$\begin{aligned} & {w_i}^p {w_j}^q {m_i}^p \otimes {m_j}^q \\ & = \Big(\Big\langle \Big[l_{g-g(1-g(\alpha_{bi}/g)^p(\alpha_{bj}/g)^q/g)^{{w_i}^p{w_j}^q}}, l_{g-g(1-g(\alpha_{ti}/g)^p(\alpha_{tj}/g)^q/g)^{{w_i}^p{w_j}^q}} \Big], \Big[l_{g(g-g(1-\beta_{bi}/g)^p(1-\beta_{bj}/g)^q/g)^{{w_i}^p{w_j}^q}}, l_{g(g-g(1-\beta_{ti}/g)^p(1-\beta_{tj}/g)^q/g)^{{w_i}^p{w_j}^q}} \Big], \\ & \Big[l_{g(g-g(1-\gamma_{bi}/g)^p(1-\gamma_{bj}/g)^q/g)^{{w_i}^p{w_j}^q}}, l_{g(g-g(1-\gamma_{ti}/g)^p(1-\gamma_{tj}/g)^q/g)^{{w_i}^p{w_j}^q}} \Big] \Big\rangle, \\ & \Big\langle l_{g-g(1-g(\alpha i/g)^p(\alpha j/g)^q/g)^{{w_i}^p{w_j}^q}}, l_{g(g-g(1-\beta i/g)^p(1-\beta j/g)^q/g)^{{w_i}^p{w_j}^q}}, l_{g(g-g(1-\gamma i/g)^p(1-\gamma j/g)^q/g)^{{w_i}^p{w_j}^q}} \Big\rangle \Big) \\ & = \Big(\Big\langle \Big[l_{g-g(1-(\alpha_{bi}/g)^p(\alpha_{bj}/g)^q)^{{w_i}^p{w_j}^q}}, l_{g-g(1-(\alpha_{ti}/g)^p(\alpha_{tj}/g)^q)^{{w_i}^p{w_j}^q}} \Big], \Big[l_{g(1-(1-\beta_{bi}/g)^p(1-\beta_{bj}/g)^q)^{{w_i}^p{w_j}^q}}, l_{g(1-(1-\beta_{ti}/g)^p(1-\beta_{tj}/g)^q)^{{w_i}^p{w_j}^q}} \Big], \\ & \Big[l_{g(1-(1-\gamma_{bi}/g)^p(1-\gamma_{bj}/g)^q)^{{w_i}^p{w_j}^q}}, l_{g(1-(1-\gamma_{ti}/g)^p(1-\gamma_{tj}/g)^q)^{{w_i}^p{w_j}^q}} \Big] \Big\rangle, \\ & \Big\langle l_{g-g(1-(\alpha i/g)^p(\alpha j/g)^q)^{{w_i}^p{w_j}^q}}, l_{g(1-(1-\beta i/g)^p(1-\beta j/g)^q)^{{w_i}^p{w_j}^q}}, l_{g(1-(1-\gamma i/g)^p(1-\gamma j/g)^q)^{{w_i}^p{w_j}^q}} \Big\rangle \Big); \end{aligned} \quad (21)$$

(5)

$$\begin{aligned} & \bigoplus_{i=1}^{n} \bigoplus_{j=i}^{n} \left({w_i}^p {w_j}^q {m_i}^p \otimes {m_j}^q \right) = \Big(\Big\langle \Big[l_{g-g\prod_{i=1}^n \prod_{j=i}^n (1-(\alpha_{bi}/g)^p(\alpha_{bj}/g)^q)^{{w_i}^p{w_j}^q}}, l_{g-g\prod_{i=1}^n \prod_{j=i}^n (1-(\alpha_{ti}/g)^p(\alpha_{tj}/g)^q)^{{w_i}^p{w_j}^q}} \Big], \\ & \Big[l_{g\prod_{i=1}^n \prod_{j=i}^n (1-(1-\beta_{bi}/g)^p(1-\beta_{bj}/g)^q)^{{w_i}^p{w_j}^q}}, l_{g\prod_{i=1}^n \prod_{j=i}^n (1-(1-\beta_{ti}/g)^p(1-\beta_{tj}/g)^q)^{{w_i}^p{w_j}^q}} \Big], \Big[l_{g\prod_{i=1}^n \prod_{j=i}^n (1-(1-\gamma_{bi}/g)^p(1-\gamma_{bj}/g)^q)^{{w_i}^p{w_j}^q}}, l_{g\prod_{i=1}^n \prod_{j=i}^n (1-(1-\gamma_{ti}/g)^p(1-\gamma_{tj}/g)^q)^{{w_i}^p{w_j}^q}} \Big] \Big\rangle, \\ & \Big\langle l_{g-g\prod_{i=1}^n \prod_{j=i}^n (1-(\alpha i/g)^p(\alpha j/g)^q)^{{w_i}^p{w_j}^q}}, l_{g\prod_{i=1}^n \prod_{j=i}^n (1-(1-\beta i/g)^p(1-\beta j/g)^q)^{{w_i}^p{w_j}^q}}, l_{g\prod_{i=1}^n \prod_{j=i}^n (1-(1-\gamma i/g)^p(1-\gamma j/g)^q)^{{w_i}^p{w_j}^q}} \Big\rangle \Big); \end{aligned} \quad (22)$$

(6)

$$\begin{aligned} & \frac{1}{\lambda} \bigoplus_{i=1}^{n} \bigoplus_{j=i}^{n} \left({w_i}^p {w_j}^q {m_i}^p \otimes {m_j}^q \right) \\ & = \Big(\Big\langle \Big[l_{g-g(\prod_{i=1}^n \prod_{j=i}^n (1-(\alpha_{bi}/g)^p(\alpha_{bj}/g)^q)^{{w_i}^p{w_j}^q})^{1/\lambda}}, \\ & l_{g-g(\prod_{i=1}^n \prod_{j=i}^n (1-(\alpha_{ti}/g)^p(\alpha_{tj}/g)^q)^{{w_i}^p{w_j}^q})^{1/\lambda}} \Big], \\ & \Big[l_{g(\prod_{i=1}^n \prod_{j=i}^n (1-(1-\beta_{bi}/g)^p(1-\beta_{bj}/g)^q)^{{w_i}^p{w_j}^q})^{1/\lambda}}, \\ & l_{g(\prod_{i=1}^n \prod_{j=i}^n (1-(1-\beta_{ti}/g)^p(1-\beta_{tj}/g)^q)^{{w_i}^p{w_j}^q})^{1/\lambda}} \Big], \\ & \Big[l_{g(\prod_{i=1}^n \prod_{j=i}^n (1-(1-\gamma_{bi}/g)^p(1-\gamma_{bj}/g)^q)^{{w_i}^p{w_j}^q})^{1/\lambda}}, \\ & l_{g(\prod_{i=1}^n \prod_{j=i}^n (1-(1-\gamma_{ti}/g)^p(1-\gamma_{tj}/g)^q)^{{w_i}^p{w_j}^q})^{1/\lambda}} \Big] \Big\rangle, \\ & \Big\langle l_{g-g(\prod_{i=1}^n \prod_{j=i}^n (1-(\alpha i/g)^p(\alpha j/g)^q)^{{w_i}^p{w_j}^q})^{1/\lambda}}, \\ & l_{g(\prod_{i=1}^n \prod_{j=i}^n (1-(1-\beta i/g)^p(1-\beta j/g)^q)^{{w_i}^p{w_j}^q})^{1/\lambda}}, \\ & l_{g(\prod_{i=1}^n \prod_{j=i}^n (1-(1-\gamma i/g)^p(1-\gamma j/g)^q)^{{w_i}^p{w_j}^q})^{1/\lambda}} \Big\rangle \Big); \end{aligned} \quad (23)$$

(7)

$$\begin{aligned} & \left(\frac{1}{\lambda} \bigoplus_{i=1}^{n} \bigoplus_{j=i}^{n} \left({w_i}^p {w_j}^q {m_i}^p \otimes {m_j}^q \right) \right)^{1/(p+q)} \\ & = \Big(\Big\langle \Big[l_{g(1-(\prod_{i=1}^n \prod_{j=i}^n (1-(\alpha_{bi}/g)^p(\alpha_{bj}/g)^q)^{{w_i}^p{w_j}^q})^{1/\lambda})^{1/(p+q)}}, \\ & l_{g(1-(\prod_{i=1}^n \prod_{j=i}^n (1-(\alpha_{ti}/g)^p(\alpha_{tj}/g)^q)^{{w_i}^p{w_j}^q})^{1/\lambda})^{1/(p+q)}} \Big], \end{aligned}$$

$$\Big[l_{g-g(1-(\prod_{i=1}^{n}\prod_{j=i}^{n}(1-(1-\beta_{bi}/g)^{p}(1-\beta_{bj}/g)^{q})^{w_i^p w_j^q})^{1/\lambda})^{1/(p+q)}},$$

$$l_{g-g(1-(\prod_{i=1}^{n}\prod_{j=i}^{n}(1-(1-\beta_{ti}/g)^{p}(1-\beta_{tj}/g)^{q})^{w_i^p w_j^q})^{1/\lambda})^{1/(p+q)}}\Big],$$

$$\Big[l_{g-g(1-(\prod_{i=1}^{n}\prod_{j=i}^{n}(1-(1-\gamma_{bi}/g)^{p}(1-\gamma_{bj}/g)^{q})^{w_i^p w_j^q})^{1/\lambda})^{1/(p+q)}},$$

$$l_{g-g(1-(\prod_{i=1}^{n}\prod_{j=i}^{n}(1-(1-\gamma_{ti}/g)^{p}(1-\gamma_{tj}/g)^{q})^{w_i^p w_j^q})^{1/\lambda})^{1/(p+q)}}\Big]\Big\rangle,$$

$$\Big\langle l_{g(1-(\prod_{i=1}^{n}\prod_{j=i}^{n}(1-(\alpha_{bi}/g)^{p}(\alpha_{bj}/g)^{q})^{w_i^p w_j^q})^{1/\lambda})^{1/(p+q)}},$$

$$l_{g-g(1-(\prod_{i=1}^{n}\prod_{j=i}^{n}(1-(1-\beta i/g)^{p}(1-\beta j/g)^{q})^{w_i^p w_j^q})^{1/\lambda})^{1/(p+q)}},$$

$$l_{g-g(1-(\prod_{i=1}^{n}\prod_{j=i}^{n}(1-(1-\gamma i/g)^{p}(1-\gamma j/g)^{q})^{w_i^p w_j^q})^{1/\lambda})^{1/(p+q)}}\Big\rangle\Big). \tag{24}$$

The proof of Theorem 11 is completed. □

Theorem 12 (idempotency). *Set* $m_i = (\langle[l_{\alpha_{bi}}, l_{\alpha_{ti}}], [l_{\beta_{bi}}, l_{\beta_{ti}}], [l_{\gamma_{bi}}, l_{\gamma_{ti}}]\rangle, \langle l_{\alpha i}, l_{\beta i}, l_{\gamma i}\rangle)$ $(i = 1, 2, \ldots, n)$ *as a collection of LNCNs in L; if* $m_i = m$, *then*

$$\begin{aligned} &LNCNGWHM^{p,q}(m_1, m_2, \ldots m_n) \\ &\quad = LNCNGWHM^{p,q}(m, m, \ldots m) = m. \end{aligned} \tag{25}$$

Proof. Since $m_i = \mathrm{m}$ for $i = 1, 2, \ldots, n$, there is the following result:

$$\begin{aligned}
&\mathrm{LNCNGWHM}^{p,q}(m_1, m_2, \ldots m_n) \\
&= \mathrm{LNCNGWHM}^{p,q}(m, m, \ldots m) \\
&= \Big(\Big\langle\Big[l_{g(1-((1-(\alpha_b/g)^{p+q})^{\sum_{i=1}^{n}\sum_{j=i}^{n} w_i^p w_j^q})^{1/\lambda})^{1/(p+q)}}, l_{g(1-((1-(\alpha_t/g)^{p+q})^{\sum_{i=1}^{n}\sum_{j=i}^{n} w_i^p w_j^q})^{1/\lambda})^{1/(p+q)}}\Big], \\
&\Big[l_{g-g(1-((1-(1-\beta_b/g)^{p+q})^{\sum_{i=1}^{n}\sum_{j=i}^{n} w_i^p w_j^q})^{1/\lambda})^{1/(p+q)}}, l_{g-g(1-((1-(1-\beta_t/g)^{p+q})^{\sum_{i=1}^{n}\sum_{j=i}^{n} w_i^p w_j^q})^{1/\lambda})^{1/(p+q)}}\Big], \\
&\Big[l_{g-g(1-((1-(1-\gamma_b/g)^{p+q})^{\sum_{i=1}^{n}\sum_{j=i}^{n} w_i^p w_j^q})^{1/\lambda})^{1/(p+q)}}, l_{g-g(1-((1-(1-\gamma_t/g)^{p+q})^{\sum_{i=1}^{n}\sum_{j=i}^{n} w_i^p w_j^q})^{1/\lambda})^{1/(p+q)}}\Big]\Big\rangle, \\
&\Big\langle l_{g(1-((1-(\alpha/g)^{p+q})^{\sum_{i=1}^{n}\sum_{j=i}^{n} w_i^p w_j^q})^{1/\lambda})^{1/(p+q)}}, l_{g-g(1-((1-(1-\beta/g)^{p+q})^{\sum_{i=1}^{n}\sum_{j=i}^{n} w_i^p w_j^q})^{1/\lambda})^{1/(p+q)}}, l_{g-g(1-((1-(1-\gamma/g)^{p+q})^{\sum_{i=1}^{n}\sum_{j=i}^{n} w_i^p w_j^q})^{1/\lambda})^{1/(p+q)}}\Big\rangle\Big) \\
&= \Big(\Big\langle\Big[l_{g((\alpha_b/g)^{p+q})^{1/(p+q)}}, l_{g((\alpha_t/g)^{p+q})^{1/(p+q)}}\Big], \Big[l_{g-g((1-\beta_b/g)^{p+q})^{1/(p+q)}}, l_{g-g((1-\beta_t/g)^{p+q})^{1/(p+q)}}\Big], \\
&\Big[l_{g-g((1-\gamma_b/g)^{p+q})^{1/(p+q)}}, l_{g-g((1-\gamma_t/g)^{p+q})^{1/(p+q)}}\Big]\Big\rangle, \Big\langle l_{g((\alpha/g)^{p+q})^{1/(p+q)}}, l_{g-g((1-\beta/g)^{p+q})^{1/(p+q)}}, l_{g-g((1-\gamma/g)^{p+q})^{1/(p+q)}}\Big\rangle\Big) \\
&= \Big(\Big\langle\Big[l_{\alpha_b}, l_{\alpha_t}\Big], \Big[l_{\beta_b}, l_{\beta_t}\Big], \Big[l_{\gamma_b}, l_{\gamma_b}\Big]\Big\rangle, \Big\langle l_\alpha, l_\beta, l_\gamma\Big\rangle\Big) = m.
\end{aligned} \tag{26}$$

The proof of Theorem 12 is completed. □

Theorem 13 (monotonicity). *Set* $m_i = (\langle[l_{\alpha_{bi}}, l_{\alpha_{ti}}], [l_{\beta_{bi}}, l_{\beta_{ti}}], [l_{\gamma_{bi}}, l_{\gamma_{ti}}]\rangle, \langle l_{\alpha i}, l_{\beta i}, l_{\gamma i}\rangle)$ *and* $k_i = (\langle[l_{\alpha_{bi}'}, l_{\alpha_{ti}'}], [l_{\beta_{bi}'}, l_{\beta_{ti}'}], [l_{\gamma_{bi}'}, l_{\gamma_{ti}'}]\rangle, \langle l_{\alpha i'}, l_{\beta i'}, l_{\gamma i'}\rangle)$ $(i = 1, 2, \ldots, n)$ *as two collections of LNCNs in L; if* $\alpha_{bi} \le \alpha_{bi}'$, $\alpha_{ti} \le \alpha_{ti}'$, $\alpha i \le \alpha i'$, $\beta_{bi} \ge \beta_{bi}'$, $\beta_{ti} \ge \beta_{ti}'$, $\beta i \ge \beta i'$ *and* $\gamma_{bi} \ge \gamma_{bi}'$, $\gamma_{ti} \ge \gamma_{ti}'$, $\gamma i \ge \gamma i'$, *then*

$$\begin{aligned} &LNCNGWHM^{p,q}(m_1, m_2, \ldots m_n) \\ &\quad \le LNCNGWHM^{p,q}(k_1, k_2, \ldots k_n). \end{aligned} \tag{27}$$

Proof. Because $\alpha_{bi} \le \alpha_{bi}'$, $\alpha_{ti} \le \alpha_{ti}'$, $\alpha i \le \alpha i'$, $\beta_{bi} \ge \beta_{bi}'$, $\beta_{ti} \ge \beta_{ti}'$, $\beta i \ge \beta i'$ and $\gamma_{bi} \ge \gamma_{bi}'$, $\gamma_{ti} \ge \gamma_{ti}'$, $\gamma i \ge \gamma i'$, we can easily obtain

$$1 - \left(\frac{\alpha_{bi}}{g}\right)^{p}\left(\frac{\alpha_{bj}}{g}\right)^{q} \ge 1 - \left(\frac{\alpha_{bi}'}{g}\right)^{p}\left(\frac{\alpha_{bj}'}{g}\right)^{q},$$

$$g\left(1 - \left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1 - \left(\frac{\alpha_{bi}}{g}\right)^{p}\left(\frac{\alpha_{bj}}{g}\right)^{q}\right)^{w_i^p w_j^q}\right)^{1/\lambda}\right)^{1/(p+q)}$$

$$\le g\left(1 - \left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1 - \left(\frac{\alpha_{bi}'}{g}\right)^{p}\left(\frac{\alpha_{bj}'}{g}\right)^{q}\right)^{w_i^p w_j^q}\right)^{1/\lambda}\right)^{1/(p+q)},$$

$$g\left(1 - \left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1 - \left(\frac{\alpha_{ti}}{g}\right)^{p}\left(\frac{\alpha_{tj}}{g}\right)^{q}\right)^{w_i^p w_j^q}\right)^{1/\lambda}\right)^{1/(p+q)}$$

$$\le g\Bigg(1$$

$$-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(\frac{\alpha_{ti}{}'}{g}\right)^{p}\left(\frac{\alpha_{tj}{}'}{g}\right)^{q}\right)^{w_i{}^{p}w_j{}^{q}}\right)^{1/\lambda}\Bigg)^{1/(p+q)},$$

$$g\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(\frac{\alpha i}{g}\right)^{p}\left(\frac{\alpha j}{g}\right)^{q}\right)^{w_i{}^{p}w_j{}^{q}}\right)^{1/\lambda}\right)^{1/(p+q)}$$

$$\leq g\Bigg(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(\frac{\alpha i'}{g}\right)^{p}\left(\frac{\alpha j'}{g}\right)^{q}\right)^{w_i{}^{p}w_j{}^{q}}\right)^{1/\lambda}\Bigg)^{1/(p+q)}. \quad (28)$$

Similarly

$$\begin{aligned}
&1-\left(1-\frac{\beta_{bi}}{g}\right)^{p}\left(1-\frac{\beta_{bj}}{g}\right)^{q} \geq 1-\left(1-\frac{\beta_{bi}{}'}{g}\right)^{p}\left(1-\frac{\beta_{bj}{}'}{g}\right)^{q},\\
&g-g\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-\frac{\beta_{bi}}{g}\right)^{p}\left(1-\frac{\beta_{bj}}{g}\right)^{q}\right)^{w_i{}^{p}w_j{}^{q}}\right)^{1/\lambda}\right)^{1/(p+q)} \geq g\\
&\quad-g\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-\frac{\beta_{bi}{}'}{g}\right)^{p}\left(1-\frac{\beta_{bj}{}'}{g}\right)^{q}\right)^{w_i{}^{p}w_j{}^{q}}\right)^{1/\lambda}\right)^{1/(p+q)},\\
&g-g\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-\frac{\beta_{ti}}{g}\right)^{p}\left(1-\frac{\beta_{tj}}{g}\right)^{q}\right)^{w_i{}^{p}w_j{}^{q}}\right)^{1/\lambda}\right)^{1/(p+q)} \geq g\\
&\quad-g\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-\frac{\beta_{ti}{}'}{g}\right)^{p}\left(1-\frac{\beta_{tj}{}'}{g}\right)^{q}\right)^{w_i{}^{p}w_j{}^{q}}\right)^{1/\lambda}\right)^{1/(p+q)},\\
&g-g\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-\frac{\beta i}{g}\right)^{p}\left(1-\frac{\beta j}{g}\right)^{q}\right)^{w_i{}^{p}w_j{}^{q}}\right)^{1/\lambda}\right)^{1/(p+q)} \geq g\\
&\quad-g\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-\frac{\beta i'}{g}\right)^{p}\left(1-\frac{\beta j'}{g}\right)^{q}\right)^{w_i{}^{p}w_j{}^{q}}\right)^{1/\lambda}\right)^{1/(p+q)}.
\end{aligned} \quad (29)$$

And

$$\begin{aligned}
&g-g\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-\frac{\gamma_{bi}}{g}\right)^{p}\left(1-\frac{\gamma_{bj}}{g}\right)^{q}\right)^{w_i{}^{p}w_j{}^{q}}\right)^{1/\lambda}\right)^{1/(p+q)}\\
&\quad\geq g-g\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-\frac{\gamma_{bi}{}'}{g}\right)^{p}\left(1-\frac{\gamma_{bj}{}'}{g}\right)^{q}\right)^{w_i{}^{p}w_j{}^{q}}\right)^{1/\lambda}\right)^{1/(p+q)},\\
&g-g\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-\frac{\gamma_{ti}}{g}\right)^{p}\left(1-\frac{\gamma_{tj}}{g}\right)^{q}\right)^{w_i{}^{p}w_j{}^{q}}\right)^{1/\lambda}\right)^{1/(p+q)}
\end{aligned}$$

$$\geq g-g\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-\frac{\gamma_{ti}{}'}{g}\right)^{p}\left(1-\frac{\gamma_{tj}{}'}{g}\right)^{q}\right)^{w_i{}^{p}w_j{}^{q}}\right)^{1/\lambda}\right)^{1/(p+q)},$$

$$g-g\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-\frac{\gamma i}{g}\right)^{p}\left(1-\frac{\gamma j}{g}\right)^{q}\right)^{w_i{}^{p}w_j{}^{q}}\right)^{1/\lambda}\right)^{1/(p+q)}$$

$$\geq g-g\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-\frac{\gamma i'}{g}\right)^{p}\left(1-\frac{\gamma j'}{g}\right)^{q}\right)^{w_i{}^{p}w_j{}^{q}}\right)^{1/\lambda}\right)^{1/(p+q)}. \quad (30)$$

So, $\mathrm{LNCNGWHM}^{p,q}(m_1, m_2, \ldots m_n) \leq \mathrm{LNCNGWHM}^{p,q}(k_1, \boldsymbol{k}_2, \ldots k_n)$. Therefore, the proof of Theorem 13 is completed. □

Theorem 14 (boundedness). *Set $m_i = (\langle[l_{\alpha_{bi}}, l_{\alpha_{ti}}], [l_{\beta_{bi}}, l_{\beta_{ti}}], [l_{\gamma_{bi}}, l_{\gamma_{ti}}]\rangle, \langle l_{\alpha i}, l_{\beta i}, l_{\gamma i}\rangle)$ $(i = 1, 2, \ldots, n)$ as a collection of LNCNs in L; letting $m^- = (\langle[\min(l_{\alpha_{bi}}), \min(l_{\alpha_{ti}})], [\max(l_{\beta_{bi}}), \max(l_{\beta_{ti}})], [\max(l_{\gamma_{bi}}), \max(l_{\gamma_{ti}})]\rangle, \langle\min(l_{\alpha i}), \max(l_{\beta i}), \max(l_{\gamma i})\rangle)$ and $m^+ = (\langle[\max(l_{\alpha_{bi}}), \max(l_{\alpha_{ti}})], [\min(l_{\beta_{bi}}), \min(l_{\beta_{ti}})], [\min(l_{\gamma_{bi}}), \min(l_{\gamma_{ti}})]\rangle, \langle\min(l_{\alpha i}), \min(l_{\beta i}), \min(l_{\gamma i})\rangle)$, then*

$$m^- \leq LNCNGWHM^{p,q}(m_1, m_2, \ldots, m_n) \leq m^+. \quad (31)$$

Proof. Based on Theorems 12 and 13, we can obtain

$$\begin{aligned} m^- &= LNCNGWHM^{p,q}(m^-, m^-, \ldots, m^-), \\ m^+ &= LNCNGWHM^{p,q}(m^+, m^+, \ldots, m^+), \\ &LNCNGWHM^{p,q}(m^-, m^-, \ldots, m^-) \\ &\leq LNCNGWHM^{p,q}(m_1, m_2, \ldots, m_n) \\ &\leq LNCNGWHM^{p,q}(m^+, m^+, \ldots, m^+). \end{aligned} \quad (32)$$

Then $m^- \leq LNCNWHM^{p,q}(m_1, m_2, \ldots, m_n) \leq m^+$. The proof of Theorem 14 is completed. □

3.2. Three-Parameter Weighted Heronian Mean Operator of LNCNs

Definition 15. Set $m_i = (\langle[l_{\alpha_{bi}}, l_{\alpha_{ti}}], [l_{\beta_{bi}}, l_{\beta_{ti}}], [l_{\gamma_{bi}}, l_{\gamma_{ti}}]\rangle, \langle l_{\alpha i}, l_{\beta i}, l_{\gamma i}\rangle)$ $(i = 1, 2, \ldots, n)$ as a collection of LNCNs in L; then the LNCNTPWHM operator can be defined as

$$LNCNTPWHM^{p,q,r}(m_1, m_2, \ldots m_n) = \left(\frac{1}{\lambda} \cdot \bigoplus_{i=1}^{n}\bigoplus_{j=i}^{n}\bigoplus_{k=j}^{n}\left(w_i{}^{p}w_j{}^{q}w_k{}^{r}m_i{}^{p} \otimes m_j{}^{q} \otimes m_k{}^{r}\right)\right)^{1/(p+q+r)}, \quad (33)$$

where $\lambda = \sum_{i=1}^{n}\sum_{j=i}^{n}\sum_{k=j}^{n} w_i{}^{p}w_j{}^{q}w_k{}^{r}$, $w_i \in [0,1]$, and $\sum_{i=1}^{n} w_i = 1$.

Then, we can use Definitions 5 and 15 to get the following theorem.

Theorem 16. *Set $m_i = (\langle[l_{\alpha_{bi}}, l_{\alpha_{ti}}], [l_{\beta_{bi}}, l_{\beta_{ti}}], [l_{\gamma_{bi}}, l_{\gamma_{ti}}]\rangle, \langle l_{\alpha i}, l_{\beta i}, l_{\gamma i}\rangle)$ $(i = 1, 2, \ldots, n)$ as a collection of LNCNs in L; then by LNCNTPWHM operator, the aggregation result of m_i is still a LNCN, which is as follows:*

$$\begin{aligned} &LNCNTPWHM^{p,q,r}(m_1, m_2, \ldots m_n) \\ &= \left(\frac{1}{\lambda}\bigoplus_{i=1}^{n}\bigoplus_{j=i}^{n}\bigoplus_{k=j}^{n}\left(w_i{}^{p}w_j{}^{q}w_k{}^{r}m_i{}^{p} \otimes m_j{}^{q} \otimes m_k{}^{r}\right)\right)^{1/(p+q+r)} \\ &= \Bigg(\Big\langle\Big[l_{g(1-(\prod_{i=1}^{n}\prod_{j=i}^{n}\prod_{k=j}^{n}(1-(\alpha_{bi}/g)^{p}(\alpha_{bj}/g)^{q}(\alpha_{bk}/g)^{r})^{w_i{}^{p}w_j{}^{q}w_k{}^{r}})^{1/\lambda})^{1/(p+q+r)}}, l_{g(1-(\prod_{i=1}^{n}\prod_{j=i}^{n}\prod_{k=j}^{n}(1-(\alpha_{ti}/g)^{p}(\alpha_{tj}/g)^{q}(\alpha_{tk}/g)^{r})^{w_i{}^{p}w_j{}^{q}w_k{}^{r}})^{1/\lambda})^{1/(p+q+r)}}\Big], \\ &\Big[l_{g-g(1-(\prod_{i=1}^{n}\prod_{j=i}^{n}\prod_{k=j}^{n}(1-(1-\beta_{bi}/g)^{p}(1-\beta_{bj}/g)^{q}(1-\beta_{bk}/g)^{r})^{w_i{}^{p}w_j{}^{q}w_k{}^{r}})^{1/\lambda})^{1/(p+q+r)}}, l_{g-g(1-(\prod_{i=1}^{n}\prod_{j=i}^{n}\prod_{k=j}^{n}(1-(1-\beta_{ti}/g)^{p}(1-\beta_{tj}/g)^{q}(1-\beta_{tk}/g)^{r})^{w_i{}^{p}w_j{}^{q}w_k{}^{r}})^{1/\lambda})^{1/(p+q+r)}}\Big], \\ &\Big[l_{g-g(1-(\prod_{i=1}^{n}\prod_{j=i}^{n}\prod_{k=j}^{n}(1-(1-\gamma_{bi}/g)^{p}(1-\gamma_{bj}/g)^{q}(1-\gamma_{bk}/g)^{r})^{w_i{}^{p}w_j{}^{q}w_k{}^{r}})^{1/\lambda})^{1/(p+q+r)}}, l_{g-g(1-(\prod_{i=1}^{n}\prod_{j=i}^{n}\prod_{k=j}^{n}(1-(1-\gamma_{ti}/g)^{p}(1-\gamma_{tj}/g)^{q}(1-\gamma_{tk}/g)^{r})^{w_i{}^{p}w_j{}^{q}w_k{}^{r}})^{1/\lambda})^{1/(p+q+r)}}\Big]\Big\rangle, \end{aligned}$$

$$\left\langle l_{g(1-(\prod_{i=1}^{n}\prod_{j=i}^{n}\prod_{k=j}^{n}(1-(\alpha i/g)^{p}(\alpha j/g)^{q}(\alpha k/g)^{r})^{w_i{}^{p}w_j{}^{q}w_k{}^{r}})^{1/\lambda})^{1/(p+q+r)}}, l_{g-g(1-(\prod_{i=1}^{n}\prod_{j=i}^{n}\prod_{k=j}^{n}(1-(1-\beta i/g)^{p}(1-\beta j/g)^{q}(1-\beta k/g)^{r})^{w_i{}^{p}w_j{}^{q}w_k{}^{r}})^{1/\lambda})^{1/(p+q+r)}}, l_{g-g(1-(\prod_{i=1}^{n}\prod_{j=i}^{n}\prod_{k=j}^{n}(1-(1-\gamma i/g)^{p}(1-\gamma j/g)^{q}(1-\gamma k/g)^{r})^{w_i{}^{p}w_j{}^{q}w_k{}^{r}})^{1/\lambda})^{1/(p+q+r)}} \right\rangle\Big), \quad (34)$$

where $\lambda = \sum_{i=1}^{n}\sum_{j=i}^{n}\sum_{k=j}^{n} w_i{}^{p} w_j{}^{q} w_k{}^{r}$, $w_i \in [0,1]$, *and* $\sum_{i=1}^{n} w_i = 1$.

Theorem 17 (idempotency). *Set* $m_i = (\langle [l_{\alpha_{bi}}, l_{\alpha_{ti}}], [l_{\beta_{bi}}, l_{\beta_{ti}}], [l_{\gamma_{bi}}, l_{\gamma_{ti}}]\rangle, \langle l_{\alpha i}, l_{\beta i}, l_{\gamma i}\rangle)$ $(i = 1, 2, \ldots, n)$ *as a collection of LNCNs in L; if* $m_i = m$, *then*

$$LNCNTPWHM^{p,q,r}(m_1, m_2, \ldots m_n) = LNCNTPWHM^{p,q,r}(m, m, \ldots m) = m. \quad (35)$$

Theorem 18 (monotonicity). *Set* $m_i = (\langle [l_{\alpha_{bi}}, l_{\alpha_{ti}}], [l_{\beta_{bi}}, l_{\beta_{ti}}], [l_{\gamma_{bi}}, l_{\gamma_{ti}}]\rangle, \langle l_{\alpha i}, l_{\beta i}, l_{\gamma i}\rangle)$ *and* $k_i = (\langle [l_{\alpha_{bi}'}, l_{\alpha_{ti}'}], [l_{\beta_{bi}'}, l_{\beta_{ti}'}], [l_{\gamma_{bi}'}, l_{\gamma_{ti}'}]\rangle, \langle l_{\alpha i'}, l_{\beta i'}, l_{\gamma i'}\rangle)$ $(i = 1, 2, \ldots, n)$ *as two collections of LNCNs in L; if* $\alpha_{bi} \le \alpha_{bi}{}'$, $\alpha_{ti} \le \alpha_{ti}{}'$, $\alpha i \le \alpha i'$, $\beta_{bi} \ge \beta_{bi}{}'$, $\beta_{ti} \ge \beta_{ti}{}'$, $\beta i \ge \beta i'$ *and* $\gamma_{bi} \ge \gamma_{bi}{}'$, $\gamma_{ti} \ge \gamma_{ti}{}'$, $\gamma i \ge \gamma i'$, *then*

$$LNCNTPWHM^{p,q,r}(m_1, m_2, \ldots m_n) \le LNCNTPWHM^{p,q,r}(k_1, k_2, \ldots k_n). \quad (36)$$

Theorem 19 (boundedness). *Set* $m_i = (\langle [l_{\alpha_{bi}}, l_{\alpha_{ti}}], [l_{\beta_{bi}}, l_{\beta_{ti}}], [l_{\gamma_{bi}}, l_{\gamma_{ti}}]\rangle, \langle l_{\alpha i}, l_{\beta i}, l_{\gamma i}\rangle)$ $(i = 1, 2, \ldots, n)$ *as a collection of LNCNs in L; letting* $m^- = (\langle [\min(l_{\alpha_{bi}}), \min(l_{\alpha_{ti}})], [\max(l_{\beta_{bi}}), \max(l_{\beta_{ti}})], [\max(l_{\gamma_{bi}}), \max(l_{\gamma_{ti}})]\rangle, \langle \min(l_{\alpha i}), \max(l_{\beta i}), \max(l_{\gamma i})\rangle)$ *and* $m^+ = (\langle [\max(l_{\alpha_{bi}}), \max(l_{\alpha_{ti}})], [\min(l_{\beta_{bi}}), \min(l_{\beta_{ti}})], [\min(l_{\gamma_{bi}}), \min(l_{\gamma_{ti}})]\rangle, \langle \min(l_{\alpha i}), \min(l_{\beta i}), \min(l_{\gamma i})\rangle)$, *then*

$$m^- \le LNCNTPWHM^{p,q,r}(m_1, m_2, \ldots m_n) \le m^+. \quad (37)$$

The proofs of Theorems 16–19 are similar to those of Theorems 11–14, so we do not repeat them again.

4. MADM Methods Based on the LNCNGWHM or LNCNTPWHM Operator

This section uses the LNCNGWHM or LNCNTPWHM operator to deal with the MADM problems with LNCN information.

Let $A = \{A_1, A_2, \ldots, A_m\}$ be a discrete set of alternatives with a set of attributes $C = \{C_1, C_2, \ldots, C_n\}$, and the weight vector of $C_j (j = 1, 2, \ldots, n)$ is $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$, $\lambda_j \ge 0$ and $\sum_{j=1}^{n} \lambda_j = 1$. A linguistic assessment set $L = \{l_j \mid j \in [0, g]\}$ is given. Some experts use LNCN to evaluate the alternatives $A_i (i = 1, 2, \ldots, m)$ under the attributes $C_j (j = 1, 2, \ldots, n)$. The assessed values of the experts for A_i with attribute C_j are $m_{ij} = (\langle [l_{\alpha_{bij}}, l_{\alpha_{tij}}], [l_{\beta_{bij}}, l_{\beta_{tij}}], [l_{\gamma_{bij}}, l_{\gamma_{tij}}]\rangle, \langle l_{\alpha ij}, l_{\beta ij}, l_{\gamma ij}\rangle)(i = 1, 2, \ldots, m; j = 1, 2, \ldots, n)$. Then, we can get the neutrosophic linguistic cubic decision evaluation matrix $(m_{ij})_{m\times n}$.

Then, the decision-making method based on the LNCNGWHM or LNCNTPWHM operator is described as follows.

Step 1. According to the weight vector $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ and the LNCNGWHM or LNCNTPWHM operator, we can calculate $m_i = LNCNGWHM(m_{i1}, m_{i2}, \ldots, m_{ij})$ or $m_i = LNCNTPWHM(m_{i1}, m_{i2}, \ldots, m_{ij})$.

Step 2. Calculate the score value of $S(m_i)$ (accuracy value of $A(m_i)$ and certain value of $C(m_i)$ if necessary) of the LNCN $A_i (i = 1, 2, \ldots, m)$ according to formula (11) (formula (12) and formula (13) if necessary).

Step 3. According to ranking method of Definition 7, we can rank the attributes corresponding to the values of $S(m_i)$ (accuracy value $A(m_i)$ and certain value $C(m_i)$ if necessary).

Step 4. End.

5. Illustrative Example

This section considers a decision-making problem adapted from the literature [20]. A mechanical designer wants to design press machine; then he should consider the design of the reducing mechanism and the working mechanism. According to the press machine's functional requirements, there are four design schemes/alternatives A_1, A_2, A_3 and A_4 to be proposed by the designers, which are shown in Table 1. The four design schemes must satisfy the requirements of four attributes while being evaluated: the manufacturing cost (C_1), the mechanical structure (C_2), the transmission effectiveness (C_3), and the reliability (C_4). The importance of four attributes is given as a weight vector $\lambda = (0.3, 0.25, 0.25, 0.2)$. Then, the experts define the linguistic term set $L = \{l_j \mid j \in [0, 8]\}$, where $L = \{l_0 =$ extremely low, $l_1 =$ very low, $l_2 =$ low, $l_3 =$ slightly low, $l_4 =$ medium , $l_5 =$ slightly high, $l_6 =$ high, $l_7 =$ very high, $l_8 =$ extremely high$\}$. Afterwards, they evaluate the four design schemes/alternatives under the four attributes by the form of LNCNs based on L. Thus, the LNCN decision matrix$(m_{ij})_{4\times 4}$ can be established, which are shown in Table 2.

Next, the decision-making methods proposed in Section 4 are employed to deal with the decision problem; the description of decision procedures is shown as follows:

5.1. *The Decision-Making Process Based on LNCNGWHM Operator or LNCNTPWHM Operator*

Step 1. By using (17) (suppose $p = q = 1$) and the weight vector $\lambda = (0.3, 0.25, 0.25, 0.2)$ of attributes, we can obtain the comprehensive evaluation values $m_i (i = 1, 2, 3, 4)$ of alternative A_i as follows:

$$m_1 = (\langle [l_{4.0000}, l_{6.4628}], [l_{1.0000}, l_{2.4379}], [l_{1.1967}, l_{2.8039}]\rangle, \langle l_{5.2546}, l_{1.5999}, l_{2.3849}\rangle),$$

Table 1: Four design schemes/alternatives for press machine.

Design scheme	A_1	A_2	A_3	A_4
Reducing mechanism	Gear reducer	Gear head motor	Gear reducer	Gear head motor
Working mechanism	Flywheel-crank-slider mechanism		Flywheel-screw-driving mechanism	

$$\begin{aligned} m_2 &= (\langle [l_{3.9271}, l_{5.8066}], [l_{1.1967}, l_{2.3849}], [l_{1.0000}, l_{2.0000}] \rangle, \langle l_{4.9402}, l_{1.5297}, l_{1.1967} \rangle), \\ m_3 &= (\langle [l_{4.9402}, l_{7.0000}], [l_{1.0000}, l_{2.4379}], [l_{1.5372}, l_{3.0000}] \rangle, \langle l_{5.7560}, l_{2.0000}, l_{2.5578} \rangle), \\ m_4 &= (\langle [l_{5.1800}, l_{6.6362}], [l_{1.4814}, l_{2.5028}], [l_{1.3041}, l_{2.3243}] \rangle, \langle l_{6.2161}, l_{1.5499}, l_{2.0274} \rangle). \end{aligned} \quad (38)$$

Step 2. Calculate the score values of $S(m_i)$ according to (11) for $m_i (i = 1, 2, 3, 4)$:

$$\begin{aligned} S(m_1) &= 0.7263; \\ S(m_2) &= 0.7412; \\ S(m_3) &= 0.7384; \\ S(m_4) &= 0.7617. \end{aligned} \quad (39)$$

According to the results of $S(m_i)$ $(i = 1, 2, 3, 4)$, we can rank the alternatives $A_4 \succ A_2 \succ A_3 \succ A_1$, and then the design scheme A_4 is the best among all the alternatives.

On the other hand, we can use the LNCNTPWHM operator (set p=1, q=1) to deal with this problem.

Step 1′. By using (34) (suppose $p = q = 1$) and the weight vector $\lambda = (0.3, 0.25, 0.25, 0.2)$of attributes, we can obtain the comprehensive evaluation values $m_i (i = 1, 2, 3, 4)$ of alternative A_i as follows:

$$\begin{aligned} m_1 &= (\langle [l_{4.0000}, l_{6.4368}], [l_{1.0000}, l_{2.4328}], [l_{1.1994}, l_{2.8180}] \rangle, \langle l_{5.2461}, l_{1.5880}, l_{2.3749} \rangle), \\ m_2 &= (\langle [l_{3.9051}, l_{5.7709}], [l_{1.1994}, l_{2.3749}], [l_{1.0000}, l_{2.0000}] \rangle, \langle l_{4.9108} l_{1.5319}, l_{1.1994} \rangle), \\ m_3 &= (\langle [l_{4.9108}, l_{7.0000}], [l_{1.0000}, l_{2.4328}], [l_{1.5632}, l_{3.0000}] \rangle, \langle l_{5.7142} l_{2.0000}, l_{2.5777} \rangle), \\ m_4 &= (\langle [l_{5.1989}, l_{6.6406}], [l_{1.5015}, l_{2.5170}], [l_{1.3314}, l_{2.3472}] \rangle, \langle l_{6.2253} l_{1.6109}, l_{2.0674} \rangle). \end{aligned} \quad (40)$$

Step 2′. Calculate the score values of $S(m_i)$ according to (11) for $m_i (i = 1, 2, 3, 4)$:

$$\begin{aligned} S(m_1) &= 0.7260; \\ S(m_2) &= 0.7400; \\ S(m_3) &= 0.7368; \\ S(m_4) &= 0.7596. \end{aligned} \quad (41)$$

According to the results of $S(m_i)$ $(i = 1, 2, 3, 4)$, we can rank the alternatives $A_4 \succ A_2 \succ A_3 \succ A_1$, so the design scheme A_4 is the best among all the alternatives.

5.2. Analyzing the Effect of the Parameters p, q, and r. Different parameters p, q, and r may have different effects on the decision results. Therefore, this section takes different values of p, q, and r to sort the various alternatives, and then Tables 3-4 present the results.

From Tables 3 and 4, we can see that LNCNTPWHM operator can get more stable sorting than the LNCNGWHM operator, and, in addition, when the parameters p, q, and r take different values, the best design scheme/alternative using either the LNCNGWHM or LNCNTPWHM operator is always A_4. Therefore, the parameters p, q, and r in the LNCNGWHM or LNCNTPWHM operator have little influence on decision-making.

5.3. Comparing with the Related Methods. Firstly, compared with the literature [20], this paper used the decision information under LNCN environment, while the literature [20] used the decision information under intuitionistic fuzzy environment. In the literature [20], only incomplete information can be effectively expressed, and the indeterminate and inconsistent information cannot be described effectively, while LNCN is composed of uncertain linguistic neutrosophic number and linguistic neutrosophic number, where the uncertain linguistic neutrosophic number is represented by the truth, the indeterminacy, and the falsity uncertain linguistic variables, respectively, and the linguistic neutrosophic number is represented by the truth, indeterminacy, and falsity linguistic variables, respectively. So LNCN contains more information than the intuitionistic fuzzy number in [20].

Second, compared with the existing related methods based on the LNCNWAA and LNCNWGA operators in literature [18], all the ranking results have been shown in Table 5.

The results given in Table 5 show that all the aggregated values of the LNCNGWHM and LNCNTPWHM operators are more or less close to moderate values between the aggregated values of the LNCNWAA and LNCNWGA operators. Then, all the ranking orders based on the LNCNWAA LNCNWGA, LNCNGWHM, and LNCNTPWHM operators are identical. However, the LNCNGWHM and LNCNTPWHM operators embody the interaction between attributes and consider the different p, q, and r values to make the decision-making results more persuasive and comprehensive than the LNCNWAA and LNCNWGA operators in literature [18].

6. Conclusions

This paper proposed MADM methods based on the LNCNGWHM and LNCNTPWHM operators for LNCNs. First, a LNCN generalized weight Heronian mean (LNCNGWHM) operator and a LNCN three-parameter weighted Heronian mean (LNCNTPWHM) operator were proposed and the

Table 2: The decision matrix $(m_{ij})_{4\times 4}$.

	C_1	C_2	C_3	C_4
A_1	$(\langle [l_4,l_6],[l_1,l_2],[l_1,l_3]\rangle,\langle l_5,l_1,l_2\rangle)$	$(\langle [l_4,l_6],[l_1,l_3],[l_1,l_3]\rangle,\langle l_5,l_2,l_2\rangle)$	$(\langle [l_4,l_7],[l_1,l_3],[l_2,l_3]\rangle,\langle l_6,l_2,l_3\rangle)$	$(\langle [l_4,l_7],[l_1,l_2],[l_1,l_2]\rangle,\langle l_5,l_2,l_3\rangle)$
A_2	$(\langle [l_3,l_5],[l_1,l_2],[l_1,l_2]\rangle,\langle l_4,l_1,l_1\rangle)$	$(\langle [l_5,l_7],[l_1,l_2],[l_1,l_2]\rangle,\langle l_6,l_1,l_2\rangle)$	$(\langle [l_4,l_6],[l_2,l_3],[l_1,l_2]\rangle,\langle l_5,l_3,l_1\rangle)$	$(\langle [l_4,l_5],[l_1,l_3],[l_1,l_2]\rangle,\langle l_5,l_2,l_1\rangle)$
A_3	$(\langle [l_4,l_7],[l_1,l_2],[l_2,l_3]\rangle,\langle l_5,l_2,l_3\rangle)$	$(\langle [l_6,l_7],[l_1,l_3],[l_1,l_3]\rangle,\langle l_7,l_2,l_2\rangle)$	$(\langle [l_5,l_7],[l_1,l_3],[l_2,l_3]\rangle,\langle l_5,l_2,l_3\rangle)$	$(\langle [l_5,l_7],[l_1,l_2],[l_1,l_3]\rangle,\langle l_6,l_2,l_2\rangle)$
A_4	$(\langle [l_6,l_7],[l_2,l_3],[l_2,l_3]\rangle,\langle l_7,l_3,l_3\rangle)$	$(\langle [l_5,l_7],[l_1,l_2],[l_1,l_2]\rangle,\langle l_6,l_1,l_2\rangle)$	$(\langle [l_4,l_6],[l_1,l_2],[l_1,l_2]\rangle,\langle l_5,l_1,l_1\rangle)$	$(\langle [l_5,l_6],[l_2,l_3],[l_1,l_2]\rangle,\langle l_6,l_1,l_2\rangle)$

Table 3: The ranking based on the LNCNGWHM operator with different values of p and q.

p, q	LNCNGWHM	Ranking
$P=1, q=0$	$S(m_1)=0.7285, S(m_2)=0.7479, S(m_3)=0.7377, S(m_4)=0.7638$	$A_4 \succ A_2 \succ A_3 \succ A_1$
$P=1, q=0.5$	$S(m_1)=0.7263, S(m_2)=0.7420, S(m_3)=0.7377, S(m_4)=0.7609$	$A_4 \succ A_2 \succ A_3 \succ A_1$
$P=1, q=2$	$S(m_1)=0.7277, S(m_2)=0.7427, S(m_3)=0.7388, S(m_4)=0.7645$	$A_4 \succ A_2 \succ A_3 \succ A_1$
$P=0, q=1$	$S(m_1)=0.7269, S(m_2)=0.7411, S(m_3)=0.7470, S(m_4)=0.7688$	$A_4 \succ A_3 \succ A_2 \succ A_1$
$P=0.5, q=1$	$S(m_1)=0.7257, S(m_2)=0.7404, S(m_3)=0.7410, S(m_4)=0.7635$	$A_4 \succ A_3 \succ A_2 \succ A_1$
$P=2, q=1$	$S(m_1)=0.7283, S(m_2)=0.7424, S(m_3)=0.7350, S(m_4)=0.7592$	$A_4 \succ A_2 \succ A_3 \succ A_1$
$P=2, q=2$	$S(m_1)=0.7290, S(m_2)=0.7428, S(m_3)=0.7352, S(m_4)=0.7609$	$A_4 \succ A_2 \succ A_3 \succ A_1$

Table 4: The ranking based on the LNCNTPWHM operator with different values of p, q, and r.

p, q, r	LNCNTPWHM	Ranking
$p=1, q=1, r=2$	$S(m_1)=0.7269, S(m_2)=0.7406, S(m_3)=0.7376, S(m_4)=0.7621$	$A_4 \succ A_2 \succ A_3 \succ A_1$
$p=1, q=2, r=1$	$S(m_1)=0.7166, S(m_2)=0.7342, S(m_3)=0.7279, S(m_4)=0.7772$	$A_4 \succ A_2 \succ A_3 \succ A_1$
$p=1, q=2, r=2$	$S(m_1)=0.7282, S(m_2)=0.7431, S(m_3)=0.7362, S(m_4)=0.7612$	$A_4 \succ A_2 \succ A_3 \succ A_1$
$p=1, q=2, r=3$	$S(m_1)=0.7299, S(m_2)=0.7444, S(m_3)=0.7363, S(m_4)=0.7650$	$A_4 \succ A_2 \succ A_3 \succ A_1$
$p=2, q=1, r=1$	$S(m_1)=0.7278, S(m_2)=0.7401, S(m_3)=0.7333, S(m_4)=0.7556$	$A_4 \succ A_2 \succ A_3 \succ A_1$
$p=2, q=1, r=2$	$S(m_1)=0.7282, S(m_2)=0.7404, S(m_3)=0.7342, S(m_4)=0.7578$	$A_4 \succ A_2 \succ A_3 \succ A_1$
$p=2, q=2, r=1$	$S(m_1)=0.7290, S(m_2)=0.7419, S(m_3)=0.7328, S(m_4)=0.7569$	$A_4 \succ A_2 \succ A_3 \succ A_1$
$p=3, q=2, r=1$	$S(m_1)=0.7302, S(m_2)=0.7412, S(m_3)=0.7303, S(m_4)=0.7531$	$A_4 \succ A_2 \succ A_3 \succ A_1$

Table 5: Decision results based on four aggregation operators.

Aggregation operator	*Aggregated result*	*Score value*	*Ranking*
LNCNWAA	$m_1 = (\langle[l_{4.0000}, l_{6.5359}], [l_{1.0000}, l_{2.4495}], [l_{1.1892}, l_{2.7663}]\rangle, \langle l_{5.2892}, l_{1.6245}, l_{2.4003}\rangle)$,	$S(m_1)=0.7277$,	$A_4 \succ A_2 \succ A_3 \succ A_1$
	$m_2 = (\langle[l_{4.0199}, l_{5.9402}], [l_{1.1892}, l_{2.4003}], [l_{1.0000}, l_{2.0000}]\rangle, \langle l_{5.0448} l_{1.5118}, l_{1.1892}\rangle)$,	$S(m_2)=0.7460$,	
	$m_3 = (\langle[l_{5.0448}, l_{7.0000}], [l_{1.0000}, l_{2.4495}], [l_{1.4641}, l_{3.0000}]\rangle, \langle l_{5.8980} l_{2.0000}, l_{2.4997}\rangle)$,	$S(m_3)=0.7435$,	
	$m_4 = (\langle[l_{5.1455}, l_{6.6340}], [l_{1.4142}, l_{2.4495}], [l_{1.2311}, l_{2.2587}]\rangle, \langle l_{6.2022} l_{1.3904}, l_{1.8993}\rangle)$	$S(m_4)=0.7686$	
LNCNWGA	$m_1 = (\langle[l_{4.0000}, l_{6.4310}], [l_{1.0000}, l_{2.5228}], [l_{1.2646}, l_{2.8143}]\rangle, \langle l_{5.2332}, l_{1.7160}, l_{2.4726}\rangle)$,	$S(m_1)=0.7205$,	$A_4 \succ A_3 \succ A_2 \succ A_1$
	$m_2 = (\langle[l_{3.8798}, l_{5.6924}], [l_{1.2646}, l_{2.4726}], [l_{1.0000}, l_{2.0000}]\rangle, \langle l_{4.8943} l_{1.7601}, l_{1.2646}\rangle)$,	$S(m_2)=0.7320$,	
	$m_3 = (\langle[l_{4.8943}, l_{7.0000}], [l_{1.0000}, l_{2.5228}], [l_{1.5690}, l_{3.0000}]\rangle, \langle l_{5.6408} l_{2.0000}, l_{2.5725}\rangle)$,	$S(m_3)=0.7343$,	
	$m_4 = (\langle[l_{4.9946}, l_{6.5309}], [l_{1.5193}, l_{2.5228}], [l_{1.3163}, l_{2.3194}]\rangle, \langle l_{6.0040} l_{1.6721}, l_{2.0962}\rangle)$	$S(m_4)=0.7512$	
LNCNGWHM ($p=q=1$)	$m_1 = (\langle[l_{4.0000}, l_{6.4628}], [l_{1.0000}, l_{2.4379}], [l_{1.1967}, l_{2.8039}]\rangle, \langle l_{5.2546}, l_{1.5999}, l_{2.3849}\rangle)$,	$S(m_1)=0.7263$;	$A_4 \succ A_2 \succ A_3 \succ A_1$
	$m_2 = (\langle[l_{3.9271}, l_{5.8066}], [l_{1.1967}, l_{2.3849}], [l_{1.0000}, l_{2.0000}]\rangle, \langle l_{4.9402}, l_{1.5297}, l_{1.1967}\rangle)$,	$S(m_2)=0.7412$;	
	$m_3 = (\langle[l_{4.9402}, l_{7.0000}], [l_{1.0000}, l_{2.4379}], [l_{1.5372}, l_{3.0000}]\rangle, \langle l_{5.7560}, l_{2.0000}, l_{2.5578}\rangle)$,	$S(m_3)=0.7384$;	
	$m_4 = (\langle[l_{5.1800}, l_{6.6362}], [l_{1.4814}, l_{2.5028}], [l_{1.3041}, l_{2.3243}]\rangle, \langle l_{6.2161}, l_{1.5499}, l_{2.0274}\rangle)$	$S(m_4)=0.7617$	
LNCNTPWHM ($p=q=1$)	$m_1 = (\langle[l_{4.0000}, l_{6.4368}], [l_{1.0000}, l_{2.4328}], [l_{1.1994}, l_{2.8180}]\rangle, \langle l_{5.2461}, l_{1.5880}, l_{2.3749}\rangle)$,	$S(m_1)=0.7260$;	$A_4 \succ A_2 \succ A_3 \succ A_1$
	$m_2 = (\langle[l_{3.9051}, l_{5.7709}], [l_{1.1994}, l_{2.3749}], [l_{1.0000}, l_{2.0000}]\rangle, \langle l_{4.9108} l_{1.5319}, l_{1.1994}\rangle)$,	$S(m_2)=0.7400$;	
	$m_3 = (\langle[l_{4.9108}, l_{7.0000}], [l_{1.0000}, l_{2.4328}], [l_{1.5632}, l_{3.0000}]\rangle, \langle l_{5.7142} l_{2.0000}, l_{2.5777}\rangle)$,	$S(m_3)=0.7368$;	
	$m_4 = (\langle[l_{5.1989}, l_{6.6406}], [l_{1.5015}, l_{2.5170}], [l_{1.3314}, l_{2.3472}]\rangle, \langle l_{6.2253} l_{1.6109}, l_{2.0674}\rangle)$	$S(m_4)=0.7596$	

related properties of these two operators were discussed. Second, the two methods of MADM in a LNCN setting were put forward based on the LNCNGWHM operator and the LNCNTPWHM operator. Finally, these two methods are used to solve a practical problem. In order to make the decision-making result more convincing, the different values of the parameters p, q, and r were taken to observe the sorting results. From the sorting results, we found that the influence of three parameters on the decision results was very small. Furthermore, compared with the relative method, the proposed methods in this paper can get the same selection result as the existing method. Therefore, the proposed methods demonstrate potential applications in handling MADM problems under LNCN environment.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare no conflicts of interest.

Authors' Contributions

Changxing Fan proposed the LNCNGWHM and LNCNTP-WHM operators and investigated their properties, Changxing Fan presented the organization and decision-making method of this paper, and Jun Ye provided the calculation and analysis of the illustrative example; the authors wrote the paper together.

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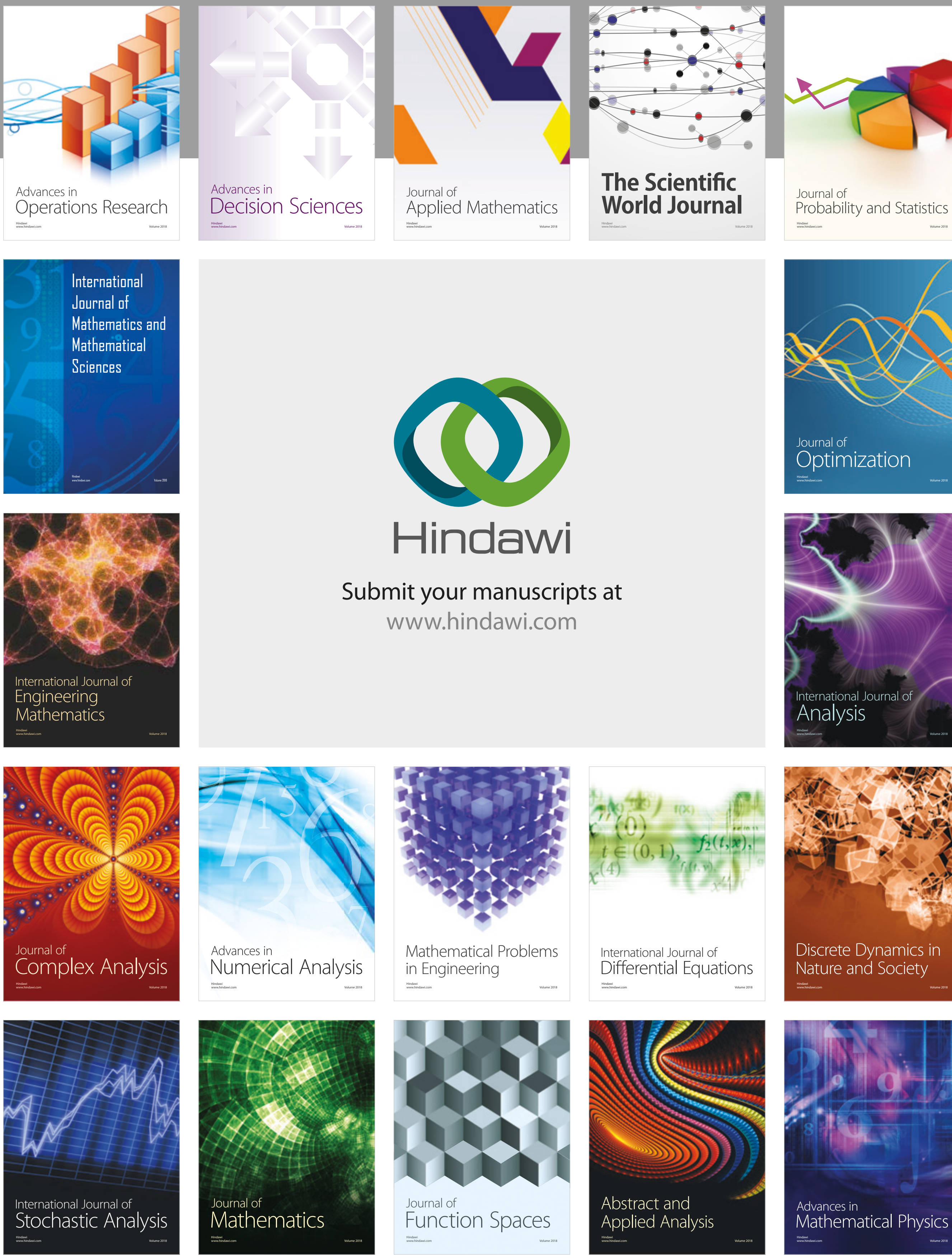
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