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Research Article

An Exponential Jensen Fuzzy Divergence Measure with Applications in Multiple Attribute Decision-Making

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A divergence measure plays an important role in discriminating two probability distributions and drawing inferences based on such discrimination. This communication introduces one such divergence measure based on Jensen inequality and exponential entropy introduced by Pal and Pal in the settings of probability theory. Further, the idea has been generalized to fuzzy sets to introduce a new fuzzy divergence measure. Besides establishing the validity, some of its major properties are also studied. At last, the application of proposed fuzzy divergence measure is given in strategic decision-making.

1. Introduction

Uncertainty in real world is ubiquitous. Before the discovery of fuzzy sets by Zadeh [1], probability was the only way to measure the uncertainty. Introduction of fuzzy set (FS) theory by Zadeh [1] provided the researchers with an important tool to handle the uncertainty/fuzziness. Contrary to the classical definition of a set in which a particular either belongs to the set or not, in fuzzy set each member of the set is assigned a membership degree. Fuzzy entropy is an important concept related to FS theory. Fuzzy entropy quantifies the fuzziness arising due to the vagueness of being or not being the member of the set. It was Zadeh again who pioneered in the introduction of fuzzy entropy as weighted Shannon entropy [2]. The axiomatization of the concept of fuzzy entropy by De Luca and Termini [3] has caused several researchers to introduce fuzzy entropies from their viewpoints and gave their applications in distinct fields [4–6]. Recently, Joshi and Kumar [7, 8] have introduced two new fuzzy entropies and applied them in decision-making. Amongst various applications, one important application of fuzzy entropies has been found in determining the divergence between two distributions. A divergence measure is a measure which is used to quantify the distance between two distributions. To develop new fuzzy divergence measures and seek their applications in different areas is a hot topic amongst the researchers nowadays. Thus, many authors are contributing towards the development of fuzzy divergence measures and their applications in distinct fields like pattern recognition, image segmentation, etc. [9–18]. The above discussion highlights the importance of divergence measures based on fuzzy sets.

Vagueness in scientific studies poses a problem. FSs introduced by Zadeh [1] have been proved to be more useful to characterizing such vagueness. Therefore, it is natural to develop such measures which can not only measure vagueness but also quantify the divergence between underlying characterizing FSs. The present communication is a sequel in this direction. Using the notion of Jensen-Shannon divergence proposed by Lin [19], a new concept called Jensen-Exponential divergence measure based on famous Jensen inequality and well-known exponential entropy introduced by Pal and Pal [5] has been put forward. One of the major benefits of using Jensen-Shannon divergence measure [19] is that we can give any weight-age to a probability distribution. While introducing the concept of exponential entropy by Pal and Pal [5], the authors claimed that exponential entropy has several advantages over logarithmic entropy like upper bound, etc. Exploring the properties of exponential entropy introduced by Pal and Pal [5], a new divergence measure has been proposed in this communication. The prime aims of introducing this manuscript are as follows: (1) to propose a new divergence measure based on exponential entropy introduced by Pal and Pal [5] and Jensen inequality; (2) to extend this idea to fuzzy settings to introduce a new fuzzy divergence measure; (3) to introduce a new multiple attribute decision-making method based on proposed fuzzy divergence measure; (4) to apply the proposed method in strategic decision-making problem.

The paper is managed as follows: The contribution of earlier researchers to the field and the aim of the manuscript are given in Section 1. Some basic concepts and definitions related to the topic under consideration are given in Section 2. In Section 3, a new divergence measure and its extension to fuzzy sets are given in Section 3. Besides this, the validity of proposed fuzzy divergence measures is also established in this section. Some major properties of proposed fuzzy divergence are discussed in Section 4. Section 5 is devoted to the application of proposed fuzzy divergence measure in decision-making. At last, the paper is summed up with "conclusions" in Section 6.

The next section contains some relevant concepts and definitions necessary to understand this communication.

2. Preliminaries

Definition 1 (See [1]). Let $X = (g_1, g_2, \ldots, g_n)$ be a finite universe of discourse. A fuzzy set (FS) S in X is defined as

$$S = \{\langle g_i, \mu_S(g_i)\rangle \mid g_i \in X\}, \quad (1)$$

where $\mu_S : X \longrightarrow [0,1]$ denotes the membership function and $\mu_S(g_i)$ represents the membership degree or measure of belongingness of $g_i \in X$ to S.

From here onwards, $FS(X)$ will denote the set of all fuzzy sets defined on X.

Now, we define some basic operations on FSs.

Definition 2 (See [1]). Let $S, T \in FS(X)$. Then

(1) $S \subseteq T$ if and only if $\mu_S(g_i) \leq \mu_T(g_i)$ for all $g_i \in X$;

(2) $S = T$ if and only if $S \subseteq T$ and $T \subseteq S$;

(3) $S^c = \{\langle g_i, (1 - \mu_S(g_i))\rangle \mid g_i \in X\}$;

(4) $S \cup T = \{\langle g_i \max(\mu_S(g_i), \mu_T(g_i))\rangle \mid g_i \in X\}$;

(5) $S \cap T = \{\langle g_i \min(\mu_S(g_i), \mu_T(g_i))\rangle \mid g_i \in X\}$.

3. A New Fuzzy Divergence Measure

3.1. Background. Let $\triangle_n = \{C = (c_1, c_2, \ldots, c_n); c_i \geq 0, \sum_{i=1}^{n} c_i = 1\}$, $n \geq 2$, be a set of complete probability distributions. For some $C \in \triangle_n$, the exponential entropy

$$H_{PP}(C) = \sum_{i=1}^{n} c_i \left(e^{1-c_i} - 1\right), \quad (2)$$

proposed by Pal and Pal [5] is one of the well-known index of diversity. The concavity of measure (2) bifurcates the mixed distribution $((C + D)/2)$ into two parts as follows:

$$H_{PP}\left(\frac{C+D}{2}\right) = \frac{1}{2}\left(H_{PP}(C) + H_{PP}(D)\right) + \frac{1}{2} J_n^{PP}(C, D). \quad (3)$$

The first term in (3), that is, $((H_{PP}(C) + H_{PP}(D))/2)$, is the average diversity within distributions C and D and the second term, that is,

$$J_n^{PP}(C, D) = \left(-H_{PP}(C) - H_{PP}(D)\right) - 2\left(-H_{PP}\left(\frac{C+D}{2}\right)\right), \quad (4)$$

is called Exponential Jensen difference. Convexity of $-H_{PP}(C)$ implies the convexity of Jensen difference given by (4). Again, $J_n^{PP}(C, D) = 0$ at $C = D$ confirms that (4) is a natural measure of divergence [20, 21]. Besides this, the measure (4) also meets the intuitive requirement that average divergence between (C, D) and (E, F) is not less than their convex combination; that is, $\chi_1 H_{PP}(C, D) + \chi_2 H_{PP}(E, F)$, where $\chi_1, \chi_2 \geq 0$ satisfying $\chi_1 + \chi_2 = 1$. The convex behavior of Jensen difference (4) is an additional feature of exponential entropy (2).

The authors claim that (2) has some advantages over Shannon entropy [22] particularly in image processing. This is due to fact that entropy (2) has a fixed upper bound, that is, for an uniform distribution $(1/n, 1/n, \ldots, 1/n)$; $\lim_{n \longrightarrow \infty} H(C) = e - 1$ as compared to infinite limit (as $n \longrightarrow \infty$) in case of Shannon entropy.

Based on the concept of Jensen-Shannon divergence [19], corresponding to (2), for any two probability distributions $C, D \in \triangle_n$ with δ_1 and δ_2 as their respective weights satisfying $\delta_1 + \delta_2 = 1$, we define a new divergence measure as

$$\widehat{\widehat{EJD}}(C, D) = \sum_{i=1}^{n} \Big((\delta_1 c_i + \delta_2 d_i) e^{1-(\delta_1 c_i + \delta_2 d_i)} + (\delta_2 c_i + \delta_1 d_i) e^{1-(\delta_2 c_i + \delta_1 d_i)} - \left(c_i e^{1-c_i}\right) - \left(d_i e^{1-d_i}\right) \Big). \quad (5)$$

To justify the existence, we now prove that proposed measure satisfies the following properties.

Properties of $\widehat{\widehat{EJD}}(C, D)$. Some major properties of proposed divergence measure are

(1) $\widehat{\widehat{EJD}}(C, D) \geq 0$ with equality when $C = D$;

(2) $\widehat{\widehat{EJD}}(C, D)$ is a convex function of C and D.

Proof. First we prove that $\widehat{\widehat{EJD}}(C, D) \geq 0$.

Since (2) represents a concave function, therefore, for any $C, D \in \triangle_n$, using Jensen's inequality, we have

$$H(\delta_1 C + \delta_2 D) \geq \delta_1 H(C) + \delta_2 H(D). \quad (6)$$

This implies

$$\sum_{i=1}^{n}\left(\delta_1 c_i+\delta_2 d_i\right)e^{(1-(\delta_1 c_i+\delta_2 d_i))} \geq \delta_1\sum_{i=1}^{n}c_i e^{1-c_i}+\delta_2\sum_{i=1}^{n}d_i e^{1-d_i}. \tag{7}$$

Similarly,

$$\sum_{i=1}^{n}\left(\delta_1 d_i+\delta_2 c_i\right)e^{(1-(\delta_1 d_i+\delta_2 c_i))} \geq \delta_1\sum_{i=1}^{n}d_i e^{1-d_i}+\delta_2\sum_{i=1}^{n}c_i e^{1-c_i}. \tag{8}$$

Adding (7) and (8), we get $\widehat{\widehat{EJD}}(C,D)\geq 0$.

Now, we prove the convexity of (5). □

For this, we consider a function given by

$$f(x,y)=\left(\delta_1 x+\delta_2 y\right)e^{(1-(\delta_1 x+\delta_2 y))}-\delta_1 x e^{1-x} - \delta_2 y e^{1-y}. \tag{9}$$

Definition 3. The Hessian matrix of a function f of two variables x and y is defined as

$$Hessian(f)=\begin{bmatrix}\dfrac{\delta^2 f}{\delta x^2} & \dfrac{\delta^2 f}{\delta x\delta y}\\ \dfrac{\delta^2 f}{\delta x\delta y} & \dfrac{\delta^2 f}{\delta y^2}\end{bmatrix}. \tag{10}$$

The function f is said to be convex at any point in its domain if $Hessian(f)$ is positive semidefinite and concave if $Hessian(f)$ is negative semidefinite at that point.

Now, differentiating (9) partially with respect to x and y, we get

$$\frac{\delta f}{\delta x}=\delta_1 e^{(1-\delta_1 x-\delta_2 y)}\left(1-\delta_1 x-\delta_2 y\right)+\delta_1 e^{1-x}(x-1), \tag{11}$$

$$\frac{\delta f}{\delta y}=\delta_1 e^{(1-\delta_2 x-\delta_1 y)}\left(1-\delta_2 x-\delta_1 y\right)+e^{1-y}(y-1). \tag{12}$$

To determine stationary point, we substitute $\delta f/\delta x = 0$ and $\delta f/\delta y = 0$. This gives $x = y$ as a stationary point. Now, computing the Hessian matrix of f at $x = y$ and using $\delta_1+\delta_2=1$, we have

$$Hessian(f)=e^{1-x}(2-x)\,\delta_1\delta_2\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}, \tag{13}$$

which is positive semidefinite. This confirms the convex character of f.

Extending the idea from probabilistic settings to fuzzy settings, in the next subsection, we introduce a new fuzzy divergence measure based on measure (5).

3.2. Definition

Definition 4. Let $X=\{g_1,g_2,\ldots,g_n\}$ be a finite universe of discourse. For any $S,T\in FS(X)$, we define a fuzzy divergence measure $\widehat{EJD}(S,T)$ as

$$\begin{aligned}\widehat{EJD}(S,T)={}&\frac{1}{n}\\ &\cdot\sum_{i=1}^{n}\Big(\left(\delta_1\mu_S(g_i)+\delta_2\mu_T(g_i)\right)e^{(1-(\delta_1\mu_S(g_i)+\delta_2\mu_T(g_i)))}\\ &+\left(1-\delta_1\mu_S(g_i)-\delta_2\mu_T(g_i)\right)e^{(\delta_1\mu_S(g_i)+\delta_2\mu_T(g_i))}\\ &+\left(\delta_1\mu_T(g_i)+\delta_2\mu_S(g_i)\right)e^{(1-(\delta_1\mu_T(g_i)+\delta_2\mu_S(g_i)))}\\ &+\left(1-\delta_1\mu_T(g_i)-\delta_2\mu_S(g_i)\right)e^{(\delta_1\mu_T(g_i)+\delta_2\mu_S(g_i))}\\ &-\left(\mu_S(g_i)e^{(1-\mu_S(g_i))}+\left(1-\mu_S(g_i)\right)e^{\mu_S(g_i)}\right)\\ &-\left(\mu_T(g_i)e^{(1-\mu_T(g_i))}+\left(1-\mu_T(g_i)\right)e^{\mu_T(g_i)}\right)\Big).\end{aligned} \tag{14}$$

3.3. Justification. Now, a very natural question that comes in mind is about the validation of the proposed measure. We answer this query by showing that the proposed measure satisfies (1) nonnegativity and (2) convexity.

Proof. We start with proving the nonnegativity of (14).

Using (6), we have

$$\begin{aligned}&\sum_{i=1}^{n}\left(\delta_1\mu_S(g_i)+\delta_2\mu_T(g_i)\right)e^{(1-(\delta_1\mu_S(g_i)+\delta_2\mu_T(g_i)))}\\ &\quad+\left(\delta_1\left(1-\mu_S(g_i)\right)+\delta_2\left(1-\mu_T(g_i)\right)\right)\\ &\quad\cdot e^{(\delta_1\mu_S(g_i)+\delta_2\mu_T(g_i))}\\ &\quad\geq\sum_{i=1}^{n}\delta_1\left(\mu_S(g_i)e^{(1-\mu_S(g_i))}+\left(1-\mu_S(g_i)\right)e^{\mu_S(g_i)}\right)\\ &\quad+\delta_2\left(\mu_T(g_i)e^{(1-\mu_T(g_i))}+\left(1-\mu_T(g_i)\right)e^{\mu_T(g_i)}\right).\end{aligned} \tag{15}$$

Similarly,

$$\begin{aligned}&\sum_{i=1}^{n}\left(\delta_1\mu_T(g_i)+\delta_2\mu_S(g_i)\right)e^{(1-(\delta_1\mu_T(g_i)+\delta_2\mu_S(g_i)))}\\ &\quad+\left(\delta_1\left(1-\mu_T(g_i)\right)+\delta_2\left(1-\mu_S(g_i)\right)\right)\\ &\quad\cdot e^{(\delta_1\mu_T(g_i)+\delta_2\mu_S(g_i))}\\ &\quad\geq\sum_{i=1}^{n}\delta_1\left(\mu_T(g_i)e^{(1-\mu_T(g_i))}+\left(1-\mu_T(g_i)\right)e^{\mu_T(g_i)}\right)\\ &\quad+\delta_2\left(\mu_S(g_i)e^{(1-\mu_S(g_i))}+\left(1-\mu_S(g_i)\right)e^{\mu_S(g_i)}\right).\end{aligned} \tag{16}$$

Adding (15) and (16), we have $\widehat{EJD}(S,T)\geq 0$.

Since $\mu_S(g_i)+(1-\mu_S(g_i))=1$ and $\mu_T(g_i)+(1-\mu_T(g_i))=1$ for all $S,T\in FS(X)$ and $g_i\in X$, therefore, convexity of (14) follows directly from the convexity of (5).

This, justifies the existence of (14). □

In the next Section, we study some major properties of $\widehat{EJD}(S,T)$.

4. Properties of Proposed Divergence Measure

Measure (14) possesses the following properties.

Theorem 5. *For $S,T,R \in FS(X)$,*

(1) $\widehat{EJD}(S,T) = \widehat{EJD}(T,S)$;

(2) $\widehat{EJD}(S,T) = 0$ *if and only if* $S = T$;

(3) $\widehat{EJD}(S,S^c) = 0$ *if and only if* $\mu_{S^c}(g_i) = 1 - \mu_S(g_i)$ *for all* $g_i \in X$;

(4) $\widehat{EJD}(S, S\cup T) = \widehat{EJD}(S\cap T, T) \le \widehat{EJD}(S,T)$;

(5) $\widehat{EJD}(S\cup T, S\cap T) = \widehat{EJD}(S\cap T, S\cup T) = \widehat{EJD}(S,T)$;

(6) $\widehat{EJD}(S\cap T, S\cup T) = \widehat{EJD}(T,S)$;

(7) $\widehat{EJD}(S, S\cup T) + \widehat{EJD}(S, S\cap T) = \widehat{EJD}(S,T)$;

(8) $\widehat{EJD}(T, S\cup T) + \widehat{EJD}(T, S\cap T) = \widehat{EJD}(T,S)$;

(9) $\widehat{EJD}(S\cup T, R) \le \widehat{EJD}(S,R) + \widehat{EJD}(T,R)$;

(10) $\widehat{EJD}(S\cap T, R) \le \widehat{EJD}(S,R) + \widehat{EJD}(T,R)$;

(11) $\widehat{EJD}(S\cup T, R) + \widehat{EJD}(S\cap T, R) = \widehat{EJD}(S,R) + \widehat{EJD}(T,R)$.

(12) $\widehat{EJD}(S,T) = \widehat{EJD}(S^c,T^c)$;

(13) $\widehat{EJD}(S,T^c) = \widehat{EJD}(S^c,T)$;

(14) $\widehat{EJD}(S,T) + \widehat{EJD}(S^c,T) = \widehat{EJD}(S^c,T^c) + \widehat{EJD}(S,T^c)$, *where S^c and T^c are the complements of FSs S and T, respectively.*

Proof. Proofs of above properties have been provided in the Appendix. □

In the next section, we give an application of proposed divergence measure in multiple attribute decision-making problem (MADM) with the help of an illustrative example.

5. Application of Proposed Fuzzy Divergence Measure in Decision-Making

In recent years, fuzzy divergence measures have been extensively applied by many researchers in many fields like in image thresholding, bioinformatics, etc. [5, 8–11]. In this paper, we apply the proposed fuzzy divergence measure in strategic decision-making. In strategic decision-making problems, we have a set of strategies say $\Lambda_1, \Lambda_2, \ldots, \Lambda_n$ and another set of inputs say $\Omega_1, \Omega_2, \ldots, \Omega_m$. Our aim is to find a particular strategy which if applied with a specific input produces optimum results. Degree of effectiveness of a particular strategy corresponding to specific input can be yielded by the method suggested by Liu and Wang [23] as follows.

Consider a team comprised of N-number of experts. Team members are expected to respond in the form "*yes*" if he/she supports a particular strategy coupled with a specific input and "*no*" in case he/she does not feel the particular strategy to be applied with a specific input. Let the number of experts who support the strategy Λ_j; $(j = 1,2,\ldots,n)$ to be applied with input "Ω_i; $(i = 1,2,\ldots,m)$" be denoted by "$n_{yes}(i,j)$". Then, the effectiveness of strategy "Λ_j" with input Ω_i' can be calculated as

$$\mu_{\Omega_i}(\Lambda_j) = \frac{n_{yes}(i,j)}{N}. \tag{17}$$

Now, let the degree of effectiveness of Λ_j's $(j = 1,2,\ldots,n)$ at uniform inputs be denoted by fuzzy set P given by

$$P = \left\{\Lambda_j, \mu_P(\Lambda_j);\ j = 1,2,\ldots,n\right\}; \tag{18}$$

where $\mu_P(\Lambda_j)$ denotes the membership degree satisfying $0 \le \mu_P(\Lambda_j) \le 1$.

Further, let the degree of effectiveness of Λ_j's $j = 1,2,\ldots,n$ at particular inputs be denoted by the fuzzy set Q given by

$$Q = \left\{\Lambda_j, \mu_Q(\Lambda_j);\ j = 1,2,\ldots n\right\}. \tag{19}$$

5.1. A New MADM Method. TOPSIS (Technique of Order Preference by Similarity to Ideal Solutions) method is a well-known technique employed for solving multiple attribute decision-making (MADM) problems. In this technique, an option nearest to the positive ideal solution and farthest from the negative ideal solution is chosen as most suitable option. Using the concept of TOPSIS initially introduced by Xu and Yager [24], we now propose a new MADM method to solve the above strategic decision-making problem. The procedural steps of proposed method are listed as follows:

(1) Determine the best strategy as follows:

$$\widetilde{\Lambda}_z = \max\left(\mu_P(\Lambda_j)\right);\quad j = 1,\ldots,n. \tag{20}$$

(2) Construct the fuzzy decision matrix as

$$\begin{array}{cccccc} & \Lambda_1 & \Lambda_2 & \cdots & \Lambda_n \\ \Omega_1 & s_{11} & s_{12} & \cdots & s_{1n} \\ \Omega_2 & s_{21} & s_{22} & \cdots & s_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \Omega_m & s_{m1} & s_{m2} & \cdots & s_{mn} \end{array} \tag{21}$$

and $U = [u_1, u_2, \ldots, u_n]$. The number "$(s_{ij})_{1\le i\le m, 1\le j\le n}$" in the above matrix denotes the degree of effectiveness of strategy Λ_j corresponding to input Ω_i using (17).

(3) Determine the normalized fuzzy decision matrix $[n_{ij}]$ as

$$n_{ij} = \frac{s_{ij}}{\sqrt{\sum_{i=1}^{m} s_{ij}^2}},\quad i = 1,2,\ldots,m;\ j = 1,2,\ldots,n. \tag{22}$$

(4) Determine the weighted normalized fuzzy decision matrix $[w_{ij}]$ as

$$w_{ij} = u_i n_{ij},\quad i = 1,2,\ldots,m;\ j = 1,2,\ldots,n. \tag{23}$$

where $U = [1, 1, \ldots, 1]$ denote the weight matrix and u_j is the weight of jth strategy.

(5) Determine the positive ideal solution (Ω^+) and negative ideal solution (Ω^-) as

$$\Omega^+ = \{w_1^+, w_2^+, \ldots, w_m^+\}, \quad (24)$$

$$\Omega^- = \{w_1^-, w_2^-, \ldots, w_m^-\}, \quad (25)$$

where $w_i^+ = \max_j (w_{ij})$, $w_i^- = \min_j (w_{ij})$, $i = 1, 2, \ldots, m$, and $j = 1, 2, \ldots, n$.

(6) Determine the separation of each alternative Ω_i ($i = 1, 2, \ldots, m$) from Ω^+ and Ω^-, respectively, by using (14). Let these be denoted by $\widehat{EJD}(\Omega_i, \Omega^+)$ and $\widehat{EJD}(\Omega_i, \Omega^-)$.

(7) Determine the relative closeness coefficients of each alternative Ω_i ($i = 1, 2, \ldots, m$) from Ω^+ and Ω^- as

$$S_i = \frac{\widehat{EJD}(\Omega_i, \Omega^-)}{\widehat{EJD}(\Omega_i, \Omega^-) + \widehat{EJD}(\Omega_i, \Omega^+)}, \quad i = 1, 2, \ldots, m. \quad (26)$$

(8) Rank the alternatives according to relative closeness coefficients obtained in Step (7).

Therefore, if the strategy $\widetilde{\Lambda}_z$ coupled with input Ω_t is applied, then we get optimal output.

An Illustrative Example. This example is adapted from Joshi and Kumar [8].

Consider an example of a company that wants to launch a new product in the market. For an effective marketing of the product, company has conducted a survey to know the opinion of people about the product before its launching. The whole compiled views are divided into five categories called inputs denoted by Ω_i; $i = 1, 2, 3, 4, 5$. Depending upon these views, company's think tank has prepared five strategies to launch the product in the market denoted by Λ_j; $j = 1, 2, 3, 4, 5$. Now, our aim is to find a particular strategy which if coupled with a specific input returns maximum gains. As a part of full-proof success of the product in the market, company has constituted a committee of 100 experts (in real world problem, no. of experts may increase or decrease). Tables 1 and 2 represent the degrees of effectiveness of different strategies at uniform inputs and at particular inputs.

Now, we employ the proposed measure (14) to compute the above example using TOPSIS method defined above.

(1) Applying Step (1) in Table 1, the best strategy so obtained is given by Λ_3.

(2) The calculated fuzzy decision matrix using Step (2) is given in Table 2.

(3) The computed weighted normalized fuzzy decision matrix using (22) and (23) is shown in Table 3.

(4) Computed values of positive ideal solution (Ω^+) and negative ideal solution (Ω^-) using Step (5) are given in Table 4.

Table 1: Efficiencies of Λ_j's at uniform inputs.

$\mu_P(\Lambda_1)$	$\mu_P(\Lambda_2)$	$\mu_P(\Lambda_3)$	$\mu_P(\Lambda_4)$	$\mu_P(\Lambda_5)$
.7	.3	.8	.5	.6

Table 2: Efficiencies of strategies at particular inputs.

Ω_j	$\mu_Q(\Lambda_1)$	$\mu_Q(\Lambda_2)$	$\mu_Q(\Lambda_3)$	$\mu_Q(\Lambda_4)$	$\mu_Q(\Lambda_5)$
Ω_1	.5	.2	.6	.9	.4
Ω_2	.4	.6	.8	.3	.6
Ω_3	.8	.3	.5	.7	.2
Ω_4	.6	.4	.2	.8	.9
Ω_5	.9	.5	.3	.4	.7

Table 3: Normalized/weighted normalized fuzzy decision matrix.

	Λ_1	Λ_2	Λ_3	Λ_4	Λ_5
Ω_1	.3356	.2108	.5108	.6082	.2933
Ω_2	.2685	.6325	.6810	.2027	.4399
Ω_3	.5369	.3162	.4256	.4730	.1466
Ω_4	.4027	.4216	.1703	.5406	.6599
Ω_5	.6040	.5270	.2554	.2703	.5133

Table 4: Fuzzy positive and negative ideal solutions.

Ω^+	.6082	.6810	.5369	.6599	.6040
Ω^-	.2108	.2027	.1466	.1703	.2554

(5) Separation measures of Ω_i's from Ω^+ and Ω^- using (14) are displayed in Table 5.

(6) Calculated values of relative closeness coefficients using Step (7) are presented in Table 6.

Arranging the alternatives according to values of relative closeness coefficients in descending order, the preferential sequence so obtained is given by the following.

For $\delta_1 = .1, \delta_2 = .9$; $\Omega_4 > \Omega_2 > \Omega_5 > \Omega_3 > \Omega_1$.

For $\delta_1 = .5, \delta_2 = .5$; $\Omega_4 > \Omega_2 > \Omega_5 > \Omega_1 > \Omega_3$.

Thus, for different values of δ_1 and δ_2, the best alternative Ω_4 remains unaltered.

Now, we compute the above Example by fuzzy MOORA (Multiobjective Optimization on the basis of Ratio Analysis) method suggested by Brauers and Zavadskas [25] for the sake of comparison. This method refers to a matrix of responses of alternatives to objectives, to which ratios are applied. The procedural steps are briefed as follows.

Fuzzy MOORA Method [25]. Steps (1)-(4) of fuzzy MOORA method are same as Steps (2)-(4) of the proposed method. The remaining steps are as follows:

(5) Compute the overall performance index $\widehat{EJD}(\Omega^+, \Omega^-)$ for all Ω_i's; $i = 1, 2, 3, 4, 5$. The calculated values are given in Table 7.

Table 5: Distances Ω_i's from Ω^+ and Ω^-.

$\widehat{EJD}(\Omega_i,\Omega^+)$		$\widehat{EJD}(\Omega_i,\Omega^-)$	
$\delta_1 = .1,\ \delta_2 = .9$	$\delta_1 = .5,\ \delta_2 = .5$	$\delta_1 = .1,\ \delta_2 = .9$	$\delta_1 = .5,\ \delta_2 = .5$
.0179	.0494	.0158	.0436
.0170	.0469	.0233	.0643
.0180	.0497	.0140	.0388
.0119	.0329	.0177	.0489
.0119	.0329	.0160	.0442

Table 6: Coefficients of closeness and ranking.

Alternative (Inputs)	$\delta_1 = .1,\ \delta_2 = .9$	Ranking	$\delta_1 = .5,\ \delta_2 = .5$	Ranking
Ω_1	.4688	IV	.4687	IV
Ω_2	.5782	II	.5781	II
Ω_3	.4375	V	.4383	V
Ω_4	.5980	I	.5978	I
Ω_5	.5735	III	.5731	III

Table 7: Computed ranks using MOORA method.

	$\widehat{EJD}(\Omega^+,\Omega^-)$			
Alternative Inputs	$\delta_1 = .1,\ \delta_2 = .9$	Ranking	$\delta_1 = .5,\ \delta_2 = .5$	Ranking
Ω_1	.0359	III	.0993	III
Ω_2	.0521	II	.1436	II
Ω_3	.0353	IV	.0977	IV
Ω_4	.0548	I	.1512	I
Ω_5	.0275	V	.0760	V

(6) Rank the Ω_i's $i = 1, 2, 3, 4, 5$ according to the values of $\widehat{EJD}(\Omega^+,\Omega^-)$ obtained in Step (5) in descending order. The ranking results so obtained for different values of δ_1 and δ_2 are shown in Table 7.

The ranking order of alternatives is as follows:

For $\delta_1 = .1, \delta_2 = .9$; $\Omega_4 > \Omega_2 > \Omega_1 > \Omega_3 > \Omega_5$.

For $\delta_1 = .5, \delta_2 = .5$; $\Omega_4 > \Omega_2 > \Omega_1 > \Omega_3 > \Omega_5$.

Thus, the best alternative, that is, Ω_4, remains unaltered for different values of δ_1 and δ_2.

5.2. A Comparative Analysis. On comparing the output obtained by using proposed method with that of fuzzy MOORA method proposed by Brauers and Zavadskas [25], we find that the best alternative generated by two methods coincide. But there is a big difference in computational procedures adopted in two methods. In fuzzy MOORA method, the best alternative is decided on the basis of overall performance score values of positive ideal solution and negative ideal solution. In this way, each attribute may not get its due weight-age during the selection of the best alternative, whereas, in proposed method, the best alternative is decided on the basis of relative closeness coefficients; that is, each attribute is given due weight-age in decision-making process [8]. Secondly, in proposed method there is a particular strategy with a specific input whereas this is not the case with fuzzy MOORA method. Thus, the performance of proposed method is considerably good.

6. Concluding Remarks

In this communication, we have successfully introduced an exponential divergence measure based on the concept of Jensen-Shannon divergence measure. Further, the proposed divergence measure has been extended to fuzzy sets to introduce a new fuzzy divergence measure. The validation of proposed divergence measure is well established along with a discussion on some major properties. In the end, the application of proposed dissimilarity measure is given in decision-making. The proposed fuzzy divergence can be further generalized to intuitionistic fuzzy settings. This work is under consideration and will be reported somewhere else.

Appendix

Proofs of Properties

Proof. To prove the properties, we bifurcate X into two parts X_1 and X_2 such that

$$X_1 = \{g_i \in X \mid S \subseteq T\},$$
$$X_2 = \{g_i \in X \mid S \supseteq T\}. \tag{A.1}$$

From (A.1), it is clear that, for all $g_i \in X_1$,

$$\mu_S(g_i) \le \mu_T(g_i); \tag{A.2}$$

and, for all $g_i \in X_2$,

$$\mu_S(g_i) \ge \mu_T(g_i). \tag{A.3}$$

(1) Proof of Properties (1), (2), and (3) follows directly from definition (9).

(2) Proof of Property (4)

$$\begin{aligned}
\widehat{EJD}(S, S\cup T) &= \frac{1}{n}\sum_{i=1}^{n}\Big((\delta_1\mu_S(g_i) + \delta_2\mu_{S\cup T}(g_i)) \\
&\cdot e^{(1-(\delta_1\mu_S(g_i)+\delta_2\mu_{S\cup T}(g_i)))} + (1 - \delta_1\mu_S(g_i) \\
&- \delta_2\mu_{S\cup T}(g_i))\, e^{(\delta_1\mu_S(g_i)+\delta_2\mu_{S\cup T}(g_i))} + (\delta_1\mu_{S\cup T}(g_i) \\
&+ \delta_2\mu_S(g_i))\, e^{(1-(\delta_1\mu_{S\cup T}(g_i)+\delta_2\mu_S(g_i)))} + (1 \\
&- \delta_1\mu_{S\cup T}(g_i) - \delta_2\mu_S(g_i))\, e^{(\delta_1\mu_{S\cup T}(g_i)+\delta_2\mu_S(g_i))} \\
&- \left(\mu_S(g_i)\, e^{(1-\mu_S(g_i))} + (1-\mu_S(g_i))\, e^{\mu_S(g_i)}\right) \\
&- \left(\mu_{S\cup T}(g_i)\, e^{(1-\mu_{S\cup T}(g_i))}\right. \\
&\left. + (1-\mu_{S\cup T}(g_i))\, e^{\mu_{S\cup T}(g_i)}\right)\Big).
\end{aligned} \tag{A.4}$$

This implies

$$\begin{aligned}
\widehat{EJD}(S, S\cup T) &= \frac{1}{n} \\
&\cdot \sum_{X_1}\Big((\delta_1\mu_S(g_i) + \delta_2\mu_T(g_i))\, e^{(1-(\delta_1\mu_S(g_i)+\delta_2\mu_T(g_i)))} \\
&+ (1 - \delta_1\mu_S(g_i) - \delta_2\mu_T(g_i))\, e^{(\delta_1\mu_S(g_i)+\delta_2\mu_T(g_i))} \\
&+ (\delta_1\mu_T(g_i) + \delta_2\mu_S(g_i))\, e^{(1-(\delta_1\mu_T(g_i)+\delta_2\mu_S(g_i)))} \\
&+ (1 - \delta_1\mu_T(g_i) - \delta_2\mu_S(g_i))\, e^{(\delta_1\mu_T(g_i)+\delta_2\mu_S(g_i))} \\
&- \left(\mu_S(g_i)\, e^{(1-\mu_S(g_i))} + (1-\mu_S(g_i))\, e^{\mu_S(g_i)}\right) \\
&- \left(\mu_T(g_i)\, e^{(1-\mu_T(g_i))} + (1-\mu_T(g_i))\, e^{\mu_T(g_i)}\right)\Big) + \frac{1}{n} \\
&\cdot \sum_{X_2}\Big((\delta_1\mu_S(g_i) + \delta_2\mu_S(g_i))\, e^{(1-(\delta_1\mu_S(g_i)+\delta_2\mu_S(g_i)))} \\
&+ (1 - \delta_1\mu_S(g_i) - \delta_2\mu_S(g_i))\, e^{(\delta_1\mu_S(g_i)+\delta_2\mu_S(g_i))} \\
&+ (\delta_1\mu_S(g_i) + \delta_2\mu_S(g_i))\, e^{(1-(\delta_1\mu_S(g_i)+\delta_2\mu_S(g_i)))} \\
&+ (1 - \delta_1\mu_S(g_i) - \delta_2\mu_S(g_i))\, e^{(\delta_1\mu_S(g_i)+\delta_2\mu_S(g_i))} \\
&- \left(\mu_S(g_i)\, e^{(1-\mu_S(g_i))} + (1-\mu_S(g_i))\, e^{\mu_S(g_i)}\right) \\
&- \left(\mu_S(g_i)\, e^{(1-\mu_S(g_i))} + (1-\mu_S(g_i))\, e^{\mu_S(g_i)}\right)\Big)
\end{aligned} \tag{A.5}$$

$$\begin{aligned}
&\le \frac{1}{n}\sum_{i=1}^{n}\Big((\delta_1\mu_S(g_i) + \delta_2\mu_T(g_i))\, e^{(1-(\delta_1\mu_S(g_i)+\delta_2\mu_T(g_i)))} \\
&+ (1 - \delta_1\mu_S(g_i) - \delta_2\mu_T(g_i))\, e^{(\delta_1\mu_S(g_i)+\delta_2\mu_T(g_i))} \\
&+ (\delta_1\mu_T(g_i) + \delta_2\mu_S(g_i))\, e^{(1-(\delta_1\mu_T(g_i)+\delta_2\mu_S(g_i)))} \\
&+ (1 - \delta_1\mu_T(g_i) - \delta_2\mu_S(g_i))\, e^{(\delta_1\mu_T(g_i)+\delta_2\mu_S(g_i))} \\
&- \left(\mu_S(g_i)\, e^{(1-\mu_S(g_i))} + (1-\mu_S(g_i))\, e^{\mu_S(g_i)}\right) \\
&- \left(\mu_T(g_i)\, e^{(1-\mu_T(g_i))} + (1-\mu_T(g_i))\, e^{\mu_T(g_i)}\right)\Big) \\
&= \widehat{EJD}(S, T).
\end{aligned} \tag{A.6}$$

Similarly, we can prove that $\widehat{EJD}(S\cap T, T) \le \widehat{EJD}(S, T)$.

(3) Proof of Property (5)

$$\begin{aligned}
\widehat{EJD}(S\cup T, S\cap T) &= \sum_{X_1}\widehat{EJD}(T, S) + \sum_{X_2}\widehat{EJD}(S, T) \\
&= \widehat{EJD}(S, T).
\end{aligned} \tag{A.7}$$

Similarly, we may prove that

$$\begin{aligned}
\widehat{EJD}(S\cap T, S\cup T) &= \sum_{X_1}\widehat{EJD}(T, S) + \sum_{X_2}\widehat{EJD}(S, T) \\
&= \widehat{EJD}(S, T).
\end{aligned} \tag{A.8}$$

(4) Proof of Property (6). Proof follows from Property (5) and Property (1).

(5) Proof of Property (7)

$$\begin{aligned}
&\widehat{EJD}(S, S\cup T) + \widehat{EJD}(S, S\cap T) \\
&\quad = \sum_{X_1}\widehat{EJD}(S, T) + \sum_{X_2}\widehat{EJD}(S, S) + \sum_{X_1}\widehat{EJD}(S, S) \\
&\quad\quad + \sum_{X_2}\widehat{EJD}(S, T); \\
&\quad = \widehat{EJD}(S, T).
\end{aligned} \tag{A.9}$$

(6) Proof of Property (8). Proof of Property (8) is similar to that of Property (7).

(7) Proof of Property (9). Consider

$$\begin{aligned}
&\widehat{EJD}(S, R) + \widehat{EJD}(T, R) - \widehat{EJD}(S\cup T, R) = \frac{1}{n} \\
&\quad \cdot \sum_{i=1}^{n}\Big((\delta_1\mu_S(g_i) + \delta_2\mu_R(g_i))\, e^{(1-(\delta_1\mu_S(g_i)+\delta_2\mu_R(g_i)))} \\
&\quad + (1 - \delta_1\mu_S(g_i) - \delta_2\mu_R(g_i))\, e^{(\delta_1\mu_S(g_i)+\delta_2\mu_R(g_i))} \\
&\quad + (\delta_1\mu_R(g_i) + \delta_2\mu_S(g_i))\, e^{(1-(\delta_1\mu_R(g_i)+\delta_2\mu_S(g_i)))} \\
&\quad + (1 - \delta_1\mu_R(g_i) - \delta_2\mu_S(g_i))\, e^{(\delta_1\mu_R(g_i)+\delta_2\mu_S(g_i))}
\end{aligned}$$

$$
\begin{aligned}
&- \left(\mu_S(g_i) e^{(1-\mu_S(g_i))} + (1-\mu_S(g_i)) e^{\mu_S(g_i)}\right) \\
&- \left(\mu_R(g_i) e^{(1-\mu_R(g_i))} + (1-\mu_R(g_i)) e^{\mu_R(g_i)}\right)\Big) + \frac{1}{n} \\
&\cdot \sum_{i=1}^{n} \Big((\delta_1\mu_T(g_i) + \delta_2\mu_R(g_i)) e^{(1-(\delta_1\mu_T(g_i)+\delta_2\mu_R(g_i)))} \\
&+ (1-\delta_1\mu_T(g_i) - \delta_2\mu_R(g_i)) e^{(\delta_1\mu_T(g_i)+\delta_2\mu_R(g_i))} \\
&+ (\delta_1\mu_R(g_i) + \delta_2\mu_T(g_i)) e^{(1-(\delta_1\mu_R(g_i)+\delta_2\mu_T(g_i)))} \\
&+ (1-\delta_1\mu_R(g_i) - \delta_2\mu_T(g_i)) e^{(\delta_1\mu_R(g_i)+\delta_2\mu_T(g_i))} \\
&- \left(\mu_T(g_i) e^{(1-\mu_T(g_i))} + (1-\mu_T(g_i)) e^{\mu_T(g_i)}\right) \\
&- \left(\mu_R(g_i) e^{(1-\mu_R(g_i))} + (1-\mu_R(g_i)) e^{\mu_R(g_i)}\right)\Big) - \frac{1}{n} \\
&\cdot \sum_{i=1}^{n} \Big((\delta_1\mu_{S\cup T}(g_i) + \delta_2\mu_R(g_i)) \\
&\cdot e^{(1-(\delta_1\mu_{S\cup T}(g_i)+\delta_2\mu_R(g_i)))} + (1-\delta_1\mu_{S\cup T}(g_i) \\
&- \delta_2\mu_R(g_i)) e^{(\delta_1\mu_{S\cup T}(g_i)+\delta_2\mu_R(g_i))} + (\delta_1\mu_R(g_i) \\
&+ \delta_2\mu_{S\cup T}(g_i)) e^{(1-(\delta_1\mu_R(g_i)+\delta_2\mu_{S\cup T}(g_i)))} + (1 \\
&- \delta_1\mu_R(g_i) - \delta_2\mu_{S\cup T}(g_i)) e^{(\delta_1\mu_R(g_i)+\delta_2\mu_{S\cup T}(g_i))} \\
&- \left(\mu_{S\cup T}(g_i) e^{(1-\mu_{S\cup T}(g_i))}\right. \\
&\left.+ (1-\mu_{S\cup T}(g_i)) e^{\mu_{S\cup T}(g_i)}\right) - \left(\mu_R(g_i) e^{(1-\mu_R(g_i))}\right. \\
&\left.+ (1-\mu_R(g_i)) e^{\mu_R(g_i)}\right)\Big).
\end{aligned}
\tag{A.10}
$$

This implies

$$
\begin{aligned}
&\widehat{EJD}(S,R) + \widehat{EJD}(T,R) - \widehat{EJD}(S\cup T,R) = \frac{1}{n} \\
&\cdot \sum_{i=1}^{n} \Big((\delta_1\mu_S(g_i) + \delta_2\mu_R(g_i)) e^{(1-(\delta_1\mu_S(g_i)+\delta_2\mu_R(g_i)))} \\
&+ (1-\delta_1\mu_S(g_i) - \delta_2\mu_R(g_i)) e^{(\delta_1\mu_S(g_i)+\delta_2\mu_R(g_i))} \\
&+ (\delta_1\mu_R(g_i) + \delta_2\mu_S(g_i)) e^{(1-(\delta_1\mu_R(g_i)+\delta_2\mu_S(g_i)))} \\
&+ (1-\delta_1\mu_R(g_i) - \delta_2\mu_S(g_i)) e^{(\delta_1\mu_R(g_i)+\delta_2\mu_S(g_i))} \\
&- \left(\mu_S(g_i) e^{(1-\mu_S(g_i))} + (1-\mu_S(g_i)) e^{\mu_S(g_i)}\right) \\
&- \left(\mu_R(g_i) e^{(1-\mu_R(g_i))} + (1-\mu_R(g_i)) e^{\mu_R(g_i)}\right)\Big) + \frac{1}{n} \\
&\cdot \sum_{i=1}^{n} \Big((\delta_1\mu_T(g_i) + \delta_2\mu_R(g_i)) e^{(1-(\delta_1\mu_T(g_i)+\delta_2\mu_R(g_i)))} \\
&+ (1-\delta_1\mu_T(g_i) - \delta_2\mu_R(g_i)) e^{(\delta_1\mu_T(g_i)+\delta_2\mu_R(g_i))} \\
&+ (\delta_1\mu_R(g_i) + \delta_2\mu_T(g_i)) e^{(1-(\delta_1\mu_R(g_i)+\delta_2\mu_T(g_i)))} \\
&+ (1-\delta_1\mu_R(g_i) - \delta_2\mu_T(g_i)) e^{(\delta_1\mu_R(g_i)+\delta_2\mu_T(g_i))} \\
&- \left(\mu_T(g_i) e^{(1-\mu_T(g_i))} + (1-\mu_T(g_i)) e^{\mu_T(g_i)}\right) \\
&- \left(\mu_R(g_i) e^{(1-\mu_R(g_i))} + (1-\mu_R(g_i)) e^{\mu_R(g_i)}\right)\Big) - \frac{1}{n} \\
&\cdot \sum_{X_1} \Big((\delta_1\mu_T(g_i) + \delta_2\mu_R(g_i)) e^{(1-(\delta_1\mu_T(g_i)+\delta_2\mu_R(g_i)))} \\
&+ (1-\delta_1\mu_T(g_i) - \delta_2\mu_R(g_i)) e^{(\delta_1\mu_T(g_i)+\delta_2\mu_R(g_i))} \\
&+ (\delta_1\mu_R(g_i) + \delta_2\mu_T(g_i)) e^{(1-(\delta_1\mu_R(g_i)+\delta_2\mu_T(g_i)))} \\
&+ (1-\delta_1\mu_R(g_i) - \delta_2\mu_T(g_i)) e^{(\delta_1\mu_R(g_i)+\delta_2\mu_T(g_i))} \\
&- \left(\mu_T(g_i) e^{(1-\mu_T(g_i))} + (1-\mu_T(g_i)) e^{\mu_T(g_i)}\right) \\
&- \left(\mu_R(g_i) e^{(1-\mu_R(g_i))} + (1-\mu_R(g_i)) e^{\mu_R(g_i)}\right)\Big) - \frac{1}{n} \\
&\cdot \sum_{X_2} \Big((\delta_1\mu_S(g_i) + \delta_2\mu_R(g_i)) e^{(1-(\delta_1\mu_S(g_i)+\delta_2\mu_R(g_i)))} \\
&+ (1-\delta_1\mu_S(g_i) - \delta_2\mu_R(g_i)) e^{(\delta_1\mu_S(g_i)+\delta_2\mu_R(g_i))} \\
&+ (\delta_1\mu_R(g_i) + \delta_2\mu_S(g_i)) e^{(1-(\delta_1\mu_R(g_i)+\delta_2\mu_S(g_i)))} \\
&+ (1-\delta_1\mu_R(g_i) - \delta_2\mu_S(g_i)) e^{(\delta_1\mu_R(g_i)+\delta_2\mu_S(g_i))} \\
&- \left(\mu_S(g_i) e^{(1-\mu_S(g_i))} + (1-\mu_S(g_i)) e^{\mu_S(g_i)}\right) \\
&- \left(\mu_R(g_i) e^{(1-\mu_R(g_i))} + (1-\mu_R(g_i)) e^{\mu_R(g_i)}\right)\Big); \\
&\geq 0.
\end{aligned}
\tag{A.11}
$$

Therefore, $\widehat{EJD}(S,R) + \widehat{EJD}(T,R) \geq \widehat{EJD}(S\cup T,R)$.

(8) Proof for Property (10). Similarly, we can prove Property (10).

(9) Proof of Property (11). Consider

$$
\begin{aligned}
&\widehat{EJD}(S\cup T,R) + \widehat{EJD}(S\cap T,R) = \frac{1}{n} \\
&\cdot \sum_{i=1}^{n} \Big((\delta_1\mu_{S\cup T}(g_i) + \delta_2\mu_R(g_i)) \\
&\cdot e^{(1-(\delta_1\mu_{S\cup T}(g_i)+\delta_2\mu_R(g_i)))} + (1-\delta_1\mu_{S\cup T}(g_i) \\
&- \delta_2\mu_R(g_i)) e^{(\delta_1\mu_{S\cup T}(g_i)+\delta_2\mu_R(g_i))} + (\delta_1\mu_R(g_i) \\
&+ \delta_2\mu_{S\cup T}(g_i)) e^{(1-(\delta_1\mu_R(g_i)+\delta_2\mu_{S\cup T}(g_i)))} + (1 \\
&- \delta_1\mu_R(g_i) - \delta_2\mu_{S\cup T}(g_i)) e^{(\delta_1\mu_R(g_i)+\delta_2\mu_{S\cup T}(g_i))} \\
&- \left(\mu_{S\cup T}(g_i) e^{(1-\mu_{S\cup T}(g_i))}\right. \\
&\left.+ (1-\mu_{S\cup T}(g_i)) e^{\mu_{S\cup T}(g_i)}\right) - \left(\mu_R(g_i) e^{(1-\mu_R(g_i))}\right.
\end{aligned}
$$

$$
\begin{aligned}
&+\left(1-\mu_R\left(g_i\right)\right) e^{\mu_R(g_i)}\Big)\Big)+\frac{1}{n}\sum_{i=1}^{n}\Big(\Big(\delta_1\mu_{S\cap T}\left(g_i\right)\\
&+\delta_2\mu_R\left(g_i\right)\Big) e^{(1-(\delta_1\mu_{S\cap T}(g_i)+\delta_2\mu_R(g_i)))}+\Big(1\\
&-\delta_1\mu_{S\cap T}\left(g_i\right)-\delta_2\mu_R\left(g_i\right)\Big) e^{(\delta_1\mu_{S\cap T}(g_i)+\delta_2\mu_R(g_i))}\\
&+\left(\delta_1\mu_R\left(g_i\right)+\delta_2\mu_{S\cap T}\left(g_i\right)\right) e^{(1-(\delta_1\mu_R(g_i)+\delta_2\mu_{S\cap T}(g_i)))}\\
&+\left(1-\delta_1\mu_R\left(g_i\right)-\delta_2\mu_{S\cap T}\left(g_i\right)\right)\\
&\cdot e^{(\delta_1\mu_R(g_i)+\delta_2\mu_{S\cap T}(g_i))}-\Big(\mu_{S\cap T}\left(g_i\right) e^{(1-\mu_{S\cap T}(g_i))}\\
&+\left(1-\mu_{S\cap T}\left(g_i\right)\right) e^{\mu_{S\cap T}(g_i)}\Big)-\Big(\mu_R\left(g_i\right) e^{(1-\mu_R(g_i))}\\
&+\left(1-\mu_R\left(g_i\right)\right) e^{\mu_R(g_i)}\Big)\Big).
\end{aligned}
\tag{A.12}
$$

Using (A.1) to (A.3), we get

$$
\begin{aligned}
&=\sum_{X_1}\widehat{EJD}\left(T,R\right)+\sum_{X2}\widehat{EJD}\left(S,R\right)+\sum_{X_1}\widehat{EJD}\left(S,R\right)\\
&\quad+\sum_{X_2}\widehat{EJD}\left(T,R\right); \qquad (A.13)\\
&=\widehat{EJD}\left(S,R\right)+\widehat{EJD}\left(T,R\right).
\end{aligned}
$$

(10) Proofs of Properties (12), (13), and (14) follows directly from definition. □

Data Availability

All data sources have been cited in the article where necessary.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this manuscript.

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