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Research Article

Second-Order Asymptotics of the Risk Concentration of a Portfolio with Deflated Risks

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The quantification of diversification benefits due to risk aggregation has received more attention in the recent literature. In this paper, we establish second-order asymptotics of the risk concentration based on several risk measures for a portfolio of n identically distributed but dependent deflated risks $X_j = R_j S$, $j = 1, 2, \dots, n$ under the assumptions of second-order regular variation on the survival functions of the risks R_j and the deflator S, where $R_1, R_2, \dots, R_n$ are n independent and identically distributed random variables with a common survival function and S is a random variable being independent of $R_1, R_2, \dots, R_n$. Examples are also given to illustrate our main results.

1. Introduction

The quantification of diversification benefits due to risk aggregation plays a prominent role in the (regulatory) capital management of large firms within the financial industry. Measuring a risk and quantifying its diversification benefits have become an important task. Especially when the underlying risk factors show a heavy-tailed pattern, many papers discussed diversification benefits; see, for instance, Degen et al. [1] (2010), Ibragimov and Walden [2], Ibragimov et al. [3], Mao et al. [4], Lv et al. [5, 6], Hashorva et al. [7], and references therein.

Risk measure is understood as a function that can assign a nonnegative real number to a risk. Consider a portfolio of n loss random variables $X_1, X_2, \dots, X_n$. The risk concentration based on the risk measure $\varrho[\cdot]$ is defined as

$$C_\varrho = \frac{\varrho\left[\sum_{i=1}^n X_i\right]}{\sum_{i=1}^n \varrho\left[X_i\right]}. \quad (1)$$

Here, $1 - C_\varrho$ refers to the diversification benefit. In recent years, empirical work has argued that financial variables often exhibit stronger dependence, while the existing work usually assumes that the risks $X_1, X_2, \dots, X_n$ are independent and identically distributed; see Embrechts et al. [8, 9], Degen et al. [10], Mao and Hu [11], Mao and Hu [12], Lv et al. [6], and so on. We focus on the asymptotic of risk concentration for a portfolio of n identically distributed but correlated deflated risks $X_j = R_j S$, $j = 1, \dots, n$ under assumptions of second-order regular variation on the survival functions of the risk $R_1, \dots, R_n$ and deflator S.

In the present paper we study mathematical properties of diversification effects under the different risk measures $\varrho[\cdot]$. Several popular risk measures have been introduced to measure tail risk, such as the Value-at-Risk (VaR), the conditional tail expectation (CTE), and the Haezendonck-Goovaerts risk measure. These risk measures have been used extensively in insurance and finance as a tool of risk management; see Denuit et al. [13], Artzner et al. [14], Cheung and Lo [15], Zhu et al. [16], and references therein. The Value-at-Risk (VaR) of X at the level p is defined as

$$\mathrm{VaR}_p[X] = \inf\{x \in \mathbb{R} : F(x) \ge p\}, \quad p \in (0,1), \quad (2)$$

and the conditional tail expectation (CTE) of X at the level p is defined as

$$\mathrm{CTE}_p[X] = E\left[X \mid X > \mathrm{VaR}_p[X]\right], \quad p \in (0,1). \quad (3)$$

The Haezendonck-Goovaerts risk measure, which was introduced by Haezendonck and Goovaerts [17], is defined via an increasing and convex Young function ϕ and a parameter $p \in (0,1)$ representing the confidence level. More precisely, let ϕ be a nonnegative and convex function on $[0,\infty)$ with $\phi(0)=0$, $\phi(1)=1$, and $\phi(\infty)=\infty$. This function is called a normalized Young function. Assume we have a real-valued random variable X with distribution function F such that

$$E[\phi(cX)] < \infty \quad \forall c > 0, \tag{4}$$

and let $H_p[X,t]$ be the unique solution h to the equation

$$E\left[\phi\left(\frac{(X-t)_+}{h}\right)\right] = 1-p, \quad p \in (0,1), \tag{5}$$

if $\overline{F}(t) > 0$ and 0 if $\overline{F}(t) = 0$, where $x_+ = \max\{x,0\}$ for any real number x. Then the Haezendonck-Goovaerts risk measure of X at the confidence level p is defined as

$$\mathrm{HG}_p[X] = \inf_{t\in\mathbb{R}} \left(t + H_p[X,t]\right). \tag{6}$$

Some important properties and connections with other risk measures are given in Goovaerts et al. [18]. It is well known that the simplest case of the Haezendonck-Goovaerts risk measure $\mathrm{HG}_p[X]$ with $\phi(x) = x$ reduces to $\mathrm{CTE}_p[X]$. Even for a power Young function, the explicit solution to (5) is generally not available. Now, in this paper, we instead considered the asymptotic behavior of risk concentration based on the Haezendonck-Goovaerts risk measure $\mathrm{HG}_p[X]$ with $\phi(t) = t^k$ for $k \ge 1$ as $p \uparrow 1$.

Another family of risk measures, introduced by Wang [19], is defined by using the concept of the distortion function. A distortion function is an increasing function $g : [0,1] \to [0,1]$ such that $g(0) = 0$ and $g(1) = 1$. Then for any risk X with distribution function F, the corresponding distortion risk measure $D_g[\cdot]$ is defined as follows:

$$D_g[X] = \int_0^\infty g\left(\overline{F}(t)\right)dt - \int_{-\infty}^0 \left[1 - g\left(\overline{F}(t)\right)\right]dt, \tag{7}$$

where $\overline{F}(t) = 1 - F(t)$ denotes the survival function of X. The distortion risk measure has several useful properties such as positive homogeneity, translation invariance, additivity for comonotonic risks, and monotonicity. For more details, see Denuit et al. [13], Dhaene et al. [20], and Balbás et al. [21]. Several popular risk measures belong to the family of distortion risk measures. For example, the Value-at-Risk (VaR) of X at the level p corresponds to the distortion function $g(x) = 1_{(1-p,1]}(x)$, $x \in [0,1]$, where 1_E is the indicator function of E; the conditional tail expectation (CTE) of X at level p corresponds to the distortion function $g(x) = \min\{x/(1-p), 1\}$, $x \in (0,1)$.

The tail distortion risk measure, first introduced by Zhu and Li [22], was reformulated by Yang [23] as follows: for a distortion function g, the tail distortion risk measure at level p of a loss variable X is defined as $T_{g_p}[X] = D_{g_p}[X]$, $p \in (0,1)$, where

$$g_p(u) = \begin{cases} g\left(\dfrac{u}{1-p}\right), & 0 \le u \le 1-p, \\ 1, & 1-p < u \le 1. \end{cases} \tag{8}$$

Since the risk is always heavy-tailed and often obeys a law of regular variation, we choose $\varrho[\cdot]$ as $\mathrm{VaR}_p[\cdot]$, $\mathrm{HG}_p[\cdot]$, and $T_{g_p}[\cdot]$ at the level $0 < p < 1$, respectively, in (1). We denote risk concentration C_ϱ at the level p by $C_\varrho(p)$.

Because risk managers become more and more concerned with tail area of risk, we will focus on the second-order approximations of the risk concentrations based on the different risk measures as $p \uparrow 1$, such as $C_{\mathrm{VaR}}(p)$, $C_{\mathrm{HG}}(p)$, $C_{\mathrm{CTE}}(p)$, and $C_{T_g}(p)$ as $p \uparrow 1$ for a portfolio of n loss random variables $X_1, X_2, \ldots, X_n$. In this paper, we assume that random variables $X_1, X_2, \ldots, X_n$ are identically distributed but not independent; that is,

$$(X_1, X_2, \ldots, X_n) = (R_1 S, R_2 S, \ldots, R_n S), \tag{9}$$

where $R_1, R_2, \ldots, R_n$ are n i.i.d random variables with a common survival function possessing the property of second-order regular variation, and the deflator S is a random variable which is independent of $R_1, R_2, \ldots, R_n$.

The first-order approximations of $C_{\mathrm{VaR}}(p)$ as $p \uparrow 1$ were studied by Embrechts et al. [8, 9] under the model assumption that the underlying risks $X_1, X_2, \ldots, X_n$ have identically distributed and regularly varying margins and have two forms of dependent structure, respectively. Degen et al. [10] derived second-order approximations of $C_{\mathrm{VaR}}(p)$ for n independent and identically distributed (i.i.d) loss variables with a common survival function possessing the property of second-order regular variation (2RV). Second-order approximations of the risk concentrations $C_{\mathrm{CTE}}(p)$ and $C_{T_g}(p)$ as $p \uparrow 1$ for n i.i.d loss random variables were derived by Mao et al. [4], Mao and Hu [12], Lv et al. [6], and Hashorva et al. [7]. For a portfolio of n i.i.d. risks, the second-order approximations of the risk concentrations $C_{\mathrm{VaR}}(p)$, $C_{\mathrm{CTE}}(p)$ as $p \uparrow 1$ have been discussed by Hashorva et al. [24], while Mao and Yang [25] consider the case with a portfolio of n dependent risks under FGM copula. Ling and Peng [26] derived higher-order approximations under some conditions.

The paper is organized as follows. In Section 2, we describe the definition of the second-order regular variation and some useful propositions of it. In Section 3, we obtain our main results, that is, the second-order approximations of the risk concentrations $C_{\mathrm{VaR}}(p)$, $C_{\mathrm{HG}}(p)$, and $C_{T_g}(p)$ as $p \uparrow 1$, and present their proofs. In Section 4, some examples are provided to illustrate the performance of our approximations. Throughout, the notation "∼" means asymptotic equivalence, that is, for functions $f(x)$ and $g(x)$,

$$f(x) \sim g(x), \quad x \longrightarrow x_0 \Longleftrightarrow \lim_{x\to x_0} \frac{f(x)}{g(x)} = 1. \tag{10}$$

2. Preliminaries

Regular variation is one of the basic concepts which appears in different contexts of applied probability. A function h is said to be of regular variation with index $\alpha \in \mathbb{R}$, denoted by $h \in \mathrm{RV}_{-\alpha}$, if

$$\lim_{x\to\infty} \frac{h(xy)}{h(x)} = y^{-\alpha} \quad (11)$$

holds for any $y > 0$. Next we recall the definition of the second-order regular variation from de Haan and Ferreira [27] and de Haan and Stadtmüller [28]. Suppose that $h \in \mathrm{RV}_{-\alpha}$ for some $\alpha \in \mathbb{R}$; then h is said to be of second-order regular variation with first-order parameter α and second-order parameter $\rho \le 0$, denoted by $h \in 2\mathrm{RV}_{-\alpha}, \rho$, if there exists some ultimately positive or negative function $A(x)$ with $A(x) \to 0$ as $x \to \infty$ such that

$$\lim_{x\to\infty} \frac{h(xy)/h(x) - y^{-\alpha}}{A(x)} = y^{-\alpha}\int_1^y u^{\rho-1}du, \quad \forall x > 0. \quad (12)$$

Here, $A(x)$ is referred to as an auxiliary function of h and $|A| \in \mathrm{RV}_{\rho}$. Several classes of parametric survival functions are shown to possess 2RV properties; see Hashorva et al. [7]. For more details on RV and 2RV, see Hua and Joe [29] and Lv et al. [5].

The function which possesses the property of second-order regular variation (2RV) plays an important role in this article. The following proposition gives a characterization of any function $h \in 2\mathrm{RV}_{-\alpha}, \rho$ with auxiliary function $A(x)$, $\alpha \in \mathbb{R}$ and $\rho < 0$, which is from Hua and Joe [29].

Proposition 1. *Let $\alpha \in \mathbb{R}$, $\rho < 0$, and $A(x) \in RV_{\rho}$. Then $h \in 2RV_{-\alpha}, \rho$ with auxiliary function $A(x)$ if and only if*

$$h(x) = cx^{-\alpha}\left[1 + \frac{A(x)}{\rho} + o(A(x))\right], \quad x \longrightarrow \infty, \quad (13)$$

where $c = \lim_{x\to\infty} x\overline{F}(x) \in (0,\infty)$.

The next two propositions give first- and second-order approximations of Haezendonck-Goovaerts risk measure $\mathrm{HG}_p[X]$ of X at the confidence level p and tail distortion risk measure $T_{g_p}[X]$ of X at confidence level p for a distortion function g, which will be used in the proofs of our main results.

Proposition 2. *Let X be a random variable with survival function $\overline{F} \in RV_{-\alpha}$, $\alpha > 0$, and let $\phi(t) = t^k$ for some $\alpha > k \ge 1$. Then one has the following:*

(i) *The first-order asymptotic (see [30]; Mao and Hu, 2012a):*

$$HG_p[X] \sim C_\alpha F^{\leftarrow}(p), \quad p \uparrow 1. \quad (14)$$

(ii) *The second-order asymptotic (see Mao and Hu, 2012a): if $\overline{F} \in 2RV_{-\alpha,\rho}$, $\rho \le 0$, with auxiliary function $A(x)$, then*

$$HG_p[X] = C_\alpha F^{\leftarrow}(p)\left[1 + H_{\alpha,\rho,k}A(F^{\leftarrow}(p))(1+o(1))\right], \quad p \uparrow 1, \quad (15)$$

where

$$C_\alpha = \frac{\alpha(\alpha-k)^{k/\alpha-1}}{k^{k-1/\alpha}}(B(\alpha-k,k))^{1/\alpha},$$
$$H_{\alpha,\rho,k} = \frac{1}{\alpha\rho}\left[(\alpha-k)^{\rho k/\alpha}k^{(\rho/\alpha)(1-k)}\xi_{k,0}^{\rho/\alpha-1}\xi_{k,\rho/\alpha} - 1\right] \quad (16)$$

with

$$\xi_{k,t} = B(\alpha(1-t)-k,k) \quad (17)$$

and $B(\cdot,\cdot)$ is the Beta function as usual; that is,

$$B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}dx, \quad a,b > 0. \quad (18)$$

Proposition 3. *Let X be a random variable with survival function $\overline{F} \in RV_{-\alpha}$, $\alpha > 0$, and let g be any distortion function with*

$$\int_0^1 x^{-1/\alpha-\delta}dg(x) < \infty \quad \text{for some } \delta > 0. \quad (19)$$

We have the follwoing:

(i) *The first-order asymptotic (see [22, 23]):*

$$T_{g_p}[X] \sim J_\alpha F^{\leftarrow}(p), \quad p \uparrow 1. \quad (20)$$

(ii) *The second-order asymptotic (see [23]): if $\overline{F} \in 2RV_{-\alpha,\rho}$, $\rho \le 0$, with auxiliary function $A(x)$, then*

$$T_{g_p}[X] = J_\alpha F^{\leftarrow}(p) + F^{\leftarrow}(p)A(F^{\leftarrow}(p))\left[I_{\alpha,\rho} + o(1)\right], \quad p \uparrow 1, \quad (21)$$

where

$$J_\alpha = \int_0^1 x^{-1/\alpha}dg(x),$$
$$I_{\alpha,\rho} = \int_0^1 x^{-1/\alpha}\frac{x^{-\rho/\alpha}-1}{\alpha\rho}dg(x). \quad (22)$$

Propositions 2(ii) and 3(ii) are, respectively, modified from Theorem 4.5 in Mao and Hu (2012a) and Corollary 4.1 in Yang [23] by using the fact that $\overline{F} \in 2\mathrm{RV}_{-\alpha,\rho}$ with auxiliary function $A(t)$ if and only if its tail quantile function $U \in 2\mathrm{RV}_{1/\alpha,\rho/\alpha}$ with auxiliary function $\alpha^{-2}A(U(t))$ (see Theorem 2.3.9 in de Haan and Ferreira [27]).

3. Main Results and Their Proofs

3.1. Main Results. In this section, we give some results establishing the second-order approximations of the risk concentration $C_{\varrho}(p)$ as $p \to 1$ for a portfolio of n random variables that satisfy (9). The first one is about the risk concentration $C_{\mathrm{VaR}}(p)$.

Theorem 4. *Let* $(X_1, X_2, \dots, X_n) = (R_1 S, R_2 S, \dots, R_n S)$*, where* $\{R_1, R_2, \dots, R_n\}$ *are i.i.d nonnegative random variables with common continuous distribution function* F *and* S *is a nonnegative random variable independent of* $\{R_1, R_2, \dots, R_n\}$*. If* $\overline{F} \in 2RV_{-\alpha,\rho}$, $\alpha > 0$, $\rho < 0$, *with auxiliary function* $A(x)$ *and* $ES^{\alpha-\rho+\epsilon} < \infty$ *for some* $\epsilon > 0$*, then*

(i) *for* $\rho < -\alpha$ *and* $0 < \alpha < 1$,

$$C_{VaR}(p) = n^{1/\alpha-1}\left[1 + \frac{(n-1)\xi_\alpha}{2n\alpha} \cdot \frac{ES^{2\alpha}}{(ES^\alpha)^2}(1-p)\right] + o(1-p), \quad p \uparrow 1; \tag{23}$$

(ii) *for* $\rho < -1$,

$$C_{VaR}(p) = n^{1/\alpha-1}\left[1 + n^{-1/\alpha}(n-1)\frac{ES^{\alpha+1}}{(ES^\alpha)^{1+1/\alpha}} \cdot \frac{\mu_F}{F^{\leftarrow}(p)}\right] + o\left(\frac{1}{F^{\leftarrow}(p)}\right), \quad p \uparrow 1 \tag{24}$$

when $\alpha > 1$,

$$C_{VaR}(p) = 1 + c\left(1 - \frac{1}{n}\right)\frac{ES^{\alpha+1}}{(ES^\alpha)^{1+1/\alpha}} \cdot \frac{\log(F^{\leftarrow}(p))}{F^{\leftarrow}(p)}(1 + o(1)), \quad p \uparrow 1 \tag{25}$$

with $c = \lim_{x\to\infty} x\overline{F}(x) \in (0,\infty)$, *when* $\alpha = 1$;

(iii) *for* $\rho > -(1 \wedge \alpha)$,

$$C_{VaR}(p) = n^{1/\alpha-1}\left[1 + \frac{n^{\rho/\alpha}-1}{\alpha\rho} \cdot \frac{ES^{\alpha-\rho}}{(ES^\alpha)^{1-\rho/\alpha}} A(F^{\leftarrow}(p))\right] + o(A(F^{\leftarrow}(p))), \quad p \uparrow 1. \tag{26}$$

In the following theorem, we derive the second-order asymptotic of risk concentration for Haezendonck-Goovaerts risk measure $C_{\mathrm{HG}}(p)$ at level p.

Theorem 5. *Let* $(X_1, X_2, \dots, X_n) = (R_1 S, R_2 S, \dots, R_n S)$*, where* $\{R_1, R_2, \dots, R_n\}$ *are i.i.d. nonnegative random variables with common continuous distribution function* F *and* S *is a nonnegative random variable independent of* $\{R_1, R_2, \dots, R_n\}$*. If* $\overline{F} \in 2RV_{-\alpha,\rho}$, $\alpha > 0$, $\rho < 0$, *with auxiliary function* $A(x)$ *and* $ES^{\alpha-\rho+\epsilon} < \infty$ *for some* $\epsilon > 0$ *and if* $\phi(t) = t^k$ *for some* $\alpha > k \geq 1$*, then*

(i) *for* $\rho < -\alpha$ *and* $0 < \alpha < 1$,

$$C_{HG}(p) = n^{1/\alpha-1}\left[1 + \frac{n-1}{2n\alpha} \cdot \frac{ES^{2\alpha}}{(ES^\alpha)^2} \cdot (1 - \alpha^2 H_{\alpha,-\alpha,k})\xi_\alpha(1-p)\right] + o(1-p), \quad p \uparrow 1; \tag{27}$$

(ii) *for* $\rho < -1$,

$$C_{HG}(p) = n^{1/\alpha-1}\left[1 + n^{-1/\alpha}(n-1)\frac{ES^{\alpha+1}}{(ES^\alpha)^{1+1/\alpha}} \cdot (1 - \alpha H_{\alpha,-1,k}) \cdot \frac{\mu_F}{F^{\leftarrow}(p)}\right] + o\left(\frac{1}{F^{\leftarrow}(p)}\right), \quad p \uparrow 1 \tag{28}$$

when $\alpha > 1$, *and*

$$C_{HG}(p) = 1 + c\left(1 - \frac{1}{n}\right)\frac{ES^{\alpha+1}}{(ES^\alpha)^{1+1/\alpha}} \cdot (1 - H_{1,-1,k}) \cdot \frac{\log(F^{\leftarrow}(p))}{F^{\leftarrow}(p)}(1 + o(1)), \quad p \uparrow 1 \tag{29}$$

with $c = \lim_{x\to\infty} x\overline{F}(x) \in (0,\infty)$ *when* $\alpha = 1$;

(iii) *for* $\rho > -(1 \wedge \alpha)$,

$$C_{HG}(p) = n^{1/\alpha-1}\left[1 + \frac{n^{\rho/\alpha}-1}{\alpha\rho} \cdot \frac{ES^{\alpha-\rho}}{(ES^\alpha)^{1-\rho/\alpha}} \cdot (1 + \alpha\rho H_{\alpha,\rho,k}) A(F^{\leftarrow}(p))\right] + o(A(F^{\leftarrow}(p))), \quad p \uparrow 1. \tag{30}$$

The last theorem gives the second-order asymptotic of risk concentration for tail distortion risk measure $C_{T_g}(p)$ at level p.

Theorem 6. *Let* $(X_1, X_2, \dots, X_n) = (R_1 S, R_2 S, \dots, R_n S)$*, where* $\{R_1, R_2, \dots, R_n\}$ *are i.i.d. nonnegative random variables with common continuous distribution function* F*, and* S *is a nonnegative random variable independent of* $\{R_1, R_2, \dots, R_n\}$*. Further assume that* $\overline{F} \in 2RV_{-\alpha,\rho}$, $\alpha > 0$, $\rho < 0$, *with auxiliary function* $A(x)$ *and* $ES^{\alpha-\rho+\epsilon} < \infty$ *for some* $\epsilon > 0$*. Let* g *be a distortion function with*

$$\int_0^1 x^{-1/\alpha-\delta}dg(x) < \infty \quad \text{for some } \delta > 0. \tag{31}$$

Then

(i) *for* $\rho < -\alpha$ *and* $0 < \alpha < 1$,

$$C_{T_g}(p) = n^{1/\alpha-1}\left[1 + \frac{n-1}{2n\alpha} \cdot \frac{ES^{2\alpha}}{(ES^\alpha)^2} \cdot \left(1 - \frac{\alpha^2 I_{\alpha,-\alpha}}{J_\alpha}\right)\xi_\alpha(1-p)\right] + o(1-p), \quad p \uparrow 1; \tag{32}$$

(ii) *for* $\rho < -1$,

$$C_{T_g}(p) = n^{1/\alpha-1}\left[1 + n^{-1/\alpha}(n-1)\frac{ES^{\alpha+1}}{(ES^\alpha)^{1+1/\alpha}} \cdot \left(1 - \frac{\alpha I_{\alpha,-1}}{J_\alpha}\right) \cdot \frac{\mu_F}{F^{\leftarrow}(p)}\right] + o\left(\frac{1}{F^{\leftarrow}(p)}\right), \quad p \uparrow 1 \tag{33}$$

when $\alpha > 1$, *and*

$$C_{T_g}(p) = 1 + c\left(1 - \frac{1}{n}\right)\frac{ES^{\alpha+1}}{(ES^{\alpha})^{1+1/\alpha}} \cdot \left(1 - \frac{I_{1,-1}}{J_1}\right) \cdot \frac{\log(F^{\leftarrow}(p))}{F^{\leftarrow}(p)}(1 + o(1)), \quad p \uparrow 1 \tag{34}$$

with $c = \lim_{x\to\infty} x\overline{F}(x) \in (0, \infty)$ *when* $\alpha = 1$;

(iii) *for* $\rho > -(1 \wedge \alpha)$,

$$C_{T_g}(p) = n^{1/\alpha-1}\left[1 + \frac{n^{\rho/\alpha} - 1}{\alpha\rho} \cdot \frac{ES^{\alpha-\rho}}{(ES^{\alpha})^{1-\rho/\alpha}} \cdot \left(1 + \frac{\alpha\rho I_{\alpha,\rho}}{J_\alpha}\right)A(F^{\leftarrow}(p))\right] + o(A(F^{\leftarrow}(p))), \quad p \uparrow 1. \tag{35}$$

Thus, we immediately obtain the following corollary which establishes the second-order asymptotic of risk concentration for conditional tail expectation $C_{\mathrm{CTE}}(p)$. And this corollary can also be obtained easily by Lemma 8.

Corollary 7. *Let* $(X_1, X_2, \ldots, X_n) = (R_1S, R_2S, \ldots, R_nS)$ *be a continuous random vector, where* $\{R_1, R_2, \ldots, R_n\}$ *are i.i.d. nonnegative random variables with common continuous distribution function* F *and* S *is a nonnegative random variable independent of* $\{R_1, R_2, \ldots, R_n\}$. *If* $\overline{F} \in 2RV_{-\alpha,\rho}$, $\alpha > 1$, $\rho < 0$, *with auxiliary function* $A(x)$ *and* $ES^{\alpha-\rho+\epsilon} < \infty$ *for some* $\epsilon > 0$. *Then*

(i) *for* $\rho < -1$,

$$C_{CTE}(p) = n^{1/\alpha-1} + \frac{n-1}{n}\left(1 - \frac{1}{\alpha}\right) \cdot \frac{ES^{\alpha+1}}{(ES^{\alpha})^{1+1/\alpha}} \cdot \frac{\mu_F}{F^{\leftarrow}(p)} + o\left(\frac{1}{F^{\leftarrow}(p)}\right), \quad p \uparrow 1; \tag{36}$$

(ii) *for* $\rho > -1$,

$$C_{CTE}(p) = n^{1/\alpha-1}\left[1 + \frac{n^{\rho/\alpha} - 1}{\rho/\alpha} \cdot \frac{\alpha - 1}{\alpha^2(\alpha - 1 - \rho)} \cdot \frac{ES^{\alpha-\rho}}{(ES^{\alpha})^{1-\rho/\alpha}}A(F^{\leftarrow}(p))\right] + o(A(F^{\leftarrow}(p))), \quad p \uparrow 1. \tag{37}$$

3.2. Proofs. Before proving the above results, we introduce some lemmas. The first one gives a second-order form of Breiman's theorem (see Breiman [31]), which is from Hashorva et al. [7].

Lemma 8. *Let* R *be a random variable with survival function* $\overline{F} \in 2RV_{-\alpha,\rho}$, $\alpha > 0$, $\rho < 0$, *with auxiliary function* $A(x)$, *and let* S *be a nonnegative random variable satisfying* $ES^{\alpha-\rho+\epsilon} < \infty$ *for some* $\epsilon > 0$, *independent of* X. *Then*

$$\frac{P(RS > x)}{\overline{F}(x)} = ES^{\alpha}[1 + \varepsilon(x)], \tag{38}$$

where $\varepsilon(x) = (1/\rho)(ES^{\alpha-\rho}/ES^{\alpha} - 1)A(x)(1 + o(1))$ *as* $x \to \infty$, *and thus* $RS \in 2RV_{-\alpha,\rho}$ *with auxiliary function*

$$A^*(x) = \frac{ES^{\alpha-\rho}}{ES^{\alpha}}A(x). \tag{39}$$

The second lemma talks about the first- and second-order asymptotic of Value-at-Risk of the product RS at the level p, which was proved by Hashorva et al. [7].

Lemma 9. *Let* R *be a random variable with survival function* $\overline{F} \in RV_{-\alpha}$, $\alpha > 0$, *and let* S *be a nonnegative random variable satisfying* $ES^{\alpha+\epsilon} < \infty$ *for some* $\epsilon > 0$, *independent of* X. *Then one has the following:*

(i) *The first-order asymptotic:*

$$VaR_p(RS) \sim (ES^{\alpha})^{1/\alpha}F^{\leftarrow}(p), \quad p \uparrow 1. \tag{40}$$

(ii) *The second-order asymptotic: if* $\overline{F} \in 2RV_{-\alpha,\rho}$, $\rho < 0$, *with auxiliary function* $A(x)$ *and* $ES^{\alpha-\rho+\epsilon} < \infty$, *then*

$$VaR_p(RS) = (ES^{\alpha})^{1/\alpha}F^{\leftarrow}(p)\left[1 + \zeta_{\alpha,\rho}A(F^{\leftarrow}(p))(1 + o(1))\right], \quad p \uparrow 1, \tag{41}$$

where $\zeta_{\alpha,\rho} = (1/\alpha\rho)(ES^{\alpha-\rho}/(ES^{\alpha})^{1-\rho/\alpha} - 1)$.

First, we introduce two definitions. Let F be a distribution function of a nonnegative random variable. We introduce the truncated mean of F:

$$\mu_F(t) = \int_0^t x\,dF(x), \quad t > 0. \tag{42}$$

Obviously, if the mean of F, μ_F, exists, then $\mu_F(t) \to \mu_F$ as $t \to \infty$. For $0 < \alpha < 1$, define

$$\xi_\alpha = 2^{2\alpha} - 2^{\alpha+1} + 2\alpha\int_0^1 \left((1 - x)^{-\alpha} - 1\right)x^{-\alpha-1}dx. \tag{43}$$

The following lemma from Mao and Hu [12] states that the 2RV property is preserved by the formation of sum of n i.i.d random variables.

Lemma 10. *Let* F *be the distribution function of a nonnegative random variable satisfying* $\overline{F} \in 2RV_{-\alpha,\rho}$, $\rho \leq 0$, *with auxiliary function* $A(x)$. *We denoted by* F^{*n} *the n-fold convolution of* F. *Then* $\overline{F^{*n}} \in 2RV_{-\alpha,\gamma}$ *with auxiliary function* $B(x)$, *where*

$$\gamma = -\min\{1, \alpha, -\rho\}, \tag{44}$$

and $B(x)$ *is given by*

$$B(x) = \begin{cases} -\frac{n-1}{2}\alpha\xi_\alpha \overline{F}(x), & \rho < -\alpha,\ 0 < \alpha < 1, \\ -(n-1)\alpha x^{-1}\mu_F(x), & \rho < -1,\ \alpha \geq 1, \\ A(x), & \rho > -(1 \wedge \alpha). \end{cases} \quad (45)$$

The last lemma from Mao et al. [4] establishes the second-order asymptotic of the risk concentration $C_{\text{VaR}}(p)$ for n i.i.d random variables with the underlying distribution possessing the 2RV property.

Lemma 11. *Let* $R_1, R_2, \ldots, R_n$ *be i.i.d. nonnegative random variables with common continuous distribution function* F*, and assume that* $\overline{F} \in 2RV_{-\alpha,\rho}$*,* $\rho \leq 0$*, with auxiliary function* $A(x)$*. Then*

(i) *for* $\rho < -\alpha$ *and* $0 < \alpha < 1$*,*

$$\frac{VaR_p\left(\sum_{i=1}^n R_i\right)}{n\,VaR_p(R_i)} = n^{1/\alpha-1}\left[1 + \frac{n-1}{2n\alpha}\xi_\alpha(1-p)\right] + o(1-p), \quad p \uparrow 1; \quad (46)$$

(ii) *for* $\rho < -1$ *and* $\alpha \geq 1$*,*

$$\frac{VaR_p\left(\sum_{i=1}^n R_i\right)}{n\,VaR_p(R_i)} = n^{1/\alpha-1}\left[1 + n^{-1/\alpha}(n-1)\frac{\mu_F(F^{\leftarrow}(p))}{F^{\leftarrow}(p)}\right] + o\left(\frac{\mu_F(F^{\leftarrow}(p))}{F^{\leftarrow}(p)}\right), \quad p \uparrow 1; \quad (47)$$

(iii) *for* $\rho > -(1 \wedge \alpha)$*,*

$$\frac{VaR_p\left(\sum_{i=1}^n R_i\right)}{n\,VaR_p(R_i)} = n^{1/\alpha-1}\left[1 + \frac{n^{\rho/\alpha}-1}{\alpha\rho}A(F^{\leftarrow}(p))\right] + o(A(F^{\leftarrow}(p))), \quad p \uparrow 1. \quad (48)$$

Now we turn to prove our theorems.

Proof of Theorem 4. Define $Y = \sum_{i=1}^n R_i$, and denote by F_Y the distribution function of Y. By Lemma 10, $\overline{F}_Y \in 2\text{RV}_{-\alpha,\gamma}$ with auxiliary function $B(x)$ with γ and $B(x)$ given by (44) and (45); we have

$$\begin{aligned} \text{VaR}_p(YS) &= (ES^\alpha)^{1/\alpha} F_Y^{\leftarrow}(p)\left[1 + \zeta_{\alpha,\gamma}B(F_Y^{\leftarrow}(p))(1+o(1))\right] \\ &= (ES^\alpha)^{1/\alpha} F_Y^{\leftarrow}(p)\left[1 + \eta_1(F_Y^{\leftarrow}(p))(1+o(1))\right], \quad p \uparrow 1, \end{aligned} \quad (49)$$

where $\eta_1(t) = \zeta_{\alpha,\gamma}B(t)$ and $|\eta_1| \in \text{RV}_\gamma$.

Similarly,

$$\begin{aligned} \text{VaR}_p(R_iS) &= (ES^\alpha)^{1/\alpha} F^{\leftarrow}(p)\left[1 + \zeta_{\alpha,\rho}A(F^{\leftarrow}(p))(1+o(1))\right] \\ &= (ES^\alpha)^{1/\alpha} F^{\leftarrow}(p)\left[1 + \eta_2(F^{\leftarrow}(p))(1+o(1))\right], \quad p \uparrow 1, \end{aligned} \quad (50)$$

for $i = 1, 2, \ldots, n$, where $\eta_2(t) = \zeta_{\alpha,\rho}A(t)$ and $|\eta_2| \in \text{RV}_\rho$. From Lemma 11, it follows that

$$\frac{F_Y^{\leftarrow}(p)}{F^{\leftarrow}(p)} \longrightarrow n^{1/\alpha}, \quad p \uparrow 1. \quad (51)$$

In view of $|\eta_1| \in \text{RV}_\gamma$ and Theorem B.1.4 of de Haan and Ferreira [27], we have

$$\eta_1(F_Y^{\leftarrow}(p)) = n^{\gamma/\alpha}\eta_1(F^{\leftarrow}(p)) \cdot (1+o(1)), \quad p \uparrow 1, \quad (52)$$

where we use the fact that $B(x)$ is ultimately positive or negative. Thus,

$$\begin{aligned} C_{\text{VaR}}(p) &= \frac{F_Y^{\leftarrow}(p)}{nF^{\leftarrow}(p)} \cdot \frac{1 + \eta_1(F_Y^{\leftarrow}(p))(1+o(1))}{1 + \eta_2(F^{\leftarrow}(p))(1+o(1))} \\ &= \frac{\text{VaR}_p(Y)}{n\,\text{VaR}_p(R_i)} \cdot \frac{1 + n^{\gamma/\alpha}\eta_1(F^{\leftarrow}(p))(1+o(1))}{1 + \eta_2(F^{\leftarrow}(p))(1+o(1))}, \quad p \uparrow 1. \end{aligned} \quad (53)$$

Next, we consider three cases.

Case 1 ($\rho < -\alpha$ and $0 < \alpha < 1$). In this case, from (44) and (45), it follows that $\gamma = -\alpha$ and

$$\begin{aligned} \eta_1(F^{\leftarrow}(p)) &= -\frac{(n-1)\alpha\zeta_{\alpha,-\alpha}}{2}\xi_\alpha(1-p) \\ &= \frac{n-1}{2\alpha}\left(\frac{ES^{2\alpha}}{(ES^\alpha)^2} - 1\right)\xi_\alpha(1-p). \end{aligned} \quad (54)$$

Note that $A(F^{\leftarrow}(p)) = o(1-p)$ as $p \uparrow 1$. So, by Lemma 11, we have

$$\begin{aligned} C_{\text{VaR}}(p) &= \left(n^{1/\alpha-1}\left(1 + \frac{n-1}{2n\alpha}\right)\xi_\alpha(1-p) + o(1-p)\right)\left(1 + \frac{n-1}{2n\alpha}\left(\frac{ES^{2\alpha}}{(ES^\alpha)^2} - 1\right)\xi_\alpha(1-p) + o(1-p)\right)(1+o(1)) \\ &= n^{1/\alpha-1}\left[1 + \frac{(n-1)\xi_\alpha}{2n\alpha} \cdot \frac{ES^{2\alpha}}{(ES^\alpha)^2}(1-p)\right] + o(1-p), \quad p \uparrow 1. \end{aligned} \quad (55)$$

Case 2 ($\rho < -1$ and $\alpha \geq 1$). In this case, $\gamma = -1$. By Karamata's theorem, it can be proved that $\mu_F(t) \in \mathrm{RV}_0$; see (2.7) and (2.8) in Mao and Hu (2012a). Hence, $A(t) = o(\mu_F(t)/t)$ as $t \to \infty$. We have

$$\begin{aligned}\eta_1(F^{\leftarrow}(p)) &= \zeta_{\alpha,-1}B(F^{\leftarrow}(p)) \\ &= (n-1)\left(\frac{ES^{\alpha+1}}{(ES^{\alpha})^{1+1/\alpha}} - 1\right)\frac{\mu_F(F^{\leftarrow}(p))}{F^{\leftarrow}(p)}, \\ \eta_2(F^{\leftarrow}(p)) &= \zeta_{\alpha,-1}A(F^{\leftarrow}(p)) = o\left(\frac{\mu_F(F^{\leftarrow}(p))}{F^{\leftarrow}(p)}\right).\end{aligned} \tag{56}$$

Thus,

$$\begin{aligned}C_{\mathrm{VaR}}(p) &= n^{1/\alpha-1}\Bigg(1 + n^{-1/\alpha}(n-1)\frac{\mu_F(F^{\leftarrow}(p))}{F^{\leftarrow}(p)}(1+o(1))\Bigg)\Bigg(1 + n^{-1/\alpha}(n-1)\left(\frac{EY^{\alpha+1}}{(EY^{\alpha})^{1+1/\alpha}} - 1\right) \\ &\quad\cdot \frac{\mu_F(F^{\leftarrow}(p))}{F^{\leftarrow}(p)}(1+o(1))\Bigg) = n^{1/\alpha-1}\Bigg[1 + n^{-1/\alpha}(n-1)\frac{ES^{\alpha+1}}{(ES^{\alpha})^{1+1/\alpha}} \\ &\quad\cdot \frac{\mu_F(F^{\leftarrow}(p))}{F^{\leftarrow}(p)}(1+o(1))\Bigg], \quad p \uparrow 1.\end{aligned} \tag{57}$$

For $\alpha > 1$, $\mu_F(t) \to \mu_F$ as $t \to \infty$. For $\alpha = 1$, by Proposition 1,

$$\begin{aligned}\mu_F(t) &\sim \int_0^t \overline{F}(x)\,dx \sim \int_1^t cx^{-1}\left(1 + \frac{A(x)}{\rho}\right)dx \\ &\sim c\log t + c\rho^{-1}\int_1^t x^{-1}A(x)\,dx \sim c\log t, \\ &\qquad t \longrightarrow \infty,\end{aligned} \tag{58}$$

where the first equation follows from (28) in Mao and Hu [12], and the last equation follows since $\int_1^{\infty} x^{-1}|A(x)|dx < \infty$. Thus, we prove the case.

Case 3 ($\rho > -(1 \wedge \alpha)$). In this case, $\gamma = \rho$, we have

$$\begin{aligned}\eta_1(F^{\leftarrow}(p)) &= \zeta_{\alpha,\rho}A(F^{\leftarrow}(p)), \\ \eta_2(F^{\leftarrow}(p)) &= \zeta_{\alpha,\rho}A(F^{\leftarrow}(p)).\end{aligned} \tag{59}$$

Thus,

$$\begin{aligned}C_{\mathrm{VaR}}(p) &= n^{1/\alpha-1}\Bigg(1 + \frac{n^{\rho/\alpha}-1}{\alpha\rho}A(F^{\leftarrow}(p)) \\ &\quad\cdot(1+o(1))\Bigg)\frac{1 + n^{\rho/\alpha}\zeta_{\alpha,\rho}A(F^{\leftarrow}(p))(1+o(1))}{1 + \zeta_{\alpha,\rho}A(F^{\leftarrow}(p))(1+o(1))} \\ &= n^{1/\alpha-1}\left(1 + \frac{n^{\rho/\alpha}-1}{\alpha\rho}A(F^{\leftarrow}(p))(1+o(1))\right) \\ &\quad\cdot\Bigg(1 + \frac{n^{\rho/\alpha}-1}{\alpha\rho}\left(\frac{ES^{\alpha-\rho}}{(ES^{\alpha})^{1-\rho/\alpha}} - 1\right)A(F^{\leftarrow}(p)) \\ &\quad\cdot(1+o(1))\Bigg) = n^{1/\alpha-1}\Bigg[1 + \frac{n^{\rho/\alpha}-1}{\alpha\rho}\cdot\frac{ES^{\alpha-\rho}}{(ES^{\alpha})^{1-\rho/\alpha}} \\ &\quad\cdot A(F^{\leftarrow}(p))\Bigg] + o(A(F^{\leftarrow}(p))), \quad p \uparrow 1.\end{aligned} \tag{60}$$

□

Proof of Theorem 5. From Proposition 2 and Lemmas 8 and 9, we can get

$$\begin{aligned}\mathrm{HG}_p[R_iS] &= C_{\alpha}\mathrm{VaR}_p(R_iS) \\ &\quad\cdot\left[1 + H_{\alpha,\rho,k}\cdot\frac{ES^{\alpha-\rho}}{(ES^{\alpha})}A\left(\mathrm{VaR}_p(R_iS)\right)(1+o(1))\right] \\ &= C_{\alpha}\mathrm{VaR}_p(R_iS) \\ &\quad\cdot\left[1 + H_{\alpha,\rho,k}\cdot\frac{ES^{\alpha-\rho}}{(ES^{\alpha})^{1-\rho/\alpha}}A(F^{\leftarrow}(p))(1+o(1))\right], \\ &\qquad p \uparrow 1,\end{aligned} \tag{61}$$

for $i = 1, 2, \ldots, n$, where we use the fact that $|A(x)| \in \mathrm{RV}_{\rho}$ and $A(x)$ is ultimately positive or negative.

Define $Y = \sum_{i=1}^{n} R_i$, and denote by F_Y the distribution function of Y. By Lemma 10, $\overline{F}_Y \in 2\mathrm{RV}_{-\alpha,\gamma}$ with auxiliary function $B(x)$ with γ and $B(x)$ given by (44) and (45). So, similarly, from Lemma 10, we can get

$$\begin{aligned}\mathrm{HG}_p[YS] &= C_{\alpha}\mathrm{VaR}_p(YS)\Bigg[1 + H_{\alpha,\gamma,k} \\ &\quad\cdot\frac{ES^{\alpha-\gamma}}{(ES^{\alpha})}B\left(\mathrm{VaR}_p(YS)\right)(1+o(1))\Bigg] \\ &= C_{\alpha}\mathrm{VaR}_p(YS)\Bigg[1 + H_{\alpha,\gamma,k} \\ &\quad\cdot\frac{ES^{\alpha-\gamma}}{(ES^{\alpha})}B\left((ES^{\alpha})^{1/\alpha}F_Y^{\leftarrow}(p)\right)(1+o(1))\Bigg] \\ &= C_{\alpha}\mathrm{VaR}_p(YS)\Bigg[1 + n^{\gamma/\alpha}H_{\alpha,\gamma,k} \\ &\quad\cdot\frac{ES^{\alpha-\gamma}}{(ES^{\alpha})^{1-\gamma/\alpha}}B(F^{\leftarrow}(p))(1+o(1))\Bigg], \quad p \uparrow 1,\end{aligned} \tag{62}$$

where we use the fact that $F_Y^{\leftarrow}(p)/F^{\leftarrow}(p) \to n^{1/\alpha}$, as $p \uparrow 1$, $|B(x)| \in \mathrm{RV}_\gamma$ and $B(x)$ is ultimately positive or negative.

Thus,

$$C_{\mathrm{HG}}(p) = \frac{\mathrm{HG}_p[YS]}{\sum_{i=1}^{n}\mathrm{HG}_p[R_iS]} = \frac{\mathrm{VaR}_p(YS)}{n\,\mathrm{VaR}_p(R_iS)} \cdot \frac{\left(1 + n^{\gamma/\alpha}H_{\alpha,\gamma,k}\cdot\left(ES^{\alpha-\gamma}/(ES^\alpha)^{1-\gamma/\alpha}\right)B(F^{\leftarrow}(p))(1+o(1))\right)}{\left(1 + H_{\alpha,\rho,k}\cdot\left(ES^{\alpha-\rho}/(ES^\alpha)^{1-\rho/\alpha}\right)A(F^{\leftarrow}(p))(1+o(1))\right)}$$
$$= C_{\mathrm{VaR}}(p)\cdot\frac{1 + n^{\gamma/\alpha}\eta_1(F^{\leftarrow}(p))(1+o(1))}{1+\eta_2(F^{\leftarrow}(p))(1+o(1))}, \quad (63)$$

where

$$\eta_1(x) = \frac{ES^{\alpha-\gamma}}{(ES^\alpha)^{1-\gamma/\alpha}}H_{\alpha,\gamma,k}B(x),$$
$$\eta_2(x) = \frac{ES^{\alpha-\rho}}{(ES^\alpha)^{1-\rho/\alpha}}H_{\alpha,\rho,k}A(x). \quad (64)$$

Next, we consider three cases.

(i) For $\rho < -\alpha$ and $0 < \alpha < 1$. In this case, $\gamma = -\alpha$. From (45), it follows that

$$\eta_1(F^{\leftarrow}(p)) = -\frac{(n-1)\alpha\xi_\alpha}{2}\cdot\frac{ES^{2\alpha}}{(ES^\alpha)^2}H_{\alpha,-\alpha,k}(1-p). \quad (65)$$

Note that $A(F^{\leftarrow}(p)) = o(1-p)$ as $p \uparrow 1$. So, by Lemma 10, we have

$$C_{\mathrm{HG}}(p) = n^{1/\alpha-1}\left(1 + \frac{(n-1)\xi_\alpha}{2n\alpha}\cdot\frac{ES^{2\alpha}}{(ES^\alpha)^2}(1-p) + o(1-p)\right)\left(1 - \frac{(n-1)\alpha\xi_\alpha}{2n}\cdot\frac{ES^{2\alpha}}{(ES^\alpha)^2}(1-p) + o(1-p)\right)(1+o(1)) = n^{1/\alpha-1}\left[1 + \frac{n-1}{2n\alpha}\cdot\frac{ES^{2\alpha}}{(ES^\alpha)^2}\cdot\left(1-\alpha^2H_{\alpha,-\alpha,k}\right)\xi_\alpha(1-p)\right] + o(1-p),$$
$$p \uparrow 1. \quad (66)$$

(ii) $\rho < -1$ and $\alpha \geq 1$. In this case, $\gamma = -1$. By Karamata's theorem, it can be proved that $\mu_F(t) \in \mathrm{RV}_0$; see (2.7) and (2.8) in Mao and Hu (2012a). Hence, $A(t) = o(\mu_F(t)/t)$ as $t \to \infty$. Thus,

$$C_{\mathrm{HG}}(p) = n^{1/\alpha-1}\left(1 + n^{-1/\alpha}(n-1)\frac{ES^{\alpha+1}}{(ES^\alpha)^{1+1/\alpha}}\cdot\frac{\mu_F(F^{\leftarrow}(p))}{F^{\leftarrow}(p)}(1+o(1))\right)\cdot\left(1 - n^{-1/\alpha}(n-1)\alpha H_{\alpha,-1,k}\frac{ES^{\alpha+1}}{(ES^\alpha)^{1+1/\alpha}}\cdot\frac{\mu_F(F^{\leftarrow}(p))}{F^{\leftarrow}(p)}(1+o(1))\right) = n^{1/\alpha-1}\left[1 + n^{-1/\alpha}(n-1)\frac{ES^{\alpha+1}}{(ES^\alpha)^{1+1/\alpha}}\cdot\left(1-\alpha H_{\alpha,-1,k}\right)\cdot\frac{\mu_F(F^{\leftarrow}(p))}{F^{\leftarrow}(p)}\right] + o\left(\frac{\mu_F(F^{\leftarrow}(p))}{F^{\leftarrow}(p)}\right), \quad p \uparrow 1. \quad (67)$$

Considering $\mu_F(t)$ as $t \to \infty$ for $\alpha > 1$ and $\alpha = 1$ as Theorem 4, we can get the result easily.

(iii) $\rho > -(1 \wedge \alpha)$. In this case, $\gamma = \rho$. Thus,

$$C_{\mathrm{HG}}(p) = n^{1/\alpha-1}\left(1 + \frac{n^{\rho/\alpha}-1}{\alpha\rho}\cdot\frac{ES^{\alpha-\rho}}{(ES^\alpha)^{1-\rho/\alpha}}\cdot A(F^{\leftarrow}(p))(1+o(1))\right)\cdot\left(1 + \left(n^{\rho/\alpha}-1\right)H_{\alpha,\rho,k}\cdot\frac{ES^{\alpha-\rho}}{(ES^\alpha)^{1-\rho/\alpha}}A(F^{\leftarrow}(p))(1+o(1))\right) = n^{1/\alpha-1}\left[1 + \frac{n^{\rho/\alpha}-1}{\alpha\rho}\cdot\frac{ES^{\alpha-\rho}}{(ES^\alpha)^{1-\rho/\alpha}}\cdot\left(1+\alpha\rho H_{\alpha,\rho,k}\right)\cdot A(F^{\leftarrow}(p))\right] + o(A(F^{\leftarrow}(p))), \quad p \uparrow 1. \quad (68)$$

□

Proof of Theorem 6. From Proposition 3 and Lemmas 8 and 9, we can get

$$T_{g_p}[R_iS] = J_\alpha\mathrm{VaR}_p(R_iS) + \mathrm{VaR}_p(R_iS)\cdot A^*\left(\mathrm{VaR}_p(R_iS)\right)\left[I_{\alpha,\rho}+o(1)\right] = J_\alpha\mathrm{VaR}_p(R_iS)\cdot\left[1 + \frac{I_{\alpha,\rho}}{J_\alpha}\cdot\frac{ES^{\alpha-\rho}}{(ES^\alpha)^{1-\rho/\alpha}}A(F^{\leftarrow}(p))(1+o(1))\right],$$
$$p \uparrow 1, \quad (69)$$

for $i = 1, 2, \ldots, n$, where we use the fact that $|A(x)| \in \mathrm{RV}_\rho$ and $A(x)$ is ultimately positive or negative.

Define $Y = \sum_{i=1}^{n} R_i$, and denote by F_Y the distribution function of Y. By Lemma 10, $\overline{F}_Y \in 2\text{RV}_{-\alpha,\gamma}$ with auxiliary function $B(x)$ with γ and $B(x)$ given by (44) and (45). So, similarly, from Lemma 10, we can get

$$T_{g_p}[YS] = J_\alpha \text{VaR}_p(YS) + \text{VaR}_p(YS) \cdot \frac{ES^{\alpha-\rho}}{ES^\alpha} \cdot B\left(\text{VaR}_p(YS)\right)\left[I_{\alpha,\gamma} + o(1)\right] = J_\alpha \text{VaR}_p(YS)\left[1 + n^{\gamma/\alpha}\frac{I_{\alpha,\gamma}}{J_\alpha} \cdot \frac{ES^{\alpha-\gamma}}{(ES^\alpha)^{1-\gamma/\alpha}} B\left(F^{\leftarrow}(p)\right)(1+o(1))\right], \quad p \uparrow 1, \tag{70}$$

where we use the fact that $F_Y^{\leftarrow}(p)/F^{\leftarrow}(p) \to n^{1/\alpha}$, as $p \uparrow 1$, $|B(x)| \in \text{RV}_\gamma$ and $B(x)$ is ultimately positive or negative. Thus,

$$C_{T_g}(p) = \frac{T_{g_p}[YS]}{\sum_{i=1}^{n} T_{g_p}[R_i S]} = \frac{\text{VaR}_p(YS)}{n \text{VaR}_p(R_i S)} \cdot \frac{1 + n^{\gamma/\alpha}\left(I_{\alpha,\gamma}/J_\alpha\right) \cdot \left(ES^{\alpha-\gamma}/(ES^\alpha)^{1-\gamma/\alpha}\right) B\left(F^{\leftarrow}(p)\right)(1+o(1))}{1 + I_{\alpha,\rho}/J_\alpha \cdot \left(ES^{\alpha-\rho}/(ES^\alpha)^{1-\rho/\alpha}\right) A\left(F^{\leftarrow}(p)\right)(1+o(1))}$$
$$= C_{\text{VaR}}(p) \cdot \frac{1 + n^{\gamma/\alpha}\eta_1\left(\left(F^{\leftarrow}(p)\right)(1+o(1))\right)}{1 + \eta_2\left(\left(F^{\leftarrow}(p)\right)(1+o(1))\right)}, \tag{71}$$

where

$$\eta_1(x) = \frac{I_{\alpha,\gamma}}{J_\alpha} \cdot \frac{ES^{\alpha-\gamma}}{(ES^\alpha)^{1-\gamma/\alpha}} B(x),$$
$$\eta_2(x) = \frac{I_{\alpha,\rho}}{J_\alpha} \cdot \frac{ES^{\alpha-\rho}}{(ES^\alpha)^{1-\rho/\alpha}} A(x). \tag{72}$$

Next, similar to Theorems 4 and 5, we consider three cases: (i) for $\rho < -\alpha$ and $0 < \alpha < 1$; (ii) for $\rho < -1$ and $\alpha \geq 1$; (iii) for $\rho > -(1 \wedge \alpha)$ to obtain the result. Thus, we complete the proof. □

Proof of Corollary 7. Note that if the distortion function $g(x) = x$, then $T_{g,p}[X]$ reduces to $\text{CTE}_p[X]$ for continuous risk variables X, and $C_{T_g}(p)$ reduces to $C_{\text{CTE}}(p)$. It is easy to see that, for $\alpha > 1$ and $\rho < 0$,

$$J_\alpha = \frac{\alpha}{\alpha - 1},$$
$$I_{\alpha,\rho} = \frac{1}{(\alpha - \rho - 1)(\alpha - 1)}, \tag{73}$$

and, hence,

$$\frac{\alpha I_{\alpha,-1}}{J_\alpha} = \frac{1}{\alpha},$$
$$\frac{\alpha\rho I_{\alpha,\rho}}{J_\alpha} = \frac{\alpha - 1}{\alpha - 1 - \rho}. \tag{74}$$

Therefore, the result is an immediate consequence of Theorem 6. □

4. Examples

In this section, two examples are given to illustrate applications of our main results.

Example 1 (Burr distribution and Beta distribution). Let R be a random variable with Burr distribution function F given by

$$\overline{F}(x) = \left(1 + x^{-\rho}\right)^{\alpha/\rho}, \quad x > 0,\ \alpha > 0,\ \rho < 0 \tag{75}$$

denoted by $F \sim \text{Burr}(\alpha, \rho)$. It is known from Example 1 in Mao et al. [4] and Example 2 in Degen et al. [10] that $\overline{F} \in 2\text{RV}_{-\alpha,\rho}$ with auxiliary function $A(x) = \alpha x^\rho$. Suppose that $S \sim \text{Beta}(a, b)$, where $\text{Beta}(a, b)$ stands for the Beta distribution with positive parameters a and b and density function

$$g(x) = \frac{1}{B(a,b)} x^{a-1}(1-x)^{b-1}, \quad 0 < x < 1,\ a, b > 0. \tag{76}$$

It is obvious that $ES^\kappa = B(a + \kappa, b)/B(a, b)$ for all $\kappa > 0$.

By Theorem 4, we have

$$C_{\text{VaR}}(p) = \begin{cases} n^{1/\alpha-1}\left[1 + \dfrac{(n-1)\xi_\alpha}{2n\alpha} \cdot \dfrac{B(a,b) \cdot B(a+2\alpha,b)}{B^2(a+\alpha,b)}(1-p)\right] + o(1-p) & \rho < -\alpha,\ 0 < \alpha < 1 \\ n^{1/\alpha-1} + \dfrac{n-1}{n}\dfrac{ES^{\alpha+1}}{(ES^\alpha)^{1+1/\alpha}} \cdot \mu_F(1-p)^{1/\alpha} + o\left((1-p)^{1/\alpha}\right) & \rho < -1,\ \alpha > 1 \\ 1 - \dfrac{n-1}{n}(1-\alpha)\log(1-\alpha)(1+o(1)) & \rho < -1,\ \alpha = 1 \\ n^{1/\alpha-1}\left[1 + \dfrac{n^{\rho/\alpha}-1}{\rho} \cdot \dfrac{ES^{\alpha-\rho}}{(ES^\alpha)^{1-\rho/\alpha}}(1-p)^{-\rho/\alpha}\right] + o\left((1-p)^{-\rho/\alpha}\right) & \rho > -(1 \wedge \alpha). \end{cases} \tag{77}$$

Similarly, we can get risk concentration based on other risk measures. We set $n = 2$ and compare the second-order approximations with the actual true value of $C_{\mathrm{VaR}}(p)$ for Burr distribution with different parameters α and ρ and Beta distribution with $a = b = 1$ in Figure 1.

Example 2 (absolute student t_α distribution and Beta distribution). Let R be a random variable having the standard Student t_α distribution with density function

$$f_R(r) = \frac{\Gamma((\alpha+1)/2)}{\sqrt{\alpha\pi}\Gamma(\alpha/2)}\left(1+\frac{r^2}{\alpha}\right)^{-(\alpha+1)/2}, \quad r > 0. \tag{78}$$

Denote by F the distribution function of R. Suppose that $S \sim \mathrm{Beta}(a,b)$, where $\mathrm{Beta}(a,b)$ stands for the Beta distribution with positive parameters a and b and density function

$$g(x) = \frac{1}{B(a,b)}x^{a-1}(1-x)^{b-1}, \quad 0 < x < 1,\ a,b > 0. \tag{79}$$

From Example 3 in Hua and Joe [29], we know that $\overline{F} \in 2\mathrm{RV}_{-\alpha,-2}$ and the mean of F, $\mu_F = \alpha/(\alpha-1)$ for $\alpha > 1$ and $ES^\kappa = B(a+\kappa,b)/B(a,b)$ for all $\kappa > 0$.

For $0 < \alpha < 1$, by Theorems 4(i) and 3.2(i), we have

$$C_{\mathrm{VaR}}(p) = n^{1/\alpha-1}\left[1+\frac{(n-1)\xi_\alpha}{2n\alpha}\cdot\frac{B(a,b)\cdot B(a+2\alpha,b)}{B^2(a+\alpha,b)}(1-p)\right]+o(1-p), \quad p\uparrow 1, \tag{80}$$

$$C_{\mathrm{HG}}(p) = n^{1/\alpha-1}\left[1+\frac{n-1}{2n\alpha}\cdot\frac{B(a,b)\cdot B(a+2\alpha,b)}{B^2(a+\alpha,b)}\cdot\left(1-\alpha^2H_{\alpha,-\alpha,k}\right)\xi_\alpha(1-p)\right]+o(1-p), \quad p\uparrow 1.$$

Choose distortion function $g(x) = x^{1/\beta}$ with $0 < \beta < \alpha$. It is easy to see that

$$J_\alpha = \frac{\alpha}{\alpha-\beta},\quad I_{\alpha,-1} = \frac{\beta}{\alpha(\alpha-\beta)},\quad I_{\alpha,-\alpha} = \frac{\beta}{(\alpha-\beta)(\alpha-\beta+\alpha\beta)}. \tag{81}$$

By Theorem 6(i), we have

$$C_{T_g}(p) = n^{1/\alpha-1}\left[1+\frac{n-1}{2n\alpha}\cdot\frac{B(a,b)\cdot B(a+2\alpha,b)}{B^2(a+\alpha,b)}\cdot\frac{\alpha-\beta}{\alpha-\beta+\alpha\beta}\xi_\alpha(1-p)\right]+o(1-p), \quad p\uparrow 1. \tag{82}$$

For $\alpha > 1$, since $F^{\leftarrow}(p) = t_\alpha^{\leftarrow}((1+p)/2)$, the quantile function of t_α distribution is at the level $(p+1)/2$. By Theorems 4(i) and 3.2(i), we have that, as $p\uparrow 1$,

$$\begin{aligned}C_{\mathrm{VaR}}(p) &= n^{1/\alpha-1}\left[1+n^{-1/\alpha}(n-1)\frac{(B(a,b))^{1/\alpha}\cdot B(a+\alpha+1,b)}{(B(a+\alpha,b))^{1+1/\alpha}}\cdot\frac{\alpha}{\alpha-1}\cdot\frac{1}{t_\alpha^{\leftarrow}((1+p)/2)}\right]+o\left(\frac{1}{t_\alpha^{\leftarrow}((1+p)/2)}\right),\\ C_{\mathrm{HG}}(p) &= n^{1/\alpha-1}\left[1+n^{-1/\alpha}(n-1)\frac{(B(a,b))^{1/\alpha}\cdot B(a+\alpha+1,b)}{(B(a+\alpha,b))^{1+1/\alpha}}\cdot\frac{\alpha(1-\alpha H_{\alpha,-1,k})}{\alpha-1}\cdot\frac{1}{t_\alpha^{\leftarrow}((1+p)/2)}\right]+o\left(\frac{1}{t_\alpha^{\leftarrow}((1+p)/2)}\right).\end{aligned} \tag{83}$$

Considering the distortion function $g(x) = x^{1/\beta}$ with $0 < \beta < \alpha$, by Theorem 6(ii), we have that, as $p\uparrow 1$,

$$C_{T_g}(p) = n^{1/\alpha-1}\left[1+n^{-1/\alpha}(n-1)\frac{(B(a,b))^{1/\alpha}\cdot B(a+\alpha+1,b)}{(B(a+\alpha,b))^{1+1/\alpha}}\cdot\frac{\alpha-\beta}{\alpha-1}\cdot\frac{1}{t_\alpha^{\leftarrow}((1+p)/2)}\right]+o\left(\frac{1}{t_\alpha^{\leftarrow}((1+p)/2)}\right). \tag{84}$$

Choosing $\beta = 1$, $C_{T_g}(p)$ reduces to

$$C_{\mathrm{CTE}}(p) = n^{1/\alpha-1}\left[1+n^{-1/\alpha}(n-1)\frac{(B(a,b))^{1/\alpha}\cdot B(a+\alpha+1,b)}{(B(a+\alpha,b))^{1+1/\alpha}}\cdot\frac{1}{t_\alpha^{\leftarrow}((1+p)/2)}\right]+o\left(\frac{1}{t_\alpha^{\leftarrow}((1+p)/2)}\right), \quad p\uparrow 1. \tag{85}$$

We set $n = 2$ and compare the second-order approximations with the actual true value of $C_{\mathrm{VaR}}(p)$ for t_α distribution with different parameters α and Beta distribution with $a = b = 1$ in Figure 2.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

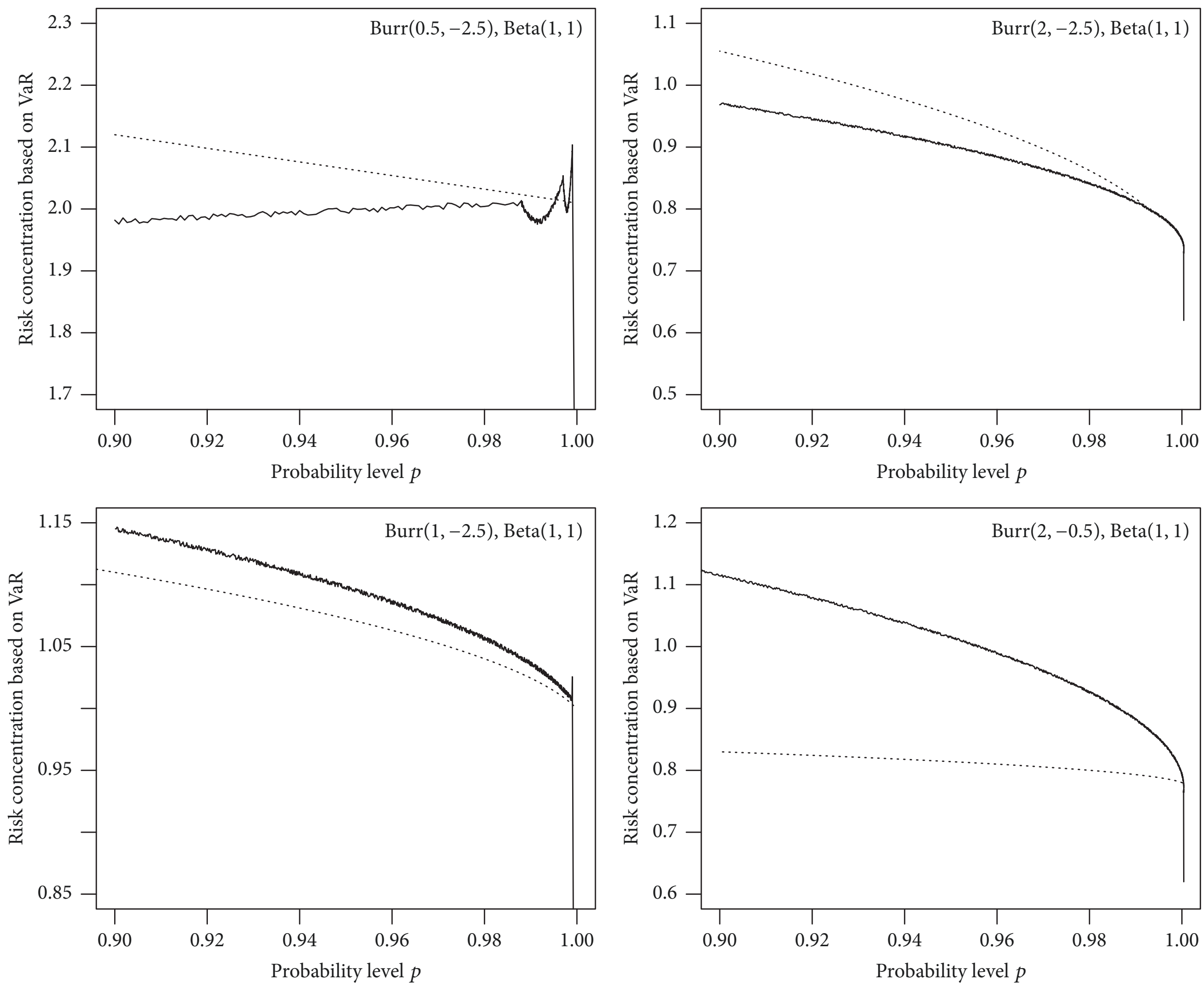

FIGURE 1: Empirical risk concentration (full, based on 10^7 simulations) together with second-order approximation (dotted) for two i.i.d. Burr(α, ρ) and Beta(1, 1) random variables based on $C_{\text{VaR}}(p)$.

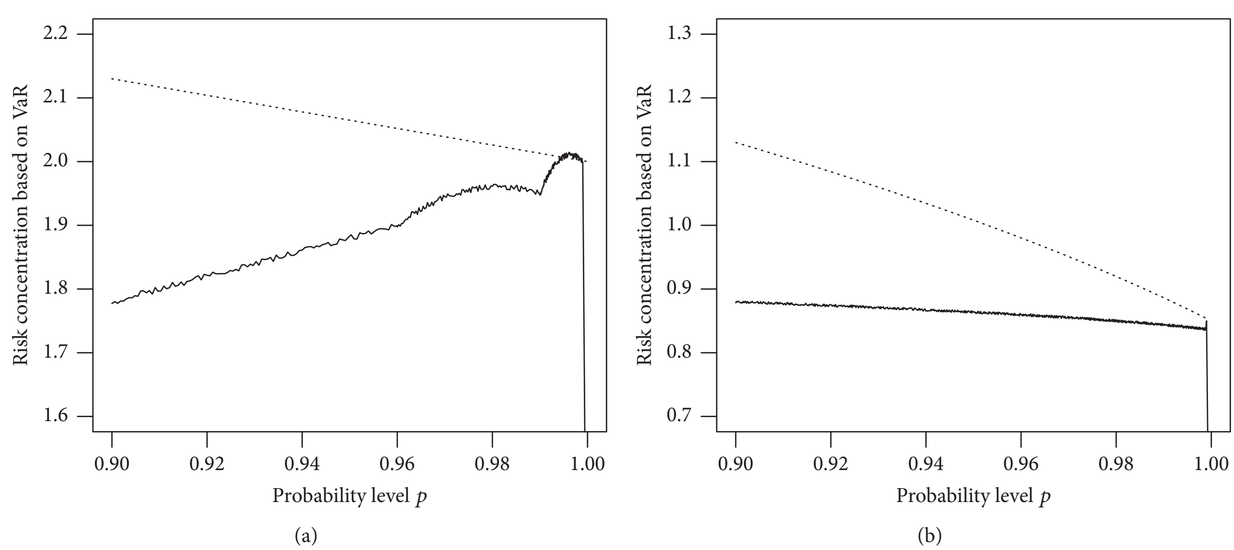

FIGURE 2: Empirical risk concentration (full, based on 10^7 simulations) together with second-order approximation (dotted) for two i.i.d. t_α and Beta(1, 1) random variables based on $C_{\text{VaR}}(p)$ with $\alpha = 0.5$ for (a) and $\alpha = 1.4$ for (b).

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