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*Research Article*

# The Application of the $\exp(-\Phi(\xi))$-Expansion Method for Finding the Exact Solutions of Two Integrable Equations

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In this article, the $\exp(-\Phi(\xi))$ method is connected to search for new hyperbolic, periodic, and rational solutions of (1 + 1)-dimensional fifth-order nonlinear integrable equation and (2 + 1)-dimensional Date-Jimbo-Kashiwara-Miwa equation. The obtained solutions consist of trigonometric, hyperbolic, rational functions and W-shaped soliton. Furthermore, 3D and 2D graphs are plotted by choosing the suitable values of the parameters involved.

## 1. Introduction

Firstly, consider the $(1+1)$-dimensional fifth-order nonlinear integrable equation. In [1], Wazwaz proposed a new (1 + 1)-dimensional fifth-order nonlinear integrable equation of the form

$$u_{ttt} - u_{txxxx} - 4\left(u_x u_t\right)_{xx} - 4\left(u_x u_{xt}\right)_x = 0, \tag{1}$$

where $u = u(x,t)$ stands for wave propagation of physical quantity and subscripts represent partial differentiation with respect to the given variable and obtained multiple soliton solutions using the simplified Hirota's method established by Hereman and Nuseir [2].

Furthermore, Yuan *et al.* [3] consider the (2 + 1)-dimensional Date-Jimbo-Kashiwara-Miwa equation. To study the (2 + 1)-dimensional Date-Jimbo-Kashiwara-Miwa equation, many researchers considered the following integrable equation:

$$\begin{aligned} &u_{xxxxy} + 4u_{xxy}u_x + 2u_{xxx}u_y + 6u_{xy}u_{xx} + u_{yyy} - 2u_{xxt} \\ &\quad = 0, \end{aligned} \tag{2}$$

where $u$ is the real function of the variables $x$, $y$, and $t$. With the help of the Hirota method and auxiliary variables, the bilinear Bäcklund transformation and N-soliton solutions are obtained.

Nonlinear evolution equations (NLEEs) are one of the fastest developing zones of research in the field of science and engineering, especially in mathematical biology, nonlinear optics, optical fiber, fluid mechanics, solid state physics, biophysics, chemical physics, chemical kinetics, *etc.* Many effective methods have been proposed to solve the NLEEs, such as the Hirota method [1], Hereman-Nuseir method [2], inverse scattering transformation [4], Painlevé technique [5], Bäcklund transformation [6], Darboux transformation [7, 8], Binary-Bell-polynomial scheme [9], first integral method [10, 11], $(G'/G)$−expansion method [12], the $\exp(-\Phi(\xi))$ expansion method [13], Exp-function method [14], ansatz method [15], sine-Gordon expansion method [16, 17], the trial equation method [18, 19], homotopy asymptotic [20], and so on.

The present paper organized as follows: In Section 2, description of the $\exp(-\Phi(\xi))$ method for finding the exact traveling wave solutions of NLEEs is presented. Section 3 illustrates the method to solve the (1 + 1)-dimensional fifth-order nonlinear integrable equation and (2 + 1)-dimensional Date-Jimbo-Kashiwara-Miwa equation. Results and discussion are presented in Section 4. Finally, in Section 5, some conclusions are given.

## 2. Traveling Wave Hypothesis

Consider a NLEE in two independent variables $x$ and $t$ of the form as

$$P\left(u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, \ldots\right) = 0, \tag{3}$$

where $u = u(x,t)$ is an unknown function and $P$ is the polynomial in $u = u(x,t)$ which have various partial derivatives. Under the wave transformation

$$u(x,t) = u(\xi), \quad \xi = \nu(x - kt), \tag{4}$$

where $\nu$ and $k$ are the wave number and wave speed, respectively. Eq. (3) can be transformed into the following nonlinear ordinary differential equation as

$$Q\left(u, \nu u', -\nu k u', \nu^2 u'', -\nu^2 k u'', k^2 \nu^2 u'', \ldots\right) = 0, \tag{5}$$

where prime denotes the derivatives with respect to $\xi$.

*2.1. The* $\exp(-\phi(\xi))$ *Expansion Method: Quick Recapitulation.* The key steps of $\exp(-\Phi(\xi))$ method are given as

*Step 1.* According to $\exp(-\Phi(\xi))$ method, the wave solution can be expressed as

$$U(\xi) = \sum_{i=0}^{N} a_i \exp(-\Phi(\xi))^i, \tag{6}$$

where $a_i (a_N \neq 0)$ are constants to be determined and $\Phi(\xi)$ satisfies the auxiliary ODE given as

$$\Phi'(\xi) = \exp(-\Phi(\xi)) + \mu \exp(\Phi(\xi)) + \lambda. \tag{7}$$

The auxiliary Eq. (7) has the general solutions given as follows.

*Case 1* (hyperbolic function solutions). When $\lambda^2 - 4\mu > 0$ and $\mu \neq 0$,

$$\Phi_1(\xi) = \ln\left(\frac{-\sqrt{\lambda^2 - 4\mu}\tanh\left(\left(\sqrt{\lambda^2 - 4\mu}/2\right)(\xi + K)\right) - \lambda}{2\mu}\right). \tag{8}$$

*Case 2* (trigonometric function solutions). When $\lambda^2 - 4\mu < 0$ and $\mu \neq 0$,

$$\Phi_2(\xi) = \ln\left(\frac{\sqrt{4\mu - \lambda^2}\tan\left(\left(\sqrt{4\mu - \lambda^2}/2\right)(\xi + K)\right) - \lambda}{2\mu}\right). \tag{9}$$

*Case 3* (hyperbolic function solutions). When $\lambda^2 - 4\mu > 0$, $\mu = 0$, and $\lambda \neq 0$,

$$\Phi_3(\xi) = -\ln\left(\frac{\lambda}{\cosh(\lambda(\xi + K)) + \sinh(\lambda(\xi + K)) - 1}\right). \tag{10}$$

*Case 4* (rational function solutions). When $\lambda^2 - 4\mu = 0$, $\mu \neq 0$, and $\lambda \neq 0$,

$$\Phi_4(\xi) = \ln\left(-\frac{2(\lambda(\xi + K)) + 2}{\lambda^2(\xi + K)}\right). \tag{11}$$

*Case 5.* When $\lambda^2 - 4\mu = 0$, $\mu = 0$, and $\lambda = 0$,

$$\Phi_5(\xi) = \ln(\xi + K), \tag{12}$$

where $K$ is integration constant.

*Step 2.* The positive integer $N$ can be determined by balancing the highest derivative term with the highest order nonlinear term in Eq. (5). Substituting Eq. (6) into Eq. (5) yields an algebraic equation involving powers of $\exp(-\Phi(\xi))^i$ $(i = 0, 1, 2, \ldots)$. Equating the coefficients of each power of $\exp(-\Phi(\xi))$ to zero gives a system of algebraic equations for $a_i$ $(i = 0, 1, 2, \ldots, n)$, $\nu$, $k$, $\lambda$, and $\mu$.

*Step 3.* Substituting $a_i$, $\nu$, $k$, $\lambda$, and $\mu$, a variety of exact solutions of Eq.(3) can be constructed.

## 3. Application of the $exp(-\Phi(\xi))$ Method

*3.1. The* (1 + 1)*-Dimensional Fifth-Order Nonlinear Integrable Equation.* The (1 + 1)-dimensional fifth-order nonlinear integrable equation is given as

$$u_{ttt} - u_{txxxx} - 4\left(u_x u_t\right)_{xx} - 4\left(u_x u_{xt}\right)_x = 0. \tag{13}$$

Applying the wave transformation $u(x,t) = V(\xi)$, $\xi = \nu(x - kt)$ to Eq. (13) yields the following nonlinear ODE:

$$-k^2 V'' + \nu^2 V^{(4)} + 12\nu V' V'' = 0. \tag{14}$$

Set $U = V'$ to get

$$-k^2 U' + \nu^2 U''' + 12\nu U U' = 0, \tag{15}$$

where the integration constant is taken as zero. Balance the linear term of highest order $U'''$ with the highest order nonlinear term $UU'$ in Eq. (15), to get balancing number as $N = 2$. Thus the solution is

$$U(\xi) = a_0 + a_1 \exp(-\Phi(\xi)) + a_2 \exp(-\Phi(\xi))^2. \quad (16)$$

Substituting Eq. (16) into Eq. (15) and collecting the coefficient of each power of $\exp(-\Phi(\xi))^i$ and then setting each of coefficients to zero give a system of algebraic equations as

$$\begin{aligned}
\exp(-\phi(\xi))^0: \quad & k^2 a_1 \mu - 12\nu a_0 a_1 \mu - \nu^2 a_1 \mu \lambda^2 \\
& - 6\nu^2 a_2 \mu^2 \lambda - 2\nu^2 a_1 \mu^2 = 0, \\
\exp(-\phi(\xi))^1: \quad & -24\nu a_0 a_2 \mu - 12\nu a_0 a_1 \lambda \\
& - 14\nu^2 a_2 \mu \lambda^2 - 8\nu^2 a_1 \mu \lambda + k^2 a_1 \lambda \\
& - 16\nu^2 a_2 \mu^2 - \nu^2 a_1 \lambda^3 + 2k^2 a_2 \mu \\
& - 12\nu a_1^2 \mu = 0, \\
\exp(-\phi(\xi))^2: \quad & k^2 a_1 - 24\nu a_0 a_2 \lambda - 36\nu a_1 a_2 \mu \\
& - 52\nu^2 a_2 \mu \lambda - 8\nu^2 a_2 \lambda^3 - 7\nu^2 a_1 \lambda^2 \\
& - 8\nu^2 a_1 \mu + 2k^2 a_2 \lambda - 12\nu a_0 a_1 \\
& - 12\nu a_1^2 \lambda = 0, \\
\exp(-\phi(\xi))^3: \quad & -12\nu a_1^2 + 2k^2 a_2 - 36\nu a_1 a_2 \lambda \\
& - 38\nu^2 a_2 \lambda^2 - 12\nu^2 a_1 \lambda - 40\nu^2 a_2 \mu \\
& - 24\nu a_2^2 \mu - 24\nu a_0 a_2 = 0, \\
\exp(-\phi(\xi))^4: \quad & -6\nu^2 a_1 - 36\nu a_1 a_2 - 54\nu^2 a_2 \lambda \\
& - 24\nu a_2^2 \lambda = 0, \\
\exp(-\phi(\xi))^5: \quad & -24\nu a_2^2 - 24\nu^2 a_2 = 0.
\end{aligned} \quad (17)$$

Solving the above system of equations yields

$$\begin{aligned}
a_0 &= a_0, \\
a_1 &= -\nu\lambda, \\
a_2 &= -\nu, \\
k &= \pm\sqrt{\nu^2\lambda^2 + 8\nu^2\mu + 12\nu a_0}.
\end{aligned} \quad (18)$$

Consequently, the following different cases are obtained for the exact solutions of (1 + 1)-dimensional fifth-order nonlinear integrable equation.

*Case 1* (hyperbolic function solutions). When $\lambda^2 - 4\mu > 0$ and $\mu \neq 0$,

$$\begin{aligned}
u_1(\xi) = {} & a_0\xi + 4\frac{\nu\lambda^3\mu \ln\left(\sqrt{\lambda^2-4\mu}\tanh\left(1/2\sqrt{\lambda^2-4\mu}(\xi+K)\right)+\lambda\right)}{(\lambda^2-4\mu)\left(\sqrt{\lambda^2-4\mu}+\lambda\right)\left(\sqrt{\lambda^2-4\mu}-\lambda\right)} \\
& - 16\frac{\nu\lambda\mu^2 \ln\left(\sqrt{\lambda^2-4\mu}\tanh\left(1/2\sqrt{\lambda^2-4\mu}(\xi+K)\right)+\lambda\right)}{(\lambda^2-4\mu)\left(\sqrt{\lambda^2-4\mu}+\lambda\right)\left(\sqrt{\lambda^2-4\mu}-\lambda\right)} - 4\frac{\nu\lambda\mu \ln\left(\tanh\left(1/2\sqrt{\lambda^2-4\mu}(\xi+K)\right)+1\right)}{\sqrt{\lambda^2-4\mu}\left(2\sqrt{\lambda^2-4\mu}-2\lambda\right)} \\
& - 4\frac{\nu\lambda\mu \ln\left(\tanh\left(1/2\sqrt{\lambda^2-4\mu}(\xi+K)\right)-1\right)}{\sqrt{\lambda^2-4\mu}\left(2\sqrt{\lambda^2-4\mu}+2\lambda\right)} \\
& + 8\frac{\nu\mu^2}{\left(\sqrt{\lambda^2-4\mu}+\lambda\right)\left(\sqrt{\lambda^2-4\mu}-\lambda\right)\left(\sqrt{\lambda^2-4\mu}\tanh\left(1/2\sqrt{\lambda^2-4\mu}(\xi+K)\right)+\lambda\right)} \\
& + 16\frac{\nu\mu^2 \ln\left(\sqrt{\lambda^2-4\mu}\tanh\left(1/2\sqrt{\lambda^2-4\mu}(\xi+K)\right)+\lambda\right)\lambda^3}{(\lambda^2-4\mu)\left(\sqrt{\lambda^2-4\mu}+\lambda\right)^2\left(\sqrt{\lambda^2-4\mu}-\lambda\right)^2} \\
& - 64\frac{\nu\mu^3 \ln\left(\sqrt{\lambda^2-4\mu}\tanh\left(1/2\sqrt{\lambda^2-4\mu}(\xi+K)\right)+\lambda\right)\lambda}{(\lambda^2-4\mu)\left(\sqrt{\lambda^2-4\mu}+\lambda\right)^2\left(\sqrt{\lambda^2-4\mu}-\lambda\right)^2} - 4\frac{\nu\mu^2 \ln\left(\tanh\left(1/2\sqrt{\lambda^2-4\mu}(\xi+K)\right)+1\right)}{\sqrt{\lambda^2-4\mu}\left(\sqrt{\lambda^2-4\mu}-\lambda\right)^2}
\end{aligned}$$

$$+4\frac{\nu\mu^2\ln\left(\tanh\left(1/2\sqrt{\lambda^2-4\mu}\,(\xi+K)\right)-1\right)}{\sqrt{\lambda^2-4\mu}\left(\sqrt{\lambda^2-4\mu}+\lambda\right)^2}. \tag{19}$$

*Case 2* (trigonometric function solutions). When $\lambda^2-4\mu<0$ and $\mu\neq 0$,

$$\begin{aligned}u_2(\xi)&=a_0\xi-\nu\lambda\\&\cdot\ln\left(\sqrt{4\mu-\lambda^2}\tan\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\,(\xi+K)\right)-\lambda\right)+2\\&\cdot\frac{\mu\nu}{\sqrt{4\mu-\lambda^2}\tan\left(1/2\sqrt{4\mu-\lambda^2}\,(\xi+K)\right)-\lambda}+2\\&\cdot\frac{\mu\nu\arctan\left(\tan\left(1/2\sqrt{4\mu-\lambda^2}\,(\xi+K)\right)\right)}{\sqrt{4\mu-\lambda^2}}+4\\&\cdot\frac{\nu\lambda\mu\ln\left(\sqrt{4\mu-\lambda^2}\tan\left(1/2\sqrt{4\mu-\lambda^2}\,(\xi+K)\right)-\lambda\right)}{4\mu-\lambda^2}\\&-\frac{\nu\lambda^3\ln\left(\sqrt{4\mu-\lambda^2}\tan\left(1/2\sqrt{4\mu-\lambda^2}\,(\xi+K)\right)-\lambda\right)}{4\mu-\lambda^2}.\end{aligned} \tag{20}$$

*Case 3* (hyperbolic function solutions). When $\lambda^2-4\mu>0$, $\mu=0$, and $\lambda\neq 0$,

$$\begin{aligned}u_3(\xi)&=a_0\xi-\nu\ln\left(\tanh\left(\frac{1}{2}\lambda\xi+\frac{1}{2}\lambda K\right)\right)\\&+\nu\ln\left(\tanh\left(\frac{1}{2}\lambda\xi+\frac{1}{2}\lambda K\right)+1\right)\\&+\frac{1}{2}\frac{\nu}{\lambda\tanh(1/2\lambda\xi+1/2\lambda K)}\\&+\frac{\nu\ln(\tanh(1/2\lambda\xi+1/2\lambda K))}{\lambda}\\&-\frac{\nu\ln(\tanh(1/2\lambda\xi+1/2\lambda K)+1)}{\lambda}.\end{aligned} \tag{21}$$

*Case 4* (rational function solutions). When $\lambda^2-4\mu=0$, $\mu\neq 0$, and $\lambda\neq 0$,

$$u_4(\xi)=a_0\xi+\frac{1}{4}\nu\lambda^2\xi+\frac{\nu\lambda}{\lambda\xi+\lambda K+2}. \tag{22}$$

*Case 5*. When $\lambda^2-4\mu=0$, $\mu=0$, and $\lambda=0$,

$$u_5(\xi)=a_0-\frac{\nu\lambda}{\xi+K}-\frac{\nu}{(\xi+K)^2}, \tag{23}$$

where $\xi=\nu(x-kt)$ and $K$ is the integration constant.

The solutions $(u_1,u_2,u_3,u_4,u_5)$ of the $(1+1)$-dimensional fifth-order nonlinear integrable equation are graphically presented by Figures 1, 2, 3, 4, and 5.

*3.2. (2 + 1)-Dimensional Date-Jimbo-Kashiwara-Miwa Equation.* The (2 + 1)-dimensional Date-Jimbo-Kashiwara-Miwa equation is given as

$$\begin{aligned}&u_{xxxxy}+4u_{xxy}u_x+2u_{xxx}u_y+6u_{xy}u_{xx}+u_{yyy}\\&\quad-2u_{xxt}=0.\end{aligned} \tag{24}$$

Applying the wave transformation $u(x,y,t)=V(\xi)$, $\xi=\nu(x+ay-kt)$ to Eq. (24) yields the following nonlinear ODE:

$$a\nu^2V^{(4)}+6a\nu V'V''+\left(a^3+2k\right)V''=0. \tag{25}$$

Set $U=V'$ to get

$$a\nu^2U'''+6a\nu UU'+\left(a^3+2k\right)U'=0, \tag{26}$$

where the integration constant is taken as zero. Balance the linear term of highest order $U'''$ with the highest order nonlinear term $UU'$ in Eq. (26), to get balancing number as $N=2$. Thus, the solution can be written as

$$U(\xi)=a_0+a_1\exp(-\Phi(\xi))+a_2\exp(-\Phi(\xi))^2. \tag{27}$$

Substituting Eq. (27) into Eq. (26) and collecting the coefficient of each power of $\exp(-\Phi(\xi))^i$ and then setting each of coefficient to zero give a system of algebraic equations as

$$\begin{aligned}\exp(-\phi(\xi))^0:\quad&-a\nu^2a_1\mu\lambda^2-6a\nu^2a_2\mu^2\lambda\\&-2a\nu^2a_1\mu^2-a^3a_1\mu-6a\nu a_0a_1\mu\\&-2ka_1\mu=0,\\\exp(-\phi(\xi))^1:\quad&-a^3a_1\lambda-2ka_1\lambda-2a^3a_2\mu\\&-4ka_2\mu-8a\nu^2a_1\mu\lambda-14a\nu^2a_2\mu\lambda^2\\&-12a\nu a_0a_2\mu-6a\nu a_0a_1\lambda-6a\nu a_1{}^2\mu\\&-a\nu^2a_1\lambda^3-16a\nu^2a_2\mu^2=0,\end{aligned}$$

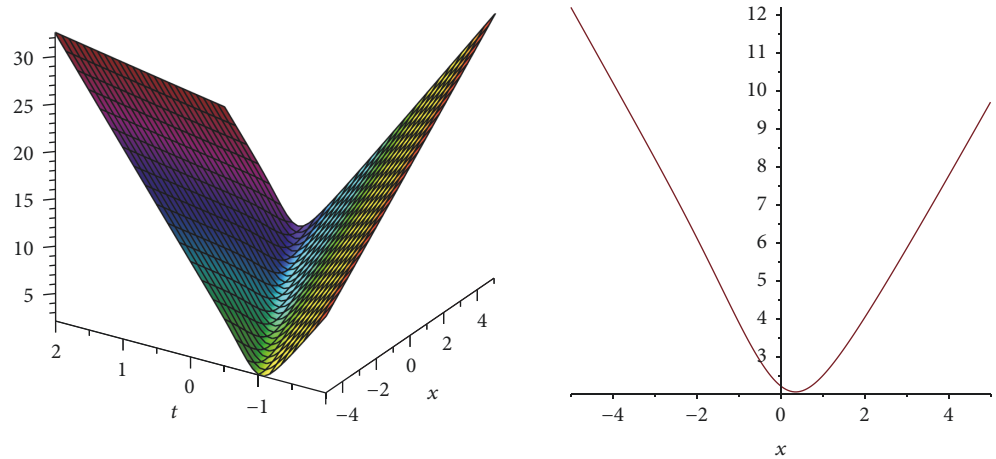


Figure 1: 3D and 2D plot of $|u_1|$ with $\mu = 1$, $\lambda = 2.5$, and $t = 0.002$ for 2D graphics.

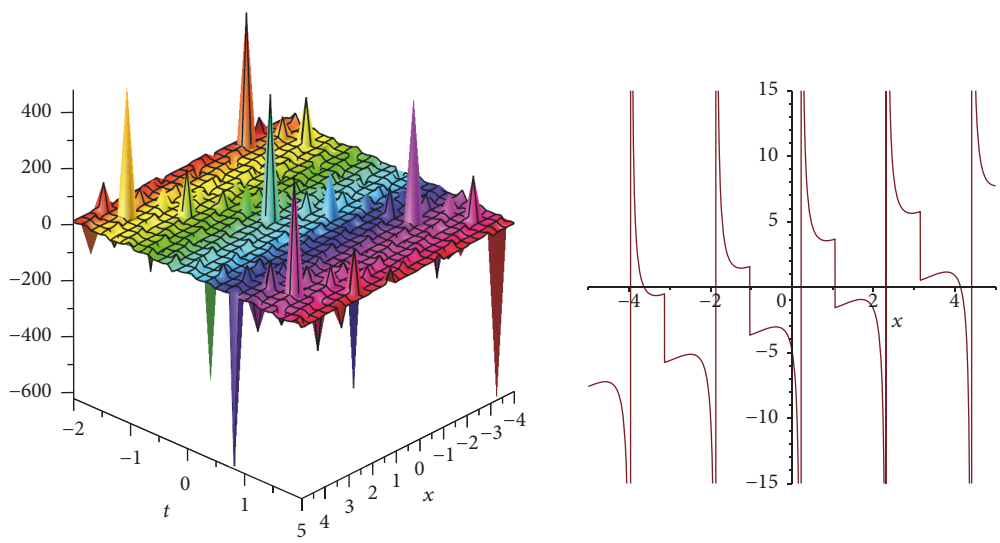


Figure 2: 3D and 2D plot of $u_2$ with $\mu = 2.5$, $\lambda = 1$, and $t = 0.002$ for 2D graphics.

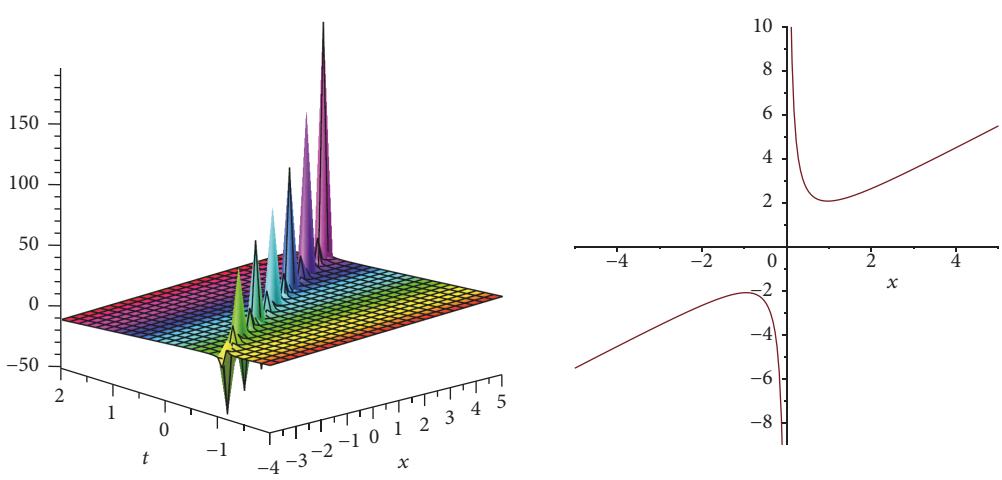


Figure 3: 3D and 2D plot of $u_3$ with $\mu = 0$, $\lambda = 1$, and $t = 0.002$ for 2D graphics.

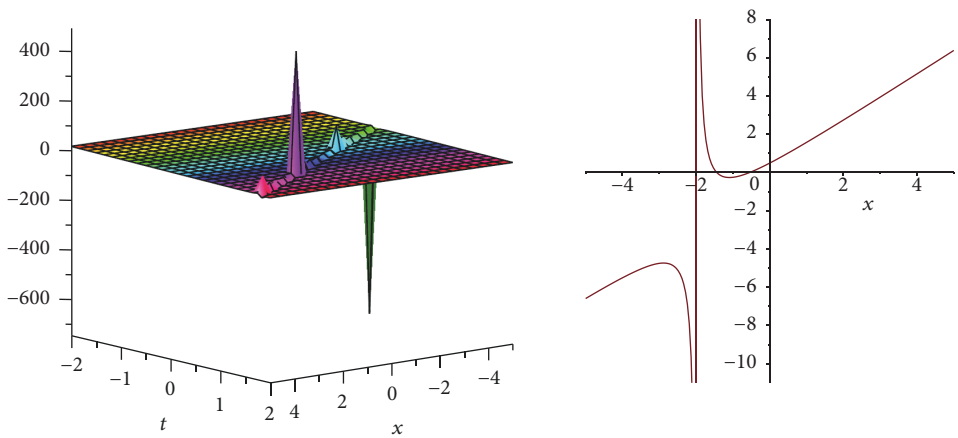


FIGURE 4: 3D and 2D plot of $u_4$ with $\mu = 0.25$, $\lambda = 1$, and $t = 0.002$ for 2D graphics.

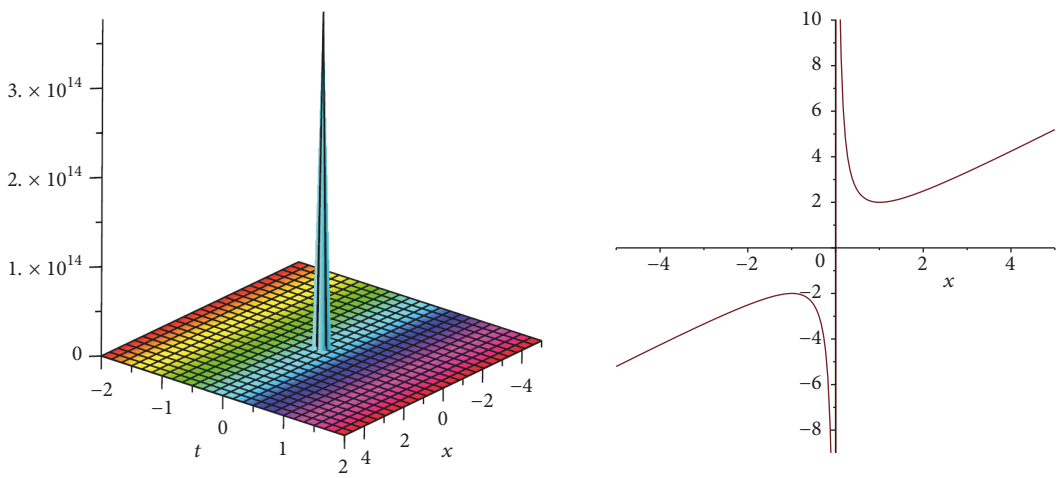


FIGURE 5: 3D and 2D-plot of $u_5$ with $\mu = 0$, $\lambda = 0$, and $t = 0.002$ for 2D graphics.

$$
\begin{aligned}
\exp(-\phi(\xi))^2: \quad & -2a^3a_2\lambda - 4ka_2\lambda - 52a\nu^2a_2\mu\lambda \\
& - 12a\nu a_0a_2\lambda - 18a\nu a_1a_2\mu \\
& - 6a\nu a_0a_1 - 6a\nu {a_1}^2\lambda - 2ka_1 - a^3a_1 \\
& - 7a\nu^2a_1\lambda^2 - 8a\nu^2a_2\lambda^3 - 8a\nu^2a_1\mu \\
& = 0, \\
\exp(-\phi(\xi))^3: \quad & -18a\nu a_1a_2\lambda - 12a\nu {a_2}^2\mu \\
& - 12a\nu a_0a_2 - 2a^3a_2 - 4ka_2 \\
& - 12a\nu^2a_1\lambda - 38a\nu^2a_2\lambda^2 \\
& - 40a\nu^2a_2\mu - 6a\nu {a_1}^2 = 0, \\
\exp(-\phi(\xi))^4: \quad & -6a\nu^2a_1 - 18a\nu a_1a_2 - 12a\nu {a_2}^2\lambda \\
& - 54a\nu^2a_2\lambda = 0, \\
\exp(-\phi(\xi))^5: \quad & -24a\nu^2a_2 - 12a\nu {a_2}^2 = 0.
\end{aligned}
\tag{28}
$$

Solving the above system of equations yields

$$
\begin{aligned}
a_0 &= -\frac{1}{6}\frac{\nu^2a\lambda^2 + 8\nu^2a\mu + a^3 + 2k}{a\nu}, \\
a_1 &= -2\nu\lambda, \\
a_2 &= -2\nu, \\
& \qquad \nu = \nu, \ k = k.
\end{aligned}
\tag{29}
$$

Consequently, the following different cases are obtained for the exact solutions of (2 + 1)-dimensional Date-Jimbo-Kashiwara-Miwa equation.

*Case 1* (hyperbolic function solutions). When $\lambda^2 - 4\mu > 0$ and $\mu \neq 0$,

$$
\begin{aligned}
u_6(\xi) = {} & -\frac{1}{6}\nu\xi\lambda^2 - \frac{4}{3}\nu\xi\mu - \frac{1}{6}\frac{a^2\xi}{\nu} - \frac{1}{3}\frac{\xi k}{a\nu} - 8\frac{\nu\lambda\mu\ln\left(\tanh\left(1/2\sqrt{\lambda^2-4\mu}\,(\xi+K)\right)+1\right)}{\sqrt{\lambda^2-4\mu}\left(2\sqrt{\lambda^2-4\mu}-2\lambda\right)} - 8 \\
& \cdot \frac{\nu\lambda\mu\ln\left(\tanh\left(1/2\sqrt{\lambda^2-4\mu}\,(\xi+K)\right)-1\right)}{\sqrt{\lambda^2-4\mu}\left(2\sqrt{\lambda^2-4\mu}+2\lambda\right)} + 8\frac{\nu\lambda^3\mu\ln\left(\sqrt{\lambda^2-4\mu}\tanh\left(1/2\sqrt{\lambda^2-4\mu}\,(\xi+K)\right)+\lambda\right)}{(\lambda^2-4\mu)\left(\sqrt{\lambda^2-4\mu}+\lambda\right)\left(\sqrt{\lambda^2-4\mu}-\lambda\right)} - 32 \\
& \cdot \frac{\nu\lambda\mu^2\ln\left(\sqrt{\lambda^2-4\mu}\tanh\left(1/2\sqrt{\lambda^2-4\mu}\,(\xi+K)\right)+\lambda\right)}{(\lambda^2-4\mu)\left(\sqrt{\lambda^2-4\mu}+\lambda\right)\left(\sqrt{\lambda^2-4\mu}-\lambda\right)} + 16 \\
& \cdot \frac{\nu\mu^2}{\left(\sqrt{\lambda^2-4\mu}+\lambda\right)\left(\sqrt{\lambda^2-4\mu}-\lambda\right)\left(\sqrt{\lambda^2-4\mu}\tanh\left(1/2\sqrt{\lambda^2-4\mu}\,(\xi+K)\right)+\lambda\right)} - 8 \\
& \cdot \frac{\nu\mu^2\ln\left(\tanh\left(1/2\sqrt{\lambda^2-4\mu}\,(\xi+K)\right)+1\right)}{\sqrt{\lambda^2-4\mu}\left(\sqrt{\lambda^2-4\mu}-\lambda\right)^2} + 8\frac{\nu\mu^2\ln\left(\tanh\left(1/2\sqrt{\lambda^2-4\mu}\,(\xi+K)\right)-1\right)}{\sqrt{\lambda^2-4\mu}\left(\sqrt{\lambda^2-4\mu}+\lambda\right)^2} + 32 \\
& \cdot \frac{\nu\mu^2\ln\left(\sqrt{\lambda^2-4\mu}\tanh\left(1/2\sqrt{\lambda^2-4\mu}\,(\xi+K)\right)+\lambda\right)\lambda^3}{(\lambda^2-4\mu)\left(\sqrt{\lambda^2-4\mu}+\lambda\right)^2\left(\sqrt{\lambda^2-4\mu}-\lambda\right)^2} - 128 \\
& \cdot \frac{\nu\mu^3\ln\left(\sqrt{\lambda^2-4\mu}\tanh\left(1/2\sqrt{\lambda^2-4\mu}\,(\xi+K)\right)+\lambda\right)\lambda}{(\lambda^2-4\mu)\left(\sqrt{\lambda^2-4\mu}+\lambda\right)^2\left(\sqrt{\lambda^2-4\mu}-\lambda\right)^2}.
\end{aligned}
\tag{30}
$$

*Case 2* (trigonometric function solutions). When $\lambda^2 - 4\mu < 0$ and $\mu \neq 0$,

$$
\begin{aligned}
u_7(\xi) = {} & -\frac{1}{6}\nu\xi\lambda^2 - \frac{4}{3}\nu\xi\mu - \frac{1}{6}\frac{b^2\xi}{\nu} - \frac{1}{3}\frac{\xi k}{b\nu} - 2\nu\lambda \\
& \cdot \ln\left(\sqrt{4\mu-\lambda^2}\tan\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\,(\xi+K)\right)-\lambda\right) + 4 \\
& \cdot \frac{\nu\mu\arctan\left(\tan\left(1/2\sqrt{4\mu-\lambda^2}\,(\xi+K)\right)\right)}{\sqrt{4\mu-\lambda^2}} + 4 \\
& \cdot \frac{\nu\mu}{\sqrt{4\mu-\lambda^2}\tan\left(1/2\sqrt{4\mu-\lambda^2}\,(\xi+K)\right)-\lambda} + 8 \\
& \cdot \frac{\nu\lambda\mu\ln\left(\sqrt{4\mu-\lambda^2}\tan\left(1/2\sqrt{4\mu-\lambda^2}\,(\xi+K)\right)-\lambda\right)}{4\mu-\lambda^2} \\
& - 2 \\
& \cdot \frac{\nu\lambda^3\ln\left(\sqrt{4\mu-\lambda^2}\tan\left(1/2\sqrt{4\mu-\lambda^2}\,(\xi+K)\right)-\lambda\right)}{4\mu-\lambda^2}.
\end{aligned}
\tag{31}
$$

*Case 3* (hyperbolic function solutions). When $\lambda^2 - 4\mu > 0$, $\mu = 0$, and $\lambda \neq 0$,

$$
\begin{aligned}
u_8(\xi) = {} & -\frac{1}{6}\nu\xi\lambda^2 - \frac{4}{3}\nu\xi\mu - \frac{1}{6}\frac{b^2\xi}{\nu} - \frac{1}{3}\frac{\xi k}{b\nu} \\
& + 2\nu\ln\left(\tanh\left(\frac{1}{2}\lambda\xi + \frac{1}{2}\lambda K\right)+1\right) \\
& - 2\nu\ln\left(\tanh\left(\frac{1}{2}\lambda\xi + \frac{1}{2}\lambda K\right)\right) \\
& + \frac{\nu}{\lambda\tanh(1/2\lambda\xi + 1/2\lambda K)} \\
& - 2\frac{\nu\ln(\tanh(1/2\lambda\xi + 1/2\lambda K)+1)}{\lambda} \\
& + 2\frac{\nu\ln(\tanh(1/2\lambda\xi + 1/2\lambda K))}{\lambda}.
\end{aligned}
\tag{32}
$$

*Case 4* (rational function solutions). When $\lambda^2 - 4\mu = 0$, $\mu \neq 0$, and $\lambda \neq 0$,

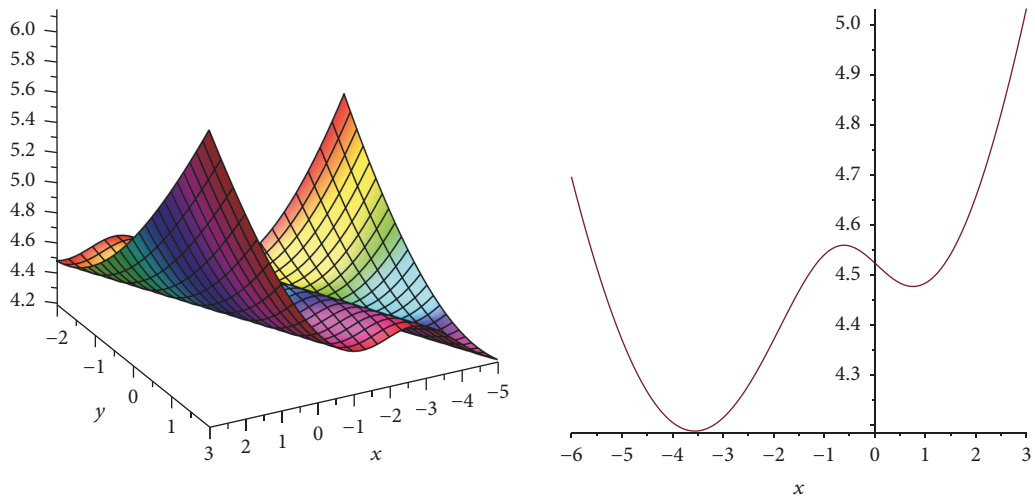


FIGURE 6: W-shaped soliton 3D and 2D plot of $|u_6|$ with $\mu = 1$, $\lambda = 2.5$, and $y = 0.002$ for 2D graphics.

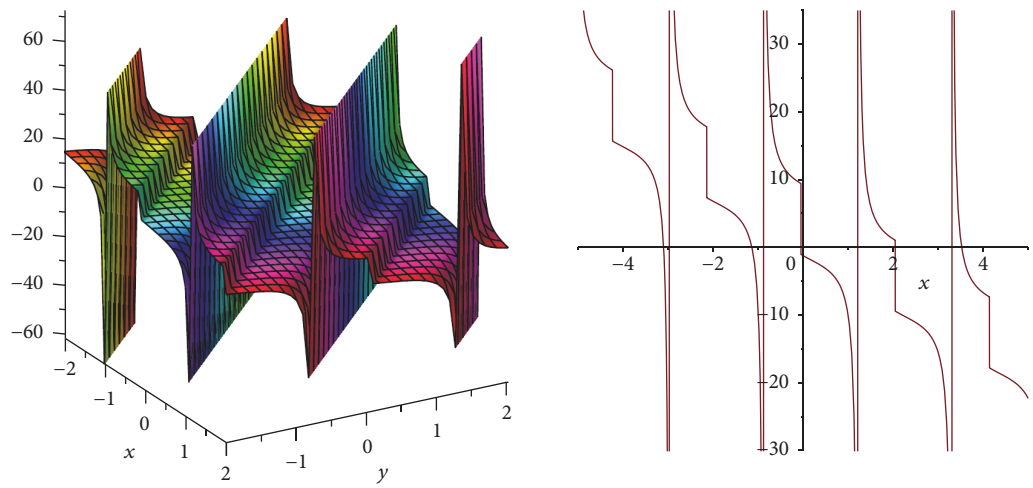


FIGURE 7: 3D and 2D plot of $u_7$ with $\mu = 2.5$, $\lambda = 1$, and $y = 0.002$ for 2D graphics.

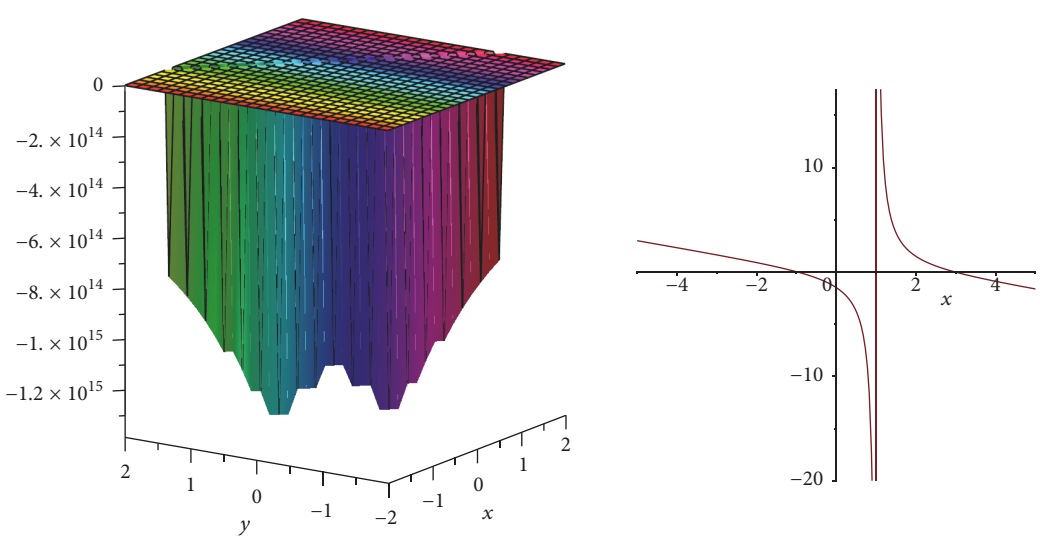


FIGURE 8: 3D and 2D plot of $u_8$ with $\mu = 0$, $\lambda = 1$, and $y = 0.002$ for 2D graphics.

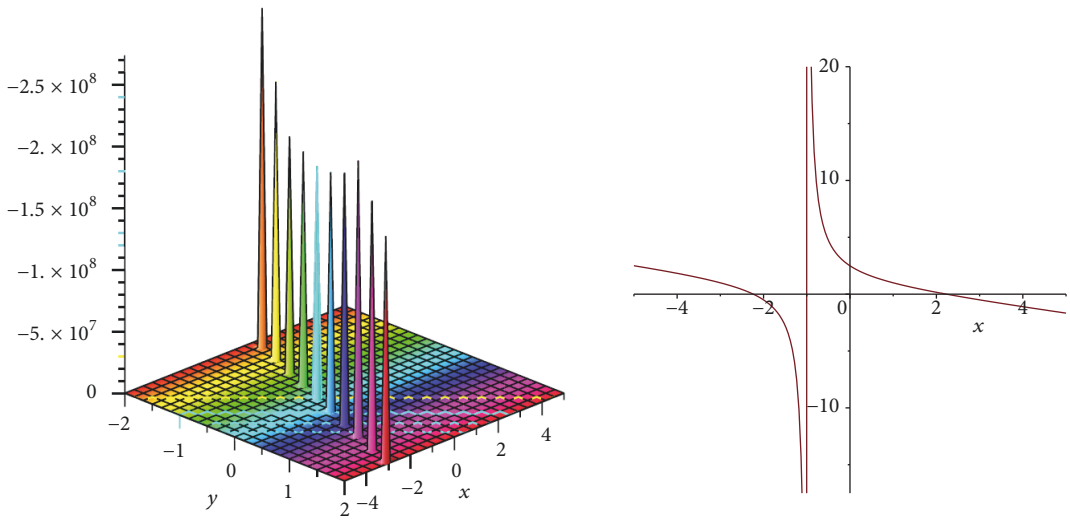


Figure 9: 3D and 2D plot of $u_9$ with $\mu = 0.25$, $\lambda = 1$, and $y = 0.002$ for 2D graphics.

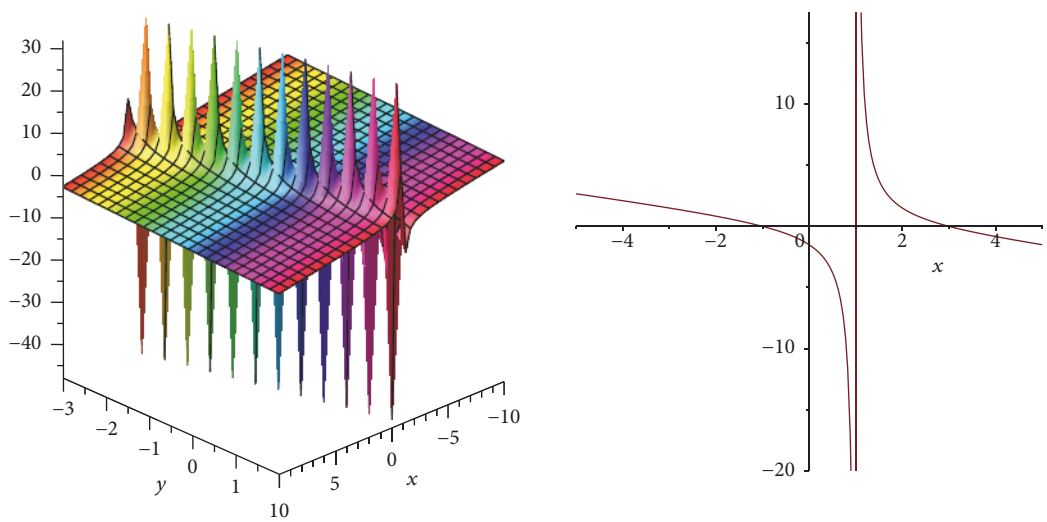


Figure 10: 3D and 2D plot of $u_{10}$ with $\mu = 0$, $\lambda = 0$, and $y = 0.002$ for 2D graphics.

$$u_9(\xi) = -\frac{1}{6}\frac{\left(\nu^2 b\lambda^2 + 8\nu^2 b\mu + b^3 + 2k\right)\xi}{b\nu} + \frac{1}{2}\nu\lambda^2\xi + 2\frac{\nu\lambda}{\lambda\xi + \lambda K + 2}. \tag{33}$$

*Case 5*. When $\lambda^2 - 4\mu = 0$, $\mu = 0$, and $\lambda = 0$,

$$u_{10}(\xi) = -\frac{1}{6}\nu\xi\lambda^2 - \frac{4}{3}\nu\xi\mu - \frac{1}{6}\frac{b^2\xi}{\nu} - \frac{1}{3}\frac{\xi k}{b\nu} - 2\nu\lambda\ln(\xi + K) + 2\frac{\nu}{\xi + K}, \tag{34}$$

where $\xi = \nu(x + ay - kt)$ and $K$ is the integration constant.

The graphical representation of the solutions $(u_6, u_7, u_8, u_9, u_{10})$ for the (2 + 1)-dimensional Date-Jimbo-Kashiwara-Miwa equation is shown by Figures 6–10.

## 4. Results and Discussion

In this manuscript, several traveling wave solutions are developed. A new kind of W-shaped soliton solution is demonstrated and the other obtained solutions consist of trigonometric, hyperbolic, rational functions which are also new. On comparing our results with the well-known results obtained in [1, 3], it may be concluded that the obtained results for Eq. (13) and Eq. (24) are newly constructed. Wazwaz [1] reported some multiple soliton solutions to Eq. (13) using simplified Hirota's method and Yuan *et al.* [2] acquired

N-soliton solutions to Eq. (24) using Hirota method and auxiliary variables. In both equations for case 1, the absolute behavior of the solution $|u_1(x,t)|$ and $|u_6(x,t)|$ is shown in Figures 1 and 6, respectively. The graphical representation of the other solutions $(u_2, u_3, u_4, u_5)$ for the $(1+1)$-dimensional fifth-order nonlinear integrable equation and $(u_7, u_8, u_9, u_{10})$ for the $(2+1)$-dimensional Date-Jimbo-Kashiwara-Miwa equation is shown by Figures 2–5 and 7–10, respectively.

## 5. Conclusion

The $\exp(-\Phi(\xi))$ method is used to investigate the exact solutions of the $(1+1)$-dimensional fifth-order nonlinear integrable equation and $(2+1)$-dimensional Date-Jimbo-Kashiwara-Miwa equation. With the implementation of the $\exp(-\Phi(\xi))$ method, many exact traveling wave solutions are obtained including trigonometric, hyperbolic, rational, and W-shaped soliton. The reported solutions in this article may be useful in explaining the physical meaning of the studied models and other nonlinear models arising in the field of fiber optics and other related fields. These results indicate that the proposed method is very useful and effective in performing solution to the NLEEs.

## Data Availability

The data used to support the findings of this study are included within the article.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

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