Research Article

Novel Delay-Dependent Stability Criteria for Discrete-Time Neural Networks with Time-Varying Delay

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The delay-dependent stability problem is investigated for discrete-time neural networks with time-varying delays. A new augmented Lyapunov-Krasovskii functional (LKF) with single and double summation terms and several augmented vectors is proposed by decomposing the time-delay interval into two nonequidistant subintervals to derive less conservative stability conditions. Then, by using Wirtinger-based inequality, reciprocally, and extended reciprocally convex combination lemmas, tight estimations for sum terms in the forward difference of the LKF are given. Several zero equalities are introduced to further relax the existing results. Less conservative stability criteria are proposed in terms of linear matrix inequalities (LMIs). Finally, numerical examples are proposed to show the effectiveness and less conservativeness of the proposed method.

1. Introduction

During the past few decades, neural networks (NNs) have received great attention because of their wide applications in various fields such as image processing, signal processing, pattern recognition, associative memory, parallel computation, optimization, and error diagnosis [1, 2]. One of the most important questions in theoretical analysis of NNs is dynamical behaviors of the NNs, such as their stability, periodic oscillatory, and chaos. Among them, analysis of the stability has received much attention and various stability conditions have been obtained in [3, 4].

It is well known that a time delay is inherent in various systems, including NNs, owing to the finite speed of signal transmission and conversion rate of the processors. Delays in a system may cause oscillation and divergence and further degrade the performance. It is of great importance to determine the admissible maximal delay bound such that the neural networks with a delay less than this upper bound

remain stable. Hence, it is essential to investigate the stability of the neural networks with time delay [5–10].

Since most systems use a digital processor to acquire information from computers at discrete instants of time, it is essential to formulate discrete-time neural networks (DNNs) that are an analogue of continuous ones. Therefore, it is significant to study the dynamics of the DNNs and many results for the DNNs with time delay has received increasing attention; see [11–16] and references therein.

The stability criteria for the DNNs with time-varying delay can be classified into two categories: delay-independent stability criteria and delay-dependent ones [17, 18]. As is well known, delay-dependent stability criteria, which take advantage of the information on the size of time delays, are less conservative than delay-independent ones. The main aim of delay-dependent stability criteria is to get maximum delay bounds guaranteeing the addressed neural networks to be stable. In order to improve results regarding this problem, various methods have been applied to the delay-dependent

category, such as augmented LKF method [13, 19–22], free-weighting matrix method [18, 23], summation inequality method [16, 24–27], delay-partitioning method [5, 28, 29], and reciprocally convex approach [20, 30, 31].

It is known that the simple LKFs with the single and/or double sum terms lead to the criteria with high conservatism, while more augmented LKFs reduce conservatism but increase the number of decision variables. By constructing an augmented LKF with the sum terms of a state vector as well as the activation function, an improved stability condition for the asymptotic stability of DNNs was derived in [13] but with an enlarged number of decision variables. In recent years, the LKF including triple summation terms has also been applied to study the stability analysis of discrete-time DNNs for a further improvement of the results [19, 32]. For the purpose of reducing conservatism, the delay-partitioning LKFs, in which the delay interval was separated into several subintervals, were introduced [5, 28, 29].

Various free-weighting matrix-based stability criteria [18, 23] were established to improve the results obtained by the Jensen-based summation inequality. However, introducing the free-weighting matrices may result in increased number of decision variables.

After the computational complexity became one of the crucial aspects of a research in the area of the system stability, the direct bounding method based on summation inequalities once again becomes the most popular method [17, 19, 32, 33]. Very recently, various types of the Wirtinger-based summation inequalities, tighter than the Jensen-based summation inequality, have been proposed for discrete-time linear time-delay systems [25, 34, 35] and have also been used for the study of the discrete-time DNNs [36]. As one of useful methods to deal with the stability of delayed systems, the reciprocally convex approach was developed in [30] and has been extensively used to study the dynamical behaviours of time-delay systems since then (see [32, 37–39] and references therein).

Not long ago, an extended reciprocally convex combination lemma (ERCCL) has been developed [26, 31] to replace the popular reciprocally convex combination lemma (RCCL). It has a potential to reduce the conservatism of the RCCL-based criteria without introducing any extra decision variable due to its advantage of reduced estimation gap using the same decision variables.

Motivated by the aforementioned discussions, in this paper, the problem of delay-dependent stability for DNNs with interval time-varying delay is considered. The objective of the paper is to derive simple but efficient stability conditions for DNNs with interval time-varying delay. The major contributions of this paper are as follows. Delay-decomposition method is used with new augmented Lyapunov–Krasovskii functional, which contains several augmented vectors in single and double summation terms with a free parameter α that divides interval $[k - h_2, k - h_1]$ into two asymmetric subintervals $[k - h_2, k - \alpha - 1]$ and $[k - \alpha, k - h_1]$. The value of parameter α is determined by solving linear matrix inequalities (LMIs) that are defined in the proposed stability criterion. In this way, a greater degree of freedom is enabled in estimating the stability of

the DDN, which leads to a smaller conservative stability criterion. The convenient existing summations inequalities, as well as extended reciprocally convex combination lemma, have been used for calculation difference of LKF. In order to improve the stability criteria, the LKF is extended with an additional double summation term and zero equalities (ZEs) are introduced in calculation difference of LKF. As a result of applying the mentioned techniques, it is shown that the derived results are less conservative than the existing ones [24, 31, 40, 41].

Notations. Throughout the paper, $\mathbb{Z}^+$ denotes the set of positive integers and $\mathfrak{R}^n$ denotes the n -dimensional Euclidean space and $\mathfrak{R}^{n \times m}$ the set of all $n \times m$ real matrices. For the positive integers a and b ($b > a$), $\mathbb{Z}[a, b]$ denotes the set of all positive integers z satisfying $a \leq z \leq b$. I_n and $0_{n \times m}$ denote $n \times n$ identity matrix and $n \times m$ zero matrix, respectively. X^T denotes the matrix transpose of X and $*$ represent the elements below the main diagonal of a symmetric matrix. $\text{diag}\{a, b, \dots, z\}$ denotes the block-diagonal matrix with elements $a, b, \dots, z$ in the diagonal entries and $\text{Sym}\{X\} = X + X^T$. For any symmetric matrix $X \in \mathfrak{R}^{n \times n}$ the notation $X > 0$ ($X \geq 0$) means that X is positive definite (positive semidefinite) matrix and $X < 0$ ($X \leq 0$) means that X is negative definite (negative semidefinite) matrix. For the matrices $A_i \in \mathfrak{R}^{n \times m}$, $i = 1, 2, \dots, l$, $\text{Col}\{A_1, A_2, \dots, A_l\}$ denotes the column block matrix $[A_1^T A_2^T \dots A_l^T]^T$. Matrices, if their dimensions are not explicitly stated, are assumed to be compatible for algebraic operations.

2. Preliminaries

Consider the discrete-time neural networks (DNNs) with interval time-varying delay of the form:

$$\begin{aligned} y(k+1) &= Cy(k) + Ag(y(k)) + A_d g(y(k-h(k))) \\ &\quad + J, \quad k \geq 0, \\ y(j) &= \varphi(j), \quad j \in \{-h_2, -h_2+1, \dots, -1, 0\}, \end{aligned} \quad (1)$$

where $k \in \mathbb{Z}^+$, $y(k) \in \mathfrak{R}^n$ is the neural state vector, $g(y(k)) = [g_1(y_1(k)), g_2(y_2(k)), \dots, g_n(y_n(k))]^T$ denotes the neural activation functions, $\varphi(j)$ is the initial condition, $C = \text{diag}\{c_1, c_2, \dots, c_n\}$, $c_i > 0$ represents the state feedback coefficient matrix, i.e., the rate with which the i -th neuron will reset its potential to the resting state in isolation when disconnected from the networks and external inputs, $A \in \mathfrak{R}^{n \times n}$ and $A_d \in \mathfrak{R}^{n \times n}$ are the connection weight matrix and the delay connection weight matrix, respectively, and $J = [J_1, J_2, \dots, J_n]^T$ is the external input vector. Integer $h(k)$ is time-varying delay that satisfies

$$0 < h_1 \leq h(k) \leq h_2, \quad (2)$$

where h_1 and h_2 are known positive integers representing the lower and upper bounds of $h(k)$.

In this paper, the following assumption on the system (1) is made.

Assumption 1 (see [18]). For any $a, b \in \mathfrak{R}$ and $a \neq b$, the continuous and bounded activation functions $g_i(\cdot)$ satisfy

$$k_i^- \leq \frac{g_i(a) - g_i(b)}{a - b} \leq k_i^+, \quad i = 1, 2, \dots, n, \quad (3)$$

where k_i^- and k_i^+ are known constants.

Since the activation function $g_i(\cdot)$ satisfies (3), the system (1) has an equilibrium point. Let us denote the equilibrium point by $y^* = [y_1^*, \dots, y_n^*]^T$, and then $y^* = Cy^* + Ag(y^*) + A_d g(y^*) + J$. In order to simplify our consideration, we shift the equilibrium point of the system (1) to the origin. Let $x_i(k) = y_i(k) - y_i^*$ and $f_i(x_i(k)) = g_i(x_i(k) + y_i^*) - g_i(y_i^*)$. Then, system (1) can be transformed to the following form:

$$x(k+1) = Cx(k) + Af(x(k)) + A_d f(x(k-h(k))), \quad k \geq 0, \quad (4)$$

$$x(j) = \phi(j), \quad j \in \{-h_2, -h_2 + 1, \dots, -1, 0\},$$

where $x(k) \in \mathfrak{R}^n$ is the state vector of the transformed system and $\phi(j)$ is the initial condition.

After applying Assumption 1, one can check that the function $f(\cdot)$ with $f(0) = 0$ satisfies

$$k_i^- \leq \frac{f_i(a) - f_i(b)}{a - b} \leq k_i^+, \quad i = 1, 2, \dots, n, \quad (5)$$

for all $a \neq b$. If $f = 0$, we have

$$k_i^- \leq \frac{f_i(a)}{a} \leq k_i^+, \quad i = 1, 2, \dots, n, \quad (6)$$

for all $a \neq 0$.

The following lemmas will be used in the sequel to establish the main results.

Lemma 2 (Jensen's inequality [42]). For any positive definite symmetric matrix $R \in \mathfrak{R}^{n \times n}$ and two positive integers a and $b > a$, the sum term $\sum_{j=a}^{b-1} x^T(j)Rx(j)$ is estimated as

$$(b-a) \sum_{j=a}^{b-1} x^T(j)Rx(j) \geq \sum_{j=a}^{b-1} x^T(j)R \sum_{j=a}^{b-1} x(j). \quad (7)$$

Lemma 3 (Wirtinger-based inequality [25]). For a given positive definite symmetric matrix R , two positive integers a and $b > a$, and any sequence of discrete-time variable $x : \mathbb{Z}[a, b] \rightarrow \mathfrak{R}^n$, the following inequality holds:

$$(b-a) \sum_{i=a}^{b-1} \eta^T(i)R\eta(i) \geq \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}^T \begin{bmatrix} R & 0 \\ * & 3\rho(a, b)R \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} \quad (8)$$

$$\geq \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}^T \begin{bmatrix} R & 0 \\ * & 3R \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}, \quad (9)$$

where

$$\eta(k) = x(k+1) - x(k),$$

$$\theta_1 = x(b) - x(a),$$

$$\theta_2 = x(b) + x(a) - \frac{2}{b-a+1} \sum_{i=a}^b x(i), \quad (10)$$

$$\rho(a, b) = \begin{cases} \frac{(b-a+1)}{(b-a-1)}, & b-a \neq 1 \\ 1, & b-a = 1. \end{cases}$$

Remark 4. In [27], stability criteria for discrete-time delay systems are considered by using a new summation inequality with three slack matrix variables. It has been shown that inequality (8) is obtained as a special case of the new summation inequality with convenient selected values of the slack matrix variables. Therefore, by using the mentioned summation inequality, the conservatism can be reduced, but the introduction of three slack matrix variables can increase the computation complexity of the obtained stability criterion.

The reciprocally convex combination lemma (RCCL) plays an important role in the estimation of the forward difference of LKF.

Lemma 5 (reciprocally convex combination lemma [30, 31, 43]). For a real scalar $\delta \in (0, 1)$, a symmetric matrix $R > 0$, and any matrix S satisfying

$$\begin{bmatrix} R & S \\ * & R \end{bmatrix} \geq 0, \quad (11)$$

the following inequality holds:

$$\begin{bmatrix} \frac{1}{\delta}R & 0 \\ \delta & \frac{1}{1-\delta}R \end{bmatrix} \geq \begin{bmatrix} R & S \\ * & R \end{bmatrix}. \quad (12)$$

An extended reciprocally convex combination lemma (ERCCL), which estimates the sum terms in the forward difference of the LKF tightly than RCCL, is presented as follows.

Lemma 6 (extended reciprocally convex combination lemma [31, 40]). For a real scalar $\delta \in (0, 1)$, a symmetric matrix $R > 0$, and any matrix S , the following inequality holds:

$$\begin{bmatrix} \frac{1}{\delta}R & 0 \\ \delta & \frac{1}{1-\delta}R \end{bmatrix} \geq \begin{bmatrix} R + (1-\delta)T_1 & S \\ * & R + \delta T_2 \end{bmatrix}, \quad (13)$$

where $T_1 = R - SR^{-1}S^T$ and $T_2 = R - S^T R^{-1}S$.

Remark 7. Unlike RCCL, the condition (11) is not used in ERCCL. Thus, the matrix S in ERCCL can be chosen more freely than the one in RCCL such that the feasibility criterion

based on (13) is better than one based on (11). Further, if (11) holds then $T_i \geq 0$, $i = 1, 2$, and

$$\begin{bmatrix} R + (1 - \delta) T_1 & S \\ * & R + \delta T_2 \end{bmatrix} \geq \begin{bmatrix} R & S \\ * & R \end{bmatrix}. \quad (14)$$

Thus, the terms $(1 - \delta)T$ and δT_2 , which appear in (13), reduce estimation gap between two sides of (13) and inequality (13) in ERCCL is tighter than inequality (12) in RCCL. Note that the RCCL and ERCCL require the same number of decision variables (R and S).

Remark 8. In [20], the specific reciprocally convex inequality is proposed. For comparison, we rewrite the result of Theorem 1 in [20] in a block form as

$$\begin{bmatrix} \frac{1}{\delta} R_1 & 0 \\ * & \frac{1}{1 - \delta} R_2 \end{bmatrix} \geq \begin{bmatrix} R_1 + (1 - \delta) T_1 & \delta S_1 + (1 - \delta) S_2 \\ * & R_2 + \delta T_2 \end{bmatrix}, \quad (15)$$

where $T_1 = R_1 - Y_1 R_2^{-1} Y_1^T$ and $T_2 = R_2 - Y_2^T R_1^{-1} Y_2$. If taking $R_1 = R_2 = R$ and $S_1 = S_2 = S$, inequality (15) immediately reduces to (13), which means that ERCCL (Lemma 6) is a special case of Theorem 1 in [20]. During the calculation of

the difference of LKF in (31), the inequality (13) is used instead of (15), because the same matrix $R = \bar{Z}_2$ appears twice on the block diagonal in (31), so that the condition $R_1 = R_2 = R$ is automatically satisfied.

Further, by applying inequality (13) to (32), only one matrix S was used instead of two (S_1 and S_2), as in the case of application of inequality (15). Thereby, a computation complexity of the proposed stability criterion (13) is decreased, but there is a possibility of increasing conservatism (13) in regard to (15).

Lemma 9 (see [40]). *For a positive definite symmetric matrix R and matrices Y and Ξ , the following statements are equivalent:*

- (i) $\Xi - Y^T R Y < 0$
- (ii) *There exists a matrix Ψ with appropriate dimension such that*

$$\begin{bmatrix} \Xi + \text{Sym} \{ Y^T \Psi \} & \Psi^T \\ * & -R \end{bmatrix} < 0. \quad (16)$$

3. Main Results

In this section, we will establish a new criterion on asymptotical stability for the discrete-time neural networks with interval time-varying delay (1).

We introduce the following notations for later use:

$$\xi(k)$$

$$= \begin{bmatrix} x^T(k) & x^T(k - h_1) & x^T(k - h(k)) & x^T(k - h_2) & x^T(k - \alpha) & v_1^T(k) & v_2^T(k) & v_3^T(k) & v_4^T(k) & f^T(x(k)) & f^T(x(k - h(k))) \end{bmatrix}^T,$$

$$\eta(k) = x(k + 1) - x(k),$$

$$\zeta_1(k) = \left[x^T(k) \sum_{j=k-h_1}^{k-1} x^T(j) \sum_{j=k-h_2}^{k-h_1-1} x^T(j) \sum_{j=k-\alpha}^{k-h_1-1} x^T(j) \right]^T,$$

$$\zeta_2(k) = \begin{bmatrix} x^T(k) & f^T(x(k)) \end{bmatrix}^T,$$

$$\xi_3(k) = \begin{bmatrix} x^T(k) & \eta^T(k) \end{bmatrix}^T,$$

$$v_1(k) = \sum_{j=k-h_1}^k \frac{x(j)}{h_1 + 1},$$

$$v_2(k) = \sum_{j=k-h(k)}^{k-h_1} \frac{x(j)}{h(k) - h_1 + 1},$$

$$v_3(k) = \sum_{j=k-h_2}^{k-h(k)} \frac{x(j)}{h_2 - h(k) + 1},$$

$$v_4(k) = \sum_{j=k-\alpha}^{k-h_1} \frac{x(j)}{\alpha - h_1 + 1},$$

$$\omega_1 = (e_1 - e_2) \xi(k),$$

$$\omega_2 = (e_1 + e_2 - 2e_6) \xi(k),$$

$$\omega_3 = (e_2 - e_5) \xi(k),$$

$$\omega_4 = (e_2 + e_5 - 2e_9) \xi(k),$$

$$e_i = [0_{n \times (i-1)n}, I_n, 0_{n \times (11-i)n}], \quad i = 1, 2, \dots, 11,$$

$$e_s = Ce_1 + Ae_{10} + A_d e_{11},$$

$$\rho(a, b) = \begin{cases} \frac{b-a+1}{b-a-1}, & b-a > 1 \\ 1, & b-a = 1, \end{cases}$$

$$\delta_h = \frac{h(k) - h_1}{h_{12}}, \quad h_{12} = h_2 - h_1,$$

$$\Sigma(h(k), \alpha) = \Sigma_{11} + \Sigma_{12}(h(k), \alpha) + \Sigma_2 + \Sigma_3 + \Sigma_{41} - \Sigma_{42} - \Sigma_{43}(h(k)) + \Sigma_{51}(\alpha) - \Sigma_{52}(\alpha) - \Sigma_S,$$

$$\Sigma_{11} = \Pi_1^T P \Pi_1 - \Pi_2^T P \Pi_2,$$

$$\Pi_1 = \begin{bmatrix} e_s \\ (h_1 + 1)e_6 - e_2 \\ -e_3 - e_4 \\ -e_5 \end{bmatrix},$$

$$\Pi_2 = \begin{bmatrix} e_1 \\ (h_1 + 1)e_6 - e_1 \\ -e_2 - e_3 \\ -e_2 \end{bmatrix},$$

$$\Sigma_{12}(h(k), \alpha) = \text{Sym} \{ (\Pi_1 - \Pi_2)^T P \Pi(h(k), \alpha) \},$$

$$\Pi(h(k), \alpha) = \begin{bmatrix} 0 \\ 0 \\ (h(k) - h_1 + 1)e_7 + (h_2 - h(k) + 1)e_8 \\ (\alpha - h_1 + 1)e_9 \end{bmatrix},$$

$$\Sigma_2 = e_1^T Q_1 e_1 + e_2^T (Q_3 - Q_1) e_2 + e_5^T (Q_2 - Q_3) e_5 - e_4^T Q_2 e_4,$$

$$\Sigma_3 = \begin{bmatrix} e_1 \\ e_{10} \end{bmatrix}^T (h_{12} + 1) R \begin{bmatrix} e_1 \\ e_{10} \end{bmatrix} - \begin{bmatrix} e_3 \\ e_{11} \end{bmatrix}^T R \begin{bmatrix} e_3 \\ e_{11} \end{bmatrix},$$

$$\Sigma_4(h(k)) = \Sigma_{41} - \Sigma_{42} - \Sigma_{43}(h(k)),$$

$$\Sigma_{41} = (e_s - e_1)^T (h_1^2 Z_1 + h_{12}^2 Z_2) (e_s - e_1),$$

$$\Sigma_{42} = E_1^T \begin{bmatrix} Z_1 & 0 \\ * & 3\rho(0, h_1) Z_1 \end{bmatrix} E_1, \quad E_1 = \begin{bmatrix} e_1 - e_2 \\ e_1 + e_2 - 2e_6 \end{bmatrix},$$

$$\Sigma_{43}(h(k)) = \begin{bmatrix} E_2 \\ E_3 \end{bmatrix}^T \left(\begin{bmatrix} \tilde{Z}_2 & S \\ * & \tilde{Z}_2 \end{bmatrix} + \begin{bmatrix} (1 - \delta_h) T_1 & 0 \\ * & \delta_h T_2 \end{bmatrix} \right) \begin{bmatrix} E_2 \\ E_3 \end{bmatrix},$$

$$T_1 = \tilde{Z}_2 - S \tilde{Z}_2^{-1} S^T,$$

$$T_2 = \tilde{Z}_2 - S^T \tilde{Z}_2^{-1} S,$$

$$\tilde{\Sigma}_{43}(h_1) = \begin{bmatrix} E_2 \\ E_3 \end{bmatrix}^T \begin{bmatrix} \tilde{Z}_2 & S \\ * & \tilde{Z}_2 \end{bmatrix} \begin{bmatrix} E_2 \\ E_3 \end{bmatrix} + E_2^T \tilde{Z}_2 E_2,$$

$$\begin{aligned}
\bar{\Sigma}_{43}(h_2) &= \begin{bmatrix} E_2 \\ E_3 \end{bmatrix}^T \begin{bmatrix} \bar{Z}_2 & S \\ * & \bar{Z}_2 \end{bmatrix} \begin{bmatrix} E_2 \\ E_3 \end{bmatrix} + E_3^T \bar{Z}_2 E_3, \\
\bar{Z}_2 &= \begin{bmatrix} Z_2 & 0 \\ * & 3Z_2 \end{bmatrix}, \\
E_2 &= \begin{bmatrix} e_2 - e_3 \\ e_2 + e_3 - 2e_7 \end{bmatrix}, \\
E_3 &= \begin{bmatrix} e_3 - e_4 \\ e_3 + e_4 - 2e_8 \end{bmatrix}, \\
E_4 &= \begin{bmatrix} e_2 - e_5 \\ e_2 + e_5 - 2e_9 \end{bmatrix}, \\
E_5(h(k)) &= \begin{bmatrix} (h(k) - h_1 + 1)e_7 - e_2 \\ e_2 - e_3 \end{bmatrix}, \\
E_6(h(k)) &= \begin{bmatrix} (h_2 - h(k) + 1)e_8 - e_3 \\ e_3 - e_4 \end{bmatrix}, \\
E_{56}(h(k)) &= \begin{bmatrix} E_5(h(k)) \\ E_6(h(k)) \end{bmatrix}, \\
\Sigma_{51}(\alpha) &= (e_s - e_1)^T (\alpha - h_1)^2 Z_3 (e_s - e_1), \\
\Sigma_{52}(\alpha) &= E_4^T \begin{bmatrix} Z_3 & 0 \\ * & 3\rho(h_1, \alpha) Z_3 \end{bmatrix} E_4, \\
\Sigma_S &= \text{Sym} \left\{ (e_{10} - K_1 e_1)^T M_1 (e_{10} - K_2 e_1) \right\} + \text{Sym} \left\{ (e_{11} - K_1 e_3)^T M_2 (e_{11} - K_2 e_3) \right\} + \text{Sym} \left\{ \begin{array}{l} (e_{10} - e_{11} - K_1 (e_1 - e_3))^T M_3 \times \\ \times (e_{10} - e_{11} - K_2 (e_1 - e_3)) \end{array} \right\}, \\
\Sigma_6 &= \Sigma_{61} - \Sigma_{62}(h(k)), \\
\Sigma_{61} &= h_{12}^2 \begin{bmatrix} e_1 \\ e_s - e_1 \end{bmatrix}^T G \begin{bmatrix} e_1 \\ e_s - e_1 \end{bmatrix} + h_{12} e_2^T H_1 e_2 - h_{12} e_4^T H_2 e_4 - h_{12} e_3^T (H_1 - H_2) e_3, \\
\Sigma_{62}(h(k)) &= E_{56}^T(h(k)) G_{12} E_{56}(h(k)), \\
G_i &= G + \begin{bmatrix} 0 & H_i \\ * & H_i \end{bmatrix}, \quad i = 1, 2, \\
G_{12} &= \begin{bmatrix} G_1 & X \\ * & G_2 \end{bmatrix}, \\
K_1 &= \text{diag} \{k_1^-, k_2^-, \dots, k_n^-\}, \\
K_2 &= \text{diag} \{k_1^+, k_2^+, \dots, k_n^+\}.
\end{aligned} \tag{17}$$

Theorem 10. For given positive integers h_1 and h_2 , system (4) with interval time-varying delay satisfying condition (2) is asymptotically stable, if there exist positive-definite matrices $P \in \mathfrak{R}^{4n \times 4n}$, $Q_i \in \mathfrak{R}^{n \times n}$, $i = 1, 2, 3$, $R \in \mathfrak{R}^{2n \times 2n}$, $Z_i \in \mathfrak{R}^{n \times n}$, $i = 1, 2, 3$, positive definite diagonal matrices $M_i \in \mathfrak{R}^{n \times n}$,

$i = 1, 2, 3$, and any matrix $S \in \mathfrak{R}^{2n \times 2n}$ such that the following LMIs hold:

$$\begin{bmatrix} \Phi(h_1, \alpha) & E_2^T S \\ * & -\bar{Z}_2 \end{bmatrix} < 0, \tag{18}$$

$$\begin{bmatrix} \Phi(h_2, \alpha) & E_3^T S^T \\ * & -\tilde{Z}_2 \end{bmatrix} < 0, \quad (19)$$

where

$$\begin{aligned} \Phi(h_i, \alpha) &= \Sigma_{11} + \Sigma_{12}(h_i, \alpha) + \Sigma_2 + \Sigma_3 + \Sigma_{41} - \Sigma_{42} \\ &\quad - \tilde{\Sigma}_{43}(h_i) + \Sigma_{51}(\alpha) - \Sigma_{52}(\alpha) - \Sigma_s, \\ i &= 1, 2. \end{aligned} \quad (20)$$

Proof. Construct a LKF for the DNN (4) as follows:

$$V(k) = \sum_{i=1}^5 V_i(k), \quad (21)$$

where

$$\begin{aligned} V_1(k) &= \zeta_1^T(k) P \zeta_1(k), \\ V_2(k) &= \sum_{j=k-h_1}^{k-1} x^T(j) Q_1 x(j) + \sum_{j=k-h_2}^{k-\alpha-1} x^T(j) Q_2 x(j) \\ &\quad + \sum_{j=k-\alpha}^{k-h_1-1} x^T(j) Q_3 x(j), \\ V_3(k) &= \sum_{j=k-h(k)}^{k-1} \zeta_2^T(j) R \zeta_2(j) \\ &\quad + \sum_{i=-h_2+1}^{-h_1} \sum_{j=k+i}^{k-1} \zeta_2^T(j) R \zeta_2(j), \\ V_4(k) &= h_1 \sum_{i=-h_1}^{-1} \sum_{j=k+i}^{k-1} \eta^T(j) Z_1 \eta(j) \\ &\quad + h_{12} \sum_{i=-h_2}^{-h_1-1} \sum_{j=k+i}^{k-1} \eta^T(j) Z_2 \eta(j), \\ V_5(k) &= (\alpha - h_1) \sum_{i=-\alpha}^{-h_1-1} \sum_{j=k+i}^{k-1} \eta^T(j) Z_3 \eta(j). \end{aligned} \quad (22)$$

The forward difference of LKF (21), $\Delta V(k) = V(k+1) - V(k)$, along the trajectories of the DNN (4) gives

$$\Delta V_1(k) = \zeta_1^T(k+1) P \zeta_1(k+1) - \zeta_1^T(k) P \zeta_1(k). \quad (23)$$

Since

$$\begin{aligned} &\zeta_1(k) \\ &= Col \left\{ \begin{array}{c} e_1, (h_1+1)e_6 - e_1, \\ (h(k) - h_1 + 1)e_7 + (h_2 - h(k) + 1)e_8 \\ -e_2 - e_3, (\alpha - h_1 + 1)e_9 - e_2 \end{array} \right\} \xi(k) \\ &= (\Pi_2 + \Pi(h(k), \alpha)) \xi(k), \end{aligned} \quad (24)$$

$$\zeta_1(k+1)$$

$$\begin{aligned} &= Col \left\{ \begin{array}{c} e_s, (h_1+1)e_6 - e_2, \\ (h(k) - h_1 + 1)e_7 \\ + (h_2 - h(k) + 1)e_8 - e_3 - e_4, \\ (\alpha - h_1 + 1)e_9 - e_5 \end{array} \right\} \xi(k) \\ &= (\Pi_1 + \Pi(h(k), \alpha)) \xi(k), \end{aligned} \quad (25)$$

it follows that

$$\begin{aligned} \Delta V_1(k) &= \xi^T(k) (\Pi_1 + \Pi(h(k), \alpha))^T P \\ &\quad \times (\Pi_1 + \Pi(h(k), \alpha)) \xi(k) \\ &\quad - \xi^T(k) (\Pi_2 + \Pi(h(k), \alpha))^T P \\ &\quad \times (\Pi_2 + \Pi(h(k), \alpha)) \xi(k) \\ &= \xi^T(k) \left\{ \begin{array}{c} \Pi_1^T P \Pi_1 - \Pi_2^T P \Pi_2 \\ + 2(\Pi_1 - \Pi_2)^T P \Pi(h(k), \alpha) \end{array} \right\} \xi(k) \\ &= \xi^T(k) (\Sigma_{11} + \Sigma_{12}(h(k), \alpha)) \xi(k). \end{aligned} \quad (26)$$

The forward differences of $V_2(k)$ and $V_3(k)$ can be obtained as

$$\begin{aligned} \Delta V_2(k) &= \sum_{j=k+1-h_1}^k x^T(j) Q_1 x(j) - \sum_{j=k-h_1}^{k-1} x^T(j) \\ &\quad \cdot Q_1 x(j) + \sum_{j=k+1-h_2}^{k-\alpha} x^T(j) Q_2 x(j) - \sum_{j=k-h_2}^{k-\alpha-1} x^T(j) \\ &\quad \cdot Q_2 x(j) + \sum_{j=k+1-\alpha}^{k-h_1} x^T(j) Q_3 x(j) - \sum_{j=k-\alpha}^{k-h_1-1} x^T(j) \\ &\quad \cdot Q_3 x(j) = x^T(k) Q_1 x(k) - x^T(k-h_1) Q_1 x(k-h_1) \\ &\quad + x^T(k-\alpha) Q_2 x(k-\alpha) - x^T(k-h_2) \\ &\quad \cdot Q_2 x(k-h_2) + x^T(k-h_1) Q_3 x(k-h_1) - x^T(k-\alpha) \\ &\quad \cdot Q_3 x(k-\alpha) = x^T(k) Q_1 x(k) + x^T(k-h_1) \\ &\quad \cdot (Q_3 - Q_1) x(k-h_1) + x^T(k-\alpha) (Q_2 - Q_3) x(k-\alpha) \\ &\quad - x^T(k-h_2) Q_2 x(k-h_2) = \xi^T(k) \end{aligned}$$

$$\begin{aligned}
& \cdot \left\{ e_1^T Q_1 e_1 + e_2^T (Q_3 - Q_1) e_2 \right\} \xi(k) = \xi^T(k) \\
& \cdot \left\{ +e_5^T (Q_2 - Q_3) e_5 - e_4^T Q_2 e_4 \right\} \xi(k) = \xi^T(k) \\
& \cdot \Sigma_2 \xi(k), \\
& \Delta V_3(k) = \sum_{j=k+1-h(k+1)}^k \zeta_2^T(j) R \zeta_2(j) - \sum_{j=k-h(k)}^{k-1} \zeta_2^T(j) \\
& \cdot R \zeta_2(j) + \sum_{i=-h_2+1}^{-h_1} \sum_{j=k+1+i}^k \zeta_2^T(j) R \zeta_2(j) \\
& - \sum_{i=-h_2+1}^{-h_1} \sum_{j=k+i}^{k-1} \zeta_2^T(j) R \zeta_2(j) = \sum_{j=k+1-h(k+1)}^{k-h_1} \zeta_2^T(j) \\
& \cdot R \zeta_2(j) + \sum_{j=k-h_1+1}^{k-1} \zeta_2^T(j) R \zeta_2(j) + \zeta_2^T(k) R \zeta_2(k) \\
& - \sum_{j=k-h(k)}^{k-1} \zeta_2^T(j) R \zeta_2(j) \\
& + \sum_{i=-h_2+1}^{-h_1} \left(\sum_{j=k+i}^{k-1} \zeta_2^T(j) R \zeta_2(j) + \zeta_2^T(k) R \zeta_2(k) \right. \\
& \quad \left. - \zeta_2^T(k+i) R \zeta_2(k+i) - \sum_{j=k+i}^{k-1} \zeta_2^T(j) R \zeta_2(j) \right) \\
& \leq \sum_{j=k+1-h_2}^{k-h_1} \zeta_2^T(j) R \zeta_2(j) + \sum_{j=k-h(k)+1}^{k-1} \zeta_2^T(j) R \zeta_2(j) \\
& + \zeta_2^T(k) R \zeta_2(k) - \sum_{j=k-h(k)+1}^{k-1} \zeta_2^T(j) R \zeta_2(j) - \zeta_2^T(k) \\
& \cdot h(k) R \zeta_2(k-h(k)) + \sum_{i=-h_2+1}^{-h_1} \zeta_2^T(k) R \zeta_2(k) \\
& - \sum_{i=-h_2+1}^{-h_1} \zeta_2^T(k+i) R \zeta_2(k+i) = \sum_{j=k+1-h_2}^{k-h_1} \zeta_2^T(j) \\
& \cdot R \zeta_2(j) + \zeta_2^T(k) R \zeta_2(k) - \zeta_2^T(k-h(k)) R \zeta_2(k) \\
& - h(k) + h_{12} \zeta_2^T(k) R \zeta_2(k) - \sum_{j=k+1-h_2}^{k-h_1} \zeta_2^T(j) R \zeta_2(j) \\
& = (h_{12} + 1) \zeta_2^T(k) R \zeta_2(k) - \zeta_2^T(k-h(k)) R \zeta_2(k) \\
& - h(k) = \xi^T(k) \begin{bmatrix} e_1 \\ e_{10} \end{bmatrix}^T (h_{12} + 1) R \begin{bmatrix} e_1 \\ e_{10} \end{bmatrix} \xi(k)
\end{aligned} \tag{27}$$

$$- \xi^T(k) \begin{bmatrix} e_3 \\ e_{11} \end{bmatrix}^T R \begin{bmatrix} e_3 \\ e_{11} \end{bmatrix} \xi(k) = \xi^T(k) \Sigma_3 \xi(k), \tag{28}$$

$$\begin{aligned}
\Delta V_4(k) &= h_1 \sum_{i=-h_1}^{-1} \left(\sum_{j=k+1+i}^k \eta^T(j) Z_1 \eta(j) \right. \\
&\quad \left. - \sum_{j=k+i}^{k-1} \eta^T(j) Z_1 \eta(j) \right) \\
&+ h_{12} \sum_{i=-h_2}^{-h_1-1} \left(\sum_{j=k+1+i}^k \eta^T(j) Z_2 \eta(j) \right. \\
&\quad \left. - \sum_{j=k+i}^{k-1} \eta^T(j) Z_2 \eta(j) \right) = h_1 \sum_{i=-h_1}^{-1} (\eta^T(k) Z_1 \eta(k) \\
&\quad - \eta^T(k+i) Z_1 \eta(k+i)) \\
&+ h_{12} \sum_{i=-h_2}^{-h_1-1} (\eta^T(k) Z_2 \eta(k) - \eta^T(k+i) Z_2 \eta(k+i)) \\
&= \eta^T(k) (h_1^2 Z_1 + h_{12}^2 Z_2) \eta(k) - h_1 \sum_{i=-h_1}^{-1} \eta^T(k+i) \\
&\cdot Z_1 \eta(k+i) - h_{12} \sum_{i=-h_2}^{-h_1-1} \eta^T(k+i) Z_2 \eta(k+i) \\
&= \xi^T(k) (e_s - e_1)^T (h_1^2 Z_1 + h_{12}^2 Z_2) (e_s - e_1) \xi(k) \\
&- h_1 \sum_{j=k-h_1}^{k-1} \eta^T(j) Z_1 \eta(j) - h_{12} \sum_{j=k-h_2}^{k-h_1-1} \eta^T(j) Z_2 \eta(j).
\end{aligned} \tag{29}$$

Using inequality (8) from Lemma 3 to estimate Z_1 -dependent summation term yields

$$\begin{aligned}
& h_1 \sum_{j=k-h_1}^{k-1} \eta^T(j) Z_1 \eta(j) \\
& \geq \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix}^T \begin{bmatrix} Z_1 & 0 \\ * & 3\rho(0, h_1) Z_1 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} \\
& = \xi^T(k) \Sigma_{42} \xi(k).
\end{aligned} \tag{30}$$

Similarly, by using (9) we estimate Z_2 -dependent summation term

$$\begin{aligned}
& h_{12} \sum_{j=k-h_2}^{k-h_1-1} \eta^T(j) Z_2 \eta(j) \\
& = h_{12} \sum_{j=k-h_2}^{k-h(k)-1} \eta^T(j) Z_2 \eta(j) \\
& + h_{12} \sum_{j=k-h(k)}^{k-h_1-1} \eta^T(j) Z_2 \eta(j)
\end{aligned}$$

$$\begin{aligned}
&\geq \frac{1}{\delta_h} \xi^T(k) E_2^T \begin{bmatrix} Z_2 & 0 \\ 0 & 3Z_2 \end{bmatrix} E_2 \xi(k) \\
&\quad + \frac{1}{1-\delta_h} \xi^T(k) E_3^T \begin{bmatrix} Z_2 & 0 \\ 0 & 3Z_2 \end{bmatrix} E_3 \xi(k) \\
&= \xi^T(k) \begin{bmatrix} E_2 \\ E_3 \end{bmatrix}^T \begin{bmatrix} \frac{1}{\delta_h} \tilde{Z}_2 & 0 \\ 0 & \frac{1}{1-\delta_h} \tilde{Z}_2 \end{bmatrix} \begin{bmatrix} E_2 \\ E_3 \end{bmatrix} \xi(k).
\end{aligned} \tag{31}$$

Based on Lemma 6, for any matrix $S \in \mathfrak{R}^{2n \times 2n}$ we have

$$\begin{aligned}
h_{12} \sum_{j=k-h_2}^{k-h_1-1} \eta^T(j) Z_2 \eta(j) &\geq \xi^T(k) \begin{bmatrix} E_2 \\ E_3 \end{bmatrix}^T \\
&\cdot \begin{bmatrix} \tilde{Z}_2 + (1-\delta_h) T_1 & S \\ * & \tilde{Z}_2 + \delta_h T_2 \end{bmatrix} \begin{bmatrix} E_2 \\ E_3 \end{bmatrix} \xi(k) \\
&= \xi^T(k) \Sigma_{43}(h(k)) \xi^T(k).
\end{aligned} \tag{32}$$

Then, we can get the upper bound of $\Delta V_4(k)$ as

$$\begin{aligned}
\Delta V_4(k) &\leq \xi^T(k) \{ \Sigma_{41} - \Sigma_{42} - \Sigma_{43}(h(k)) \} \xi(k) \\
&= \xi^T(k) \Sigma_4(h(k)) \xi(k).
\end{aligned} \tag{33}$$

Calculating $\Delta V_5(k)$ gives

$$\begin{aligned}
\Delta V_5(k) &= (\alpha - h_1) \sum_{i=-\alpha}^{-h_1-1} \left(\begin{aligned} &\sum_{j=k+i}^{k-1} \eta^T(j) Z_3 \eta(j) \\ &+ \eta^T(k) Z_3 \eta(k) \\ &- \eta^T(k+i) Z_3 \eta(k+i) \\ &- \sum_{j=k+i}^{k-1} \eta^T(j) Z_3 \eta(j) \end{aligned} \right) \\
&= (\alpha - h_1)^2 \eta^T(k) Z_3 \eta(k) \\
&\quad - (\alpha - h_1) \sum_{i=-\alpha}^{-h_1-1} \eta^T(k+i) Z_3 \eta(k+i) \\
&= \xi^T(k) (e_s - e_1)^T (\alpha - h_1)^2 Z_3 (e_s - e_1) \xi(k) \\
&\quad - (\alpha - h_1) \sum_{j=k-\alpha}^{k-h_1-1} \eta^T(j) Z_3 \eta(j).
\end{aligned} \tag{34}$$

By inequality (8), the second term of $\Delta V_5(k)$ can be written as

$$\begin{aligned}
&(\alpha - h_1) \sum_{j=k-\alpha}^{k-h_1-1} \eta^T(j) Z_3 \eta(j) \\
&\geq \begin{bmatrix} \omega_3 \\ \omega_4 \end{bmatrix}^T \begin{bmatrix} Z_3 & 0 \\ * & 3\rho(h_1, \alpha) Z_3 \end{bmatrix} \begin{bmatrix} \omega_3 \\ \omega_4 \end{bmatrix}.
\end{aligned} \tag{35}$$

Then, we can get the upper bound of $\Delta V_5(k)$ as

$$\begin{aligned}
\Delta V_5(k) &\leq \xi^T(k) \left(\begin{array}{c} \Sigma_{51}(\alpha) \\ -E_4^T \begin{bmatrix} Z_3 & 0 \\ * & 3\rho(h_1, \alpha) Z_3 \end{bmatrix} E_4 \end{array} \right) \xi(k) \\
&= \xi^T(k) (\Sigma_{51}(\alpha) - \Sigma_{52}(\alpha)) \xi(k).
\end{aligned} \tag{36}$$

Under the assumption on the activation functions (5) and (6), for any positive definite diagonal matrices $M_j = \text{diag}\{m_{j1}, m_{j2}, \dots, m_{jn}\}$, $j = 1, 2, 3$, the following inequality holds:

$$\begin{aligned}
0 &\geq 2 \sum_{i=1}^n m_{1i} [f_i(x_i(k)) - k_i^- x_i(k)] \\
&\quad \times [f_i(x_i(k)) - k_i^+ x_i(k)] \\
&\quad + 2 \sum_{i=1}^n m_{2i} [f_i(x_i(k-h(k))) - k_i^- x_i(k-h(k))] \\
&\quad \times [f_i(x_i(k-h(k))) - k_i^+ x_i(k-h(k))] \\
&\quad + 2 \sum_{i=1}^n m_{3i} \begin{bmatrix} f_i(x_i(k)) - f_i(x_i(k-h(k))) \\ -k_i^- (x_i(k) - x_i(k-h(k))) \end{bmatrix} \\
&\quad \times \begin{bmatrix} f_i(x_i(k)) - f_i(x_i(k-h(k))) \\ -k_i^+ (x_i(k) - x_i(k-h(k))) \end{bmatrix}.
\end{aligned} \tag{37}$$

Then

$$\begin{aligned}
0 &\geq 2 [f(x(k)) - K_1 x(k)] M_1 [f(x(k)) - K_2 x(k)] \\
&\quad + 2 [f(x(k-h(k))) - K_1 x(k-h(k))] \\
&\quad \times M_2 [f(x(k-h(k))) - K_2 x(k-h(k))] \\
&\quad + 2 \begin{bmatrix} f(x(k)) - f(x(k-h(k))) \\ -K_1 (x(k) - x(k-h(k))) \end{bmatrix} M_3 \\
&\quad \times \begin{bmatrix} f(x(k)) - f(x(k-h(k))) \\ -K_2 (x(k) - x(k-h(k))) \end{bmatrix}, \\
0 &\geq 2 \xi^T(k) (e_{10} - K_1 e_1)^T M_1 (e_{10} - K_2 e_1) \xi^T(k) \\
&\quad + 2 \xi^T(k) (e_{11} - K_1 e_3)^T M_2 (e_{11} - K_2 e_3) \xi^T(k) \\
&\quad + 2 \xi^T(k) (e_{10} - e_{11} - K_1 (e_1 - e_3))^T M_3 \\
&\quad \times (e_{10} - e_{11} - K_2 (e_1 - e_3)) \xi^T(k).
\end{aligned} \tag{38}$$

So, we have

$$\xi^T(k) \Sigma_5 \xi(k) \leq 0. \tag{40}$$

Therefore, by combining (26)-(28), (33), (36), and (40), the forward difference of $V(k)$ is obtained as

$$\Delta V(k) \leq \xi^T(k) \Sigma(h(k), \alpha) \xi(k). \tag{41}$$

Since the matrix $\Sigma(h(k), \alpha)$ is affine with respect to the delay $h(k)$, the condition $\Sigma(h(k), \alpha) < 0$ is satisfied if and only if

$$\Sigma(h_i, \alpha) < 0, \quad i = 1, 2. \quad (42)$$

By calculating, we have

$$\Sigma_{12}(h_1, \alpha) = \text{Sym} \left\{ (\Pi_1 - \Pi_2)^T P \Pi(h_1, \alpha) \right\}, \quad (43)$$

$$\Sigma_{12}(h_2, \alpha) = \text{Sym} \left\{ (\Pi_1 - \Pi_2)^T P \Pi(h_2, \alpha) \right\},$$

$$\begin{aligned} \Sigma_{43}(h_1) &= \begin{bmatrix} E_2 \\ E_3 \end{bmatrix}^T \left(\begin{bmatrix} \tilde{Z}_2 & S \\ * & \tilde{Z}_2 \end{bmatrix} + \begin{bmatrix} T_1 & 0 \\ * & 0 \end{bmatrix} \right) \begin{bmatrix} E_2 \\ E_3 \end{bmatrix} \\ &= \begin{bmatrix} E_2 \\ E_3 \end{bmatrix}^T \begin{bmatrix} \tilde{Z}_2 & S \\ * & \tilde{Z}_2 \end{bmatrix} \begin{bmatrix} E_2 \\ E_3 \end{bmatrix} + E_2^T \tilde{Z}_2 E_2 \\ &\quad - E_2^T S \tilde{Z}_2^{-1} S^T E_2 \\ &= \tilde{\Sigma}_{43}(h_1) - E_2^T S \tilde{Z}_2^{-1} S^T E_2, \end{aligned} \quad (44)$$

$$\begin{aligned} \Sigma_{43}(h_2) &= \begin{bmatrix} E_2 \\ E_3 \end{bmatrix}^T \left(\begin{bmatrix} \tilde{Z}_2 & S \\ * & \tilde{Z}_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ * & T_2 \end{bmatrix} \right) \begin{bmatrix} E_2 \\ E_3 \end{bmatrix} \\ &= \begin{bmatrix} E_2 \\ E_3 \end{bmatrix}^T \begin{bmatrix} \tilde{Z}_2 & S \\ * & \tilde{Z}_2 \end{bmatrix} \begin{bmatrix} E_2 \\ E_3 \end{bmatrix} + E_3^T \tilde{Z}_2 E_3 \\ &\quad - E_3^T S^T \tilde{Z}_2^{-1} S E_3 \\ &= \tilde{\Sigma}_{43}(h_2) - E_3^T S^T \tilde{Z}_2^{-1} S E_3, \end{aligned} \quad (45)$$

$$\Sigma(h_1, \alpha) = \Phi(h_1, \alpha) + E_2^T S \tilde{Z}_2^{-1} S^T E_2 < 0, \quad (46)$$

$$\Sigma(h_2, \alpha) = \Phi(h_2, \alpha) + E_3^T S^T \tilde{Z}_2^{-1} S E_3 < 0. \quad (47)$$

Then, by taking Schur complement, it can be seen that (46) and (47) are equivalent to (18) and (19), respectively. Therefore, when (18) and (19) hold, $\Delta V(k) < 0$, which shows that system (4) is asymptotically stable. $\square$

Remark 11. In Theorem 10, the Wirtinger-based summation inequality (8) is applied to summation with the constant lower and upper bound ($h_1 \sum_{j=k-h_1}^{k-1} \eta^T(j) Z_1 \eta(j)$). However, in the case of the summation with the time-varying lower or upper bound

$$h_{12} \sum_{j=k-h_2}^{k-h(k)-1} \eta^T(j) Z_2 \eta(j) + h_{12} \sum_{j=k-h(k)}^{k-h_1-1} \eta^T(j) Z_2 \eta(j) \quad (48)$$

combination of the Wirtinger-based inequality (9) and the reciprocally convex approach (13) is applied.

By introducing an augmented LKF and zero equations, a further improved stability condition of system (4) can be obtained as follows.

Theorem 12. For given positive integers h_1 and h_2 , system (4) with interval time-varying delay satisfying condition (2)

is asymptotically stable, if there exist positive-definite matrices $P \in \mathfrak{R}^{4n \times 4n}$, $Q_i \in \mathfrak{R}^{n \times n}$, $i = 1, 2, 3$, $R \in \mathfrak{R}^{2n \times 2n}$, $Z_i \in \mathfrak{R}^{n \times n}$, $i = 1, 2, 3$, $G \in \mathfrak{R}^{2n \times 2n}$, positive definite diagonal matrices $M_i \in \mathfrak{R}^{n \times n}$, $i = 1, 2, 3$, symmetric matrices $H_i \in \mathfrak{R}^{n \times n}$, $i = 1, 2$, and any matrices $S \in \mathfrak{R}^{2n \times 2n}$ and $X \in \mathfrak{R}^{2n \times 2n}$ such that the following LMIs hold:

$$\begin{bmatrix} G_1 & X \\ * & G_2 \end{bmatrix} > 0, \quad (49)$$

$$\begin{bmatrix} \Phi(h_1, \alpha) + \Sigma_{61} - \Sigma_{62}(h_1) & E_2^T S \\ * & -\tilde{Z}_2 \end{bmatrix} < 0, \quad (50)$$

$$\begin{bmatrix} \Phi(h_2, \alpha) + \Sigma_{61} - \Sigma_{62}(h_2) & E_3^T S^T \\ * & -\tilde{Z}_2 \end{bmatrix} < 0, \quad (51)$$

where

$$\begin{aligned} \Phi(h_i, \alpha) &= \Sigma_{11} + \Sigma_{12}(h_i, \alpha) + \Sigma_2 + \Sigma_3 + \Sigma_{41} - \Sigma_{42} \\ &\quad - \tilde{\Sigma}_{43}(h_i) + \Sigma_{51}(\alpha) - \Sigma_{52}(\alpha) - \Sigma_s, \\ &\quad i = 1, 2. \end{aligned} \quad (52)$$

Proof. Construct the following LKF candidate:

$$\tilde{V}(k) = V(k) + V_6(k), \quad (53)$$

where

$$V_6(k) = h_{12} \sum_{i=-h_2}^{-h_1-1} \sum_{j=k+i}^{k-1} \zeta_3^T(j) G \xi_3(j). \quad (54)$$

The forward difference of (54) along the trajectories of the DNN (4) gives

$$\begin{aligned} \Delta V_6(k) &= h_{12}^2 \zeta_3^T(k) G \xi_3(k) \\ &\quad - h_{12} \sum_{j=k-h_2}^{k-h(k)-1} \zeta_3^T(j) G \xi_3(j) \\ &\quad - h_{12} \sum_{j=k-h(k)}^{k-h_1-1} \zeta_3^T(j) G \xi_3(j). \end{aligned} \quad (55)$$

For any symmetric matrices H_i , $i = 1, 2$, we have

$$\begin{aligned} &\sum_{j=k-l_2}^{k-l_1-1} (x^T(j+1) H_i x(j+1) - x^T(j) H_i x(j)) \\ &= x^T(k-l_1) H_i x(k-l_1) \\ &\quad - x^T(k-l_2) H_i x(k-l_2). \end{aligned} \quad (56)$$

Since

$$\begin{aligned}
 & x^T(j+1)H_i x(j+1) - x^T(j)H_i x(j) \\
 &= (x(j) + \eta(k))^T H_i (x(j) + \eta(k)) \\
 &\quad - x^T(j)H_i x(j) \\
 &= \eta^T(j)H_i \eta(j) + 2x^T(j)H_i \eta(j) \\
 &= \begin{bmatrix} x(j) \\ \eta(j) \end{bmatrix}^T \begin{bmatrix} 0 & H_i \\ * & H_i \end{bmatrix} \begin{bmatrix} x(j) \\ \eta(j) \end{bmatrix} \\
 &= \zeta_3^T(k) \begin{bmatrix} 0 & H_i \\ * & H_i \end{bmatrix} \xi_3(j),
 \end{aligned} \tag{57}$$

then the following zero equation holds:

$$\begin{aligned}
 EQ(l_1, l_2, H_i) &= x^T(k-l_1)H_i x(k-l_1) \\
 &\quad - x^T(k-l_2)H_i x(k-l_2) \\
 &\quad - \sum_{j=k-l_2}^{k-l_1-1} \zeta_3^T(k) \begin{bmatrix} 0 & H_i \\ * & H_i \end{bmatrix} \xi_3(j) = 0.
 \end{aligned} \tag{58}$$

Now, from (58) we get the following two-zero equations:

$$\begin{aligned}
 EQ(h_1, h(k), H_1) &= x^T(k-h_1)H_1 x(k-h_1) \\
 &\quad - x^T(k-h(k))H_1 x(k-h(k)) \\
 &\quad - \sum_{j=k-h(k)}^{k-h_1-1} \zeta_3^T(k) \begin{bmatrix} 0 & H_1 \\ * & H_1 \end{bmatrix} \xi_3(j) \\
 &= 0,
 \end{aligned} \tag{59}$$

$$\begin{aligned}
 EQ(h(k), h_2, H_2) &= x^T(k-h(k))H_2 x(k-h(k)) \\
 &\quad - x^T(k-h_2)H_2 x(k-h_2) \\
 &\quad - \sum_{j=k-h_2}^{k-h(k)-1} \zeta_3^T(k) \begin{bmatrix} 0 & H_2 \\ * & H_2 \end{bmatrix} \xi_3(j) \\
 &= 0.
 \end{aligned} \tag{60}$$

Adding the inequalities (59) and (60) to (55) implies

$$\begin{aligned}
 \Delta V_6(k) &= h_{12}^2 \zeta_3^T(k) G \xi_3(k) \\
 &\quad + h_{12} x^T(k-h_1)H_1 x(k-h_1) \\
 &\quad - h_{12} x^T(k-h(k))(H_1 - H_2)x(k-h(k)) \\
 &\quad - h_{12} x^T(k-h_2)H_2 x(k-h_2)
 \end{aligned}$$

$$\begin{aligned}
 & - h_{12} \sum_{j=k-h(k)}^{k-h_1-1} \zeta_3^T(j) G_1 \xi_3(j) \\
 & - h_{12} \sum_{j=k-h_2}^{k-h(k)-1} \zeta_3^T(j) G_2 \xi_3(j).
 \end{aligned} \tag{61}$$

Applying Jensen inequality (7), $\Delta V_6(k)$ is estimated as

$$\begin{aligned}
 \Delta V_6(k) &\leq \xi^T(k) \left\{ h_{12}^2 \begin{bmatrix} e_1 \\ e_s - e_1 \end{bmatrix}^T G \begin{bmatrix} e_1 \\ e_s - e_1 \end{bmatrix} \right. \\
 &\quad \left. + h_{12} e_2^T H_1 e_2 - h_{12} e_4^T H_2 e_4 \right. \\
 &\quad \left. - h_{12} e_3^T (H_1 - H_2) e_3 \right\} \xi(k) \\
 &\quad - \frac{1}{\delta_h} \sum_{j=k-h(k)}^{k-h_1-1} \zeta_3^T(j) G_1 \sum_{j=k-h(k)}^{k-h_1-1} \xi_3(j) \\
 &\quad - \frac{1}{1-\delta_h} \sum_{j=k-h_2}^{k-h(k)-1} \zeta_3^T(j) G_2 \sum_{j=k-h_2}^{k-h(k)-1} \xi_3(j) \\
 &= \xi^T(k) \Sigma_{61} \xi(k) \\
 &\quad - \frac{1}{\delta_h} \xi^T(k) E_5^T(h(k)) G_1 E_5(h(k)) \xi(k) \\
 &\quad - \frac{1}{1-\delta_h} \xi^T(k) E_6^T(h(k)) G_2 E_6(h(k)) \xi(k).
 \end{aligned} \tag{62}$$

Using Lemma 5, for any matrix $X \in \mathfrak{R}^{2n \times 2n}$ we have

$$\begin{aligned}
 \Delta V_6(k) &\leq \xi^T(k) \\
 &\quad \cdot \left(\Sigma_{61} - E_{56}^T(h(k)) \begin{bmatrix} G_1 & X \\ * & G_2 \end{bmatrix} E_{56}(h(k)) \right) \xi(k) \\
 &= \xi^T(k) (\Sigma_{61} - E_{56}^T(h(k)) G_{12} E_{56}(h(k))) \xi(k) \\
 &= \xi^T(k) [\Sigma_{61} - \Sigma_{62}(h(k))] \xi(k) = \xi^T(k) \Sigma_6(h(k)) \\
 &\quad \cdot \xi(k).
 \end{aligned} \tag{63}$$

Finally

$$\begin{aligned}
 \Delta \tilde{V}(k) &\leq \xi^T(k) [\Sigma(h(k), \alpha) + \Sigma_{61} - \Sigma_{62}(h(k))] \xi(k).
 \end{aligned} \tag{64}$$

Based on Lemma 9, the equivalent condition of $\Sigma(h(k), \alpha) + \Sigma_{61} - E_{56}^T(h(k)) G_{12} E_{56}(h(k)) < 0$ is that matrix Ψ exists such that

$$\begin{aligned}
 \Theta(h(k), \alpha) &= \left[\begin{array}{c|c} \Sigma(h(k), \alpha) + \Sigma_{61} & \Psi^T \\ \hline +\text{Sym}\{E_{56}^T(h(k)) \Psi\} & -G_{12} \end{array} \right] \\
 &\quad < 0.
 \end{aligned} \tag{65}$$

TABLE 1: MAUB $h_{2\max}$ for given h_1 in Example 1.

Method	2	4	6	h_1 8	10	15	20	NDVs
Th. 2 [40]	19	19	20	21	22	24	27	$17.5n^2 + 8.5n$
Th. 1 [40]	20	20	21	21	22	24	27	$53.5n^2 + 8.5n$
Th. 1 [31]	-	20	20	21	22	24	-	$13.5n^2 + 11.5n$
Th. 3.1 [24]	19	19	20	20	21	24	27	$11n^2 + 6n$
Cor. 2 [41]	19	20	20	21	21	24	27	$154n^2 + 6n$
Theorem 10	19	20	20	21	22	24	27	$17n^2 + 9n$
Theorem 12	20	20	21	21	22	24	27	$23n^2 + 11n$

Since the matrix $\Theta(h(k), \alpha)$ is affine with respect to the delay $h(k)$, the condition $\Theta(h(k), \alpha) < 0$ is satisfied if and only if

$$\Theta(h_i, \alpha) < 0, \quad i = 1, 2. \quad (66)$$

Based on Lemma 9, (66) is equivalent to

$$\begin{aligned} \Sigma(h_i, \alpha) + \Sigma_{61} - E_{56}^T(h_i) G_{12} E_{56}(h_i) \\ = \Sigma(h_i, \alpha) + \Sigma_{61} - \Sigma_{62}(h_i) < 0, \quad i = 1, 2, \end{aligned} \quad (67)$$

i.e.,

$$\Phi(h_1, \alpha) + E_2^T S \tilde{Z}_2^{-1} S^T E_2 + \Sigma_{61} - \Sigma_{62}(h_1) < 0, \quad (68)$$

$$\Phi(h_2, \alpha) + E_3^T S^T \tilde{Z}_2^{-1} S E_3 + \Sigma_{61} - \Sigma_{62}(h_2) < 0. \quad (69)$$

Then, by taking Schur complement, it can be seen that (68) and (69) are equivalent to (50) and (51), respectively. Therefore, when (50) and (51) hold, $\Delta \tilde{V}(k) < 0$, which shows that system (4) is asymptotically stable. $\square$

Remark 13. In order to improve the stability criterion proposed in Theorem 10, a new double summation term $V_6(k)$ with vector $\xi_3(k) = [x^T(k) \ \eta^T(k)]^T$ and free matrix G is introduced in LKF. Further, in the proof of Theorem 12, two zero equalities with symmetric matrices H_1 and H_2 are introduced in $\Delta V_6(k)$. In this way, the conservatism of the criterion is reduced.

Remark 14. In order to reduce the number of decision variables, Lemma 9 is used twice. First, by using the lemma, nonaffine term

$$\Sigma(h(k), \alpha) + \Sigma_{61} - E_{56}^T(h(k)) G_{12} E_{56}(h(k)) < 0 \quad (70)$$

with respect to $h(k)$ is transformed into the affine term $\Theta(h(k), \alpha)$. On that occasion, additional matrix $\Psi \in \mathcal{R}^{11n \times 2n}$ is introduced in (65) and the number of decision variables has increased significantly (for $22n^2$). Second, if we apply Lemma 9 on (66), then the matrix $\Psi \in \mathcal{R}^{11n \times 2n}$ can be eliminated from $\Theta(h_i, \alpha)$. In this way, the significant reduction of decision variables was performed.

Remark 15. The proposed stability criterion depends on the lower and upper bounds of time-delay, h_1 and h_2 . In order to compare our results with existing ones, a maximum allowable

upper bound (MAUB) of the time-delay, $h_{2\max}$, is adopted, such that the concerned system is asymptotically stable for any delay size less than the MAUB. Note that a criterion that gives a lower value of MAUB is less conservative with respect to other criteria.

4. Demonstrative Examples

In the section bellow, we provide three numerical examples to illustrate the effectiveness of the stability criteria proposed in this paper.

Example 1. Consider the DNN (4) with the following parameters:

$$\begin{aligned} C &= \begin{bmatrix} 0.8 & 0 \\ 0 & 0.9 \end{bmatrix}, \\ A &= \begin{bmatrix} 0.001 & 0 \\ 0 & 0.005 \end{bmatrix}, \\ A_d &= \begin{bmatrix} -0.1 & 0.01 \\ -0.2 & -0.1 \end{bmatrix}, \end{aligned} \quad (71)$$

which was used to check the feasible region of stability criteria in [24, 31, 40, 41]. The activation function is in the form of $f(x) = \tanh(x)$ and satisfies Assumption 1 with $k_1^- = k_2^- = 0$ and $k_1^+ = k_2^+ = 1$. By using Theorem 10, the MAUB $h_{2\max}$ is computed with different lower bounds h_1 and the obtained results are given in Table 1. The number of decision variables (NDVs) is also given to show computation complexity. From this table, we can see that our results are less conservative and/or have less decision variables than [24, 31, 40, 41].

The time-varying delay and the state trajectories of the DNN (4) with parameters (71) for initial value $\phi(j) = [3 \ 1]^T$, $j \in \{-22, -21, \dots, -1, 0\}$ are shown in Figures 1 and 2, which shows that the DNN (4) is asymptotically stable for $10 \leq h(k) \leq 22$.

Example 2. Consider the DNN (4) with the following parameters [31]:

$$C = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.3 \end{bmatrix},$$

TABLE 2: MAUB $h_{2\max}$ for given h_1 in Example 2.

Method	h_1						NDVs
	2	4	6	8	10	20	
Th. 1 [31]	99	101	103	105	107	117	$13.5n^2 + 11.5n$
Theorem 10	2028	2030	2032	2034	2036	2046	$17n^2 + 9n$
Theorem 12	3140	3142	3144	3146	3148	3158	$23n^2 + 11n$

FIGURE 1: The time-varying delay.

FIGURE 2: State trajectories of the DNN.

$$A = \begin{bmatrix} 0.02 & 0 \\ 0 & 0.004 \end{bmatrix},$$

$$A_d = \begin{bmatrix} -0.01 & 0.01 \\ -0.02 & -0.01 \end{bmatrix},$$

(72)

and activation functions that satisfy Assumption 1 with $k_1^- = k_2^- = 0$ and $k_1^+ = k_2^+ = 1$. In this example, the MAUB $h_{2\max}$ is computed by using Theorems 10 and 12 for different lower bounds h_1 and results are shown in Table 2. From this table, it can be seen that the proposed stability criteria, Theorems 10 and 12, provide significantly larger MAUB than the previous result [31], while the numbers of decision variables are slightly higher.

Example 3. Consider the DNN (4) with the following [40]:

$$C = \begin{bmatrix} 0.4 & 0 & 0 \\ 0 & 0.3 & 0 \\ 0 & 0 & 0.3 \end{bmatrix},$$

$$A = \begin{bmatrix} 0.2 & -0.2 & 0.1 \\ 0 & -0.3 & 0.3 \\ -0.2 & -0.1 & -0.2 \end{bmatrix},$$

$$A_d = \begin{bmatrix} -0.2 & 0.1 & 0 \\ -0.2 & 0.3 & 0.1 \\ 0.1 & -0.2 & 0.3 \end{bmatrix}, \quad (73)$$

$$k_1^- = 0,$$

$$k_2^- = -0.4,$$

$$k_3^- = -0.2,$$

$$k_1^+ = 0.6,$$

$$k_2^+ = 0,$$

$$k_3^+ = 0.$$

By applying Theorems 10 and 12 to the DNN (4) with parameters (73), the MAUB $h_{2\max}$ is computed for different lower bounds h_1 and results are shown in Table 3. From this table it can be seen that stability criterion proposed in Theorem 12 gives approximately the same results but with significantly smaller number of decision variables than [40].

5. Conclusion

In this study, a novel Lyapunov-Krasovskii functional with single and double summation terms and several augmented vectors is constructed by decomposing the time-delay interval into two nonequidistant subintervals. Using Wirtinger-based inequality, extended reciprocally convex approach, and

TABLE 3: MAUB $h_{2\max}$ for given h_1 in Example 3.

Method	h_1						NDVs
	2	4	6	10	15	20	
Th. 2 [40]	13	15	17	21	26	31	$17.5n^2 + 8.5n$
Th. 1 [40]	18	20	22	26	31	36	$53.5n^2 + 8.5n$
Theorem 10	13	15	16	19	24	29	$17n^2 + 9n$
Theorem 12	19	20	22	26	31	36	$23n^2 + 11n$

several zero equalities, new stability conditions are developed in the form of linear matrix inequalities. It is shown that the proposed results are less conservative than the existing ones. Several numerical examples are presented to show the effectiveness and less conservativeness of the proposed method.

Data Availability

The [tables and figures] data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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