Research Article

Resonant Integral Boundary Value Problems for Caputo Fractional Differential Equations

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This paper deals with the following Caputo fractional differential equations with Riemann-Stieltjes integral boundary conditions ${}^c D_{0+}^\alpha u(t) = f(t, u(t), u'(t), u''(t))$, $t \in [0, 1]$, $u(0) = u''(0) = 0$, $u(1) = \int_0^1 u(t) dA(t)$, where ${}^c D_{0+}^\alpha$ denotes the standard Caputo derivative, $\alpha \in (2, 3]$; $\int_0^1 x(t) dA(t)$ denotes the Riemann-Stieltjes integrals of x with respect to A . By mean of coincidence degree theory, we obtain the existence of solutions for the above fractional BVP at resonance. In the end, according to the main results, we give a typical example.

1. Introduction

Fractional differential equations have various applications in chemical engineering, blood flow phenomena, physics, signal and image processing, fitting of experimental data, control theory, etc. Fractional differential equations also are a fundamental tool for the description of process of natural phenomena. In past years, many authors studied the existence of solutions or positive solutions of various fractional differential equations such as fractional boundary value problem at resonance [1–9], fractional boundary value problem at nonresonance [10–23], Caputo fractional derivative problem [24, 25], impulsive problem [26], conjugate boundary value problems [27, 28], integral boundary value problem [29], variational structure problem [30], fractional p -Laplace problem [31], fractional lower and upper solution problem [32, 33], solitons [34, 35], etc.

For example, Alberto et al. [11] considered the following fractional differential equations:

$$\begin{aligned} {}^c D^\alpha u(t) + f(t, u(t)) &= 0, \quad t \in (0, 1), \\ u(0) &= u''(0) = 0, \\ u(1) &= \lambda \int_0^1 u(s) ds, \end{aligned} \quad (1)$$

where $\lambda \in (0, 2)$, $\alpha \in (2, 3)$, ${}^c D_{0+}^\alpha$ denotes the Caputo fractional derivative, and $f: [0, 1] \times [0, \infty) \rightarrow [0, \infty)$ is a continuous function. The existence of solutions is obtained by using the Guo-Krasnoselskii fixed point theorem. Similar arguments for related issues are treated in [12].

In [7], by mean of coincidence degree theory, the existence of solution of integral BVP for nonlinear differential equation is proved:

$$\begin{aligned} -D_{0+}^p x(t) &= f\left(t, x(t), D_{0+}^{p-1} x(t), D_{0+}^{p-2} x(t)\right), \\ t &\in (0, 1), \\ x(0) &= x'(0) = 0, \end{aligned} \quad (2)$$

$$x(1) = \int_0^1 x(t) dA(t),$$

where $p \in (2, 3)$, D_{0+}^p is the Riemann-Liouville derivative, $f: [0, 1] \times \mathbb{R}^3 \rightarrow \mathbb{R}$, and f satisfies Caratheodory conditions. Similar method for different nonlinear differential equations, with either Caputo or Riemann-Liouville derivatives, and using suitable coincidence degree theory in each case, has allowed proving the existence of solutions in many papers; see also [1–5, 24, 25, 33].

In [5], the authors studied nonlocal fractional differential equation:

$$\begin{aligned} D_{0+}^{\alpha} u(t) + f(t, u(t)) &= 0, \quad t \in (0, 1), \\ u(0) &= 0, \\ u(1) &= \eta u(\xi), \end{aligned} \quad (3)$$

where $\eta \xi^{\alpha-1} = 1$, $\xi \in (0, 1)$, $\alpha \in (1, 2)$, D_{0+}^{α} is the Riemann-Liouville derivative, and $f : [0, 1] \times [0, \infty) \rightarrow R$ is continuous function. They solved the above differential equation by using iterative technique and the fixed point index theory. We can also refer to [16, 17].

In this paper, we are concerned with the existence of solution for the following differential equation with boundary value conditions:

$$\begin{aligned} {}^c D_{0+}^{\alpha} u(t) &= f(t, u(t), u'(t), u''(t)), \quad t \in [0, 1], \\ u(0) &= u''(0) = 0, \\ u(1) &= \int_0^1 u(t) dA(t), \end{aligned} \quad (4)$$

where ${}^c D_{0+}^{\alpha}$ denotes the Caputo fractional derivative, $\alpha \in (2, 3]$, and $f : [0, 1] \times R^3 \rightarrow R$ is a continuous function. The BVP (4) is a resonance problem, because $\int_0^1 t dA(t) = 1$, if and only if its associated linear homogeneous BVP

$$\begin{aligned} {}^c D_{0+}^{\alpha} x(t) &= 0, \quad t \in [0, 1], \\ x(0) &= x''(0) = 0, \\ x(1) &= \int_0^1 x(t) dA(t) \end{aligned} \quad (5)$$

have nontrivial solution. Here, we assume that the following two conditions hold:

$$\begin{aligned} (A_1) \quad \int_0^1 t dA(t) &= 1, \quad \int_0^1 t^{\alpha} dA(t) \neq 1. \\ (A_2) \quad f : [0, 1] \times R^3 &\rightarrow R \text{ is a continuous function.} \end{aligned}$$

As is known to all, there are few studies on such boundary value problems. Therefore, by mean of coincidence degree theory, we consider the existence of solutions for fractional differential equations (4) with Riemann-Stieltjes integral boundary conditions at resonance.

2. Preliminaries

For the convenience of the reading, we introduce the theorem and lemma of fractional calculus theory.

Lemma 1 (see [36]). *Put $\alpha \in (0, +\infty)$, $f \in L(0, 1) \cap C(0, 1)$; the fractional differential equation*

$${}^c D_{0+}^{\alpha} f(x) = 0 \quad (6)$$

has the general solution $f(x) = A_0 + A_1 x + \dots + A_{n-1} x^{n-1}$, $A_k \in R$, $k = 0, 1, \dots, n-1$, where n is the smallest integer greater than or equal to α .

Lemma 2 (see [36]). *Given that $f \in C^n[0, 1]$, then*

$$\begin{aligned} I_{0+}^{\alpha} {}^c D_{0+}^{\alpha} f(x) &= f(x) - f(0) - f'(0)x - \dots \\ &\quad - \frac{f^{(n-1)}(0)}{(n-1)!} x^{n-1}, \end{aligned} \quad (7)$$

where n is the smallest integer greater than or equal to α .

Let Y, Z be two Banach spaces and $L : D(L) \subset Y \rightarrow Z$ be a linear operator. Assume that L is a Fredholm operator of index zero; $P : Y \rightarrow Y$, $Q : Z \rightarrow Z$ are two continuous projectors such that

$$\begin{aligned} \text{Im } P &= \ker L, \\ \ker Q &= \text{Im } L, \\ Y &= \ker L \oplus \ker P, \\ Z &= \text{Im } L \oplus \text{Im } Q. \end{aligned} \quad (8)$$

Thus

$$L|_{\text{Ker } P \cap D(L)} : \text{Ker } P \cap D(L) \rightarrow \text{Im } L \quad (9)$$

is invertible. We denote the inverse of the mapping by K_P (generalized inverse operator of L). If Ω is an open bounded subset of Y such that $D(L) \cap \Omega \neq \emptyset$, $N : Y \rightarrow Z$ will be called L -compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_P(I-Q)N : \overline{\Omega} \rightarrow Y$ is compact.

Theorem 3 (see [37]). *$L : D(L) \subset Y \rightarrow Z$ is a linear operator, and L is a Fredholm operator of index zero, and N is L -compact on $\overline{\Omega}$. The equation $L\psi = N\psi$ has at least one solution in $D(L) \cap \overline{\Omega}$ if and only if the following three conditions are satisfied:*

- (1) $L\psi \neq \lambda N\psi$ for every $(\psi, \lambda) \in [(D(L) \setminus \ker L) \cap \partial\Omega] \times (0, 1)$.
- (2) $N\psi \notin \text{Im } L$ for every $\psi \in \text{Ker } L \cap \partial\Omega$.
- (3) $\deg(QN|_{\text{Ker } L}, \Omega \cap \text{Ker } L, 0) \neq 0$, where $Q : Y \rightarrow Y$ is a projection such that $\text{Im } L = \text{Ker } Q$.

In this paper, let $Y = C^2[0, 1]$ with the norm $\|x\| = \max\{\|x\|_{\infty}, \|x'\|_{\infty}, \|x''\|_{\infty}\}$ in which $\|x\|_{\infty} = \max_{t \in [0, 1]} |x(t)|$. Let $Z = C[0, 1]$ with the norm $\|x\|_{\infty} = \max_{t \in [0, 1]} |x(t)|$. Operator L is defined as $L : Lu(t) = {}^c D_{0+}^{\alpha} u(t)$ with

$$\begin{aligned} D(L) &= \left\{ u \in Y : {}^c D_{0+}^{\alpha} u(t) \in C[0, 1], \quad u(0) = u''(0) \right. \\ &\quad \left. = 0, \quad u(1) = \int_0^1 u(t) dA(t) \right\}. \end{aligned} \quad (10)$$

Define operator $N : Y \rightarrow Z$ as follows:

$$(Nu)(x) = f(t, u(t), u'(t), u''(t)). \quad (11)$$

And so problem (4) becomes $Lu = Nu, u \in D(L)$.

3. Main Results

In this section, we continue to give the following assumptions.

(A₃) There exist functions $a(t), b(t), c(t), d(t) \in C[0, 1]$ with $\|b\|_\infty + \|c\|_\infty + \|d\|_\infty < \Gamma(\alpha - 1)$ such that

$$|f(t, u, v, w)| \leq a(t) + b(t)|u| + c(t)|v| + d(t)|w|, \quad t \in [0, 1]. \quad (12)$$

(A₄) There exists positive constant F such that, for $y \in \text{Dom} L$, if $|y'(t)| > F$ for $t \in [0, 1]$, then

$$\int_0^1 (1-s)^{\alpha-1} (Ny)(s) ds - \int_0^1 \int_0^t (t-s)^{\alpha-1} (Ny)(s) ds dA(t) \neq 0. \quad (13)$$

(A₅) There exists constant $a_0 > 0$ such that either

$$c \left[\int_0^1 (1-s)^{\alpha-1} f(s, cs, c, 0) ds - \int_0^1 \int_0^t (t-s)^{\alpha-1} f(s, cs, c, 0) ds dA(t) \right] < 0 \quad (14)$$

or

$$c \left[\int_0^1 (1-s)^{\alpha-1} f(s, cs, c, 0) ds - \int_0^1 \int_0^t (t-s)^{\alpha-1} f(s, cs, c, 0) ds dA(t) \right] > 0 \quad (15)$$

holds if $|c| > a_0$.

Based on the above, we will prove the following important theorem.

Theorem 4. Suppose that (A₁)-(A₅) is satisfied; there exists at least one solution of the fractional differential equation (4) in Y .

For convenience, we first give some important lemmas as follows.

Lemma 5. Assume that (A₁) is satisfied; then $L : D(L) \subset Y \rightarrow Z$ is Fredholm operator with index zero, and a linear continuous projector $Q : Z \rightarrow Z$ is defined by

$$(Qy)(t) = \frac{\alpha}{1 - \int_0^1 t^\alpha dA(t)} \left[\int_0^1 (1-s)^{\alpha-1} y(s) ds - \int_0^1 \int_0^t (t-s)^{\alpha-1} y(s) ds dA(t) \right]. \quad (16)$$

Furthermore, define a linear operator $K_p : \text{Im } L \rightarrow D(L) \cap \ker P$ as follows:

$$(K_p y)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) ds - \frac{t}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} y(s) ds \quad (17)$$

such that $K_p = (L|_{D(L) \cap \ker P})^{-1}$.

Proof. It follows from Lemma 1 and $Lu = 0$ that

$$u(t) = c_1 + c_2 t + c_3 t^2, \quad c_1, c_2, c_3 \in \mathbb{R}. \quad (18)$$

According to $u(0) = u''(0) = 0$, $u(1) = \int_0^1 u(t) dA(t)$, we have

$$\ker L = \{ct : c \in \mathbb{R}\}. \quad (19)$$

Moreover, we can obtain that

$$\text{Im } L = \left\{ y \in Z : \int_0^1 (1-s)^{\alpha-1} y(s) ds - \int_0^1 \int_0^t (t-s)^{\alpha-1} y(s) ds dA(t) = 0 \right\}. \quad (20)$$

On one hand, suppose $y \in \text{Im } L$; then there exist $u \in D(L)$:

$$y = Lu \in Z. \quad (21)$$

Then we have

$$u(t) = I_{0+}^\alpha y(t) + c_1 + c_2 t + c_3 t^2. \quad (22)$$

This together with (A₁), $u(0) = u''(0) = 0$, $u(1) = \int_0^1 u(t) dA(t)$, we can get $c_1 = c_3 = 0$ and

$$\int_0^1 (1-s)^{\alpha-1} y(s) ds - \int_0^1 \int_0^t (t-s)^{\alpha-1} y(s) ds dA(t) = 0. \quad (23)$$

And so we obtain that

$$\text{Im } L \subset \left\{ y \in Z : \int_0^1 (1-s)^{\alpha-1} y(s) ds - \int_0^1 \int_0^t (t-s)^{\alpha-1} y(s) ds dA(t) = 0 \right\}. \quad (24)$$

On the other hand, if $y \in Z$ satisfies

$$\int_0^1 (1-s)^{\alpha-1} y(s) ds - \int_0^1 \int_0^t (t-s)^{\alpha-1} y(s) ds dA(t) = 0, \quad (25)$$

we let

$$\begin{aligned} u(t) &= I_{0+}^\alpha y(t) - t I_{0+}^\alpha y(1) \\ &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) ds \\ &\quad - \frac{t}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} y(s) ds. \end{aligned} \quad (26)$$

Then we conclude that

$$\begin{aligned} u(0) &= u''(0) = 0, \\ u(1) &= I_{0+}^\alpha y(1) - \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} y(s) ds = 0 \end{aligned} \quad (27)$$

and

$$\begin{aligned}
 & \int_0^1 u(t) dA(t) \\
 &= \int_0^1 \frac{1}{\Gamma(\alpha)} \int_0^t (1-s)^{\alpha-1} y(s) ds dA(t) \\
 &\quad - \int_0^1 t dA(t) \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} y(s) ds \\
 &= \int_0^1 \frac{1}{\Gamma(\alpha)} \int_0^t (1-s)^{\alpha-1} y(s) ds dA(t) \\
 &\quad - \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} y(s) ds = 0.
 \end{aligned} \tag{28}$$

That is, $u \in D(L)$, then $y \in \text{Im } L$. In conclusion,

$$\begin{aligned}
 \text{Im } L = \left\{ y \in Z : \int_0^1 (1-s)^{\alpha-1} y(s) ds \right. \\
 \left. - \int_0^1 \int_0^t (t-s)^{\alpha-1} y(s) ds dA(t) = 0 \right\}.
 \end{aligned} \tag{29}$$

We give projection operator $P : Y \rightarrow Y$ as

$$(Pu)(t) = u(1)t. \tag{30}$$

We get $\text{Ker } P = \{u \in Y \mid u(1) = 0\}$, $P^2u = Pu$, and $\text{Im } P = \text{ker } L$. For any $u \in Y$, together with $u = (u - Pu) + Pu$, we have $Y = \text{ker } P + \text{ker } L$. It is easy to obtain that $\text{ker } L \cap \text{ker } P = \{0\}$ which implies

$$Y = \text{ker } P \oplus \text{ker } L. \tag{31}$$

Next operator $Q : Z \rightarrow Z$ is defined as follows:

$$\begin{aligned}
 (Qy)(t) &= \frac{\alpha}{1 - \int_0^1 t^\alpha dA(t)} \left[\int_0^1 (1-s)^{\alpha-1} y(s) ds \right. \\
 &\quad \left. - \int_0^1 \int_0^t (t-s)^{\alpha-1} y(s) ds dA(t) \right].
 \end{aligned} \tag{32}$$

Noting that

$$\begin{aligned}
 (Q^2y)(t) &= (Q(Qy))(t) \\
 &= \frac{\alpha}{1 - \int_0^1 t^\alpha dA(t)} \left[\int_0^1 (1-s)^{\alpha-1} Qy(s) ds \right. \\
 &\quad \left. - \int_0^1 \int_0^t (t-s)^{\alpha-1} Qy(s) ds dA(t) \right] = Qy(t) \\
 &\quad \cdot \frac{\alpha}{1 - \int_0^1 t^\alpha dA(t)} \left[\int_0^1 (1-s)^{\alpha-1} ds \right. \\
 &\quad \left. - \int_0^1 \int_0^t (t-s)^{\alpha-1} ds dA(t) \right] = (Qy)(t)
 \end{aligned} \tag{33}$$

means Q is a projection operator. And obviously, $\text{ker } Q = \text{Im } L$. For any $y \in Z$, let $y_1 = y - Qy$,

$$\begin{aligned}
 (Qy_1)(t) &= (Q(y - Qy))(t) \\
 &= \frac{\alpha}{1 - \int_0^1 t^\alpha dA(t)} \left[\int_0^1 (1-s)^{\alpha-1} \right. \\
 &\quad \cdot (y(s) - Qy(s)) ds \\
 &\quad \left. - \int_0^1 \int_0^t (t-s)^{\alpha-1} (y(s) - Qy(s)) ds dA(t) \right] \\
 &= \frac{\alpha}{1 - \int_0^1 t^\alpha dA(t)} \left[\int_0^1 (1-s)^{\alpha-1} y(s) ds \right. \\
 &\quad \left. - \int_0^1 \int_0^t (t-s)^{\alpha-1} y(s) ds dA(t) \right] \\
 &\quad + \frac{\alpha}{1 - \int_0^1 t^\alpha dA(t)} \left[\int_0^1 (1-s)^{\alpha-1} Qy(s) ds \right. \\
 &\quad \left. - \int_0^1 \int_0^t (t-s)^{\alpha-1} Qy(s) ds dA(t) \right] = (Qy)(t) \\
 &\quad - (Q^2y)(t) = 0,
 \end{aligned} \tag{34}$$

and we have $y_1 \in \text{Im } L$, so $Z = \text{Im } L + \text{Im } Q$. Moreover, by simple calculation, we can get $\text{Im } Q \cap \text{Im } L = \{0\}$. Above all, $Z = \text{Im } L \oplus \text{Im } Q$.

To summarize, we can know that $\text{Im } L$ is closed subspace of Z ; $\dim \text{ker } L = \text{codim } \text{Im } L = 1$; and so L is a Fredholm operator of index zero.

Next we will prove that K_P is the inverse of $L|_{\text{dom } L \cap \text{ker } P}$. It is clear that

$$(L_P K_P y)(t) = {}^c D_{0+}^\alpha K_P y(t) = y(t), \quad y \in \text{Im } L. \tag{35}$$

For each $u \in \text{dom } L \cap \text{ker } P$, we have $u(0) = u'(0) = 0$, $u(1) = 0$ and

$$\begin{aligned}
 (K_P L_P u)(t) &= I_{0+}^\alpha {}^c D_{0+}^\alpha u(t) - t I_{0+}^\alpha {}^c D_{0+}^\alpha u(1) \\
 &= u(t) + u'(0)t - t(u(1) + u'(0)) \\
 &= u(t).
 \end{aligned} \tag{36}$$

This implies that $K_P L_P u = u$. So $K_P = (L|_{\text{dom } L \cap \text{ker } P})^{-1}$. $\square$

Lemma 6. Assume that Ω is a bounded open subset of Y , then N is L -compact on $\overline{\Omega}$ if $\text{dom } L \cap \overline{\Omega} \neq \emptyset$.

Proof. By the continuity of f , we obtain $QN(\overline{\Omega})$ and $(I - Q)N(\overline{\Omega})$ are bounded, so there exists constant $M > 0$, such that $|(I - Q)Nu(t)| \leq M$ for $u \in \overline{\Omega}$ and $t \in [0, 1]$. $\forall t_1, t_2 \in$

$[0, 1]$, $t_1 < t_2$, $u \in \overline{\Omega}$, let $(Ou)(s) = (I - Q)Nu(s)$, and we have

$$\begin{aligned}
 & |(K_P(I - Q)Nu)(t_2) - (K_P(I - Q)Nu)(t_1)| \\
 & \leq \frac{1}{\Gamma(\alpha)} \left| \int_0^{t_2} (t_2 - s)^{\alpha-1} (Ou)(s) ds \right. \\
 & \quad \left. - \int_0^{t_1} (t_1 - s)^{\alpha-1} (Ou)(s) ds \right| \\
 & + \frac{t_2 - t_1}{\Gamma(\alpha)} \left| \int_0^1 (1 - s)^{\alpha-1} (Ou)(s) ds \right| \\
 & \leq \frac{M}{\Gamma(\alpha)} \left| \int_0^{t_1} [(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}] ds \right. \\
 & \quad \left. + \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} ds \right| + \frac{M(t_2 - t_1)}{\Gamma(\alpha)} \int_0^1 (1 - s)^{\alpha-1} ds \quad (37) \\
 & = \frac{M}{\Gamma(\alpha + 1)} (t_2^\alpha - t_1^\alpha + t_2 - t_1), \\
 & |(K_P(I - Q)Nu)'(t_2) - (K_P(I - Q)Nu)'(t_1)| \\
 & = \frac{1}{\Gamma(\alpha - 1)} \left| \int_0^{t_2} (t_2 - s)^{\alpha-2} (Ou)(s) ds \right. \\
 & \quad \left. - \int_0^{t_1} (t_1 - s)^{\alpha-2} (Ou)(s) ds \right| \\
 & \leq \frac{M}{\Gamma(\alpha - 1)} \left| \int_0^{t_1} [(t_2 - s)^{\alpha-2} - (t_1 - s)^{\alpha-2}] ds \right. \\
 & \quad \left. + \int_{t_1}^{t_2} (t_2 - s)^{\alpha-2} ds \right| = \frac{M}{\Gamma(\alpha)} (t_2^{\alpha-1} - t_1^{\alpha-1}),
 \end{aligned}$$

and

$$\begin{aligned}
 & |(K_P(I - Q)Nu)''(t_2) - (K_P(I - Q)Nu)''(t_1)| \\
 & = \frac{1}{\Gamma(\alpha - 2)} \left| \int_0^{t_2} (t_2 - s)^{\alpha-3} (Ou)(s) ds \right. \\
 & \quad \left. - \int_0^{t_1} (t_1 - s)^{\alpha-3} (Ou)(s) ds \right| \quad (38) \\
 & \leq \frac{M}{\Gamma(\alpha - 2)} \left| \int_0^{t_1} [(t_2 - s)^{\alpha-3} - (t_1 - s)^{\alpha-3}] ds \right. \\
 & \quad \left. + \int_{t_1}^{t_2} (t_2 - s)^{\alpha-3} ds \right| = \frac{M}{\Gamma(\alpha - 1)} (t_2^{\alpha-2} - t_1^{\alpha-2}).
 \end{aligned}$$

As a result of $t^\alpha, t^{\alpha-1}, t^{\alpha-2}$ are uniformly continuous on $[0, 1]$. So we have that $K_P(I - Q)N(\overline{\Omega}) \subset C[0, 1]$, $(K_P(I - Q)N)'(\overline{\Omega}) \subset C[0, 1]$, $(K_P(I - Q)N)''(\overline{\Omega}) \subset C[0, 1]$ are equicontinuous, and, by Ascoli-Arzelà theorem, we have that $K_P(I - Q)N : \overline{\Omega} \rightarrow Y$ is compact. Thus, N is L -compact. $\square$

Lemma 7. Set $\Omega_1 = \{u \in \ker L : \lambda u + \beta(1 - \lambda)QNu = 0, \lambda \in [0, 1]\}$ is bounded if conditions (A_1) , (A_2) , and (A_5) are satisfied, and

$$\beta = \begin{cases} -1, & \text{if (14) holds;} \\ 1, & \text{if (15) holds.} \end{cases} \quad (39)$$

Proof. Suppose that $u \in \Omega_1$ we have $u(t) = ct$, $c \in R$ and

$$\begin{aligned}
 & \lambda ct + \beta(1 - \lambda) \frac{\alpha}{1 - \int_0^1 t^\alpha dA(t)} \\
 & \cdot \left[\int_0^1 (1 - s)^{\alpha-1} f(s, cs, c, 0) ds \right. \\
 & \quad \left. - \int_0^1 \int_0^t (t - s)^{\alpha-1} f(s, cs, c, 0) ds dA(t) \right] = 0 \quad (40)
 \end{aligned}$$

If $\lambda = 0$, by condition (A_5) , we have $|c| \leq a_0$. If $\lambda = 1$, then $c = 0$. If $\lambda \in (0, 1)$, we suppose $|c| \geq a_0$, then

$$\begin{aligned}
 & \lambda c^2 t = -\beta(1 - \lambda) c \frac{\alpha}{1 - \int_0^1 t^\alpha dA(t)} \\
 & \cdot \left[\int_0^1 (1 - s)^{\alpha-1} f(s, cs, c, 0) ds \right. \\
 & \quad \left. - \int_0^1 \int_0^t (t - s)^{\alpha-1} f(s, cs, c, 0) ds dA(t) \right] < 0 \quad (41)
 \end{aligned}$$

which contradicts with $\lambda c^2 > 0$. $\square$

Lemma 8. Set $\Omega_2 = \{u \in D(L) \setminus \ker L : Lu = \lambda Nu, \lambda \in [0, 1]\}$ is bounded if (A_1) – (A_4) are satisfied.

Proof. Take $u \in \Omega_2$, then $Nu \in \text{Im } L$. Thus we have

$$\begin{aligned}
 & \int_0^1 (1 - s)^{\alpha-1} (Ny)(s) ds \\
 & - \int_0^1 \int_0^t (t - s)^{\alpha-1} (Ny)(s) ds dA(t) = 0. \quad (42)
 \end{aligned}$$

According to (A_4) , there exists constant $\xi \in (0, 1)$ such that

$$|u'(\xi)| \leq F. \quad (43)$$

Because $|u'(t)| \leq |u'(\xi)| + \left| \int_\xi^t u''(s) ds \right| \leq F + \|u''\|_\infty$,

$$\|u'\|_\infty \leq F + \|u''\|_\infty. \quad (44)$$

If $u \in D(L)$, we have $u(0) = 0$, then

$$|u(t)| = \left| u(0) + \int_0^t u'(s) ds \right| \leq \|u'\|_\infty, \quad (45)$$

and therefore $\|u\|_\infty \leq \|u'\|_\infty$.

By $Lu = \lambda Nu$, we have

$$\begin{aligned} u(t) &= \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, u(s), u'(s), u''(s)) ds \\ &\quad - u(0) - u'(0)t - \frac{u''(0)}{2}t^2. \end{aligned} \quad (46)$$

Then we obtain that

$$\begin{aligned} u''(t) &= \frac{\lambda}{\Gamma(\alpha-2)} \int_0^t (t-s)^{\alpha-3} f(s, u(s), u'(s), u''(s)) ds \\ &\quad - u''(0). \end{aligned} \quad (47)$$

According to the boundary condition $u''(0) = 0$, one has

$$\begin{aligned} u''(t) &= \frac{\lambda}{\Gamma(\alpha-2)} \\ &\quad \cdot \int_0^t (t-s)^{\alpha-3} f(s, u(s), u'(s), u''(s)) ds. \end{aligned} \quad (48)$$

So we get

$$\begin{aligned} |u''(t)| &\leq \frac{1}{\Gamma(\alpha-2)} \int_0^t (t-s)^{\alpha-3} \\ &\quad \cdot |f(s, u(s), u'(s), u''(s))| ds \leq \frac{1}{\Gamma(\alpha-2)} \\ &\quad \cdot \int_0^1 (1-s)^{\alpha-3} |f(s, u(s), u'(s), u''(s))| ds \\ &\leq \frac{1}{\Gamma(\alpha-2)} \int_0^1 (1-s)^{\alpha-3} (a(s) + b(s)|u(s)| \\ &\quad + c(s)|u'(s)| + d(s)|u''(s)|) ds \\ &\leq \frac{1}{\Gamma(\alpha-1)} (\|a\|_\infty + \|b\|_\infty \|u\|_\infty + \|c\|_\infty \|u'\|_\infty \\ &\quad + \|d\|_\infty \|u''\|_\infty) \leq \frac{1}{\Gamma(\alpha-1)} (\|a\|_\infty + \|b\|_\infty (F \\ &\quad + \|u''\|_\infty) + \|c\|_\infty (F + \|u''\|_\infty) + \|d\|_\infty \|u''\|_\infty) \\ &\leq \frac{1}{\Gamma(\alpha-1)} (\|a\|_\infty + \|b\|_\infty F + \|c\|_\infty F + (\|b\|_\infty \\ &\quad + \|c\|_\infty + \|d\|_\infty) \|u''\|_\infty). \end{aligned} \quad (49)$$

Thus, from $\|b\|_\infty + \|c\|_\infty + \|d\|_\infty < \Gamma(\alpha-1)$, we obtain that

$$\|u''\|_\infty \leq \frac{\|a\|_\infty + F[\|b\|_\infty + \|c\|_\infty]}{\Gamma(\alpha-1) - \|b\|_\infty - \|c\|_\infty - \|d\|_\infty} = M_1. \quad (50)$$

So we have

$$\|u\| = \max \{ \|u\|_\infty, \|u'\|_\infty, \|u''\|_\infty \} \leq M_1 + F. \quad (51)$$

Therefore, Ω_2 is bounded. $\square$

Lemma 9. Set $\Omega_3 = \{u : u \in \ker L, Nu \in \text{Im } L\}$ is bounded if (A_1) , (A_2) , and (A_5) hold.

Proof. Let $u \in \Omega_3$, then $u(t) = ct$, $c \in \mathbb{R}$, $Nu \in \text{Im } L$, and we can get

$$\begin{aligned} &\int_0^1 (1-s)^{\alpha-1} f(s, cs, c, 0) ds \\ &\quad - \int_0^1 \int_0^t (t-s)^{\alpha-1} f(s, cs, c, 0) ds dA(t) = 0. \end{aligned} \quad (52)$$

According to (A_5) we have $|c| \leq a_0$, that is to say, Ω_3 is bounded. $\square$

In summary, we can prove Theorem 4.

Proof of Theorem 4. Assume that $\Omega \supset \bigcup_{i=1}^3 \overline{\Omega}_i \cup \{0\}$ is bounded open subset of X , and, according to Lemma 6, we get that N is L -compact on $\overline{\Omega}$. Form Lemmas 8 and 9, we have

- (1) $Lu \neq \lambda Nu$, for all $(u, \lambda) \in [(D(L) \setminus \ker L) \cap \partial\Omega] \times (0, 1)$;
- (2) $Nu \notin \text{Im } L$, for all $u \in \text{Ker } L \cap \partial\Omega$.

Set $H(u, \lambda) = \lambda u + \beta(1-\lambda)QNu$. It follows from Lemma 7 that we get $H(u, \lambda) \neq 0$, $u \in \partial\Omega \cap \text{Ker } L$, so we have

$$\begin{aligned} \deg(QN|_{\text{Ker } L}, \Omega \cap \text{Ker } L, 0) &= \deg(\beta J, \Omega \cap \text{Ker } L, 0) \\ &\neq 0. \end{aligned} \quad (53)$$

According to the above proof and Theorem 3, Theorem 4 holds. $\square$

Example 10. Considering the following fractional differential equation,

$$\begin{aligned} {}^c D_{0+}^{5/2} u(t) &= t^2 + t \frac{u^2(t)}{1+u^2(t)} + \frac{1}{4\pi} u'(t) \arctan u'(t) \\ &\quad + \frac{1}{4} \sin u''(t), \quad t \in [0, 1], \end{aligned} \quad (54)$$

$$u(0) = u''(0) = 0,$$

$$u(1) = 2 \int_0^1 u(t) dt.$$

Obviously, $\alpha = 5/2$, $A(t) = 2t$, then $\int_0^1 t dA(t) = 1$, and thus it is at resonance. Let

$$f(t, u, v, w) = t^2 + t \frac{u^2}{1+u^2} + \frac{1}{4\pi} v \arctan v + \frac{1}{4} \sin w, \quad (55)$$

then

$$|f(t, u, v, w)| \leq 1 + t^2 + \frac{1}{2} |v| + \frac{1}{4} |w|, \quad (56)$$

where

$$\|c\|_\infty + \|d\|_\infty = \frac{1}{2} + \frac{1}{4} = \frac{3}{4} < \frac{\sqrt{\pi}}{2} = \Gamma\left(\frac{3}{2}\right). \quad (57)$$

Taking $F = 16$, then we have $|u'(t)| \geq 16$,

$$f(t, u(t), u'(t), u''(t)) \geq -\frac{1}{2} + \frac{4}{\pi} \arctan 1 - \frac{1}{4} > 0. \quad (58)$$

and

$$\begin{aligned} & \int_0^1 (1-s)^{\alpha-1} (Nu)(s) ds \\ & - \int_0^1 \int_0^t (t-s)^{\alpha-1} (Nu)(s) ds dA(t) \\ & = \int_0^1 (1-s)^{\alpha-1} (Nu)(s) ds \\ & - 2 \int_0^1 \int_s^1 (t-s)^{\alpha-1} dt (Nu)(s) ds \\ & = \int_0^1 \left((1-s)^{\alpha-1} - \frac{2(1-s)^\alpha}{\alpha} \right) (Nu)(s) ds \\ & = \int_0^1 (1-s)^{\alpha-1} \\ & \cdot \frac{\alpha-2+2s}{\alpha} f(s, u(s), u'(s), u''(s)) ds > 0. \end{aligned} \quad (59)$$

That is, (A_4) holds. Finally, taking $a_0 = 16$, when $|c| > 16$, we conclude that

$$\begin{aligned} & \int_0^1 (1-s)^{\alpha-1} f(s, cs, c, 0) ds \\ & - \int_0^1 \int_0^t (t-s)^{\alpha-1} f(s, cs, c, 0) ds dA(t) > 0 \end{aligned} \quad (60)$$

then (A_5) is satisfied. It follows from Theorem 4 that (54) has at least one solution.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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