Hindawi
Mathematical Problems in Engineering
Volume 2018, Article ID 5946924, 8 pages
https://doi.org/10.1155/2018/5946924

Research Article

A New Linear Difference Scheme for Generalized Rosenau-Kawahara Equation

Tao Chen , Kaili Xiang, Peimin Chen , and Xumei Luo

School of Economic Mathematics, Southwestern University of Finance and Economics, Chengdu 610074, China

Correspondence should be addressed to Tao Chen; shuxue_ch@sina.com

Received 20 December 2017; Revised 3 April 2018; Accepted 18 April 2018; Published 24 May 2018

Academic Editor: Francesca Vipiana

We introduce in this paper a new technique, a semiexplicit linearized Crank-Nicolson finite difference method, for solving the generalized Rosenau-Kawahara equation. We first prove the second-order convergence in L_∞-norm of the difference scheme by an induction argument and the discrete energy method, and then we obtain the prior estimate in L_∞-norm of the numerical solutions. Moreover, the existence, uniqueness, and satiability of the numerical solution are also shown. Finally, numerical examples show that the new scheme is more efficient in terms of not only accuracy but also CPU time in implementation.

1. Introduction

In the study of the dynamics of dense discrete systems, Rosenau in [1, 2] proposed the so-called Rosenau equation:

$$u_t + u_x + u_{xxxxt} + uu_x = 0, \tag{1}$$

as the KdV equation cannot describe the phenomena of wave-wave and wave-wall interactions. In [3], the existence and the uniqueness of the solution of (1) have been studied. For further consideration of nonlinear waves, adding the viscous terms u_{xxx} and $-u_{xxxxx}$ to (1), respectively, gives the following equations:

$$u_t + u_{xxxxt} + u_x + \alpha u_{xxx} + \gamma \left(u^p\right)_x = 0, \tag{2}$$

$$u_t + u_{xxxxt} + u_x + \alpha u_{xxx} - \beta u_{xxxxx} + \gamma \left(u^p\right)_x = 0, \tag{3}$$

where α, β, and γ are all real constants, while for the exponent, we assume that $p \geq 2$ is an integer. Equations (2) and (3) are called the generalized Rosenau-RdV equation [4–6] and the generalized Rosenau-Kawahara equation [7, 8]. When $p = 2$, they can be considered as Rosenau-KdV equation and the Rosenau-Kawahara equation, respectively.

As we all know, most of the time, we need to think of the numerical solution of nonlinear evolution equations. Many scholars in this field have carried out good work. In [9–22], the nonlinear Crank-Nicolson scheme (see [9, 10, 13, 16, 19, 21]) and the linear finite difference scheme (see [11–15, 18, 20, 22]) have been used for the generalized Rosenau-type equations, including the Rosenau-Burgers equation, the Rosenau-RLW equation, the Rosenau-KdV equation, and the Rosenau-Kawahara equation. Those classical finite difference schemes often use the formula $(u^p)_x = (p/(1+p))[u^{p-1}u_x + (u^p)_x]$ to construct second-order convergent linear finite difference scheme [11–15, 18, 20, 22]. In [23], the authors used the formula $(u^2)_x = 2\theta uu_x + (1-\theta)(u^2)_x$ and proposed a linear finite difference scheme, but the proof of the prior estimate in L_∞-norm is not perfect.

In this paper, using technique $[(U_j^n)^p]_x = p((3/2)U_j^n - (1/2)U_j^{n-1})^{p-1}(U_j^{n+1/2})_{\hat{x}}$, we propose a semiexplicit linearized Crank-Nicolson finite difference method for the generalized Rosenau-Kawahara equation (3) with an initial condition

$$u(x,0) = u_0(x), \quad x \in [x_L, x_R], \tag{4}$$

and boundary conditions

$$\begin{gathered} u(x_L,t) = u(x_R,t) = 0, \\ u(x_L,t)_x = u(x_R,t)_x = 0, \\ u(x_L,t)_{xx} = u(x_R,t)_{xx} = 0, \\ t \in (0,T]. \end{gathered} \tag{5}$$

By comparison with the classical second-order accuracy finite difference scheme on a test problem [15, 20–22], our new scheme improves the CPU time and gives a better maximal error $\|e^n\|_\infty$ of numerical solutions. But the prior estimate in L_∞-norm of the numerical solutions is very hard to obtain directly; the proofs of convergence and stability are difficult for our new schemes (7). So, the discrete energy analysis method [24] and an induction argument (see [25–27]) are used to prove the second-order convergence and stability. Furthermore, our new method has a wide range of applications for the generalized Rosenau-type equations including the Rosenau-Burgers equation, the Rosenau-KdV equation, and the Rosenau-RLW equation.

The content of this paper is organized as follows. In the following, we propose a semiexplicit linearized Crank-Nicolson finite difference scheme for initial boundary value problems (3)–(5). In Section 3, we prove the second-order convergence in L_∞-norm of the difference scheme by an induction argument and the discrete energy method, and then we obtain the prior estimate in L_∞-norm of the numerical solutions. Moreover, based on the prior estimate, the existence and uniqueness of the numerical solution are also shown. Section 4 is devoted to the numerical tests of the new scheme and shows that our scheme has reliable accuracy and spends less CPU time than the classical schemes in implementation. Finally, we finish our paper by concluding remarks in the last section.

2. Finite Difference Schemes

In this section, we give a complete description of our numerical method for the initial value problem (3)–(5). We first give some notation which will be used in this paper. As usual, we let J, N be any positive integers. Let $h = (x_R - x_L)/J$, $\tau = T/N$; the domain of solution is defined to be $\Omega_{h\tau} = \Omega_h \times \Omega_\tau$, which is covered by uniform grid $\Omega_h = \{x_j = x_L + jh;\ j = 0, 1, \ldots, J\}$, $\Omega_\tau = \{t_n = n\tau; n = 0, 1, \ldots, N\}$. Suppose $U = \{U_j^n \mid j = 0, 1, \ldots, J,\ n = 0, 1, \ldots, N\}$ is a discrete function on $\Omega_{h\tau}$ and denote $u(x_j, t_n) \equiv u_j^n, u(x_j, t_n) \approx U_j^n$, $Z_h^0 = \{U = (U_j) \mid U_{-2} = U_{-1} = U_0 = U_J = U_{J+1} = U_{J+2} = 0, j = -2, -1, 0, 1, \ldots, \ldots, J, J+1, J+2\}$, C as a general positive constant independent of step sizes h and τ which may have different values at different occurrences. Introduce the following notations:

$$
\begin{aligned}
\left(U_j^n\right)_x &= \frac{U_{j+1}^n - U_j^n}{h}, \\
\left(U_j^n\right)_{\bar{x}} &= \frac{U_j^n - U_{j-1}^n}{h}, \\
\left(U_j^n\right)_{\hat{x}} &= \frac{U_{j+1}^n - U_{j-1}^n}{2h}, \\
\left(U_j^n\right)_t &= \frac{U_j^{n+1} - U_j^n}{\tau}, \\
\left(U_j^{n+1/2}\right) &= \frac{U_j^{n+1} + U_j^n}{2}, \\
\|U^n\|^2 &= \langle U^n, U^n \rangle, \\
\langle U^n, V^n \rangle &= h \sum_{j=0}^{J-1} U_j^n V_j^n, \\
\|U^n\|_\infty &= \max_{0 \le j \le J-1} \left|U_j^n\right|.
\end{aligned} \tag{6}
$$

In this paper, we propose a new linear finite difference scheme for the generalized Rosenau-Kawahara equation (3)–(5) which is written as

$$
\begin{aligned}
&\left(U_j^n\right)_t + \left(U_j^n\right)_{xx\bar{x}\bar{x}t} + \left(U_j^{n+1/2}\right)_{\hat{x}} + \alpha \left(U_j^{n+1/2}\right)_{x\bar{x}\hat{x}} \\
&\quad - \beta \left(U_j^{n+1/2}\right)_{xx\bar{x}\bar{x}\hat{x}} \\
&\quad + p\left((3/2) U_j^n - (1/2) U_j^{n-1}\right)^{p-1} \left(U_j^{n+1/2}\right)_{\hat{x}} = 0, \\
&U_j^0 = u_0\left(x_j\right), \quad (j = 0, 1, 2, \ldots, J) \\
&U^n \in Z_h^0, \\
&\left(U_0^n\right)_{\hat{x}} = \left(U_J^n\right)_{\hat{x}}, \\
&\left(U_0^n\right)_{x\bar{x}} = \left(U_J^n\right)_{x\bar{x}}, \\
&\qquad\qquad (n = 0, 1, 2, \ldots, N).
\end{aligned} \tag{7}
$$

This scheme is a semiexplicit linearized Crank-Nicolson scheme; the truncation error of this scheme is of order $O(h^2 + \tau^2)$. In this scheme, complicated nonlinear term $[(U_j^n)^p]_x$ is extrapolated by $p((3/2)U_j^n - (1/2)U_j^{n-1})^{p-1}(U_j^{n+1/2})_{\hat{x}}$. Thus, we only need to solve a linear system of equations in computing U_j^{n+1}. Hence, scheme (7) can be expected to be more efficient.

3. Convergence and Stability

In this section, we prove the convergence and stability of scheme (7). Let $e_j^n = u_j^n - U_j^n$, where u_j^n and U_j^n are the solutions of (3)–(5) and (7), respectively. We then obtain the following error equation:

$$
\begin{aligned}
r_j^n = {} & \left(e_j^n\right)_t + \left(e_j^n\right)_{xx\bar{x}\bar{x}t} + \left(e_j^{n+1/2}\right)_{\hat{x}} + \alpha \left(e_j^{n+1/2}\right)_{x\bar{x}\hat{x}} \\
& - \beta \left(e_j^{n+1/2}\right)_{xx\bar{x}\bar{x}\hat{x}} + R_j^n,
\end{aligned} \tag{8}
$$

where

$$
\begin{aligned}
R_j^n = {} & p\left((3/2) u_j^n - (1/2) u_j^{n-1}\right)^{p-1} \left(u_j^{n+1/2}\right)_{\hat{x}} \\
& - p\left((3/2) U_j^n - (1/2) U_j^{n-1}\right)^{p-1} \left(U_j^{n+1/2}\right)_{\hat{x}},
\end{aligned} \tag{9}
$$

and r_j^n denotes the truncation error. By using Taylor expansion at $(x_j, t_{n+1/2})$, we easily obtain that the truncation error of scheme satisfies

$$
r_j^n = O\left(h^2 + \tau^2\right), \quad \tau, h \longrightarrow 0. \tag{10}
$$

The following lemma is a property of scheme (7); we can obtain that directly from the boundary conditions and notations. This is a well known result, which is essential for existence, uniqueness, convergence, and stability of our numerical solution.

Lemma 1 (see [18, 24]). *For any two discrete functions $U, V \in Z_h^0$, one has*

$$\begin{aligned}\langle U_x, V\rangle &= -\langle U, V_{\bar{x}}\rangle,\\ \langle U_{\hat{x}}, V\rangle &= -\langle U, V_{\hat{x}}\rangle,\\ \langle U_{x\bar{x}}, V\rangle &= -\langle U_x, V_x\rangle,\end{aligned} \tag{11}$$

and then one has

$$\begin{aligned}\langle U, U_{xx}\rangle &= -\|U_x\|,\\ \langle U, U_{\hat{x}}\rangle &= 0,\\ \langle U, U_{x\bar{x}\hat{x}}\rangle &= 0,\\ \langle U, U_{xx\bar{x}\bar{x}\hat{x}}\rangle &= 0.\end{aligned} \tag{12}$$

Furthermore, if $(U_0)_{xx} = (U_J)_{xx} = 0$, then

$$\begin{aligned}\langle U, U_{xx\bar{x}\bar{x}}\rangle &= \|U_{xx}\|,\\ \langle 2U^{n+1/2}, U_{xx\bar{x}\bar{x}t}\rangle &= \frac{1}{\tau}\left(\left\|U_{x\bar{x}}^{n+1}\right\|^2 - \left\|U_{x\bar{x}}^{n}\right\|^2\right).\end{aligned} \tag{13}$$

The following lemmas including Lemmas 2 and 3 are well known and useful for the proofs of the convergence and stability. Lemma 4 can be deduced directly from the Cauchy-Schwarz inequality and Sobolev inequality.

Lemma 2 (discrete Sobolev inequality [24]). *For any discrete function $U = \{U_j^n \mid j = 0, 1, \ldots, J,\ n = 0, 1, \ldots, N\}$ on the finite interval $[x_L, x_R]$, there is the inequality*

$$\|U^n\|_\infty \le C_0 \sqrt{\|U^n\|}\sqrt{\|U_x^n\| + \|U^n\|}, \tag{14}$$

or

$$\|U^n\|_\infty \le \varepsilon \|U_x^n\| + K(\varepsilon)\|U^n\|. \tag{15}$$

Here, ε, C_0, and $K(\varepsilon)$ are three constants independent of $U = \{U_j^n \mid j = 0, 1, \ldots, J,\ n = 0, 1, \ldots, N\}$ and τ, h. ε can be any small and $K(\varepsilon)$ is a constant dependent on ε.

Lemma 3 (discrete Gronwall's inequality [24]). *Suppose that the discrete function $\{\omega^n \mid n = 0, 1, \ldots, N;\ N\tau = T\}$ satisfies the inequality*

$$\omega^n \le A + \tau\sum_{l=1}^{n} B_l \omega^l, \tag{16}$$

where A and B_l $(l = 0, 1, \ldots, N)$ are nonnegative constants. Then,

$$\max_{1\le n\le N}|\omega^n| \le A\exp\left(2\tau\sum_{l=1}^{N} B_l\right), \tag{17}$$

where τ is sufficiently small, such that $\tau \cdot \max_{1\le n\le N} B_l \le 1/2$.

Lemma 4 (see [22]). *Suppose that $u_0 \in H_0^2[x_L, x_R]$. Then, the solution of problems (3)–(5) satisfies*

$$\begin{aligned}\|u\|_{L_2} &\le C,\\ \|u_x\|_{L_2} &\le C,\\ \|u_{xx}\|_{L_2} &\le C,\\ \|u\|_{L_\infty} &\le C,\\ \|u_x\|_{L_\infty} &\le C.\end{aligned} \tag{18}$$

It should be pointed out that the induction argument is very useful for proving the convergence of a difference scheme whose prior estimate is difficult to obtain directly (see [25–27]). The following theorem shows the convergence of our scheme (7) with the convergence rate $O(\tau^2 + h^2)$ in the L_∞-norm by an induction argument.

Theorem 5. *Suppose $u_0 \in H_0^2[x_L, x_R]$, and $u(x,t) \in C_{x,t}^{7,3}$; then the solution of the difference problem (7) converges to the solution of problem (3)–(5) with order $O(\tau^2 + h^2)$ in the L_∞-norm, if*

$$\begin{aligned}&\frac{3}{2}C_0 \cdot \max\left(C_{n-1}, C_n\right)\left(\tau^2 + h^2\right) \le 1,\\ &\tau \le \frac{1}{2^{p+1}3p(p-1)(C_u+1)^{p-1}}.\end{aligned} \tag{19}$$

Proof. We use the mathematical induction to prove it. First, from (10) and Lemma 4, we have

$$\begin{aligned}&\left\|r_j^n\right\|_\infty \le C_r,\\ &\left\|u_j^n\right\|_\infty \le C_u,\\ &\qquad n = 0, 1, 2, \ldots, N,\end{aligned} \tag{20}$$

where C_r and C_u are two constants independent of τ and h. It follows from the initial conditions that

$$\begin{aligned}\left\|e^0\right\| &= 0,\\ \left\|e_{xx}^0\right\| &= 0,\\ \left\|U^0\right\|_\infty &= 0.\end{aligned} \tag{21}$$

We also can get U^1 by the C-N scheme. Hence, the following estimate holds:

$$\left\|e^1\right\| + \left\|e_{xx}^1\right\| \le \left(\tau^2 + h^2\right)C_1. \tag{22}$$

Now, assume that

$$\begin{aligned}&\left\|e^l\right\| + \left\|e_{xx}^l\right\| \le \left(\tau^2 + h^2\right)C_l,\\ &\qquad l = 0, 1, 2, \ldots, n,\ (0 \le n \le \mathrm{N}),\end{aligned} \tag{23}$$

where C_l is a constant independent of τ and h. Using Lemma 2 and Cauchy-Schwarz inequality, we get

$$\begin{aligned}\left\|e^l\right\|_\infty &\le C_0\sqrt{\left\|e^l\right\|}\sqrt{\left\|e_x^l\right\|+\left\|e^l\right\|} \le \frac{1}{2}C_0\left(2\left\|e^l\right\|+\left\|e_x^l\right\|\right)\\ &\le \frac{3}{2}\left(\tau^2+h^2\right)C_0C_l,\\ &\qquad l=0,1,2,\ldots,n,\ (0\le n\le N),\end{aligned} \tag{24}$$

$$\begin{aligned}\left\|U^l\right\|_\infty &\le \left\|u^l\right\|_\infty+\left\|e^l\right\|_\infty \le C_u+\frac{3}{2}\left(\tau^2+h^2\right)C_0C_l,\\ &\qquad l=0,1,2,\ldots,n,\ (0\le n\le N).\end{aligned}$$

Now, computing the inner product of error equation (8) with $e^{n+1/2}$ and using boundary condition (5) and Lemma 1, we obtain

$$\begin{aligned}&\frac{1}{2\tau}\left(\left\|e^{n+1}\right\|^2-\left\|e^n\right\|^2\right)+\frac{1}{2\tau}\left(\left\|e_{xx}^{n+1}\right\|^2-\left\|e_{xx}^n\right\|^2\right)\\ &=\left\langle r^n,e^{n+1/2}\right\rangle-\left\langle R^n,e^{n+1/2}\right\rangle=\left\langle r^n,e^{n+1/2}\right\rangle\\ &-ph\sum_{j=1}^{J-1}\left(\frac{3}{2}U_j^n-\frac{1}{2}U_j^{n-1}\right)^{p-1}\left(e_j^{n+1/2}\right)_{\hat{x}}e_j^{n+1/2}\\ &-ph\sum_{j=1}^{J-1}\left(\frac{3}{2}e_j^n-\frac{1}{2}e_j^{n-1}\right)\left[\sum_{k=0}^{p-2}\left(\frac{3}{2}u_j^n-\frac{1}{2}u_j^{n-1}\right)^{p-2-k}\right.\\ &\left.\cdot\left(\frac{3}{2}U_j^n-\frac{1}{2}U_j^{n-1}\right)^k\left(u_j^{n+1/2}\right)_{\hat{x}}e_j^{n+1/2}\right].\end{aligned} \tag{25}$$

Using Lemma 1, Lemma 4, Cauchy-Schwarz inequality, and (24) for

$$\frac{3}{2}C_0\cdot\max\left(C_{n-1},C_n\right)\left(\tau^2+h^2\right)\le 1, \tag{26}$$

we have

$$\begin{aligned}&-ph\sum_{j=1}^{J-1}\left(\frac{3}{2}U_j^n-\frac{1}{2}U_j^{n-1}\right)^{p-1}\left(e_j^{n+1/2}\right)_{\hat{x}}e_j^{n+1/2}\le p\left[2C_u\right.\\ &\left.+3\left(\tau^2+h^2\right)C_0\cdot\max\left(C_{n-1},C_n\right)\right]^{p-1}\\ &\cdot h\sum_{j=1}^{J-1}\left|\left(e_j^{n+1/2}\right)_{\hat{x}}\right|\cdot\left|e_j^{n+1/2}\right|\le p2^{p-2}\left(C_u+1\right)^{p-1}\\ &\cdot\left(\left\|e^{n+1/2}\right\|^2+\left\|e_x^{n+1/2}\right\|^2\right)\le p2^{p-4}\left(C_u+1\right)^{p-1}\\ &\cdot\left(3\left\|e^{n+1}\right\|^2+3\left\|e^n\right\|^2+\left\|e_{xx}^{n+1}\right\|^2+\left\|e_{xx}^n\right\|^2\right)\\ &\qquad (0\le n\le N-1),\end{aligned} \tag{27}$$

$$\begin{aligned}&-ph\sum_{j=1}^{J-1}\left(\frac{3}{2}e_j^n-\frac{1}{2}e_j^{n-1}\right)\left[\sum_{k=0}^{p-2}\left(\frac{3}{2}u_j^n-\frac{1}{2}u_j^{n-1}\right)^{p-2-k}\right.\\ &\left.\cdot\left(\frac{3}{2}U_j^n-\frac{1}{2}U_j^{n-1}\right)^k\left(u_j^{n+1/2}\right)_{\hat{x}}e_j^{n+1/2}\right]\\ &\le p\sum_{k=0}^{p-2}\left[\left(2C_u\right)^{p-2-k}\right.\\ &\cdot\left(2C_u+3\left(\tau^2+h^2\right)C_0\cdot\max\left(C_{n-1},C_n\right)\right)^k\\ &\left.\cdot C_uh\sum_{j=0}^{J-1}\left|\frac{3}{2}e_j^n-\frac{1}{2}e_j^{n-1}\right|\cdot\left|e_j^{n+1/2}\right|\right]\le p\left(p-1\right)\\ &\cdot 2^{p-2}C_u\left(C_u+1\right)^{p-2}h\sum_{j=0}^{J-1}\left|\frac{3}{2}e_j^n-\frac{1}{2}e_j^{n-1}\right|\cdot\left|e_j^{n+1/2}\right|\\ &\le p\left(p-1\right)2^{p-4}\left(C_u+1\right)^{p-1}\left(3\left\|e^n\right\|^2+4\left\|e^{n+1/2}\right\|^2\right.\\ &\left.+\left\|e^{n-1}\right\|^2\right)\le p\left(p-1\right)2^{p-4}\left(C_u+1\right)^{p-1}\left(2\left\|e^{n+1}\right\|^2\right.\\ &\left.+5\left\|e^n\right\|^2+\left\|e^{n-1}\right\|^2\right),\quad (0\le n\le N-1).\end{aligned} \tag{28}$$

On the other hand,

$$\begin{aligned}\left\langle r^n,e^{n+1/2}\right\rangle &= \frac{1}{2}\left\langle r^n,e^{n+1}+e^n\right\rangle\\ &\le \frac{1}{2}\left\|r^n\right\|^2+\frac{1}{4}\left(\left\|e^{n+1}\right\|^2+\left\|e^n\right\|^2\right).\end{aligned} \tag{29}$$

Substituting (27)-(29) into (25), we obtain

$$\begin{aligned}&\left(\left\|e^{n+1}\right\|^2-\left\|e^n\right\|^2\right)+\left(\left\|e_{xx}^{n+1}\right\|^2-\left\|e_{xx}^n\right\|^2\right)\le\tau\left\|r^n\right\|^2\\ &+\frac{\tau}{2}\left(\left\|e^{n+1}\right\|^2+\left\|e^n\right\|^2\right)+\tau p2^{p-3}\left(C_u+1\right)^{p-1}\\ &\cdot\left(3\left\|e^{n+1}\right\|^2+3\left\|e^n\right\|^2+\left\|e_{xx}^{n+1}\right\|^2+\left\|e_{xx}^n\right\|^2\right)+\tau p\left(p\right.\\ &\left.-1\right)2^{p-3}\left(C_u+1\right)^{p-1}\left(2\left\|e^{n+1}\right\|^2+5\left\|e^n\right\|^2\right.\\ &\left.+\left\|e^{n-1}\right\|^2\right)\le\tau\left\|r^n\right\|^2+\tau p\left(p-1\right)2^p\left[\left(C_u+1\right)^{p-1}\right.\\ &\left.\cdot\left(\left\|e^{n+1}\right\|^2+\left\|e^n\right\|^2+\left\|e^{n-1}\right\|^2+\left\|e_{xx}^{n+1}\right\|^2+\left\|e_{xx}^n\right\|^2\right)\right],\\ &\qquad (0\le n\le N-1).\end{aligned} \tag{30}$$

Summing up (30) from 0 to $N-1$, we get

$$\begin{aligned}&\left\|e^N\right\|^2+\left\|e_{xx}^N\right\|^2\\ &\le\left\|e^0\right\|^2+\left\|e_{xx}^0\right\|^2+\tau\sum_{n=0}^{N-1}\left\|r^n\right\|^2\\ &+\tau\sum_{n=0}^{N}3p\left(p-1\right)2^p\left(C_u+1\right)^{p-1}\left(\left\|e^n\right\|^2+\left\|e_{xx}^n\right\|^2\right).\end{aligned} \tag{31}$$

Taking small τ and h such that

$$\tau \sum_{k=0}^{N-1} \|r^n\|^2 \le \tau N \max_{1\le n\le N} \|r^n\|^2 \le T\left(\tau^2+h^2\right)^2 (C_r)^2, \quad (32)$$

then using (22) and Lemma 3, for

$$\tau \le \frac{1}{3p(p-1)2^{p+1}(C_u+1)^{p-1}}, \quad (33)$$

we have

$$\begin{aligned} &\|e^N\|^2 + \|e^N_{xx}\|^2 \\ &\quad \le \left(\tau^2+h^2\right)^2 (C_r)^2 Te^{2T(3p(p-1)2^p(C_u+1)^{p-1})} \\ &\quad \le \left(\tau^2+h^2\right)^2 (C_{n+1})^2, \quad (n=0,1,2,\ldots,N-1), \end{aligned} \quad (34)$$

where $C_{n+1} = C_r\sqrt{T}e^{3p(p-1)T2^p(C_u+1)^{p-1}}$ is a constant independent of n. By an induction argument,

$$\begin{aligned} &\|e^n\| \le O\left(\tau^2+h^2\right), \\ &\|e^n_{xx}\| \le O\left(\tau^2+h^2\right), \\ &\qquad (n=0,1,2,\ldots,N). \end{aligned} \quad (35)$$

From Cauchy-Schwarz inequality, we can obtain

$$\|e^n_x\| \le O\left(\tau^2+h^2\right), \quad (n=0,1,2,\ldots,N). \quad (36)$$

By using Lemma 2, it is shown that

$$\|e^n\|_\infty \le O\left(\tau^2+h^2\right), \quad (n=0,1,2,\ldots,N). \quad (37)$$

This completes the proof of Theorem 5. □

The following theorem guarantees that the numerical solution obtained from the difference scheme (7) is bounded. This can be proved from (24) of Theorem 5.

Theorem 6. *Suppose $u_0 \in H_0^2[x_L, x_R]$, and $u(x,t) \in C^{7,3}_{x,t}$; if τ and h are small enough, then, for $n = 1,2,\ldots,N$, the inequality*

$$\left\|U^n_j\right\|_\infty \le \widetilde{C} \quad (38)$$

holds, where $\widetilde{C}$ is a constant independent of τ and h.

By a similar proof of Theorem 5, we can obtain the following theorems. Theorem 7 shows that the solution of difference scheme (7) is stable in L_∞-norm. Theorem 8 guarantees the existence and uniqueness of numerical solution.

Theorem 7. *Under the conditions of Theorem 5, the solution of scheme (7) is stable in L_∞-norm for the initial values.*

Theorem 8. *There exists a unique solution for difference scheme (7).*

Another similar semiexplicit scheme for the generalized Rosenau-Kawahara equation (3)–(5) is written as

$$\begin{aligned} &\left(U^n_j\right)_t + \left(U^n_j\right)_{xx\bar{x}\bar{x}t} + \left(U^{n+1/2}_j\right)_{\hat{x}} + \alpha\left(U^{n+1/2}_j\right)_{x\bar{x}\hat{x}} \\ &\quad - \beta\left(U^{n+1/2}_j\right)_{xx\bar{x}\bar{x}\hat{x}} \\ &\quad + p\left(\frac{3}{2}\left(U^n_j\right)^{p-1} - \frac{1}{2}\left(U^{n-1}_j\right)^{p-1}\right)\left(U^{n+1/2}_j\right)_{\hat{x}} = 0, \\ &U^0_j = u_0\left(x_j\right), \quad (j=0,1,2,\ldots,J) \\ &U^n \in Z^0_h, \\ &\left(U^n_0\right)_{\hat{x}} = \left(U^n_J\right)_{\hat{x}}, \\ &\left(U^n_0\right)_{x\bar{x}} = \left(U^n_J\right)_{x\bar{x}}, \\ &\qquad (n=0,1,2,\ldots,N). \end{aligned} \quad (39)$$

Its second-order convergence in the L_∞-norm and stability can be proved by a similar proof to that of scheme (7) in this paper.

4. Numerical Examples

In this section, we compute three numerical examples to demonstrate and validate the effectiveness of our difference scheme. As a test problem for the scheme proposed here, we chose three test problems for which exact solution or numerical solutions have been reported previously. For the Rosenau-KdV, Rosenau-Kawahara, and generalized Rosenau-Kawahara equations, the parameters used by other researchers [15, 20, 22] to obtain their results were taken as a guiding principle for our computations. As scheme (7) is a semiexplicit scheme, which is a linear system about U^{n+1}_j, we use the Thomas algorithm to solve the system. All the numerical experiments were executed on a 3.20 GHz computer, with 8 G RAM, running Matlab 2013a.

For convenience, we denote the new linear difference scheme (7) as *New*. In [15], we denote the linear difference scheme as *Linear 1*. In [20], we denote the linear difference scheme as *Linear 2* when $p = 2$. In [22], we denote the linear difference scheme as *Linear 3* when $p = 8$. We will measure the accuracy of the proposed scheme using the maximum norm errors defined by $\|e\|_\infty = \|u^n - U^n\|_\infty$. The second-order convergence of the numerical solutions is verified directly from $\|e^n(h,\tau)\|_\infty / \|e^n(h/2,\tau/2)\|_\infty$.

4.1. Example 1. Consider the Rosenau-KdV equation with the initial condition

$$\begin{aligned} &u_0(x) \\ &\quad = \left(-\frac{35}{48} + \frac{35}{624}\sqrt{313}\right)\operatorname{sech}^4\left(\frac{1}{24}\sqrt{-26+2\sqrt{313}}x\right) \end{aligned} \quad (40)$$

TABLE 1: Comparison of the maximal errors $\|e\|_\infty$ and CPU time for the Rosenau-KdV equation at $t = 40$ with $\alpha = 1$, $\beta = 0$, and $p = 2$.

Scheme	$h=\tau=1/10$		$h=\tau=1/20$		$h=\tau=1/40$	
	$\|e\|_\infty$	CPU(s)	$\|e\|_\infty$	CPU(s)	$\|e\|_\infty$	CPU(s)
Linear 1 [15]	9.39*e* − 4	4.11	2.35*e* − 4	14.85	5.91*e* − 5	69.06
New	4.66*e* − 4	4.02	1.17*e* − 4	14.07	2.95*e* − 5	66.66

TABLE 2: The numerical verification of theoretical accuracy $O(h^2+\tau^2)$ for Rosenau-KdV equation at different time at $t = 40$ with $\alpha = 1$, $\beta = 0$, and $p = 2$.

t	$\|e^n(h,\tau)\|_\infty / \|e^n(h/2,\tau/2)\|_\infty$		
	$h=\tau=1/10$	$h=\tau=1/20$	$h=\tau=1/40$
$t = 10$	-	3.9730964	4.0264475
$t = 20$	-	3.9871826	4.0049147
$t = 30$	-	3.9940428	3.9816829
$t = 40$	-	3.9982009	3.9577326

and the boundary conditions

$$\begin{gathered} u(x_L,t) = u(x_R,t) = 0, \\ u(x_L,t)_x = u(x_R,t)_x = 0, \\ u(x_L,t)_{xx} = u(x_R,t)_{xx} = 0, \\ 0 \le t \le T. \end{gathered} \tag{41}$$

It is known that the solitary wave solution [4] is

$$u_0(x) = \left(-\frac{35}{48} + \frac{35}{624}\sqrt{313}\right)\operatorname{sech}^4\left(\frac{1}{24} \cdot \sqrt{-26+2\sqrt{313}}\left(x - \left(\frac{1}{2} + \frac{1}{26}\sqrt{313}\right)t\right)\right). \tag{42}$$

The results in terms of the maximal norm errors and CPU time at the time $T = 40$ using $\alpha = 1$, $\beta = 0$, $x_L = -80$, $x_R = 120$, and $p = 2$ are reported in Table 1. It can be seen that the computational efficiency of the present new method is slightly better than that of the method in [15], in terms of grid point number. As shown in Table 2, the second-order convergence in L_∞-norm of the new schemes verifies the correction of the theoretical analysis. In Figure 1, plots of maximal errors from the four schemes are presented when $\alpha = 1$, $\beta = 0$, $p = 2$, and $\tau = h = 1/40$. Clearly, our proposed new scheme gives smaller maximal error than the scheme in [15].

4.2. Example 2. According to [7, 8], when $\alpha = \beta = 1$, $p = 2$, the solitary wave solution of the initial boundary problem (3)–(5) is

$$u_0(x) = \left(-\frac{35}{24} + \frac{35}{312}\sqrt{205}\right)\operatorname{sech}^4\left(\frac{1}{12} \cdot \sqrt{-13+\sqrt{205}}\left(x - \left(\frac{1}{13} + \sqrt{205}\right)t\right)\right), \tag{43}$$

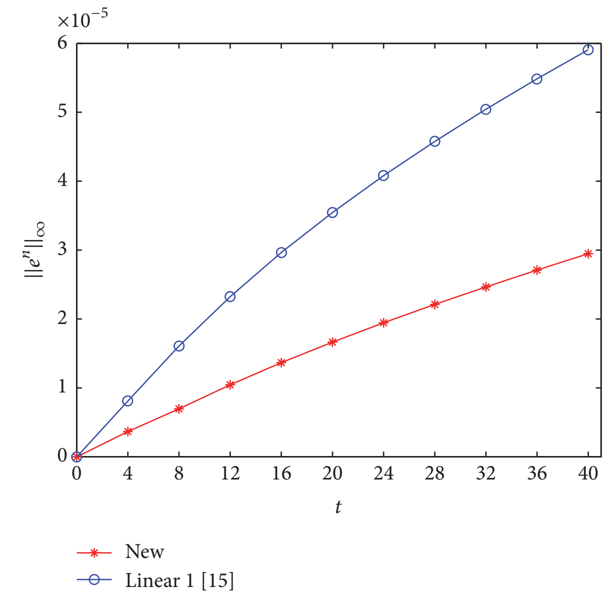

FIGURE 1: Comparison between maximal errors of numerical solutions computed by scheme [15] and new scheme for Rosenau-KdV equation with $\alpha = 1$, $\beta = 0$, $h = \tau = 1/40$, and $p = 2$.

and when $\alpha = \beta = 1$, $p = 8$, the solitary wave solution [7] is

$$u(x,t) = \left(-\frac{35}{1224}\left(-85+\sqrt{7549}\right)\right)^{1/7} \cdot \operatorname{sech}^{4/7}\left(\frac{7}{36}\sqrt{-85+\sqrt{7549}}\left(x - \frac{\sqrt{7549}}{85}t\right)\right). \tag{44}$$

The results in terms of the maximal norm errors and CPU time at the time $T = 40$ using $x_L = -80$, $x_R = 120$ are reported in Tables 3 and 4. It is clear from Tables 3 and 4 that results by our new method show improvement over the previous one reported by [20, 22]. As shown in Table 5, the second-order convergence of the numerical solutions is verified. In Figures 2 and 3, the graphs of the maximal errors $\|e\|_\infty$ are presented; the graphs show that our method gives

Table 3: Comparison of the maximal errors $\|e\|_\infty$ and CPU time for Rosenau-Kawahara equation at $t = 40$ with $\alpha = 1$, $\beta = 1$, and $p = 2$.

Scheme	$h = \tau = 1/10$		$h = \tau = 1/20$		$h = \tau = 1/40$	
	$\|e\|_\infty$	CPU(s)	$\|e\|_\infty$	CPU(s)	$\|e\|_\infty$	CPU(s)
Linear 2 [20]	$2.51e-4$	6.01	$6.28e-5$	22.60	$1.56e-5$	100.43
New	$1.19e-4$	5.87	$3.00e-5$	22.45	$7.44e-6$	97.91

Table 4: Comparison of the maximal errors $\|e\|_\infty$ and CPU time for generalized Rosenau-Kawahara equation at $t = 40$ with $\alpha = 1$, $\beta = 1$, and $p = 8$.

Scheme	$h = \tau = 1/10$		$h = \tau = 1/20$		$h = \tau = 1/40$	
	$\|e\|_\infty$	CPU(s)	$\|e\|_\infty$	CPU(s)	$\|e\|_\infty$	CPU(s)
Linear 3 [22]	$1.01e-3$	7.47	$2.53e-4$	28.84	$6.62e-5$	124.68
New	$3.97e-4$	6.89	$9.77e-5$	26.32	$2.84e-5$	114.59

Table 5: The numerical verification of theoretical accuracy $O(h^2 + \tau^2)$ for generalized Rosenau-Kawahara equation.

t	$p = 2$			$p = 8$		
	$h = 1/10$	$h = 1/20$	$h = 1/40$	$h = 1/10$	$h = 1/20$	$h = 1/40$
$t = 10$	-	3.9760462	4.0631134	-	4.0350242	3.6676336
$t = 20$	-	3.9878186	4.0811327	-	4.0699563	3.5392528
$t = 30$	-	3.9915570	4.0556430	-	4.0749519	3.7479758
$t = 40$	-	3.9933012	4.0316195	-	4.0701304	3.5400894

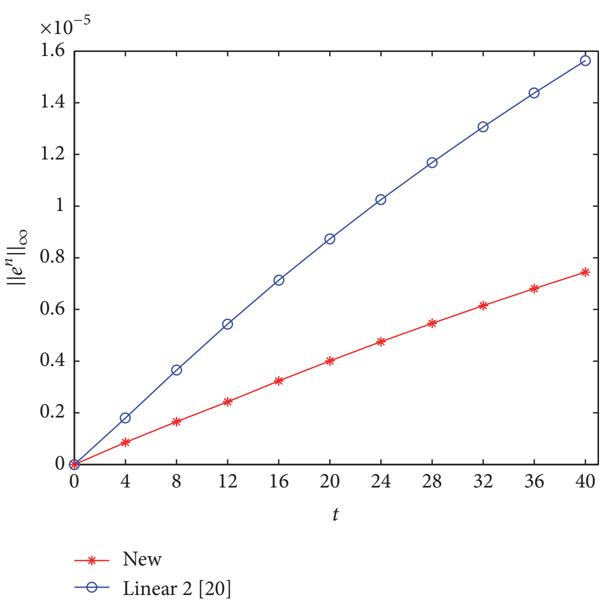

Figure 2: Comparison between maximal errors of numerical solutions computed by scheme [20] and new scheme for Rosenau-Kawahara equation with $\alpha = 1$, $\beta = 1$, $h = \tau = 1/40$, and $p = 2$.

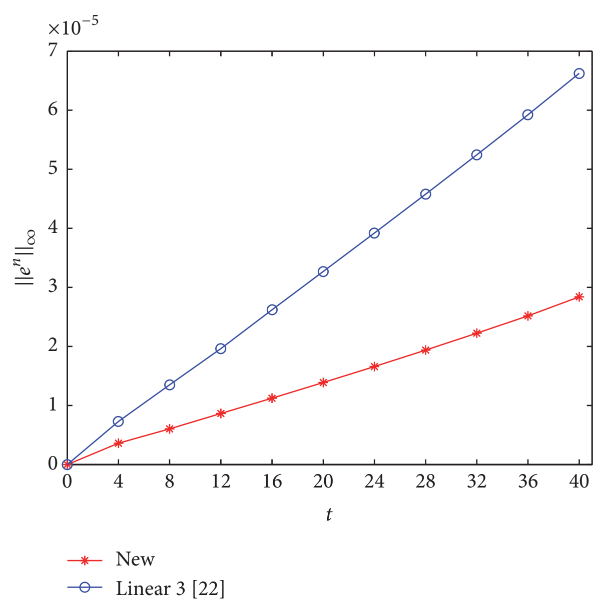

Figure 3: Comparison between maximal errors of numerical solutions computed by scheme [22] and new scheme for generalized Rosenau-Kawahara with $\alpha = 1$, $\beta = 1$, $h = \tau = 1/40$, and $p = 8$.

a better approximate solution than the scheme proposed in [20, 22]. Moreover, our proposed scheme gives less CPU time than [20, 22].

5. Conclusion

In brief, we first proposed a semiexplicit linearized Crank-Nicolson finite difference scheme for generalized Rosenau-Kawahara equation. We prove the second-order convergence in L_∞-norm of the difference scheme and then obtain the prior estimate in L_∞-norm of the numerical solutions. The stability, existence, and uniqueness of the numerical solution are also shown. Finally, some examples were given to show the efficiency of the new scheme. For future research, our new method has a wide range of applications for some nonlinear wave equations including the generalized Rosenau-type equations, Ginzburg-Landau equation, and a generalization of the BBM equation [28].

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Acknowledgments

This work was supported by the Fundamental Research Funds for the Central Universities (nos. JBK1803014, JBK170940, and JBK151126) and Applied Basic Research Programs of Sichuan Province Science and Technology Department (2016JY0133).

References

[1] P. Rosenau, "A quasi-continuous description of a nonlinear transmission line," *Physica Scripta*, vol. 34, pp. 827–829, 1986.

[2] P. Rosenau, "Dynamics of dense discrete systems," *Progress of Theoretical and Experimental Physics*, vol. 79, pp. 1028–1042, 1988.

[3] M. A. Park, "On the Rosenau equation," *Matemática Aplicada e Computacional*, vol. 9, no. 2, pp. 145–152, 1990.

[4] A. Esfahani, "Solitary wave solutions for generalized Rosenau-KdV equation," *Communications in Theoretical Physics*, vol. 55, no. 3, pp. 396–398, 2011.

[5] A. Saha, "Topological 1-soliton solutions for the generalized Rosenau-KdV equation," *Fundamental Journal of Mathematical Physics*, vol. 2, pp. 19–25, 2012.

[6] P. Razborova, H. Triki, and A. Biswas, "Perturbation of dispersive shallow water waves," *Ocean Engineering*, vol. 63, pp. 1–7, 2013.

[7] A. Biswas, H. Triki, and M. Labidi, "Bright and dark solitons of the Rosenau-Kawahara equation with power law nonlinearity," *Physics of Wave Phenomena*, vol. 19, no. 1, pp. 24–29, 2011.

[8] J.-M. Zuo, "Solitons and periodic solutions for the Rosenau-KdV and Rosenau-Kawahara equations," *Applied Mathematics and Computation*, vol. 215, no. 2, pp. 835–840, 2009.

[9] K. Omrani, F. Abidi, and T. Achouri, "A new conservative finite difference scheme for the Rosenau equation," *Applied Mathematics and Computation*, vol. 201, no. 1-2, pp. 35–43, 2008.

[10] B. Hu, Y. Xu, and J. Hu, "Crank-Nicolson finite difference scheme for the Rosenau-Burgers equation," *Applied Mathematics and Computation*, vol. 204, no. 1, pp. 311–316, 2008.

[11] X. Pan and L. Zhang, "A new finite difference scheme for the Rosenau-Burgers equation," *Applied Mathematics and Computation*, vol. 218, no. 17, pp. 8917–8924, 2012.

[12] J. Hu, B. Hu, and Y. Xu, "Average implicit linear difference scheme for generalized Rosenau-Burgers equation," *Applied Mathematics and Computation*, vol. 217, no. 19, pp. 7557–7563, 2011.

[13] N. Atouani and K. Omrani, "On the convergence of conservative difference schemes for the 2D generalized Rosenau-Korteweg de Vries equation," *Applied Mathematics and Computation*, vol. 250, pp. 832–847, 2015.

[14] M. Zheng and J. Zhou, "An average linear difference scheme for the generalized Rosenau-KdV equation," *Journal of Applied Mathematics*, vol. 2014, Article ID 202793, 9 pages, 2014.

[15] J. Hu, Y. Xu, and B. Hu, "Conservative linear difference scheme for Rosenau-KdV equation," *Advances in Mathematical Physics*, vol. 2013, Article ID 423718, 7 pages, 2013.

[16] J. S. Hu, X. P. Xie, B. Hu, and Y. C. Xu, "Conservative Crank-Nicolson difference scheme for Rosenau-KdV equation," *Numerical Mathematics. A Journal of Chinese Universities. Gaodeng Xuexiao Jisuan Shuxue Xuebao*, vol. 37, no. 4, pp. 360–369, 2015.

[17] A. Ghiloufi and T. Kadri, "Analysis of new conservative difference scheme for two-dimensional Rosenau-RLW equation," *Applicable Analysis: An International Journal*, vol. 96, no. 7, pp. 1255–1267, 2017.

[18] X. Pan and L. Zhang, "Numerical simulation for general Rosenau-RLW equation: An average linearized conservative scheme," *Mathematical Problems in Engineering*, vol. 2012, Article ID 517818, 15 pages, 2012.

[19] J.-M. Zuo, Y.-M. Zhang, T.-D. Zhang, and F. Chang, "A new conservative difference scheme for the general rosenau-RLW equation," *Boundary Value Problems*, vol. 2010, Article ID 516260, 2010.

[20] J. Hu, Y. Xu, B. Hu, and X. Xie, "Two conservative difference schemes for Rosenau-Kawahara equation," *Advances in Mathematical Physics*, vol. 2014, Article ID 217393, 11 pages, 2014.

[21] T. Chen, *Some studies on the finite difference methods for the generalized Rosenau-Kawahara equation*, Xihua University, 2015.

[22] D. He and K. Pan, "A linearly implicit conservative difference scheme for the generalized Rosenau-Kawahara-RLW equation," *Applied Mathematics and Computation*, vol. 271, pp. 323–336, 2015.

[23] B. Wongsaijai and K. Poochinapan, "A three-level average implicit finite difference scheme to solve equation obtained by coupling the Rosenau-KdV equation and the Rosenau-RLW equation," *Applied Mathematics and Computation*, vol. 245, pp. 289–304, 2014.

[24] Y. L. Zhou, *Application of Discrete Functional Analysis to the Finite Difference Methods*, International Academic Publishers, Beijing, China, 1990.

[25] B. Y. Guo, "The convergence of numerical method for nonlinear Schrödinger equation," *Journal of Computational Mathematics. An International Journal on Numerical Methods, Analysis and Applications*, vol. 4, no. 2, pp. 121–130, 1986.

[26] T. Wang and B. Guo, "A robust semi-explicit difference scheme for the Kuramoto-Tsuzuki equation," *Journal of Computational and Applied Mathematics*, vol. 233, no. 4, pp. 878–888, 2009.

[27] Q. Xu and Q. Chang, "Difference methods for computing the Ginzburg-Landau equation in two dimensions," *Numerical Methods for Partial Differential Equations*, vol. 27, no. 3, pp. 507–528, 2011.

[28] T. B. Benjamin, J. L. Bona, and J. J. Mahony, "Model equations for long waves in nonlinear dispersive systems," *Philosophical Transactions of the Royal Society of London. Series A, Mathematical and Physical Sciences*, vol. 272, no. 1220, pp. 47–78, 1972.