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Research Article

A Note on the Existence of the Location Parameter Estimate of the Three-Parameter Weibull Model Using the Weibull Plot

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The Weibull model is one of the widely used distributions in reliability engineering. For the parameter estimation of the Weibull model, there are several existing methods. The method of the maximum likelihood estimation among others is preferred because of its attractive statistical properties. However, for the case of the three-parameter Weibull model, the method of the maximum likelihood estimation has several drawbacks. To avoid the drawbacks, the method using the sample correlation from the Weibull plot is recently suggested. In this paper, we provide the justification for using this new method by showing that the location estimate of the three-parameter Weibull model exists in a bounded interval.

1. Introduction

There are several existing methods for the location parameter estimation of the three-parameter Weibull model. One can use the method of the maximum likelihood estimation (MLE) which is preferred by many statisticians due to its attractive statistical properties. However, it is well known that the MLE method has several drawbacks for the case of the three-parameter Weibull model. For example, the global maximum can reach infinity at the singularity $\mu = \min(x_1, x_2, \ldots, x_n)$ and this singularity can result in *local* maxima of the likelihood function when it is numerically computed. For more details, see Barnard [1] and Smith and Naylor [2].

This MLE method has a convergence issue and it can also have an unfeasible value so that the location estimate of the three-parameter Weibull model can be greater than the minimum value of the observations [3, 4]. Cheng and Amin [5], Cheng and Iles [6], and Liu et al. [7] also pointed out that the likelihood function has the unbounded likelihood problem and the location parameter tends to approach the smallest observation. Huzurbazar [8] also showed that no stationary point can yield a consistent estimator, which results in no local maximum. Thus, whether a global or a local maximum is sought, the MLE is bound to fail.

To avoid the above problems, several authors, including Gumbel [3] and Vogel and Kroll [4], suggest the method of estimating the parameters using an estimate for the minimum drought. However, in order to estimate the location parameter of the three-parameter Weibull model using the methods in Gumbel [3] or Vogel and Kroll [4], one has to use the special tables provided by Gumbel [3] which are available only for limited cases. Sirvanci and Yang [9] also recommend to estimate the location parameter with $x_{(1)} - 1/n$. However, it is reported that the performance of these methods is not satisfactory. For more details, the reader is referred to Park [10]. It should be noted that this Gumbel method is improved by Park [10]. He proposes to estimate only the location parameter using the ordinary Gumbel method and estimate the other shape and scale parameters using the MLE of the two-parameter Weibull model and shows that the parameter estimates are noticeably improved by the proposed method.

Park [10] also proposed a method which maximizes the sample correlation function from the Weibull plot to estimate the location parameter of the three-parameter Weibull model. Comparing the sample correlations, the p-values, and the Anderson-Darling test statistics, he shows that his method outperforms the afore-mentioned existing methods. His method is conceptually easy to understand, simple to use and convenient for practitioners. However, the existence of the location estimate is not yet proved.

In this paper, we show that the location parameter estimate of the three-parameter Weibull model should exist in the bounded interval. Thus, unlike the MLE case, the method by Park [10] does not suffer from nonconvergence, singularity, or infeasibility issues when we calculate the location parameter numerically.

2. Weibull Plot and Correlation Coefficient from the Plot

In this section, we briefly review the Weibull plot [11] and present the sample correlation coefficient from the Weibull plot. The Weibull distribution has the respective probability density function and cumulative distribution function (CDF)

$$f(x) = \frac{\kappa}{\theta}\left(\frac{x}{\theta}\right)^{\kappa-1} e^{-(x/\theta)^{\kappa}}$$
$$\text{and } F(x) = 1 - e^{-(x/\theta)^{\kappa}}. \tag{1}$$

We let $p = F(x_p)$ for convenience. Then we have

$$\log(1-p) = -\left(\frac{x_p}{\theta}\right)^{\kappa}. \tag{2}$$

It is immediate from (2) that we have

$$\log\{-\log(1-p)\} = \kappa \log x_p - \kappa \log \theta. \tag{3}$$

It is observed that the plot of $\log\{-\log(1-p)\}$ versus $\log x_p$ is ideally a straight line with slope κ and intercept $-\kappa \log\theta$ if the data are from the Weibull distribution. The widely used Weibull probability paper in engineering reliability is based on this idea.

With real data, we need to estimate $p = F(x_p)$ to draw the Weibull plot. It should be noted that the estimation of $F(x_{(i)})$ is often called the plotting position in the statistics literature. Let $x_{(1)}, x_{(2)}, \ldots, x_{(n)}$ be the order statistics from the smallest to the largest. There are several methods of estimating $F(x_{(i)})$ in the literature. Let $p_i = \widehat{F}(x_{(i)})$ be the empirical CDF value at $x_{(i)}$ for convenience. In practice, the plotting positions such as

$$p_i = \frac{i - 3/8}{n + 1/4} \quad \text{for } n \le 10$$
$$\text{and } p_i = \frac{i - 1/2}{n} \quad \text{for } n \ge 11 \tag{4}$$

are widely used due to Blom [12] and Wilk and Gnanadesikan [13].

The Weibull plot is constructed by plotting $\log\{-\log(1 - p_i)\}$ on the vertical axis and $\log x_{(i)}$ on the horizontal axis. It should be noted that the straightness of the Weibull plot can also be used to assess the goodness-of-fit of the Weibull model. See Park [10] along with the `weibullness` R package by Park [14]. The measure of the straightness in the Weibull plot can be evaluated by calculating the sample correlation coefficient of the paired points,

$$\left(\log x_{(i)}, \log\{-\log(1-p_i)\}\right). \tag{5}$$

We let $u_i = \log x_{(i)}$ and $v_i = \log\{-\log(1-p_i)\}$ for convenience. Then the sample correlation coefficient from the Weibull plot is defined as

$$R = \frac{\sum_{i=1}^{n}(u_i - \overline{u})(v_i - \overline{v})}{\left[\sum_{i=1}^{n}(u_i - \overline{u})^2 \cdot \sum_{i=1}^{n}(v_i - \overline{v})^2\right]^{1/2}}, \tag{6}$$

where $\overline{u} = \sum u_i/n$ and $\overline{v} = \sum v_i/n$.

3. Existence of the Location Parameter Estimate of the Three-Parameter Weibull Model

In many reliability applications, failures do not occur below a certain limit which is also known as a failure-free life (FFL) parameter in the engineering literature [15]. The three-parameter Weibull model with this FFL parameter has been widely used to describe the reliability of surface-mount assemblies due to wear-out failures, etc. For more details, see Wong [16], Clech et al. [17], Drapella [18], Mitchell et al. [19], and Lam et al. [20].

It is thus reasonable to consider a lower limit to the Weibull model. This Weibull model is called the three-parameter Weibull with its CDF given by

$$F(x) = 1 - \exp\left[-\left(\frac{x-\mu}{\theta}\right)^{\kappa}\right], \tag{7}$$

where $x > \mu$. This lower limit μ is often called a location parameter.

Replacing $x_{(i)}$ by $x_{(i)} - \mu$ in (6), we can obtain the sample correlation as a function of μ

$$R(\mu) = \frac{\sum_{i=1}^{n}\left(u_i^* - \overline{u^*}\right)(v_i - \overline{v})}{\left[\sum_{i=1}^{n}\left(u_i^* - \overline{u^*}\right)^2 \cdot \sum_{i=1}^{n}(v_i - \overline{v})^2\right]^{1/2}}, \tag{8}$$

where $0 \le \mu < x_{(1)}$, $u_i^* = \log(x_{(i)} - \mu)$ and $\overline{u^*} = \sum u_i^*/n$. For more details, see Section 5 of Park [10].

It is quite reasonable to impose a condition that $0 \le \mu < x_{(1)}$ for practical applications. Then the estimate of μ is given by

$$\widehat{\mu} = \underset{0 \le \mu < x_{(1)}}{\arg\max}\, R(\mu). \tag{9}$$

Lemma 1. *The function $R(\mu)$ has the limit as*

$$\lim_{\mu \longrightarrow x_{(1)}^-} R(\mu) = \frac{\overline{v} - v_1}{\left[(1 - 1/n)\sum_{i=1}^{n}(v_i - \overline{v})^2\right]^{1/2}}. \tag{10}$$

Proof. In the following, we use the Bachmann-Landau's big $O(\cdot)$ notation. See de Bruijin [21] for more details. That is, if $f(\cdot)$ and $g(\cdot)$ are defined on the domain D, then $f(x) = O(g(x))$ means that $|f(x)| \le K|g(x)|$ for all $x \in D$ where K is a constant.

For convenience, let $\delta = x_{(1)} - \mu$ and then $u_1^* = \log\delta$. It is easily seen that as $\delta \longrightarrow 0^+$ (that is, $\mu \longrightarrow x_{(1)}^-$), we have

$$\begin{aligned}\sum_{i=1}^{n}\left(u_i^* - \overline{u^*}\right)\left(v_i - \overline{v}\right) &= \sum_{i=1}^{n} u_i^*\left(v_i - \overline{v}\right)\\ &= u_1^*\left(v_i - \overline{v}\right) + \sum_{i=2}^{n} u_i^*\left(v_i - \overline{v}\right)\\ &= u_1^*\left(v_1 - \overline{v}\right) + O(1)\end{aligned} \tag{11}$$

and

$$\begin{aligned}\sum_{i=1}^{n}\left(u_i^* - \overline{u^*}\right)^2 &= \sum_{i=1}^{n} u_i^{*2} - \frac{1}{n}\left(\sum_{i=1}^{n} u_i^*\right)^2\\ &= u_1^{*2} + \sum_{i=2}^{n} u_i^{*2} - \frac{1}{n}\left(u_1^* + \sum_{i=2}^{n} u_i^*\right)^2\\ &= \left(1 - \frac{1}{n}\right)u_1^{*2} + u_1^* \cdot O(1).\end{aligned} \tag{12}$$

It is immediate upon substituting (11) and (12) into (8) that as $\delta \longrightarrow 0^+$, we have

$$R(\mu) = \frac{u_1^*\left(v_1 - \overline{v}\right) + O(1)}{\left[(1 - 1/n)\,u_1^{*2} + u_1^* \cdot O(1)\right]^{1/2} \cdot \left[\sum_{i=1}^{n}\left(v_i - \overline{v}\right)^2\right]^{1/2}}. \tag{13}$$

We have $u_1^* \longrightarrow -\infty$ as $\delta \longrightarrow 0^+$. Thus, we let $u_1^{**} = -u_1^*$ for convenience and we then have $u_1^{**} \longrightarrow \infty$ as $\delta \longrightarrow 0^+$. Rewriting (13) using u_1^{**}, we have

$$R(\mu) = \frac{u_1^{**}\left(\overline{v} - v_1\right) + O(1)}{\left[(1 - 1/n)\,u_1^{**2} + u_1^{**} \cdot O(1)\right]^{1/2} \cdot \left[\sum_{i=1}^{n}\left(v_i - \overline{v}\right)^2\right]^{1/2}}. \tag{14}$$

By dividing both the numerator and denominator of (14) by u_1^{**}, we have

$$R(\mu) = \frac{\left(v_1 - \overline{v}\right) + O(1)/u_1^{**}}{\left[(1 - 1/n) + O(1)/u_1^{**}\right]^{1/2} \cdot \left[\sum_{i=1}^{n}\left(v_i - \overline{v}\right)^2\right]^{1/2}}. \tag{15}$$

When taking the limit of (15) as $u_1^{**} \longrightarrow \infty$ (that is, as $\delta \longrightarrow 0^+$), we have

$$\lim_{\delta \longrightarrow 0^+} R(\mu) = \frac{\overline{v} - v_1}{\left[(1 - 1/n)\sum_{i=1}^{n}\left(v_i - \overline{v}\right)^2\right]^{1/2}}, \tag{16}$$

which completes the proof. □

Lemma 2. *As $\delta \longrightarrow 0^+$, we have*

$$\frac{dR(\mu)}{d\mu} = -\sqrt{\frac{n(n-2)^2}{n-1}} \cdot \frac{\operatorname{cov}\left(U^*, V\right)}{\left[\sum_{i=1}^{n}\left(v_i - \overline{v}\right)^2\right]^{1/2}} \cdot \frac{1}{\delta\log^2\delta} + o\left(\frac{1}{\delta\log^2\delta}\right), \tag{17}$$

where $\operatorname{cov}(U^*, V)$ is the sample covariance between U^* and V and a series of n observations of U^* and V is given by $u_i^* = \log(x_{(i)} - \mu)$ and $v_i = \log\{-\log(1 - p_i)\}$ for $i = 1, 2, \ldots, n$.

Proof. Differentiating (8) with respect to μ, we have

$$\frac{dR(\mu)}{d\mu} = \frac{A \cdot B - C \cdot D}{E} \tag{18}$$

where

$$\begin{aligned}A &= \sum_{i=1}^{n}\left(w_i^* - \overline{w^*}\right)\left(v_i - \overline{v}\right),\\ B &= \sum_{i=1}^{n}\left(u_i^* - \overline{u^*}\right)^2,\\ C &= \sum_{i=1}^{n}\left(u_i^* - \overline{u^*}\right)\left(v_i - \overline{v}\right),\\ D &= \sum_{i=1}^{n}\left(u_i^* - \overline{u^*}\right)\left(w_i^* - \overline{w^*}\right),\\ E &= B^{3/2} \cdot \left[\sum_{i=1}^{n}\left(v_i - \overline{v}\right)^2\right]^{1/2},\\ u_i^* &= \log\left(x_{(i)} - \mu\right),\end{aligned} \tag{19}$$

and also

$$w_i^* = \frac{(-1)}{\left(x_{(i)} - \mu\right)}. \tag{20}$$

Again, we let $\delta = x_{(1)} - \mu$ for convenience so that we have $u_1^* = \log\delta$ and $w_1^* = (-1)/\delta$. Then we can rewrite A, B, C, and D as a function of δ as follows:

$$\begin{aligned}A(\delta) &= -\left(v_1 - \overline{v}\right)\frac{1}{\delta} + K_1,\\ B(\delta) &= \left(1 - \frac{1}{n}\right)\log^2\delta - \frac{2}{n}\sum_{i=2} u_i^*\log\delta + K_2,\\ C(\delta) &= \left(v_1 - \overline{v}\right)\log\delta + K_3,\\ D(\delta) &= -\left(1 - \frac{1}{n}\right)\frac{\log\delta}{\delta} + \frac{1}{n}\sum_{i=2}^{n} u_i^*\frac{1}{\delta} - \frac{1}{n}\sum_{i=2}^{n} w_i^*\log\delta\\ &\quad + K_4,\end{aligned} \tag{21}$$

and also

$$E(\delta) = B(\delta)^{3/2} \cdot \left[\sum_{i=1}^{n}\left(v_i - \overline{v}\right)^2\right]^{1/2}, \tag{22}$$

where

$$\begin{aligned}K_1 &= \sum_{i=2}^{n} w_i^*\left(v_i - \overline{v}\right),\\ K_2 &= \sum_{i=2}^{n} u_i^{*2} - \frac{\left(\sum_{i=2}^{n} u_i^*\right)^2}{n},\\ K_3 &= \sum_{i=2}^{n} u_i^*\left(v_i - \overline{v}\right),\end{aligned} \tag{23}$$

$$K_4 = \sum_{i=2}^{n} u_i^* w_i^* - \frac{1}{n}\left(\sum_{i=2}^{n} u_i^*\right)\left(\sum_{i=2}^{n} w_i^*\right). \tag{24}$$

It should be noted that K_1, K_2, K_3, and K_4 do not include δ.

After some tedious algebra, when $\delta \longrightarrow 0^+$, we have

$$\begin{aligned} &A(\delta) \cdot B(\delta) - C(\delta) \cdot D(\delta) \\ &\quad = \frac{n-1}{n}\left[\sum_{i=2}^{n} u_i^* v_i^* - \frac{1}{n-1}\sum_{i=2}^{n} u_i^* \sum_{i=2}^{n} v_i^*\right]\frac{1}{\delta}\log\delta \\ &\qquad + O\left(\frac{1}{\delta}\right) + O\left(\log^2\delta\right) + O(\log\delta). \end{aligned} \tag{25}$$

Since $\log\delta < 0$ as $\delta \longrightarrow 0^+$, we have $\log\delta = -|\log\delta|$. The sample covariance between U^* and V is given by

$$\operatorname{cov}(U^*, V) = \frac{1}{n-2}\left[\sum_{i=2}^{n} u_i^* v_i - \frac{1}{n-1}\sum_{i=2}^{n} u_i^* \sum_{i=2}^{n} v_i\right]. \tag{26}$$

It is easily shown that

$$O\left(\frac{1}{\delta}\right) + O\left(\log^2\delta\right) + O(\log\delta) = o\left(\frac{1}{\delta}\log\delta\right), \tag{27}$$

where $o(\cdot)$ is the Bachmann-Landau's little o-notation in de Bruijin [21] for example. That is, $f(x) = o(g(x))$ implies that $f(x)/g(x) \longrightarrow 0$ as $x \longrightarrow c$.

Substituting $\log\delta = -|\log\delta|$, (26) and (27) into (25), we have

$$\begin{aligned} &A(\delta) \cdot B(\delta) - C(\delta) \cdot D(\delta) \\ &\quad = -\frac{(n-1)(n-2)}{n} \cdot \operatorname{cov}(U^*, V) \cdot \frac{1}{\delta}|\log\delta| \\ &\qquad + o\left(\frac{1}{\delta}\log\delta\right). \end{aligned} \tag{28}$$

Similarly, we can rewrite $E(\delta)$ as

$$\begin{aligned} &E(\delta) \\ &\quad = \left[\left(1-\frac{1}{n}\right)\log^2\delta + O(\log\delta)\right]^{3/2}\left[\sum_{i=1}^{n}(v_i - \overline{v})^2\right]^{1/2}. \end{aligned} \tag{29}$$

Substituting (28) and (29) into (18), we have

$$\begin{aligned} \frac{dR(\mu)}{d\mu} &= -\frac{(n-1)(n-2)/n \cdot \operatorname{cov}(U^*, V) \cdot (1/\delta)|\log\delta| + o((1/\delta)\log\delta)}{\left[(1-1/n)\log^2\delta + O(\log\delta)\right]^{3/2}\left[\sum_{i=1}^{n}(v_i - \overline{v})^2\right]^{1/2}} \\ &= -\sqrt{\frac{n(n-2)^2}{n-1}} \cdot \frac{\operatorname{cov}(U^*, V)}{\left[\sum_{i=1}^{n}(v_i - \overline{v})^2\right]^{1/2}} \cdot \frac{1}{\delta\log^2\delta} + o\left(\frac{1}{\delta\log^2\delta}\right), \end{aligned} \tag{30}$$

which completes the proof. □

Theorem 3. *The global maximum of $R(\mu)$ exists on $[0, x_{(1)})$ with $n \geq 3$.*

Proof. The function $R(\mu)$ is continuous on $[0, x_{(1)})$. Considering the result of Lemma 1, we define

$$R(x_{(1)}) = \frac{\overline{v} - v_1}{\left[(1-1/n)\sum_{i=1}^{n}(v_i - \overline{v})^2\right]^{1/2}}. \tag{31}$$

Then $R(\mu)$ is continuous on the closed bounded interval $[0, x_{(1)}]$. Thus, the function $R(\mu)$ has a global maximum and a global minimum on $[0, x_{(1)}]$, due to Theorem 4.28 in Apostol [22]. Note that $\mu = x_{(1)}$ is a singularity point. Thus, it suffices to show that the global maximum of $R(\mu)$ is not obtained at $\mu = x_{(1)}$.

Since $u_1^* < u_2^* < \cdots < u_n^*$ and $v_1 < v_2 < \cdots < v_n$, it is easily seen that the term $\operatorname{cov}(U^*, V)$ in Lemma 2 is always positive. It is immediate from the L'Hôpital's rule that we have

$$\lim_{\delta \longrightarrow 0^+} \frac{1}{\delta\log^2\delta} = \infty. \tag{32}$$

Thus, using these with Lemma 2, we have

$$\frac{dR(\mu)}{d\mu} \longrightarrow -\infty \tag{33}$$

as $\delta \longrightarrow 0^+$ (that is, as $\mu \longrightarrow x_{(1)}^-$). Since $R(\mu)$ is differentiable on $(0, x_{(1)})$, it is easily seen that $R(x_{(1)})$ cannot be a maximum from the intermediate value property of derivatives. For more details, see Lemma 6.2.11 of Bartle and Sherbert [23]. Thus, the global maximum exists on $[0, x_{(1)})$. □

It is worthwhile to mention the lower bound of the sample correlation coefficient from the Weibull plot. It is well known that the sample correlation coefficient should be in $[-1, 1]$ in general. However, in the Weibull plot, the data and plotting positions are ordered and thus the sample correlation coefficient should be positive. Also, it should be noted that with the order statistics restriction, the sample correlation coefficient is bounded below by $1/(n-1)$ which is the best lower bound due to Hwang and Hu [24].

Finally, after the location parameter is obtained, we can estimate the other shape and scale parameters by several existing methods. We recommend the MLE method of the two-parameter Weibull for the estimation of shape and scale.

Table 1: Parameter estimates, correlations, p value, and log-likelihood under consideration.

Method	$\hat{\mu}$	$\hat{\kappa}$	$\hat{\theta}$	$R(\hat{\mu})$	p-value	log-likelihood
Proposed	139.33	1.29174	1032.2	0.98135	0.6066	−156.4056
McLE	123.46	1.15000	1081.7	0.98075	0.5870	−156.8671
Min. SSE	125.23	1.15000	1081.7	0.98086	0.5906	−156.8455

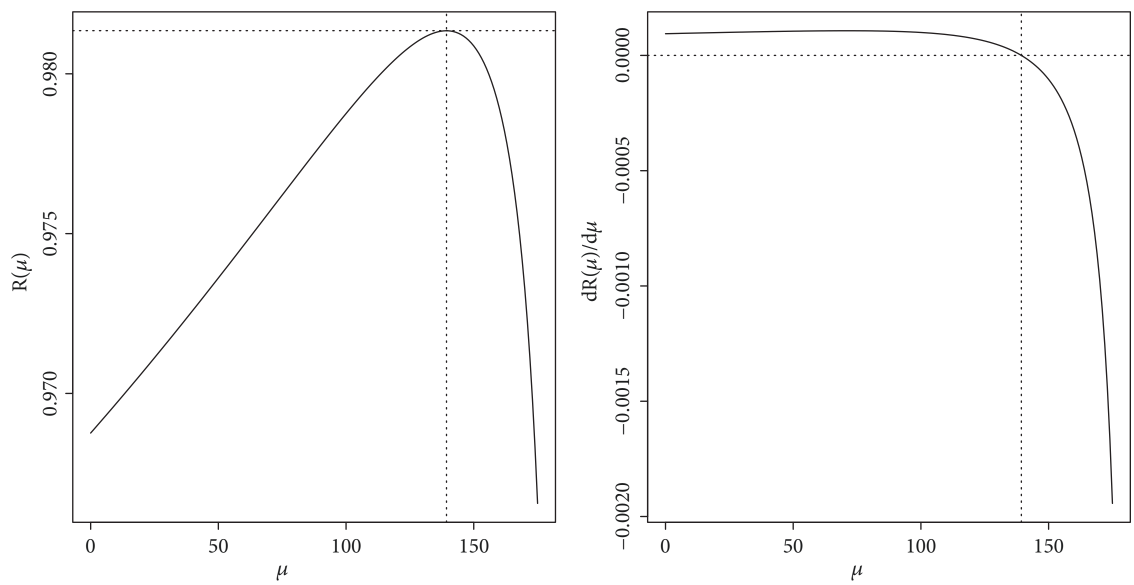

Figure 1: Correlation function, $R(\mu)$, and $dR(\mu)/d\mu$.

For more details, see Section 5 of Park [10]. Unlike the MLE of the three-parameter Weibull, the MLE of the two-parameter Weibull guarantees the existence and uniqueness due to Farnum and Booth [25].

4. An Illustrative Example

The data in this example, published in Bilikam et al. [26], are the numbers of miles to failure of a type of vehicle. This data set has since then been often used for illustration of a three-parameter Weibull distribution [6, 20].

We can estimate the location parameter by maximizing the correlation function in (8) or solving $dR(\mu)/d\mu = 0$ in (18) as shown in Figure 1 which results in $\hat{\mu} = 139.33$. As recommended earlier, we estimated the other shape and scale parameters using the MLE of the two-parameter Weibull.

In order to examine the performance of the proposed method, we compare it with other existing methods in Lam et al. [20]. They estimated the parameters using the constrained MLE (McLE) approach and the minimum SSE approach. The results are summarized in Table 1 with the corresponding $R(\hat{\mu})$, p value for Weibullness, and log-likelihood. Note that the p values for Weibullness testing were obtained using the `weibullness` R package by Park [14]. The results show that the proposed method outperforms the existing methods.

Data Availability

The article has no data used for study. So, it belongs to case 12 in the statement of the provided URL: that is, "12. No data were used to support this study".

Conflicts of Interest

The author declares that they have no conflicts of interest.

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