Research Article

Finite-Time Bounded Tracking Control for Linear Discrete-Time Systems

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A finite-time bounded tracking control problem for a class of linear discrete-time systems subject to disturbances is investigated. Firstly, by applying a difference method to constructing the error system, the problem is transformed into a finite-time boundedness problem of the output vector of the error system. In fact, this is a finite-time boundedness problem with respect to the partial variables. Secondly, based on the partial stability theory and the research methods of finite-time boundedness problem, a state feedback controller formulated in form of linear matrix inequality is proposed. Based on this, a finite-time bounded tracking controller of the original system is obtained. Finally, a numerical example is presented to illustrate the effectiveness of the controller.

1. Introduction

In 1961, Dorato proposed the concept of finite-time stability (FTS) in [1]. The main concept is that if the bound of the initial condition is given, the state of the system does not exceed a certain bound over a given finite time-interval. Since then, many scholars have conducted indepth research on FTS. In the 1960s, Kushner investigated the FTS of stochastic systems in [2]. Weiss and Infante discussed the FTS of nonlinear systems in [3, 4]. However, due to the lack of effective mathematical tools at that time, the research progress is relatively slow.

With the development of linear matrix inequality (LMI) theory, the research on FTS yielded fruitful results. In [5, 6], Amato et al. extended the concept of FTS to the linear continuous-time system with external disturbances and presented the concept of finite-time boundedness (FTB). The FTB of time-varying continuous-time systems was discussed in [7]. Subsequently, the discrete-time system was investigated in [8, 9] and further research was done in [10–12]. In

[10], the state feedback controller and output feedback controller were designed to guarantee the FTB of the discrete-time system with disturbance. In [11], the FTS of discrete systems was analyzed by using polyhedral Lyapunov function. In [12], the sufficient conditions for FTS of time-varying discrete systems were given and an output feedback controller was developed.

Following the pioneering work of Amato et al., many scholars extended the research of FTB of discrete-time systems. In [13], the finite-time control for linear discrete-time system with external disturbances was studied. The FTS of discrete-time stochastic systems with time-varying delays and its application to multiagent systems were considered in [14]. In [15], a finite-time optimal control method for a class of linear discrete-time systems with parameter variation was presented. By constructing the Lyapunov-Krasovskii functional, the FTS of discrete time-delay systems with nonlinear perturbations was studied in [16]. In [17], a robust controller was proposed to address the finite-time control problem of linear uncertain discrete systems by using an augmented LMI

method. In [18], the FTS and H_∞ control problem of discrete-time systems were discussed and a robust finite-time control scheme was provided.

On the basis of [13], the tracking control problem of linear discrete-time systems with disturbances in a finite time-interval is considered in this paper. Firstly, the error system is constructed based on the preview control theory [19, 20], and the problem is turned into a FTB problem of the output vector of the error system. Then a state feedback controller is designed for the error system via the LMI approach. Finally, a finite-time state feedback controller of the original system is derived.

Throughout this paper, the following notations are adopted. Matrix $P > 0$ (or $P < 0$) means that P is symmetric positive definite (or negative definite). $P \geq 0$ ($P \leq 0$) means that P is symmetric positive semidefinite (or negative semi-definite). $P > Q$ ($P < Q$, $P \geq Q$, and $P \leq Q$) means that $P - Q > 0$ ($P - Q < 0$, $P - Q \geq 0$, and $P - Q \leq 0$). $\lambda_{\max}(A)$ ($\lambda_{\min}(A)$) denotes the maximal (or minimal) eigenvalue of a real symmetric matrix A. diag($\ldots$) denotes a block-diagonal matrix.

2. Preliminaries and Basic Concepts

This paper considers the following linear discrete-time system:

$$x(k+1) = Ax(k) + Ew(k), \tag{1}$$

where $x(k) \in R^n$ and $w(k) \in R^p$ are the state vector and the disturbance vector of the system, respectively. $A \in R^{n \times n}$ and $E \in R^{n \times p}$ are known constant matrices.

In [10–13], the FTB problem of system (1) was investigated and its basic definition was described as follows: system (1) is said to be finite-time bounded with respect to $(\delta, d, \varepsilon, R, N)$, where $N \geq 1$, $d > 0$, $\varepsilon > 0$, $\delta > 0$, and $R > 0$, if

$$x^T(0) Rx(0) \leq \delta^2,$$

$$\sum_{k=0}^{N} w^T(k) w(k) \leq d^2 \Longrightarrow \tag{2}$$

$$x^T(k) Rx(k) \leq \varepsilon^2,$$

$$\forall k \in \{1, 2, \cdots, N\}.$$

For convenience, hereinafter, the state vector of system (1) is also said to be finite-time bounded with respect to $(\delta, d, \varepsilon, R, N)$. The object of this paper is to generalize this concept and further study the finite-time bounded tracking problem of control systems. In the following, we first propose a definition of finite-time bounded tracking. Consider the discrete-time system

$$x(k+1) = Ax(k) + Ew(k),$$

$$y(k) = Cx(k), \tag{3}$$

where $x(k) \in R^n$, $w(k) \in R^p$, $y(k) \in R^q$, $A \in R^{n \times n}$, and $E \in R^{n \times p}$.

In some practical problems, it is hoped that the output of system (3) is always located in a ε neighborhood of a reference signal under some certain conditions. This kind of problem is referred to as "finite-time bounded tracking problem." Let the reference signal be $r(k) \in R^q$. And the error signal $e(k)$ is defined as

$$e(k) = y(k) - r(k). \tag{4}$$

The concept mentioned above can be described by the following definition.

Definition 1. System (3) achieves finite-time bounded tracking of the reference signal $r(k)$ with respect to $(\delta, d, \varepsilon, R, N)$, where $N \geq 1$, $\delta > 0$, $d > 0$, $\varepsilon > 0$, and $R > 0$, if

$$e^T(0) Re(0) \leq \delta^2,$$

$$\sum_{k=0}^{N} w^T(k) w(k) \leq d^2 \Longrightarrow \tag{5}$$

$$e^T(k) Re(k) \leq \varepsilon^2,$$

$$\forall k \in \{1, 2, \cdots, N\}.$$

Remark 2. The conclusion of Definition 1 is equivalent to the fact that the error signal $e(k)$ is finite-time bounded with respect to $(\delta, d, \varepsilon, R, N)$; that is, the output $y(k)$ of system (3) is always located in the ε neighborhood of the reference signal $r(k)$ within a given time-interval $\{1, 2, \cdots, N\}$.

In [13], the sufficient conditions for FTB of system (1) with respect to $(\delta, d, \varepsilon, R, N)$ were presented in terms of LMI. In this paper, the research methods in [13] will be modified and combined with the error system method in preview control theory to study the finite-time bounded tracking problem.

The Schur complement lemma is needed to deduce an LMI feasibility problem.

Lemma 3 (see [21]). *Symmetric matrix $\begin{bmatrix} S_{11} & S_{12} \\ S_{12}^T & S_{22} \end{bmatrix} < 0$ if and only if one of the following two conditions is satisfied:*

(1) $S_{11} < 0$, $S_{22} - S_{12}^T S_{11}^{-1} S_{12} < 0$.

(2) $S_{22} < 0$, $S_{11} - S_{12} S_{22}^{-1} S_{12}^T < 0$.

3. Problem Description

Let us consider the linear discrete-time system with disturbance

$$x(k+1) = Ax(k) + Bu(k) + Ew(k),$$

$$y(k) = Cx(k), \tag{6}$$

where $x(k) \in R^n$, $u(k) \in R^m$, $w(k) \in R^p$, and $y(k) \in R^q$ are the state vector, the input vector, the disturbance vector, and the output vector of the system, respectively. $A \in R^{n \times n}$, $B \in R^{n \times m}$, $E \in R^{n \times p}$, and $C \in R^{q \times n}$ are known constant matrices.

The difference operator Δ is defined as

$$\Delta v(k) = v(k) - v(k-1). \tag{7}$$

The reference signal is $r(k) \in R^q$, and the error signal is defined by (4). The assumptions on disturbance signal and reference signal of system (7) are presented as follows:

A1: Assume that the disturbance vector satisfies the condition, $\sum_{j=1}^{N} \Delta w^T(j) \Delta w(j) \leq d_1^2$, where $d_1 > 0$.

A2: Assume that the reference signal satisfies the condition, $\sum_{j=1}^{N} \Delta r^T(j) \Delta r(j) \leq d_2^2$, where $d_2 > 0$.

The purpose of this paper is to design a controller with preview action for linear discrete-time system (6) so that the closed-loop system can achieve the finite-time bounded tracking of the reference signal $r(k)$ with respect to $(\delta, d, \varepsilon, R, N)$.

To achieve the above objective, an error system that includes the information of the error signal $e(k)$ will be first constructed. Then, the error signal is considered as the output vector of this system. By this means, the original problem is converted into a FTB problem of the output vector of the error system.

4. Derivation of the Error System

Taking the operator Δ on both sides of the first equation of (6), it follows that

$$\Delta x(k+1) = A\Delta x(k) + B\Delta u(k) + E\Delta w(k). \quad (8)$$

Applying Δ to $e(k+1) = y(k+1) - r(k+1)$ and noting that $\Delta e(k+1) = e(k+1) - e(k)$, the following is obtained:

$$
\begin{aligned}
e(k+1) &= e(k) + \Delta y(k+1) - \Delta r(k+1) \\
&= e(k) + C\Delta x(k+1) - \Delta r(k+1) \\
&= e(k) + CA\Delta x(k) + CB\Delta u(k) + CE\Delta w(k) \\
&\quad - \Delta r(k+1).
\end{aligned}
\quad (9)
$$

Introduce the formal state vector $X_0(k) = \begin{bmatrix} e(k) \\ \Delta x(k) \end{bmatrix}$ and matrices $\Phi = \begin{bmatrix} I & CA \\ 0 & A \end{bmatrix}$, $G = \begin{bmatrix} CB \\ B \end{bmatrix}$, $G_r = \begin{bmatrix} -I \\ 0 \end{bmatrix}$, and $G_w = \begin{bmatrix} CE \\ E \end{bmatrix}$. Combining (8) and (9) yields

$$
\begin{aligned}
X_0(k+1) &= \Phi X_0(k) + G\Delta u(k) + G_r \Delta r(k+1) \\
&\quad + G_w \Delta w(k).
\end{aligned}
\quad (10)
$$

Define the new output

$$e(k) = C_0 X_0(k), \quad (11)$$

where $C_0 = \begin{bmatrix} I & 0 \end{bmatrix}$. Thus, we can obtain the following error system:

$$
\begin{aligned}
X_0(k+1) &= \Phi X_0(k) + G\Delta u(k) + G_r \Delta r(k+1) \\
&\quad + G_w \Delta w(k), \\
e(k) &= C_0 X_0(k).
\end{aligned}
\quad (12)
$$

Since $y(k) = Cx(k)$ is the output equation of system (6), $y(k)$ is measurable. Moreover, the reference signal $r(k)$ is known in advance. Thus, it is reasonable to consider $e(k)$ as the output vector of system (10).

Based on the above discussion, the finite-time bounded tracking problem of system (6) is transformed into the FTB problem of the output vector $e(k)$ of the closed-loop system of error system (12).

5. Design of the Controller

Let us consider the following state feedback controller:

$$\Delta u(k) = KX_0(k), \quad (13)$$

where $K = \begin{bmatrix} K_e & K_x \end{bmatrix}$ will be determined later. Applying this controller to system (12) results in

$$
\begin{aligned}
X_0(k+1) &= (\Phi + GK)X_0(k) + G_r \Delta r(k+1) \\
&\quad + G_w \Delta w(k), \\
e(k) &= C_0 X_0(k).
\end{aligned}
\quad (14)
$$

Compared with system (3), it can be seen that system (14) is exactly same as system (3) except $G_r \Delta r(k+1)$. Hence, $\Delta r(k+1)$ can be treated as the external disturbance. Putting $G_r \Delta r(k+1)$ and $G_w \Delta w(k)$ together, a new disturbance vector $W(k) = \begin{bmatrix} \Delta w(k) \\ \Delta r(k+1) \end{bmatrix}$ is obtained. In this way, the closed-loop system (14) becomes

$$
\begin{aligned}
X_0(k+1) &= (\Phi + GK)X_0(k) + \overline{E}W(k), \\
e(k) &= C_0 X_0(k),
\end{aligned}
\quad (15)
$$

where $\overline{E} = \begin{bmatrix} G_w & G_r \end{bmatrix}$.

Remark 4. System (15) is now fully in the form of system (3), which will facilitate the controller design. Since system (15) contains disturbances $\Delta r(k+1)$ and $\Delta w(k)$, the corresponding assumptions can be relaxed to A1 and A2. For $w(k)$, it is easy to prove that A1 is much weaker than that of $\sum_{k=0}^{N} w^T(k)w(k) \leq d_1^2$. In fact, if $\sum_{k=0}^{N} w^T(k)w(k) \leq d_1^2/4$, then A1 is satisfied. This is because

$$
\begin{aligned}
\sum_{k=1}^{N} \Delta w^T(k) \Delta w(k) &= \sum_{k=1}^{N} (w(k) - w(k-1))^T (w(k) \\
&\quad - w(k-1)) \leq \sum_{k=1}^{N} (\|w(k)\| + \|w(k-1)\|)^2 \\
&= \sum_{k=1}^{N} \left(\|w(k)\|^2 + \|w(k-1)\|^2 \right. \\
&\quad + 2\|w(k)\| \|w(k-1)\| \right) \leq \sum_{k=1}^{N} \left(2\|w(k)\|^2 \right. \\
&\quad + 2\|w(k-1)\|^2 \right) = \sum_{k=1}^{N} \left(2w^T(k)w(k) \right. \\
&\quad + 2w^T(k-1)w(k-1) \right) = 2w^T(0)w(0)
\end{aligned}
$$

$$+ 2w^T(N) w(N) + 4 \sum_{k=1}^{N-1} w^T(k) w(k) \le 4 \sum_{k=0}^{N} w^T(k)$$

$$\cdot w(k) \le d_1^2.$$

(16)

So far, the original problem has been converted into a FTB problem of partial variable $e(k)$ of system (15). The conclusion

of [13] cannot be directly applied to system (15). Therefore, it is necessary to combine relative ideas on partial stability with the proof methods in [13] to obtain the results of this paper. The following Theorem 5 is the first main result of this paper.

Theorem 5. *The closed-loop system (15) achieves finite-time bounded tracking of the reference signal $r(k)$ with respect to $(\delta, d, \varepsilon, R, N)$, if for a given scalar $\gamma > 1$, there exist matrices $P_1 > 0$, $P_2 > 0$ and scalars $\lambda_1 > 0$, $\lambda_2 > 0$ such that*

$$\begin{bmatrix} (\Phi + GK)^T C_0^T P_1 C_0 (\Phi + GK) - \gamma C_0^T P_1 C_0 & (\Phi + GK)^T C_0^T P_1 C_0 \overline{E} \\ \overline{E}^T C_0^T P_1 C_0 (\Phi + GK) & \overline{E}^T C_0^T P_1 C_0 \overline{E} - \gamma P_2 \end{bmatrix} \le 0,$$

(17)

$$R < P_1 < \lambda_1 R,$$

(18)

$$0 < P_2 < \lambda_2 I,$$

(19)

$$\lambda_1 \delta^2 + \lambda_2 d^2 < \frac{\varepsilon^2}{\gamma^{N-1}}.$$

(20)

Moreover, the controller is $\Delta u(k) = KX_0(k)$.

Proof. Construct the following Lyapunov function:

$$V(e(k)) = e^T(k) P_1 e(k).$$

(21)

Due to $P_1 > 0$, $V(e)$ is a positive-definite quadratic form with respect to e. Then, with some mathematical operations, we have

$$\begin{aligned} V(e(k+1)) &= e^T(k+1) P_1 e(k+1) = (C_0 X_0(k+1))^T P_1 (C_0 X_0(k+1)) \\ &= \left[C_0 (\Phi + GK) X_0(k) + C_0 \overline{E} W(k) \right]^T P_1 \left[C_0 (\Phi + GK) X_0(k) + C_0 \overline{E} W(k) \right] \\ &= \left[X_0^T(k) \quad W^T(k) \right] \begin{bmatrix} (\Phi + GK)^T C_0^T \\ \overline{E}^T C_0^T \end{bmatrix} P_1 \left[C_0 (\Phi + GK) \quad C_0 \overline{E} \right] \begin{bmatrix} X_0(k) \\ W(k) \end{bmatrix} \\ &= \left[X_0^T(k) \quad W^T(k) \right] \begin{bmatrix} (\Phi + GK)^T C_0^T P_1 C_0 (\Phi + GK) & (\Phi + GK)^T C_0^T P_1 C_0 \overline{E} \\ \overline{E}^T C_0^T P_1 C_0 (\Phi + GK) & \overline{E}^T C_0^T P_1 C_0 \overline{E} \end{bmatrix} \begin{bmatrix} X_0(k) \\ W(k) \end{bmatrix}. \end{aligned}$$

(22)

If condition (17) holds, the following stands:

$$\begin{aligned} V(e(k+1)) &\le \gamma V(e(k)) + \gamma W^T(k) P_2 W(k) \\ &\le \gamma V(e(k)) \\ &\quad + \lambda_{\max}(P_2) \gamma W^T(k) W(k). \end{aligned}$$

(23)

Applying (23) iteratively leads to

$$V(e(k)) \le \gamma V(e(k-1)) + \lambda_{\max}(P_2) \gamma W^T(k-1)$$

$$\cdot W(k-1) \le \gamma \left[\gamma V(e(k-2)) \right.$$

$$\left. + \lambda_{\max}(P_2) \gamma W^T(k-2) W(k-2) \right] + \lambda_{\max}(P_2)$$

$$\cdot \gamma W^T(k-1) W(k-1) = \gamma^2 V(e(k-2))$$

$$+ \lambda_{\max}(P_2) \left[\gamma^2 W^T(k-2) W(k-2) \right.$$

$$\left. + \gamma W^T(k-1) W(k-1) \right] \le \cdots \le \gamma^{k-1} V(e(1))$$

$$+ \lambda_{\max}(P_2) \sum_{j=1}^{k-1} \gamma^j W^T(k-j) W(k-j).$$

(24)

Taking $\gamma > 1$ into account, it is easily obtained from (24) that

$$V(e(k)) \le \gamma^{k-1} \left[V(e(1)) \right.$$

$$\left. + \lambda_{\max}(P_2) \sum_{j=1}^{k-1} W^T(k-j) W(k-j) \right].$$

(25)

Thus for all $k \in \{1, 2, \cdots, N\}$, we get

$$V(e(k)) \le \gamma^{N-1} \left[V(e(1)) \right.$$
$$\left. + \lambda_{\max}(P_2) \sum_{j=1}^{N-1} W^T(N-j) W(N-j) \right]. \quad (26)$$

By setting $d^2 = d_1^2 + d_2^2$, the following is obtained:

$$\sum_{j=1}^{N-1} W^T(N-j) W(N-j)$$
$$= \sum_{j=1}^{N-1} \Delta w^T(N-j) \Delta w(N-j)$$
$$+ \sum_{j=1}^{N-1} \Delta r^T(N+1-j) \Delta r(N+1-j) \quad (27)$$
$$= \sum_{j=1}^{N-1} \Delta w^T(j) \Delta w(j) + \sum_{j=2}^{N} \Delta r^T(j) \Delta r(j)$$
$$\le \sum_{j=1}^{N} \Delta w^T(j) \Delta w(j) + \sum_{j=1}^{N} \Delta r^T(j) \Delta r(j)$$
$$\le d_1^2 + d_2^2 = d^2.$$

Moreover,

$$V(e(1)) = \left[R^{1/2} e(1) \right]^T \left(R^{-1/2} P_1 R^{-1/2} \right) \left[R^{1/2} e(1) \right]. \quad (28)$$

Denoting $\widetilde{P}_1 = R^{-1/2} P_1 R^{-1/2}$, it follows that

$$V(e(1)) \le \lambda_{\max}\left(\widetilde{P}_1 \right) e^T(1) \operatorname{Re}(1). \quad (29)$$

Substituting (27) and (29) into (26) yields the further estimation:

$$V(e(k))$$
$$\le \gamma^{N-1} \left[\lambda_{\max}\left(\widetilde{P}_1 \right) e^T(1) \operatorname{Re}(1) + \lambda_{\max}(P_2) d^2 \right]. \quad (30)$$

Since condition (18) is equivalent to $I < R^{-1/2} P_1 R^{-1/2} < \lambda_1 I$, i.e., $I < \widetilde{P}_1 < \lambda_1 I$, it can be obtained that

$$1 < \lambda_{\min}\left(\widetilde{P}_1 \right) \le \lambda_{\max}\left(\widetilde{P}_1 \right) < \lambda_1. \quad (31)$$

In addition, condition (19) implies

$$0 < \lambda_{\min}(P_2) \le \lambda_{\max}(P_2) < \lambda_2. \quad (32)$$

Hence, if (18) and (19) hold, it can be easily seen from (30) that

$$V(e(k)) \le \gamma^{N-1} \left(\lambda_1 \delta^2 + \lambda_2 d^2 \right). \quad (33)$$

On the other hand, because of $\lambda_{\min}(\widetilde{P}_1) > 1$, then

$$V(e(k)) = e^T(k) P_1 e(k) \ge \lambda_{\min}\left(\widetilde{P}_1 \right) e^T(k) \operatorname{Re}(k)$$
$$\ge e^T(k) \operatorname{Re}(k). \quad (34)$$

According to (33) and (34), the following is obtained:

$$e^T(k) \operatorname{Re}(k) \le \gamma^{N-1} \left(\lambda_1 \delta^2 + \lambda_2 d^2 \right). \quad (35)$$

Condition (20) implies that $\gamma^{N-1}(\lambda_1 \delta^2 + \lambda_2 d^2) < \varepsilon^2$. Then, it can be concluded that $e^T(k) \operatorname{Re}(k) \le \varepsilon^2$ ($k \in \{1, 2, \cdots, N\}$). This completes the proof. □

By observing the inequality (17) carefully, it can be seen that (17) is not an LMI. Hence, it cannot be easily solved by Matlab LMI toolbox. To this end, a tractable LMI form will be presented in the following. This is the second main theorem of this paper.

Theorem 6. *The closed-loop system (15) achieves finite-time bounded tracking of the reference signal $r(k)$ with respect to $(\delta, d, \varepsilon, R, N)$, if for a given scalar $\gamma > 1$, there exist matrices $Q_1 > 0$, $P_2 > 0$ and scalars $\lambda_1' > 0$, $\lambda_2 > 0$ such that*

$$\begin{bmatrix} -\gamma Q_1 & 0 & 0 & (Q_1 + CBL)^T \\ 0 & 0 & 0 & (CA + CBK_x)^T \\ 0 & 0 & -\gamma P_2 & \widetilde{E}^T \\ (Q_1 + CBL) & (CA + CBK_x) & \widetilde{E} & -Q_1 \end{bmatrix} \quad (36)$$
$$\le 0,$$

$$\lambda_1' R^{-1} < Q_1 < R^{-1}, \quad (37)$$

$$0 < P_2 < \lambda_2 I, \quad (38)$$

$$\begin{bmatrix} \lambda_2 d^2 - \dfrac{\varepsilon^2}{\gamma^{N-1}} & \delta \\ \delta & -\lambda_1' \end{bmatrix} < 0, \quad (39)$$

where $\widetilde{E} = [CE \ -I]$. In this case the controller is $\Delta u(k) = K_e e(k) + K_x \Delta x(k)$ with $K_e = LQ_1^{-1}$.

Proof. The key of the proof lies in that the conditions of Theorem 5 are satisfied if the condition of this theorem holds.

To convert (17) to an LMI, let $Q_1 = P_1^{-1}$; then (17) can be equivalently written as

$$\begin{bmatrix} (\Phi + GK)^T C_0^T Q_1^{-1} C_0 (\Phi + GK) - \gamma C_0^T Q_1^{-1} C_0 & (\Phi + GK)^T C_0^T Q_1^{-1} C_0 \overline{E} \\ \overline{E}^T C_0^T Q_1^{-1} C_0 (\Phi + GK) & \overline{E}^T C_0^T Q_1^{-1} C_0 \overline{E} - \gamma P_2 \end{bmatrix} \le 0. \quad (40)$$

Since the equivalent transformation of this inequality cannot yield the desired result, the matrix $K = \begin{bmatrix} K_e & K_x \end{bmatrix}$ and the expressions of C_0, Φ, G, K, $\overline{E}$, G_r, and G_w in the closed-loop system (15) are substituted into this inequality. Then the following can be obtained:

$$
\left[
\begin{array}{c|c}
\begin{bmatrix} I + CBK_e & CA + CBK_x \\ BK_e & A + BK_x \end{bmatrix}^T \begin{bmatrix} Q_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I + CBK_e & CA + CBK_x \\ BK_e & A + BK_x \end{bmatrix} - \gamma \begin{bmatrix} Q_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} I + CBK_e & CA + CBK_x \\ BK_e & A + BK_x \end{bmatrix}^T \begin{bmatrix} Q_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} CE & -I \\ E & 0 \end{bmatrix} \\
\hline
\begin{bmatrix} CE & -I \\ E & 0 \end{bmatrix}^T \begin{bmatrix} Q_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I + CBK_e & CA + CBK_x \\ BK_e & A + BK_x \end{bmatrix} & \begin{bmatrix} CE & -I \\ E & 0 \end{bmatrix}^T \begin{bmatrix} Q_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} CE & -I \\ E & 0 \end{bmatrix} - \gamma P_2
\end{array}
\right] \tag{41}
$$

$$\leq 0,$$

that is,

$$
\begin{bmatrix}
(I + CBK_e)^T Q_1^{-1} (I + CBK_e) - \gamma Q_1^{-1} & (I + CBK_e)^T Q_1^{-1} (CA + CBK_x) & (I + CBK_e)^T Q_1^{-1} CE & -(I + CBK_e)^T Q_1^{-1} \\
(CA + CBK_x)^T Q_1^{-1} (I + CBK_e) & (CA + CBK_x)^T Q_1^{-1} (CA + CBK_x) & (CA + CBK_x)^T Q_1^{-1} CE & -(CA + CBK_x)^T Q_1^{-1} \\
(CE)^T Q_1^{-1} (I + CBK_e) & (CE)^T Q_1^{-1} (CA + CBK_x) & \begin{bmatrix} (CE)^T Q_1^{-1} (CE) & -(CE)^T Q_1^{-1} \\ -Q_1^{-1} CE & Q_1^{-1} \end{bmatrix} - \gamma P_2 \\
-Q_1^{-1} (I + CBK_e) & -Q_1^{-1} (CA + CBK_x) &
\end{bmatrix} \tag{42}
$$

$$\leq 0.$$

Rewriting the left side of (42) yields

$$
\begin{bmatrix}
(I + CBK_e)^T Q_1^{-1} (I + CBK_e) - \gamma Q_1^{-1} & (I + CBK_e)^T Q_1^{-1} (CA + CBK_x) & (I + CBK_e)^T Q_1^{-1} \begin{bmatrix} CE & -I \end{bmatrix} \\
(CA + CBK_x)^T Q_1^{-1} (I + CBK_e) & (CA + CBK_x)^T Q_1^{-1} (CA + CBK_x) & (CA + CBK_x)^T Q_1^{-1} \begin{bmatrix} CE & -I \end{bmatrix} \\
\begin{bmatrix} (CE)^T \\ -I \end{bmatrix} Q_1^{-1} (I + CBK_e) & \begin{bmatrix} (CE)^T \\ -I \end{bmatrix} Q_1^{-1} (CA + CBK_x) & \begin{bmatrix} (CE)^T \\ -I \end{bmatrix} Q_1^{-1} \begin{bmatrix} CE & -I \end{bmatrix} - \gamma P_2
\end{bmatrix} \leq 0. \tag{43}
$$

By denoting $\widetilde{E} = \begin{bmatrix} CE & -I \end{bmatrix}$, the above inequality becomes

$$
\begin{bmatrix}
(I + CBK_e)^T Q_1^{-1} (I + CBK_e) - \gamma Q_1^{-1} & (I + CBK_e)^T Q_1^{-1} (CA + CBK_x) & (I + CBK_e)^T Q_1^{-1} \widetilde{E} \\
(CA + CBK_x)^T Q_1^{-1} (I + CBK_e) & (CA + CBK_x)^T Q_1^{-1} (CA + CBK_x) & (CA + CBK_x)^T Q_1^{-1} \widetilde{E} \\
\widetilde{E}^T Q_1^{-1} (I + CBK_e) & \widetilde{E}^T Q_1^{-1} (CA + CBK_x) & \widetilde{E}^T Q_1^{-1} \widetilde{E} - \gamma P_2
\end{bmatrix} \leq 0. \tag{44}
$$

Pre- and postmultiplying (44) by the symmetric matrix $\mathrm{diag}(Q_1, I, I)$ and its transpose, respectively, we obtain

$$
\begin{bmatrix}
Q_1 (I + CBK_e)^T Q_1^{-1} (I + CBK_e) Q_1 - \gamma Q_1 & Q_1 (I + CBK_e)^T Q_1^{-1} (CA + CBK_x) & Q_1 (I + CBK_e)^T Q_1^{-1} \widetilde{E} \\
(CA + CBK_x)^T Q_1^{-1} (I + CBK_e) Q_1 & (CA + CBK_x)^T Q_1^{-1} (CA + CBK_x) & (CA + CBK_x)^T Q_1^{-1} \widetilde{E} \\
\widetilde{E}^T Q_1^{-1} (I + CBK_e) Q_1 & \widetilde{E}^T Q_1^{-1} (CA + CBK_x) & \widetilde{E}^T Q_1^{-1} \widetilde{E} - \gamma P_2
\end{bmatrix} \leq 0. \tag{45}
$$

In fact, (45) can be rewritten as

$$
\begin{bmatrix} -\gamma Q_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\gamma P_2 \end{bmatrix} - \begin{bmatrix} Q_1 \left(I + CBK_e \right)^T \\ \left(CA + CBK_x \right)^T \\ \widetilde{E}^T \end{bmatrix} \left(-Q_1 \right)^{-1}
$$

$$
\cdot \left[\left(I + CBK_e \right) Q_1 \quad \left(CA + CBK_x \right) \quad \widetilde{E} \right] \leq 0.
$$

(46)

From $-Q_1 < 0$ and Lemma 3 (2), (45) is equivalent to

$$
\begin{bmatrix} -\gamma Q_1 & 0 & 0 & Q_1 \left(I + CBK_e \right)^T \\ 0 & 0 & 0 & \left(CA + CBK_x \right)^T \\ 0 & 0 & -\gamma P_2 & \widetilde{E}^T \\ \left(I + CBK_e \right) Q_1 & \left(CA + CBK_x \right) & \widetilde{E} & -Q_1 \end{bmatrix}
$$

(47)

$$
\leq 0.
$$

By setting $L = K_e Q_1$, it can be seen that (47) becomes (36) and (17) is finally converted to an equivalent LMI (36).

Because of $Q_1 = P_1^{-1}$, (18) becomes

$$
R < Q_1^{-1} < \lambda_1 R.
$$

(48)

Then the above equation leads to

$$
\frac{1}{\lambda_1} R^{-1} < Q_1 < R^{-1},
$$

(49)

which implies that Q_1 needs to meet the condition: $(1/\lambda_1)R^{-1} - Q_1 < 0$. Since $\lambda_1 > 0$ and $Q_1 > 0$ are unknown, this inequality cannot be solved by the LMI toolbox in Matlab. Thus, let $\lambda_1' = 1/\lambda_1$; then (49) is converted into a computationally tractable condition (37).

Taking $\lambda_1' = 1/\lambda_1$ into account, (20) can be written as

$$
\frac{1}{\lambda_1'} \delta^2 + \lambda_2 d^2 < \frac{\varepsilon^2}{\gamma^{N-1}}.
$$

(50)

It is also necessary to convert (50) to an LMI. For this purpose, we rewrite (50) as

$$
\left(\lambda_2 d^2 - \frac{\varepsilon^2}{\gamma^{N-1}} \right) - \delta \left(-\lambda_1' \right)^{-1} \delta < 0.
$$

(51)

Because of $-\lambda_1' < 0$, it can be obtained by Lemma 3 (2) that conditions (50) and (39) are equivalent. Besides, (19) in Theorem 5 does not require any change and it is (38). This completes the proof. □

Note that $\Delta u(k) = u(k) - u(k-1)$. By solving $u(k)$, the following result is derived instantly.

Theorem 7. *Assume that A1-A2 are satisfied. The control input of linear discrete-time system (6) is given by*

$$
u(k) = u(0) + K_e \sum_{j=1}^{k} e(j) + K_x (x(k) - x(0)),
$$

(52)

$$
k \in \{1, 2, \cdots, N\},
$$

where K_e and K_x are determined by LMIs (36)-(39). Under the controller, system (6) achieves finite-time bounded tracking of the reference signal $r(k)$ with respect to $(\delta, d, \varepsilon, R, N)$.

Remark 8. The results in this paper can be readily extended to linear discrete-time systems with state delay. In this case, we can construct a delay-free error system by applying the difference method and the discrete lifting technique [22]. Furthermore, the error vector is still taken as the output vector of the error system. Then, applying the controller design method in this paper, a finite-time bounded tracking controller of discrete time-delay systems can be obtained.

6. Simulation Example

The effectiveness of the proposed method will be shown by a numerical example, in which two different reference signals are considered.

Example 1. Consider system (6) with the following system matrices:

$$
A = \begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix},
$$

$$
B = \begin{bmatrix} -0.5 \\ 1 \end{bmatrix},
$$

(53)

$$
E = \begin{bmatrix} -0.1 \\ 0.5 \end{bmatrix},
$$

$$
C = \begin{bmatrix} 0.1 & 0.3 \end{bmatrix}.
$$

Take $R = I$, $\delta = 0.1$, $d = \sqrt{6}/2$, $\varepsilon = \sqrt{5}$, $N = 100$, and $\gamma = 1.01$. By using the LMI toolbox in Matlab to solve the LMIs (36)-(39) in Theorem 6, the feedback gain matrices are given by

$$
K_e = -4,
$$

$$
K_x = \begin{bmatrix} 0.8 & 1.6 \end{bmatrix}.
$$

(54)

Then let $x(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $u(0) = 0$; we obtain

$$
u(k) = u(0) + K_e \sum_{j=1}^{k} e(j) + K_x (x(k) - x(0))
$$

$$
= -4 \sum_{j=1}^{k} e(j) + 0.8 x_1(k) + 1.6 x_2(k).
$$

(55)

The disturbance is taken as

$$
w(k) = \frac{3 \sin 0.07 \pi k}{(0.5 + k)^{0.6}}.
$$

(56)

By calculation, we have

$$
\sum_{j=1}^{N} \Delta w^T(j) \Delta w(j) \approx 0.3917 \leq 0.5 \overset{\text{def}}{=} d_1^2.
$$

(57)

Below, two different reference signals are considered to do the numerical simulation.

(1) The reference signal is taken as

$$r(k) = \begin{cases} 0, & 0 \le k < 10, \\ 0.2, & 10 \le k < 20, \\ 0.4, & 20 \le k < 30, \\ 0.6, & 30 \le k < 40, \\ 0.8, & 40 \le k < 50, \\ 1, & 50 \le k < 60, \\ 1.2, & 60 \le k < 70, \\ 1.4, & 70 \le k < 80, \\ 1.6, & 80 \le k \le 100. \end{cases} \tag{58}$$

In this case, $r(k)$ satisfies $\sum_{j=1}^{N} \Delta r^T(j)\Delta r(j) = 0.32 \le 1 \overset{\text{def}}{=} d_2^2$. Note that

$$d_1^2 + d_2^2 = \frac{3}{2} = d^2. \tag{59}$$

In addition, from $x(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $u(0) = 0$, $e^T(1)\operatorname{Re}(1) = e^T(1)e(1) \approx 0.0052$ can be obtained. Then the following condition is guaranteed:

$$e^T(1)\operatorname{Re}(1) \le 0.01 = \delta^2. \tag{60}$$

Therefore, the tracking error between the closed-loop output and the reference signal (58) should satisfy

$$e^T(k)\operatorname{Re}(k) \le \varepsilon^2 \quad (k \in \{1, 2, \cdots, 100\}). \tag{61}$$

Figure 1 shows the output response of the closed-loop system, and Figure 2 shows the tracking error between the closed-loop output and the reference signal.

As shown in Figures 1-2, the proposed controller guarantees that the closed-loop output signal is always in the ε neighborhood of the reference signal $r(k)$ within a given time-interval $\{1, 2, \cdots, 100\}$ and the error signal is always in a given range. That is to say, the closed-loop system achieves finite-time bounded tracking of the reference signal $r(k)$ with respect to $(0.1, \sqrt{6}/2, \sqrt{5}, I, 100)$. It needs to be emphasized that the tracking error is very small even if a strong disturbance signal exists in the system.

Note that from Definition 1, if $e^T(0)\operatorname{Re}(0) \le \delta^2$ and other conditions are satisfied, $e^T(k)\operatorname{Re}(k) \le \varepsilon^2$ holds. This result has nothing to do with the selection of initial state $x(0)$. But in fact, due to $e(0) = y(0) - r(0) = Cx(0) - r(0)$, the initial state $x(0)$ is still limited. In this example, If we let $x(0) = \begin{bmatrix} -0.00083 \\ -0.07 \end{bmatrix}$ and $u(0) = 0$, this results in $e^T(1)\operatorname{Re}(1) = e^T(1)e(1) = 0.01 = \delta^2$. $\gamma = 1.01$ is still taken to solve the corresponding LMIs. In this case, the simulation results are completely consistent with the theoretical results, and they are omitted here.

Note that the reference signal (58) is very valuable in practice. In fact, the desired trajectory of a biped robot in the upslope process is usually in the form of function (58) [23].
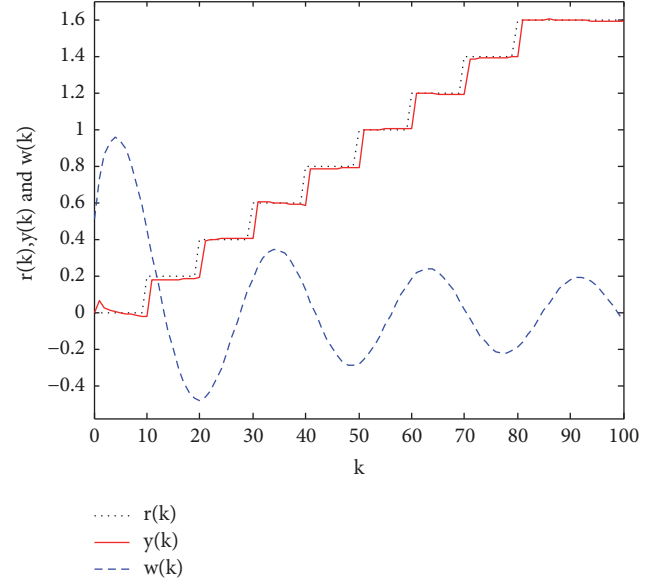

FIGURE 1: The output response of the closed-loop system to reference signal (58).
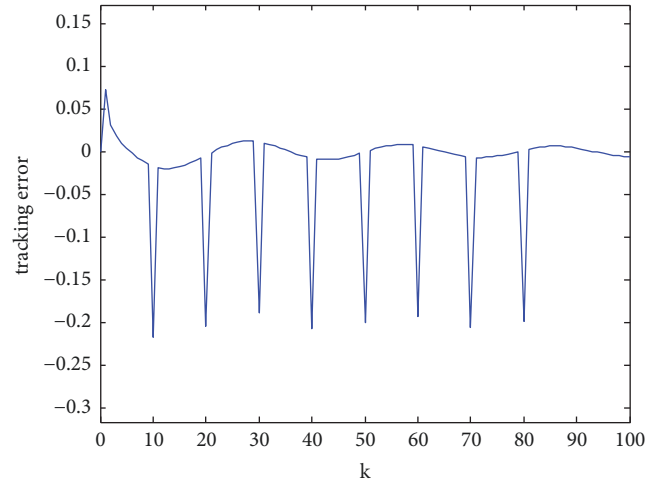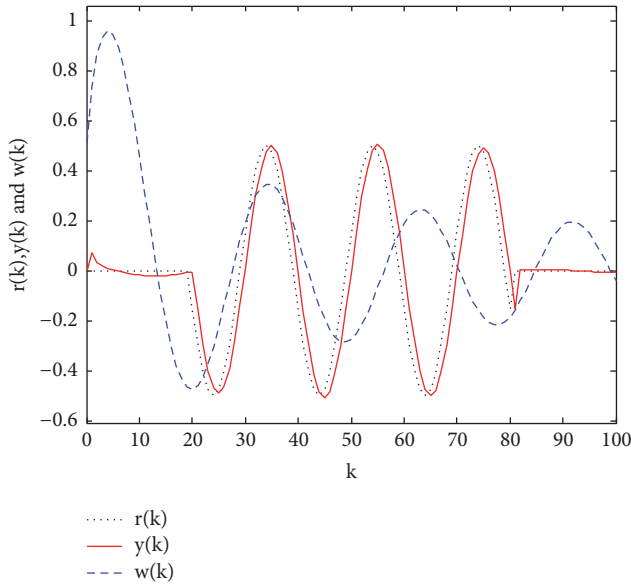

FIGURE 2: The tracking error of the closed-loop system to reference signal (58).

(2) The reference signal is taken as the periodic function given by

$$r(k) = \begin{cases} 0, & 80 < k \le 100, \\ 0.5\sin\left[\dfrac{\pi}{10(k-10)}\right], & 20 \le k \le 80, \\ 0, & 0 \le k < 20. \end{cases} \tag{62}$$

In this case, the following can be obtained:

$$\sum_{j=1}^{N} \Delta r^T(j)\Delta r(j) \approx 0.7819 \le 1 \overset{\text{def}}{=} d_2^2, \tag{63}$$

which implies $\sum_{j=1}^{N} \Delta w^T(j)\Delta w(j) + \sum_{j=1}^{N} \Delta r^T(j)\Delta r(j) \le d^2$. Thus, the condition of Theorem 6 holds.

FIGURE 3: The output response of the closed-loop system to reference signal (62).
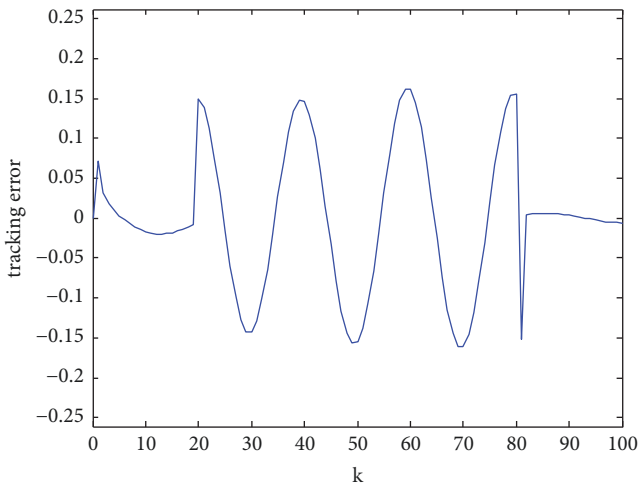

FIGURE 4: The tracking error of the closed-loop system to reference signal (62).

Figure 3 shows the output response of the closed-loop system, and Figure 4 shows the tracking error between the actual output and the desired output. It can be seen that the closed-loop system achieves finite-time bounded tracking of the reference signal $r(k)$ with respect to $(0.1, \sqrt{6}/2, \sqrt{5}, I, 100)$.

7. Conclusion

In this paper, the concept of finite-time bounded tracking control for linear discrete-time systems is proposed. Using the difference method, we construct an error system where the tracking error is only a part of the augmented state vector. Then, by constructing a Lyapunov function with respect to

the tracking error, a sufficient condition guaranteeing that the norm of tracking error is finite-time bounded is presented in terms of a set of LMIs. Based on this criterion, a feedback controller of the original system is derived, under which the closed-loop output achieves finite-time bounded tracking of the reference signal. Numerical simulation shows the effectiveness of the proposed controller.

Conflicts of Interest

No potential conflicts of interest was reported by the authors.

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