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Research Article

Multiple Nontrivial Solutions for a Class of Biharmonic Elliptic Equations with Sobolev Critical Exponent

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In this paper, we study the existence and multiplicity of nontrivial solutions for a class of biharmonic elliptic equation with Sobolev critical exponent in a bounded domain. By using the idea of the previous paper, we generalize the results and prove the existence and multiplicity of nontrivial solutions of the biharmonic elliptic equations.

1. Introduction and Main Results

In the present paper, we are concerned with the existence of multiple solutions to the following biharmonic elliptic equation with perturbation

$$\begin{aligned} \Delta^2 u &= |u|^{p-2} u + f, \quad x \in \Omega, \\ u &= \nabla u = 0, \quad x \in \partial\Omega, \end{aligned} \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 5)$, Δ^2 is the biharmonic operator, and $p = 2^{**} = 2N/(N-4)$ is the Sobolev critical exponent.

The second-order semilinear and quasilinear problems have been object of intensive research in the last years. Brezis and Nirenberg [1] have studied the existence of positive solutions of (1). Particularly, when $f = \lambda u$, where $\lambda \in \mathbb{R}$ is a constant, they have discovered the following remarkable phenomenon: the qualitative behavior of the set of solutions of (1) is highly sensitive to N, the dimension of the space. Precisely, Brezis and Nirenberg [1] have shown that, in dimension $N \geq 4$, there exists a positive solution of (1), if and only if $\lambda \in (0, \lambda_1)$; while, in dimension $N = 3$ and when $\Omega = B_1$ is the unit ball, there exists a positive solution of (1), if and only if $\lambda \in (\lambda_1/4, \lambda_1)$, where $\lambda_1 > 0$ is the first eigenvalue of $-\Delta$ in Ω. For more results on this direction we refer the readers to [2–5] and the references therein.

During the last decades many works have been orientated to the analysis of biharmonic nonlinear Schrödinger equation (BHNSE)

$$\begin{aligned} i\varphi_t - \Delta^2\varphi + g(x, |\varphi|)\varphi &= 0, \\ \varphi(0, x) = \varphi_0(x) &\in H_0^2(\Omega), \end{aligned} \tag{2}$$

$\Omega \subset \mathbb{R}^N$ is an open domain $N \geq 5$. For instance, paper [6] proved that some of the properties and characteristics for the second-order semilinear problems can be extended to BHLSE. Paper [7] proved the existence of blow-up solutions. In papers [8–10], the authors proved the existence of global solutions, in particular, looking for standing wave solutions for (2) of the form

$$\varphi(t, u) = e^{i\lambda t} u \tag{3}$$

such that u is a solution satisfying the equation

$$\Delta^2 u + \lambda u = \hat{g}(x, u), \quad u \in H_0^2(\Omega). \tag{4}$$

If $\lambda = 0$ and $\hat{g}(x, u) = |u|^{p-2}u$, we know that (4) admits no positive solutions if Ω is star shaped under the Navier or Dirichlet boundary conditions (see [11, Theorem 3.3] and [12, Corollary 1]). If $\lambda > 0$ and Ω is a ball, paper [13] proved the existence of positive radially symmetric solutions. For more general results on this direction one can refer to [14, 15, 15–21] and the references therein.

Motivated by the above results, we study the case that $\lambda = 0$, $\hat{g}(x, u) = |u|^{p-2}u + f(x)$, and $\Omega \subset \mathbb{R}^N$ is a bounded domain. Precisely, we shall generalize the results of Tarantello [22] to the biharmonic and critical exponent case. Our main tool here is the Nehari manifold method which is similar to the fibering method of Pohozaev's.

In order to state the main results, we shall give some notation and assumptions. Let $D = H_0^2(\Omega)$, and $\|u\|_p = (\int_\Omega |u|^p dx)^{1/p}$ be the usual $L^p(\Omega)$ norm. Obviously, D is a Hilbert space under the inner product $\langle u, v\rangle = \int_\Omega \Delta u \Delta v dx$. Correspondingly, the norm is denoted by $\|\cdot\|$; i.e., $\|u\|^2 = \int_\Omega |\Delta u|^2 dx$. Assume that $f \in L^q(\Omega)(q = 2N/(N+4))(f \neq 0)$ satisfies

$$\|f\|_q \leq C_N S^{(N+4)/8}, \tag{5}$$

where S is the best Sobolev embedding constant of $D \hookrightarrow L^p(\Omega)(p = 2^{**} = 2N/(N-4))$, and

$$C_N = \frac{8}{N-4}\left(\frac{N-4}{N+4}\right)^{(N+4)/8}, \quad S = \inf_{u\in D\backslash\{0\}} \frac{\|\Delta u\|_2^2}{\|u\|_p^2}. \tag{6}$$

Let

$$u_\varepsilon(x) = \frac{\left(N(N-4)\left(N^2-4\right)\right)^{(N-4)/8} \varepsilon^{(N-4)/2}}{\left(\varepsilon^2 + |x|^2\right)^{(N-4)/2}} \tag{7}$$

be an extremal function for the Sobolev inequality in $\mathbb{R}^N$. For $a \in \Omega$, let $u_{\varepsilon,a}(x) = u_\varepsilon(x-a)$ and $\xi_a \in C_0^\infty(\Omega)$ with $\xi_a \geq 0$ and $\xi_a = 1$ near a. We point out that the embedding $D \hookrightarrow L^p(\Omega)$ is not compact. This leads to the lack of compactness for the proved existence and multiplicity of nontrivial solutions of (1). Motivated by [1, 22], we recover the local compactness by dividing the Nehari manifold into three parts and give some estimates for the least energy of (1)

It is easy to see that the energy functional of (1) is denoted by

$$I(u) = \frac{1}{2}\int_\Omega |\Delta u|^2 dx - \frac{1}{p}\int_\Omega |u|^p dx - \int_\Omega f u dx, \quad u \in D. \tag{8}$$

Hence, I is well defined (under (5)) and of the class $C^2(\Omega)$. Moreover, all the critical points of I are precisely the solutions of (1). We define the Nehari manifold N associated with the functional by

$$N = \left\{u \in D \mid \left\langle I'(u), u\right\rangle = 0\right\}. \tag{9}$$

It is clear that all critical points lie in the Nehari manifold, and it is usually effective to consider the existence of critical points in this smaller subset of the Sobolev space. For fixed $u \in D \setminus \{0\}$, we set

$$\phi(t) = I(tu) = \frac{t^2}{2}\int_\Omega |\Delta u|^2 dx - \frac{t^p}{p}\int_\Omega |u|^p dx - t\int_\Omega f u dx, \quad t \geq 0. \tag{10}$$

The mapping is called fibering map. Such maps are often used to investigate Nehari manifolds for various semilinear problems. From the relationship between I and $\phi(t)$, we can divide N into three parts

$$\begin{aligned} N^+ &= \left\{u \in N \mid \|\Delta u\|_2^2 - (p-1)\|u\|_p^p > 0\right\}, \\ N^0 &= \left\{u \in N \mid \|\Delta u\|_2^2 - (p-1)\|u\|_p^p = 0\right\}, \\ N^- &= \left\{u \in N \mid \|\Delta u\|_2^2 - (p-1)\|u\|_p^p < 0\right\}. \end{aligned} \tag{11}$$

It turns out that under the assumption (5), we infer that $N^0 = \{0\}$ (see Lemma 5 below). Now the main result in this paper can be stated as follows.

Theorem 1. *Assume that $f \neq 0$ satisfies (5). Then*

$$\inf_N I = \inf_{N^+} I = c_0 \tag{12}$$

is achieved at a point $u_0 \in N$. Furthermore, u_0 is a critical point of I, and $u_0 \geq 0$ when $f \geq 0$.

In the following we study the second infimum problem

$$\inf_{N^-} I = c_1. \tag{13}$$

In this case we have the following results.

Theorem 2. *Assume that $f \neq 0$ satisfies (5). Then $c_1 > c_0$ and the infimum in (13) is achieved at a point $u_1 \in N^-$, which is a critical point of I.*

The proofs of Theorems 1–2 rely on the Ekeland's variational principle and careful estimates (see [1]) of minimizing sequence.

2. Some Preliminary Results

In this section we prove some preliminary results for the proof of Theorems 1–2. The main ideas are coming from [1, 22]. We begin with the following lemma which states the purpose of assumption (5).

Lemma 3. *Supposed that $f \neq 0$ satisfies (5). For every $u \in D\backslash\{0\}$, there exists a unique $t_1 = t_1(u) > 0$ such that $t_1 u \in N^-$. Particularly, we have*

$$t_1 > \left[\frac{\|\Delta u\|_2^2}{(p-1)\|u\|_p^p}\right]^{1/(p-2)} := t_{max} \tag{14}$$

and $I(t_1 u) = \max_{t\geq t_{max}} I(tu)$. Moreover, if $\int_\Omega f u dx > 0$, then there exists a unique $t_2 = t_2(u) > 0$ such that $t_2 u \in N^+$. In particular, one has

$$t_2 < \left[\frac{\|\Delta u\|_2^2}{(p-1)\|u\|_p^p}\right]^{1/(p-2)} \tag{15}$$

and $I(t_2 u) \leq I(tu), \forall t \in [0, t_1]$.

Proof. Recall that the fibering map is defined by

$$\phi(t) = I(tu) = \frac{t^2}{2}\int_\Omega |\Delta u|^2 dx - \frac{t^p}{p}\int_\Omega |u|^p dx - t\int_\Omega fudx. \quad (16)$$

Then

$$\phi'(t) = t\int_\Omega |\Delta u|^2 dx - t^{p-1}\int_\Omega |u|^p dx - \int_\Omega fudx \coloneqq g(t) - \int_\Omega fudx. \quad (17)$$

We deduce from $g'(t) = 0$ that

$$t = t_{max} = \left[\frac{\|\Delta u\|_2^2}{(p-1)\|u\|_p^p}\right]^{1/(p-2)}. \quad (18)$$

If $0 < t < t_{max}$, we have $\phi''(t) = g'(t) > 0$, and if $t > t_{max}$, one sees $\phi''(t) = g'(t) < 0$. A direct computation shows that $g(t)$ achieves its maximum at t_{max}, and

$$g(t_{max}) = \frac{8}{N-4}\left(\frac{N-4}{N+4}\right)^{(N+4)/8}\frac{\|\Delta u\|_2^{(N+4)/4}}{\|u\|_p^{N/4}} = C_N\frac{\|\Delta u\|_2^{(N+4)/4}}{\|u\|_p^{N/4}} > 0. \quad (19)$$

We divide the following two cases to accomplish our results.

Case 1. If $\int_\Omega fudx \le 0$, then $\phi'(t_{max}) = g(t_{max}) - \int_\Omega fudx > 0$. It is easy to see that if $t \longrightarrow +\infty$, we have $\phi'(t) < 0$. So, there exists unique $t_1 > t_{max}$ such that $\phi'(t_1) = 0$ and $g(t_1) = \int_\Omega fudx$. We infer from the monotonicity of $g(t)$ that, for $t_1 > t_{max}$,

$$\phi''(t_1) = g'(t_1) < 0, \qquad t_1^2 g'(t_1) = \|\Delta(t_1 u)\|_2^2 - (p-1)\|t_1 u\|_p^p < 0. \quad (20)$$

This shows that $t_1 u \in N^-$.

Case 2. If $\int_\Omega fudx > 0$, we infer from assumption (5) that $\int_\Omega fudx < g(t_{max}) \forall u \in D$. Then $\phi'(t_{max}) = g(t_{max}) - \int_\Omega fudx > 0$. Since $\phi'(0) = -\int_\Omega fudx < 0$, there exists a unique $t_2 \in [0, t_{max}]$ such that $\phi'(t_2) = 0$ and $g(t_2) = \int_\Omega fudx$. A direct computation shows that $t_2 u \in N^+$ and $I(t_2 u) \le I(tu), \forall t \in [0, t_1]$. □

Lemma 4. *Assume that $f \neq 0$ satisfies (5). We infer that the infimum*

$$\inf_{\|u\|_p=1}\left(C_N\|\Delta u\|_2^{(N+4)/4} - \int_\Omega fudx\right) \coloneqq \mu_0 \quad (21)$$

is achieved, where $\mu_0 > 0$.

The proof of Lemma 4 is technical and the idea of the proof is mainly motivated by paper [23]. We shall prove it in the appendix. Next we study the property of the set N^0.

Lemma 5. *Let $f \neq 0$ satisfy (5). Then for every $u \in N, u \neq 0$, we can get the conclusion that $N^0 = \{0\}$.*

Proof. We use the contradiction arguments. Assume that, for some $u \in N, u \neq 0$, we have $u \in N^0$. That is,

$$\|\Delta u\|_2^2 - (p-1)\|u\|_p^p = 0. \quad (22)$$

Since $u \in N$, it follows that $\|\Delta u\|_2^2 - \|u\|_p^p - \int_\Omega fudx = 0$. Hence, we get

$$\|u\|_p^p - \int_\Omega fudx = 0. \quad (23)$$

By Sobolev inequality, we deduce that $(p-2)\|u\|_p \ge (S/(p-1))^{1/(p-2)}$. For $u \neq 0$, we set

$$A(u) = C_N\frac{\|\Delta u\|_2^{(N+4)/4}}{\|u\|_p^{N/4}} - \int_\Omega fudx. \quad (24)$$

For $t \ge 0$ and $\|u\|_p = 1$, a direct computation shows that

$$A(tu) = t\left[C_N\|\Delta u\|_2^{(N+4)/4} - \int_\Omega fudx\right]. \quad (25)$$

We derive from Lemma 4 that, for $\gamma > 0$,

$$\inf_{\|u\|\ge\gamma} A(u) \ge \gamma\mu_0. \quad (26)$$

Let $\gamma = (S/(p-1))^{1/(p-2)} > 0$. We infer from (26) that

$$0 < \gamma\mu_0 \le A(u) = C_N\frac{\|\Delta u\|_2^{(N+4)/4}}{\|u\|_p^{N/4}} - \int_\Omega fudx = (p-2)\|u\|_p^p\left(\left[\frac{\|\Delta u\|_2^2}{(p-1)\|u\|_p^p}\right]^{(p-1)/(p-2)} - 1\right) = 0, \quad (27)$$

which is a contradiction. □

Lemma 6. *Let $f \neq 0$ satisfy (5). For each $u \in N \setminus \{0\}$, there exist $\varepsilon > 0$ and a differentiable function $t = t(w) > 0, w \in D, \|w\| < \varepsilon$, satisfying the following:*

$$t(0) = 1,$$
$$t(w)(u - w) \in N, \quad \forall \|w\| < \varepsilon,$$
$$\langle t'(0), w\rangle = \frac{2\int_\Omega \Delta u\Delta wdx - p\int_\Omega |u|^{p-2}uwdx - \int_\Omega fwdx}{\|\Delta u\|_2^2 - (p-1)\|u\|_p^p}. \quad (28)$$

Proof. We define $F : \mathbb{R} \times D \longrightarrow \mathbb{R}$ by

$$F(t, w) = t\|\Delta(u-w)\|_2^2 - t^{(p-1)}\|u-w\|_p^p - \int_\Omega f(u-w)\,dx. \quad (29)$$

Since $F(1,0) = 0$ and $F_t(1,0) = \|\Delta u\|_2^2 - (p-1)\|u\|_p^p \neq 0$ (Lemma 5), by using the implicit function theorem at the point $(1,0)$ we know that the results of the lemma hold. □

3. Proof of Theorem 1

In this part we shall give the proof of Theorem 1.

Proof of Theorem 1. We first claim that the functional I is bounded from below in N. For $u \in N$, we have $\langle I'(u), u\rangle = 0$. That is, $\|\Delta u\|_2^2 - \|u\|_p^p - \int_\Omega fu dx = 0$. One deduces from (2) and Hölder inequality that

$$\begin{aligned} I(u) &= \frac{1}{2}\int_\Omega |\Delta u|^2 dx - \frac{1}{p}\int_\Omega |u|^p dx - \int_\Omega fu dx \\ &= \left(\frac{1}{2} - \frac{1}{p}\right)\|\Delta u\|_2^2 - \left(1 - \frac{1}{p}\right)\int_\Omega fu dx \\ &\geq \frac{2}{N}\|\Delta u\|_2^2 - \left(1 - \frac{1}{p}\right)\|f\|_q \|u\|_p \\ &\geq \frac{2}{N}\|\Delta u\|_2^2 - \left(1 - \frac{1}{p}\right) C_N S^{N/8}\|\Delta u\|_2 \\ &\geq -\frac{N(1-1/p)^2 C_N^2 S^{N/4}}{8}. \end{aligned} \quad (30)$$

Hence, we know that the infimum c_0 is also bounded from below. Second, we can get an upper bound for c_0. Let $v \in D$ be the solution for $\Delta^2 u = f$. For $f \neq 0$, one obtains that

$$\int_\Omega f v dx = \|\Delta v\|_2^2 > 0. \quad (31)$$

Set $t_0 = t_2(v) > 0$ as defined by Lemma 3. Thus, we have that $t_0 v \in N^+$ and

$$\begin{aligned} c_0 \leq I(t_0 v) &= \frac{t_0^2}{2}\|\Delta v\|_2^2 - \frac{t_0^p}{p}\|v\|_p^p - t_0\int_\Omega f v dx \\ &= -\frac{t_0^2}{2}\|\Delta v\|_2^2 + \frac{p-1}{p}t_0^p\|v\|_p^p < -\frac{2t_0^2}{N}\|\Delta v\|_2^2 < 0. \end{aligned} \quad (32)$$

For any minimizing sequence $\{u_n\} \subset N$, we can use Ekeland's variational principle (see [24]) to get following properties:

(i) $I(u_n) < c_0 + 1/n$,

(ii) $I(w) \geq I(u_n) - (1/n)\|\Delta(w-u_n)\|_2, \forall w \in N$.

Hence for n large enough, we obtain

$$\begin{aligned} I(u_n) &= \left(\frac{1}{2} - \frac{1}{p}\right)\|\Delta u_n\|_2^2 - \left(1 - \frac{1}{p}\right)\int_\Omega f u_n dx \\ &< c_0 + \frac{1}{n} < -\frac{2t_0^2}{N}\|\Delta v\|_2^2. \end{aligned} \quad (33)$$

This implies

$$\int_\Omega f u_n dx \geq \frac{4t_0^2}{N+4}\|\Delta v\|_2^2 > 0, \quad \text{and } u_n \neq 0. \quad (34)$$

Since $I(u_n) < 0$, we infer from Hölder's inequality that

$$\exists M > 0, \quad \|\Delta u_n\|_2^2 \leq M. \quad (35)$$

At the same time, we observe that

$$\frac{4t_0^2}{N+4}\|\Delta v\|_2^2 \leq \int_\Omega f u_n dx. \quad (36)$$

One deduces from (5) and Hölder's and Sobolev's inequalities that

$$\exists m > 0, \quad \|\Delta u_n\|_2^2 \geq m > 0. \quad (37)$$

So we derive from (35) and (37) that

$$0 < m \leq \|\Delta u_n\|_2^2 \leq M, \quad (38)$$

where m and M only depend on f and Ω.

Next we shall prove that $\|I'(u_n)\| \longrightarrow \infty$, as $n \longrightarrow \infty$. Applying Lemma 6 with $u = u_n$ and $w = \delta(I'(u_n)/\|I'(u_n)\|)(\delta > 0)$, we can find some $t_n(\delta) = t[\delta(I'(u_n)/\|I'(u_n)\|)]$ such that

$$w_\delta = t_n(\delta)\left[u_n - \delta\frac{I'(u_n)}{\|I'(u_n)\|}\right] \in N. \quad (39)$$

By condition (ii) we have

$$\begin{aligned} \frac{1}{n}\|\Delta(w-u_n)\|_2 &\geq I(u_n) - I(w_\delta) \\ &= (1 - t_n(\delta))\langle I'(w_\delta), u_n\rangle \\ &\quad + \delta t_n(\delta)\left\langle I'(w_\delta), u\frac{I'(u_n)}{\|I'(u_n)\|}\right\rangle \\ &\quad + o(\delta). \end{aligned} \quad (40)$$

Dividing by δ and letting $\delta \longrightarrow 0$, we get

$$\begin{aligned} \frac{1}{n}\left(1 + t_n'(0)\|\Delta u_n\|_2\right) &\geq -t_n'(0)\langle I'(u_n), u_n\rangle \\ &\quad + \|I'(u_n)\| = \|I'(u_n)\|, \end{aligned} \quad (41)$$

where $t_n'(0) = \langle t'(0), I'(u_n)/\|I'(u_n)\|\rangle$. So, we conclude that

$$\|I'(u_n)\| \leq \frac{C}{n}\left(1 + |t_n'(0)|\right), \quad (42)$$

where C is a constant. In order to complete the proof we need to prove that $t_n'(0)$ is bounded uniformly on n. By Lemma 6 we can get

$$|t_n'(0)| \leq \frac{C}{\left|\|\Delta u_n\|_2^2 - (p-1)\|u_n\|_p^p\right|}. \quad (43)$$

Thus, there exists subsequence $\{u_n\}$ (still denote by $\{u_n\}$) such that

$$\|\Delta u_n\|_2^2 - (p-1)\|u_n\|_p^p = o(1). \quad (44)$$

We infer from $\{u_n\} \subset N$ that

$$\|\Delta u_n\|_2^2 - \|u_n\|_p^p = \int_\Omega f u_n dx, \\ \int_\Omega f u_n dx = (p-2)\|u_n\|_p^p + o(1). \quad (45)$$

By the estimate of $\|\Delta u_n\|_2$ from (38), we have that $\|u_n\|_p \geq \gamma > 0$ and

$$0 < \mu_0 \gamma^{5N/4} \leq \|u_n\|_p^{N/4} A(u_n) \leq C_N \|\Delta u_n\|_2^{(N+4)/4} - (p-2)\|u_n\|_p^{p+(N+4)/4} = (p-2)\left(\frac{1}{p-1}\right)^{(N+4)/8} \cdot \|\Delta u_n\|_2^{(N+4)/4} - (p-2)\|u_n\|_p^{p+(N+4)/4} = (p-2) \cdot \left[\left(\frac{\|\Delta u_n\|_2^2}{p-1}\right)^{(p-2)/(p-1)} - \left(\|u_n\|_p^p\right)^{(p-2)/(p-1)}\right] = o(1). \quad (46)$$

This is impossible. So, $\|\Delta u_n\|_2^2 - (p-1)\|u_n\|_p^p$ is away from zero. Thus, we conclude that

$$\|I'(u_n)\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \quad (47)$$

Let $u_0 \in D$ be the weak limit in D of $\{u_n\}$. From (47) we can get that u_0 is a weak solution for (1). In fact, $u_0 \in N$ and

$$c_0 \leq I(u_0) \leq \lim_{n\longrightarrow+\infty} I(u_n) = c_0. \quad (48)$$

So, we have that $u_n \longrightarrow u_0$ strongly in D and $I(u_0) = c_0 = \inf_{u\in N} I(u)$. Moreover, $u_0 \in N^+$. By using standard method, we can prove that u_0 is a global minimum for I in D (See [25]). □

4. Proof of Theorem 2

In this section, we shall give the proof of Theorem 2. Since the embedding $D \hookrightarrow L^{2N/(N-4)}(\Omega)$ is not compact, we need to find some way to recover this compactness. Motivated by previous works of [1, 22, 23], we will seek the level in which $(PS)_c$-condition will recover. Then we shall use the Mountain-Pass principle to get the second nontrivial solution of (1). The related problems have been studied in [23], and such an approach has been used. The threshold is found in the following lemma to obtain the compactness.

Lemma 7. *Assume that the sequence $\{u_n\} \subset D$ satisfying*

(i) $I(u_n) \longrightarrow c$ *with* $c < c_0 + (2/N)S^{N/4}$, *where* c_0 *is defined in (12).*
(ii) $\|I'(u_n)\| \longrightarrow 0$ *as* $n \longrightarrow \infty$.

Then $\{u_n\}$ *has a convergent subsequence.*

Proof. It is clear that $\|\Delta u_n\|_2^2$ is uniformly bounded from condition (i) and (ii). For a subsequence of u_n, we can get a $w_0 \in D$ such that

$$u_n \rightharpoonup w_0 \quad \text{in } D. \quad (49)$$

So, from (ii), we obtain that

$$\langle I'(w_0), w\rangle = 0, \quad \text{for } \forall w \in D. \quad (50)$$

Then w_0 is a weak solution of (1), $w_0 \neq 0$, and $w_0 \in N$, $I(w_0) \geq c_0$. Let $u_n = w_0 + v_n$. So, $v_n \rightharpoonup 0$ in D. By Brezis-Lieb lemma (see [24]), we conclude that

$$\|u_n\|_p^p = \|w_0 + v_n\|_p^p = \|w_0\|_p^p + \|v_n\|_p^p + o(1). \quad (51)$$

Thus, for n large enough, we get

$$c_0 + \frac{2}{N}S^{N/4} > I(w_0 + v_n) = I(w_0) + \frac{1}{2}\|\Delta v_n\|_2^2 - \frac{1}{p}\|v_n\|_p^p + o(1) \geq c_0 + \frac{1}{2}\|\Delta v_n\|_2^2 - \frac{1}{p}\|v_n\|_p^p + o(1), \quad (52)$$

which means

$$\frac{1}{2}\|\Delta v_n\|_2^2 - \frac{1}{p}\|v_n\|_p^p < \frac{2}{N}S^{N/4} + o(1). \quad (53)$$

Moreover, we infer from condition (ii) that

$$o(1) = \langle I'(u_n), u_n\rangle = \langle I'(w_0), w_0\rangle + \|\Delta v_n\|_2^2 - \|v_n\|_p^p + o(1) = \|\Delta v_n\|_2^2 - \|v_n\|_p^p + o(1), \quad (54)$$

and then we obtain

$$\|\Delta v_n\|_2^2 - \|v_n\|_p^p = o(1). \quad (55)$$

Next we shall prove that if (53) and (55) hold, then there exists the subsequence of $\{v_n\}$ (still denoted by $\{v_n\}$), which satisfies

$$\|\Delta v_n\|_2 \longrightarrow 0, \quad n \longrightarrow +\infty. \quad (56)$$

We assume $\{v_n\}$ is bounded away from 0; that is

$$\exists C > 0, \\ \|\Delta v_n\|_2 \geq C, \quad \forall n \in \mathbb{N}. \quad (57)$$

So from (55) we can get

$$\|v_n\|_p^{p-2} \geq S + o(1), \\ \|v_n\|_p^p \geq S^{N/4} + o(1). \quad (58)$$

We infer from (53) and (55) that

$$\frac{2}{N}S^{N/4} \le \frac{2}{N}\|v_n\|_p^p + o(1) = \frac{1}{2}\|\Delta v_n\|_2^2 - \frac{1}{p}\|v_n\|_p^p + o(1) < \frac{2}{N}S^{N/4} \quad (59)$$

for n large. This is contradiction. So, we can get $u_n \longrightarrow w_0$ strongly in D. □

Note that $u_0 \ne 0$. Following [23], we set $\Sigma \subset \Omega$ to be a set of positive measures such that $u_0 > 0$ on Σ. Let us define

$$U_{\varepsilon,a}(x) = \xi_a(x)\,u_{\varepsilon,a}(x), \quad x \in \mathbb{R}^N, \quad (60)$$

where $u_{\varepsilon,a}(x)$ and $\xi_a(x)$ are defined in Section 1. Without loss of generality, we take $u_{\varepsilon,a}(x) = \varepsilon^{(N-4)/2}/(\varepsilon^2 + |x-a|^2)^{(N-4)/2}$. Then we have the following estimates for $U_{\varepsilon,a}$.

Lemma 8. *$\forall R > 0$ and a.e. $a \in \Sigma$, there exists $\varepsilon_0 > 0$ such that*

$$I\left(u_0 + RU_{\varepsilon,a}\right) < c_0 + \frac{2}{N}S^{N/4} \quad (61)$$

for every $0 < \varepsilon < \varepsilon_0$.

Proof. Let $B = \int_{\mathbb{R}^N}|\Delta u_1(x)|^2dx$ and $A = \int_{\mathbb{R}^N}|u_1(x)|^2dx$. By the definition of $u_\varepsilon(x)$, we can get the Sobolev embedding exponent $S = B/A^{2/p}$. A direct computation shows that

$$\begin{aligned}
&\int_\Omega |\Delta\xi_a(x)|^2 |u_{\varepsilon,a}(x)|^2 dx = O\left(\varepsilon^{N-4}\right),\\
&\int_\Omega |\nabla\xi_a(x)|^2 |\nabla u_{\varepsilon,a}(x)|^2 dx = O\left(\varepsilon^{N-4}\right),\\
&\int_\Omega |\Delta\xi_a(x)|\,|\xi_a(x)|\,|u_{\varepsilon,a}(x)|\,|\Delta u_{\varepsilon,a}(x)|\,dx = O\left(\varepsilon^{N-4}\right),\\
&\int_\Omega |\Delta\xi_a(x)|\,|\nabla\xi_a(x)|\,|\nabla u_{\varepsilon,a}(x)|\,|u_{\varepsilon,a}(x)|\,dx = O\left(\varepsilon^{N-4}\right),\\
&\int_\Omega |\nabla\xi_a(x)|\,|\xi_a(x)|\,|\Delta u_{\varepsilon,a}(x)|\,|\nabla u_{\varepsilon,a}(x)|\,dx = O\left(\varepsilon^{N-4}\right),\\
&\int_\Omega |\xi_a(x)|^2 |\Delta u_{\varepsilon,a}(x)|^2 dx = B + O\left(\varepsilon^{N-4}\right),\\
&\|U_{\varepsilon,a}(x)\|_p^p = A + O\left(\varepsilon^N\right).
\end{aligned} \quad (62)$$

Now we take the $C_0^\infty(\Omega)$ function $\xi_a(x)$ such that

$$\begin{aligned}
&\xi_a(x) \equiv 1, \quad \text{when } |x-a| \le r_0,\\
&0 \le \xi_a(x) \le 1, \quad \text{when } r_0 \le |x-a| \le 2r_0,\\
&\xi_a(x) \equiv 0, \quad \text{when } |x-a| \ge 2r_0,\\
&|\nabla\xi_a(x)| \le \frac{C}{r_0},\\
&|\Delta\xi_a(x)| \le \frac{C}{r_0^2},
\end{aligned} \quad (63)$$

where $r_0 > 0$. On the other hand, we see that

$$\begin{aligned}
u_{\varepsilon,a}(x) &= \frac{\varepsilon^{(N-4)/2}}{\left(\varepsilon^2 + |x-a|^2\right)^{(N-4)/2}},\\
\nabla u_{\varepsilon,a} &= \frac{(N-4)\,\varepsilon^{(N-4)/2}(x-a)}{\left(\varepsilon^2 + |x-a|^2\right)^{(N-2)/2}},\\
\Delta u_{\varepsilon,a} &= \frac{(N-4)\,\varepsilon^{N/2} - (N-4)(N-3)\,\varepsilon^{(N-4)/2}|x-a|^2}{\left(\varepsilon^2 + |x-a|^2\right)^{N/2}}.
\end{aligned} \quad (64)$$

So, by direct computation we infer that

$$\begin{aligned}
&\int_\Omega |\Delta\xi_a(x)|^2 |u_{\varepsilon,a}(x)|^2 dx\\
&\le \frac{C}{r_0^4}\int_{r_0\le|x-a|\le 2r_0} \frac{\varepsilon^{N-4}}{\left(\varepsilon^2+|x-a|^2\right)^{N-4}}dx\\
&= \frac{C\omega_{N-1}}{r_0^4}\int_{r_0/\varepsilon}^{2r_0/\varepsilon} \frac{\varepsilon^4 r^{N-1}}{\left(1+r^2\right)^{N-4}}dr\\
&\le \frac{C\omega_{N-1}\varepsilon^4}{r_0^4}\int_{r_0/\varepsilon}^{2r_0/\varepsilon} r^{7-N}dr = O\left(\varepsilon^{N-4}\right),
\end{aligned} \quad (65)$$

where ω_{N-1} is the measure of the unit sphere in $\mathbb{R}^N$. Moreover, we have that

$$\begin{aligned}
&\int_\Omega |\nabla\xi_a(x)|^2 |\nabla u_{\varepsilon,a}(x)|^2 dx \le \frac{C}{r_0^2}\\
&\cdot \int_{r_0\le|x-a|\le 2r_0} \frac{\varepsilon^{N-4}|x-a|^2}{\left(\varepsilon^2+|x-a|^2\right)^{N-2}}dx = \frac{C\omega_{N-1}}{r_0^2}\\
&\cdot \int_{r_0/\varepsilon}^{2r_0/\varepsilon} \frac{\varepsilon^2 r^{N+1}}{\left(1+r^2\right)^{N-2}}dr \le \frac{C\omega_{N-1}\varepsilon^2}{r_0^2}\int_{r_0/\varepsilon}^{2r_0/\varepsilon} r^{5-N}dr\\
&= O\left(\varepsilon^{N-4}\right),\\
&\int_\Omega |\Delta\xi_a(x)|\,|\xi_a(x)|\,|u_{\varepsilon,a}(x)|\,|\Delta u_{\varepsilon,a}(x)|\,dx \le \frac{C}{r_0^2}\\
&\cdot \int_{r_0\le|x-a|\le 2r_0} \frac{\varepsilon^{N-2} + \varepsilon^{N-4}|x-a|^2}{\left(\varepsilon^2+|x-a|^2\right)^{N-2}}dx
\end{aligned}$$

$$= \frac{C\omega_{N-1}}{r_0^2}\left(\int_{r_0/\varepsilon}^{2r_0/\varepsilon}\frac{\varepsilon^2 r^{N-1}}{(1+r^2)^{N-2}}dr + \int_{r_0/\varepsilon}^{2r_0/\varepsilon}\frac{\varepsilon^2 r^{N+1}}{(1+r^2)^{N-2}}dr\right) \le \frac{C\omega_{N-1}}{r_0^2}\left(\int_{r_0/\varepsilon}^{2r_0/\varepsilon}\varepsilon^2 r^{3-N}dr + \int_{r_0/\varepsilon}^{2r_0/\varepsilon}\varepsilon^2 r^{5-N}dr\right) = O\left(\varepsilon^{N-2}\right) + O\left(\varepsilon^{N-4}\right) = O\left(\varepsilon^{N-4}\right),$$

$$\int_\Omega |\Delta\xi_a(x)||\nabla\xi_a(x)||\nabla u_{\varepsilon,a}(x)||u_{\varepsilon,a}(x)|dx \le \frac{C}{r_0^3} \cdot \int_{r_0\le|x-a|\le 2r_0}\frac{\varepsilon^{N-4}|x-a|}{\left(\varepsilon^2+|x-a|^2\right)^{N-3}}dx = \frac{C\omega_{N-1}}{r_0^3} \cdot \int_{r_0/\varepsilon}^{2r_0/\varepsilon}\frac{\varepsilon^3 r^N}{(1+r^2)^{N-3}}dr \le \frac{C\omega_{N-1}\varepsilon^3}{r_0^3}\int_{r_0/\varepsilon}^{2r_0/\varepsilon} r^{6-N}dr = O\left(\varepsilon^{N-4}\right),$$

$$\int_\Omega |\nabla\xi_a(x)||\xi_a(x)||\Delta u_{\varepsilon,a}(x)||\nabla u_{\varepsilon,a}(x)|dx \le \frac{C}{r_0} \cdot \int_{r_0\le|x-a|\le 2r_0}\frac{\varepsilon^{N-2}|x-a| + \varepsilon^{N-4}|x-a|^3}{\left(\varepsilon^2+|x-a|^2\right)^{N-1}}dx = \frac{C\omega_{N-1}}{r_0}\left(\int_{r_0/\varepsilon}^{2r_0/\varepsilon}\frac{\varepsilon r^N}{(1+r^2)^{N-1}}dr + \int_{r_0/\varepsilon}^{2r_0/\varepsilon}\frac{\varepsilon r^{N+2}}{(1+r^2)^{N-1}}dr\right) \le \frac{C\omega_{N-1}}{r_0^3}\left(\int_{r_0/\varepsilon}^{2r_0/\varepsilon}\varepsilon r^{2-N}dr + \int_{r_0/\varepsilon}^{2r_0/\varepsilon}\varepsilon r^{4-N}dr\right) = O\left(\varepsilon^{N-2}\right) + O\left(\varepsilon^{N-4}\right) = O\left(\varepsilon^{N-4}\right),$$

$$\int_\Omega |\xi_a(x)|^2|\Delta u_{\varepsilon,a}(x)|^2dx \le \int_{|x-a|\le r_0}|\Delta u_{\varepsilon,a}(x)|^2dx + \int_{r_0\le|x-a|\le 2r_0}|\Delta u_{\varepsilon,a}(x)|^2dx \le C\int_{r_0/\varepsilon}^{2r_0/\varepsilon}\left(\frac{r^{N-1}}{(1+r^2)^N} + \frac{r^{N+3}}{(1+r^2)^N}dr + \frac{r^{N+1}}{(1+r^2)^N}\right)dr + \int_{\mathbb{R}^N}\frac{|y|^2}{\left(1+|y|^2\right)^N}dy \le B + O\left(\varepsilon^N\right) + O\left(\varepsilon^{N-4}\right) + O\left(\varepsilon^{N-2}\right) = B + O\left(\varepsilon^{N-4}\right). \quad (66)$$

Thus, we infer from [23] that

$$\left\|u_0 + RU_{\varepsilon,a}\right\|_p^p = \left\|u_0\right\|_p^p + R^p\left\|U_{\varepsilon,a}\right\|_p^p + pR\int_\Omega |u_0|^{p-2}u_0U_{\varepsilon,a}dx + pR^{p-1}\int_\Omega U_{\varepsilon,a}^{p-1}u_0dx + o\left(\varepsilon^{(N-4)/2}\right). \quad (67)$$

From all of the above, noticing that $u_0 \in N$, one has that

$$I\left(u_0 + RU_{\varepsilon,a}\right) = \frac{1}{2}\int_\Omega\left|\Delta\left(u_0 + RU_{\varepsilon,a}\right)\right|^2dx - \frac{1}{p} \cdot \int_\Omega\left|u_0 + RU_{\varepsilon,a}\right|^pdx - \int_\Omega f\left(u_0 + RU_{\varepsilon,a}\right)dx = \frac{1}{2}\int_\Omega|\Delta u_0|^2 + R\int_\Omega \Delta u_0\Delta U_{\varepsilon,a}dx + \frac{1}{2} \cdot R^2\int_\Omega\left|\Delta U_{\varepsilon,a}\right|^2dx - \frac{1}{p}\int_\Omega\left|u_0 + RU_{\varepsilon,a}\right|^pdx - \int_\Omega f\left(u_0 + RU_{\varepsilon,a}\right)dx = \left(\frac{1}{2}\int_\Omega|\Delta u_0|^2 - \frac{1}{p}\int_\Omega|u_0|^pdx - \int_\Omega fu_0dx\right) + R\left(\int_\Omega \Delta u_0\Delta U_{\varepsilon,a}dx - \int_\Omega|u_0|^{p-2}u_0U_{\varepsilon,a}dx - \int_\Omega fU_{\varepsilon,a}dx\right) + \frac{R^2}{2}B - \frac{R^p}{p}A - R^{p-1}\int_\Omega U_{\varepsilon,a}^{p-1}u_0dx + o\left(\varepsilon^{(N-4)/2}\right) = I(u_0) + \frac{R^2}{2} \cdot B - \frac{R^p}{p}A - R^{p-1}\int_\Omega U_{\varepsilon,a}^{p-1}u_0dx + o\left(\varepsilon^{(N-4)/2}\right). \quad (68)$$

By using an estimate obtained by G. Folland [26] and setting $u_0 = 0$ outside Ω, one gets that

$$\int_\Omega U_{\varepsilon,a}^{p-1}u_0\,dx = \varepsilon^{(N-4)/2}u_0(a)E + o\left(\varepsilon^{(N-4)/2}\right), \quad (69)$$

where

$$E = \int_{\mathbb{R}^N}\frac{dx}{\left(1+|x|^2\right)^{(N+4)/2}}, \quad \frac{1}{\left(1+|x|^2\right)^{(N+4)/2}} \in L^1\left(\mathbb{R}^N\right). \quad (70)$$

Consequently, we have

$$I\left(u_0+RU_{\varepsilon,a}\right)=c_0+\frac{R^2}{2}B-\frac{R^p}{p}A-R^{p-1}u_0\left(a\right)E\varepsilon^{(N-4)/2}+o\left(\varepsilon^{(N-4)/2}\right). \tag{71}$$

We set

$$h\left(s\right)=\frac{B}{2}s^2-\frac{A}{p}s^p-u_0\left(a\right)E\varepsilon^{(N-4)/2}s^{p-1},\quad s>0, \tag{72}$$

and assume $h(s)$ achieves its maximum at $s_1>0$, which satisfies

$$s_1B-s_1^{p-1}A=\left(p-1\right)u_0\left(a\right)E\varepsilon^{(N-4)/2}s^{p-2}. \tag{73}$$

We define

$$s_0=\left(\frac{B}{A}\right)^{1/(p-2)}, \tag{74}$$

which is the maximum point of $h_1(s)=(B/2)s^2-(A/p)s^p$. We can conclude that $0<s_1<s_0$, and $s_1\longrightarrow s_0\ (\varepsilon\longrightarrow 0)$. Let $s_1=s_0\left(1-\delta\right)$. It is easy to see that $\delta\longrightarrow 0\ (\varepsilon\longrightarrow 0)$. From (73) we can get

$$s_0\left(1-\delta\right)B-s_0^{p-1}\left(1-\delta\right)^{p-1}A=\left(p-1\right)u_0\left(a\right)E\varepsilon^{(N-4)/2}s_0^{p-2}(\left(1-\delta\right)^{p-2} \tag{75}$$

and then expanding for δ, we can get

$$\left(p-2\right)\left(\frac{B^{(p-1)/(p-2)}}{A^{1/(p-2)}}\right)\delta=\left(p-1\right)\frac{B}{A}u_0\left(a\right)E\varepsilon^{(N-4)/2}+o\left(\varepsilon^{(N-4)/2}\right). \tag{76}$$

So, one sees that

$$\begin{aligned}I\left(u_0+RU_{\varepsilon,a}\right)&=c_0+\frac{R^2}{2}B-\frac{R^p}{p}A-R^{p-1}u_0\left(a\right)E\varepsilon^{(N-4)/2}+o\left(\varepsilon^{(N-4)/2}\right)\\&\le c_0+\frac{s_1^2}{2}B-\frac{s_1^p}{p}A-s_1^{p-1}u_0\left(a\right)E\varepsilon^{(N-4)/2}+o\left(\varepsilon^{(N-4)/2}\right)\\&=c_0+\frac{s_0^2}{2}B-\frac{s_0^p}{p}A-s_0^2B\delta+s_0^pA\delta-s_0^{p-1}u_0\left(a\right)E\varepsilon^{(N-4)/2}+o\left(\varepsilon^{(N-4)/2}\right)\\&=c_0+\frac{2}{N}S^{N/4}-s_0^{p-1}u_0\left(a\right)E\varepsilon^{(N-4)/2}+o\left(\varepsilon^{(N-4)/2}\right).\end{aligned} \tag{77}$$

When we take small $\varepsilon_0>0$, we arrive at

$$I\left(u_0+RU_{\varepsilon,a}\right)<c_0+\frac{2}{N}S^{N/4},\quad\forall 0<\varepsilon<\varepsilon_0. \tag{78}$$

This finishes the proof. □

Now we are ready to give the proof of Theorem 2.

Proof of Theorem 2. It is clear that the uniqueness of $t_1(u)$ satisfies the following condition:

$$\begin{gathered}t_1\left(u\right)u\in N^-,\\ I\left(t_1\left(u\right)u\right)=\max_{t\ge t_{max}}I\left(tu\right),\\ \text{for every } u\in D,\ \|u\|=1.\end{gathered} \tag{79}$$

At the same time, $t_1(u)$ is a continuous function of u. And N^- divides D into two components D_1 and D_2, which are disconnected from each other. Let

$$\begin{aligned}D_1&=\left\{u=0\text{ or }u:\|u\|<t_1\left(\frac{u}{\|u\|}\right)\right\},\\ D_2&=\left\{u=0\text{ or }u:\|u\|>t_1\left(\frac{u}{\|u\|}\right)\right\}.\end{aligned} \tag{80}$$

Obviously, $D-N^-=D_1\cup D_2$, and we can check $N^+\subset D_1$, $u_0\in D_1$. We can choose a constant C_0, which satisfies

$$0<t_1\left(u\right)\le C_0,\quad\forall\|u\|=1, \tag{81}$$

and claim that

$$w=u_0+R_0U_{\varepsilon,a}\in D_2, \tag{82}$$

where $R_0=((1/B)|C_0^2-\|u_0\|^2|)^{1/2}+1$. In fact, a direct computation shows that

$$\begin{aligned}\|w\|^2&=\left\|u_0\right\|^2+R_0^2\left\|U_{\varepsilon,a}\right\|^2+2R_0\int_\Omega\left|\Delta u_0\right|\left|\Delta U_{\varepsilon,a}\right|dx\\&=\left\|u_0\right\|^2+R_0^2B+o\left(1\right)>C_0^2\ge\left[t_1\left(\frac{w}{\|w\|}\right)\right]^2\end{aligned} \tag{83}$$

for $\varepsilon>0$ small enough. Thus, claim (82) holds.

We fix $\varepsilon>0$ such that both (61) and (82) hold by the choice of R_0 and $a\in\Sigma$. We set

$$\Gamma=\left\{\gamma\in C\left(\left[0,1\right],D\right):\gamma\left(0\right)=u_0,\gamma\left(1\right)=u_0+R_0U_{\varepsilon,a}\right\}, \tag{84}$$

and take $h(t)=u_0+tR_0U_{\varepsilon,a}$, which belongs to Γ. From Lemma 7, we conclude that

$$c=\inf_{h\in\Gamma}\max_{t\in[0,1]}I\left(h\left(t\right)\right)<c_0+\frac{2}{N}S^{N/4}. \tag{85}$$

Since every $h\in\Gamma$ intersects N^-, we get that

$$c_1=\inf_{N^-}I\le c<c_0+\frac{2}{N}S^{N/4}. \tag{86}$$

Next we use Mountain-Pass lemma to prove Theorem 2. Let $\{u_n\} \subset N^-$ be such that

$$I(u_n) \longrightarrow c_1, \quad \left\|I'(u_n)\right\| \longrightarrow 0. \tag{87}$$

We deduce from Lemma 7 that there exists a subsequence (still denoted by $\{u_n\}$) of $\{u_n\}$, and $u_1 \in D$ such that

$$u_n \longrightarrow u_1 \quad \text{in } D. \tag{88}$$

So, u_1 is a critical point for I, $u_1 \in N^-$ and $I(u_1) = c_1$. □

Remark 9. We point out that the results of Theorems 1–2 can be generalized to polyharmonic problem. Precisely, we can consider the semilinear polyharmonic problem

$$\begin{aligned} (-\Delta)^m u &= |u|^{p-2} u + f, \quad x \in \Omega, \\ u = Du = \cdots &= D^{m-1} u = 0, \quad x \in \partial\Omega, \end{aligned} \tag{89}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N(N \geq 2m + 1).m \in \mathbb{N}^+$, $p = 2N/(N-2m)$ denotes the critical Sobolev exponent for $(-\Delta)^m$, and $f \in L^q(\Omega)(q = 2N/(N+2m))(f \neq 0)$ is small enough. We can define the energy functional:

$$I(u) = \frac{1}{2}\|u\|_m^2 - \frac{1}{p}\int_\Omega |u|^p dx - \int_\Omega fu dx, \quad u \in H, \tag{90}$$

where

$$\begin{aligned} H &= H_0^m(\Omega) \\ &= \left\{v \in H^m(\Omega) \mid D^i v = 0 \text{ on } \partial\Omega,\ \forall 0 \leq i < m\right\}. \end{aligned} \tag{91}$$

H is Hilbert space and endowed with the scalar product

$$(u, v) = \begin{cases} \int_\Omega \left((-\Delta)^k u\right)\left((-\Delta)^k v\right) dx, & \textit{if } m = 2k \textit{ is even}, \\ \int_\Omega \left(\nabla(-\Delta)^k u\right)\left(\nabla(-\Delta)^k v\right) dx, & \textit{if } m = 2k+1 \textit{ is odd} \end{cases} \tag{92}$$

and $\|\cdot\|_m$ is the corresponding norm. Let

$$u_\varepsilon(x) = C_{N,m}\frac{\varepsilon^{(N-2m)/2}}{\left(\varepsilon^2 + |x|^2\right)^{(N-2m)/2}} \tag{93}$$

be an extremal function for the Sobolev inequality in $\mathbb{R}^N$, and the constant $C_{N,m}$ be independent of ε. By dividing the Nehari manifold, we can prove $(PS)_c$ condition when $c < c_0 + (m/N)S^{m/2N}$, where $c_0 = I(u_0)$ and u_0 is the first solution. By using the same idea of this article, one can obtain that (89) has at least two nontrivial solutions.

Appendix

In this appendix we mainly focus on the proof of Lemma 4.

Proof of Lemma 4. For $u \in D$, we define

$$G(u) = C_N \|\Delta u\|_2^{(N+4)/4} - \int_\Omega fu dx. \tag{A.1}$$

Let $\{u_n\}$ be the minimizing sequence of (21) with $\|u_n\|_p = 1$. That is, we have that

$$G(u_n) = \mu_0 + o(1), \tag{A.2}$$

and $u_n \rightharpoonup u_0$ in D, $u_n \longrightarrow u_0$ a.e in Ω and $\|u_0\|_p \leq 1$. If $\|u_0\|_p = 1$, then the conclusion holds. In the following we consider the case $\|u_0\|_p < 1$ by using contradiction argument. Let $u_n = u_0 + w_n$. So, $w_n \rightharpoonup 0$ in D. From Brezis-Lieb lemma [27], we obtain that

$$\begin{aligned} 1 = \|u_0 + w_n\|_p^p &= \|u_0\|_p^p + \|w_n\|_p^p + o(1), \\ \|w_n\|_p^2 &= \left(1 - \|u_0\|_p^p\right)^{2/p} + o(1). \end{aligned} \tag{A.3}$$

By Sobolev's inequality, we conclude that

$$\begin{aligned} &\mu_0 + o(1) \\ &= C_N \|\Delta(u_0 + w_n)\|_2^{(N+4)/4} - \int_\Omega f(u_0 + w_n) dx \\ &= C_N \left(\|\Delta u_0\|_2^2 + \|\Delta w_n\|_2^2\right)^{(N+4)/8} - \int_\Omega fu_0 dx \\ &\quad + o(1) \\ &\geq C_N \left(\|\Delta u_0\|_2^2 + S\left(1 - \|u_0\|_p^p\right)^{2/p} + o(1)\right)^{(N+4)/8} \\ &\quad - \int_\Omega fu_0 dx + o(1). \end{aligned} \tag{A.4}$$

Hence we get

$$C_N \left(\|\Delta u_0\|_2^2 + S\left(1 - \|u_0\|_p^p\right)^{2/p}\right)^{(N+4)/8} - \int_\Omega fu_0 dx \leq \mu_0. \tag{A.5}$$

From paper [23], we know that for every $u \in D$, $\|u\|_p < 1$, and $a \in \Omega$, there exists $C_\varepsilon = C_\varepsilon(a) > 0$ such that

$$\left\|u + C_\varepsilon U_{\varepsilon,a}\right\|_p = 1, \tag{A.6}$$

where $U_{\varepsilon,a}$ is defined in (60). We infer from (A.6) that

$$\begin{aligned} 1 &= \left\|u + C_\varepsilon U_{\varepsilon,a}\right\|_p^p \\ &= \|u\|_p^p + C_\varepsilon^p A + o(1), \\ C_\varepsilon^2 &= \frac{\left(1 - \|u\|_p^p\right)^{2/p}}{A^{2/p}} + o(1), \end{aligned} \tag{A.7}$$

$$\begin{aligned} \left\|\Delta\left(u + C_\varepsilon U_{\varepsilon,a}\right)\right\|_2^2 &= \|\Delta u\|_2^2 + C_\varepsilon^2 B + o(1) \\ &= \|\Delta u\|_2^2 + S\left(1 - \|u\|_p^p\right)^{2/p} \\ &\quad + o(1). \end{aligned} \tag{A.8}$$

Thus, for each $u \in D$ and $\|u\|_p < 1$, we obtain that

$$\begin{aligned}\mu_0 &\le G\left(u + C_\varepsilon U_{\varepsilon,a}\right) \\ &= C_N \left\|\Delta\left(u + C_\varepsilon U_{\varepsilon,a}\right)\right\|_2^{(N+4)/4} \\ &\quad - \int_\Omega f\left(u + C_\varepsilon U_{\varepsilon,a}\right) dx \\ &= C_N \left(\|\Delta u\|_2^2 + S\left(1 - \|u\|_p^p\right)^{2/p}\right)^{(N+4)/8} \\ &\quad - \int_\Omega f u dx + o(1).\end{aligned} \tag{A.9}$$

Combining (A.5) and (A.9), we get

$$\begin{aligned}&C_N \left(\|\Delta u_0\|_2^2 + S\left(1 - \|u_0\|_p^p\right)^{2/p}\right)^{(N+4)/8} \\ &\quad - \int_\Omega f u_0 dx = \mu_0.\end{aligned} \tag{A.10}$$

Moreover, for each $w \in D$ one has

$$\begin{aligned}&\frac{d}{dt}\left[C_N \left(\left\|\Delta\left(u_0 + tw\right)\right\|_2^2\right.\right. \\ &\quad \left. + S\left(1 - \left\|u_0 + tw\right\|_p^p\right)^{2/p}\right)^{(N+4)/8} \\ &\quad \left.- \int_\Omega f\left(u_0 + tw\right) dx\right]_{t=0} = 0.\end{aligned} \tag{A.11}$$

That is,

$$\begin{aligned}&\frac{N+4}{4} C_N \left[\|\Delta u_0\|_2^2 + S\left(1 - \|u_0\|_p^p\right)^{2/p}\right]^{(N-4)/8} \\ &\times \left[\int_\Omega \Delta u_0 \Delta w dx \right. \\ &\left. - S\left(1 - \|u_0\|_p^p\right)^{(2-p)/p} \int_\Omega |u_0|^{p-2} u_0 w dx\right] \\ &- \int_\Omega f w dx = 0.\end{aligned} \tag{A.12}$$

Let $k = ((N+4)/4) C_N [\|\Delta u_0\|_2^2 + S(1 - \|u_0\|_p^p)^{2/p})]^{(N-4)/8} > 0$ and $\lambda = S(1 - \|u_0\|_p^p)^{(2-p)/p}$. Then (A.12) implies that u_0 is the weak solution of

$$\Delta^2 u = \lambda |u|^{p-2} u + \frac{1}{k} f. \tag{A.13}$$

Since $f \ne 0$, we can conclude that $u_0 \ne 0$. Recall that $u_0(a) > 0$, $\forall a \in \Sigma$, and $\Sigma \subset \Omega$. Replace u_0 with $-u_0$, and f with $-f$ if necessarily. For $a \in \Sigma$, we take $c_\varepsilon = c_\varepsilon(a)$ such that

$$\left\|u_0 + c_\varepsilon U_{\varepsilon,a}\right\|_p = 1. \tag{A.14}$$

We obtain the contradiction if we prove that

$$G\left(u_0 + c_\varepsilon U_{\varepsilon,a}\right) < \mu_0 \tag{A.15}$$

for a suitable choice of $a \in \Sigma$ and small ε.

From (A.7), we infer that $c_\varepsilon \nearrow c_0$ as $\varepsilon \longrightarrow 0$, where $c_0 = (1 - \|u_0\|_p^p)^{1/p}/A^{1/p}$. Let $c_\varepsilon = c_0(1 - \delta_\varepsilon)$, where $\delta_\varepsilon \longrightarrow 0$ as $\varepsilon \longrightarrow 0$. A direct computation shows that

$$\begin{aligned}&c_0^p A \delta_\varepsilon \\ &\quad = \varepsilon^{(N-4)/2} \left[c_0 \int_\Omega \frac{|u_0|^{p-2} u_0 \xi_a}{|x-a|^{N-4}} dx + c_0^{p-1} u_0(a) E\right] \\ &\qquad + o\left(\varepsilon^{(N-4)/2}\right),\end{aligned} \tag{A.16}$$

where $E = \int_{\mathbb{R}^N} (dx/(\varepsilon^2 + |x|^2)^{(N+4)/2})$. We deduce from (A.10) and (A.16) and the definition of c_0 that

$$\begin{aligned}&G\left(u_0 + c_\varepsilon U_{\varepsilon,a}\right) = C_N \left\|\Delta\left(u_0 + c_\varepsilon U_{\varepsilon,a}\right)\right\|_2^{(N+4)/4} \\ &\quad - \int_\Omega f\left(u_0 + c_\varepsilon U_{\varepsilon,a}\right) dx = C_N \left[\|\Delta u_0\|_2^2\right. \\ &\quad + 2c_0 \int_\Omega \Delta u_0 \Delta U_{\varepsilon,a} dx + c_0^2\left(1 - 2\delta_\varepsilon\right) B \\ &\quad \left.+ o\left(\varepsilon^{(N-4)/2}\right)\right]^{(N+4)/8} - \int_\Omega f\left(u_0 + c_\varepsilon U_{\varepsilon,a}\right) dx \\ &\quad = C_N \left[\|\Delta u_0\|_2^2 + c_0^2 B\right]^{(N+4)/8} - \int_\Omega f u_0 dx \\ &\quad + \frac{N+4}{8} C_N \left[\|\Delta u_0\|_2^2 + c_0^2 B\right]^{(N-4)/8} \\ &\quad \cdot \left[2c_0 \int_\Omega \Delta u_0 \Delta U_{\varepsilon,a} dx - 2c_0^2 \delta_\varepsilon B\right] \\ &\quad - c_0 \int_\Omega f U_{\varepsilon,a} dx + o\left(\varepsilon^{(N-4)/2}\right) = \mu_0 \\ &\quad + k\lambda c_0 \int_\Omega |u_0|^{p-2} u_0 U_{\varepsilon,a} dx - k c_0^2 B \delta_\varepsilon \\ &\quad + o\left(\varepsilon^{(N-4)/2}\right)\end{aligned} \tag{A.17}$$

and, furthermore, we infer from (A.16) that

$$\begin{aligned}&c_0 \int_\Omega |u_0|^{p-2} u_0 U_{\varepsilon,a} dx \\ &\quad = \varepsilon^{(N-4)/2} \int_\Omega \frac{|u_0|^{p-2} u_0 \xi_a}{|x-a|^{N-4}} dx + o\left(\varepsilon^{(N-4)/2}\right) \\ &\quad = c_0^p A \delta_\varepsilon - c_0^{p-1} u_0(a) E \varepsilon^{(N-4)/2} + o\left(\varepsilon^{(N-4)/2}\right).\end{aligned} \tag{A.18}$$

Also, we notice that

$$\begin{aligned}\lambda c_0^p A &= S\left(1 - \|u_0\|_p^p\right)^{(2-p)/p} \left(1 - \|u_0\|_p^p\right) \\ &= \frac{B}{A^{2/p}} \left(1 - \|u_0\|_p^p\right)^{2/p} = c_0^2 B.\end{aligned} \tag{A.19}$$

Hence it follows that

$$\begin{aligned} G\left(u_0+c_\varepsilon U_{\varepsilon,a}\right) &= \mu_0+k\lambda\left(c_0^p A\delta_\varepsilon-c_0^{p-1}u_0(a)E\varepsilon^{(N-4)/2}\right) \\ &\quad -kc_0^2B\delta_\varepsilon+o\left(\varepsilon^{(N-4)/2}\right) \\ &= \mu_0-k\lambda c_0^{p-1}u_0(a)E\varepsilon^{(N-4)/2}+o\left(\varepsilon^{(N-4)/2}\right) \\ &< \mu_0. \end{aligned} \tag{A.20}$$

This finishes the proof. □

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

Xiaoyong Qian was devoted to prove the first solution of the equation. Jun Wang proved the existence of the second solution of the equation. Maochun Zhu participated in the proof of the section solution of the equation. All authors read and approved the final manuscript.

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