Research Article

A New Approach to Adaptive Stabilization of Stochastic High-Order Nonholonomic Systems

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This paper studies the problem of adaptive stabilization for a class of stochastic high-order nonholonomic systems. Under the weaker assumptions, by constructing the appropriate Lyapunov function and combining sign function technique, an adaptive state feedback controller is designed to guarantee global asymptotic stability in probability of the closed-loop system. The effectiveness of the controller is demonstrated by a mechanical system.

1. Introduction

Ever since the stochastic stability theory was established by [1, 2], the design and analysis of backstepping controller for stochastic nonlinear systems has achieved remarkable development in recent years; see [1, 3–12] and the references therein. But these papers do not consider stochastic nonholonomic systems.

In this paper, we consider stochastic high-order nonholonomic systems

$$dx_0 = d_0(t,\theta) u_0^{p_0} dt + f_0(x_0,\theta) dt + g_0^\top(x_0,\theta) d\omega,$$

$$dx_i = d_i(t,\theta) u_0^{q_i} x_{i+1}^{p_i} dt + f_i(x_0,x,\theta) dt$$

$$\qquad + g_i^\top(x_0,x,\theta) d\omega, \qquad (1)$$

$$dx_n = d_n(t,\theta) u_0^{q_n} u^{p_n} dt + f_n(x_0,x,\theta) dt$$

$$\qquad + g_n^\top(x_0,x,\theta) d\omega,$$

where $x_0 \in R$ and $x = (x_1,\ldots,x_n)^\top \in R^n$ are system states, u_0 and u are control inputs, $p_i \in R_{odd}^{\geq 1} \triangleq \{p/q | p$ and q are positive odd integers, and $p \geq q\}$ are odd integers, and $q_i \geq 0$ are constants, $i = 1,\ldots,n$. $\theta \in R^m$ is an unknown constant vector. ω is a r-dimensional standard Wiener process defined on a probability space $(\Omega, \mathscr{F}, P)$ with

Ω being a sample space, $\mathscr{F}$ being a filtration, and P being a probability measure. $f_0(x_0,\theta) : R \times R^m \longrightarrow R$, $g_0(x_0,\theta) : R \times R^m \longrightarrow R^r$, $f_i(x_0,x,\theta) : R \times R^n \times R^r \longrightarrow R$, $g_i(x_0,x,\theta) : R \times R^n \times R^r \longrightarrow R^r$, $i = 1,\ldots,n$, are locally Lipschitz functions. $d_i(t,\theta)$, $i = 0,\ldots,n$, are nonlinear control coefficients.

Since many mechanical systems can be modeled by system (1), a series of theoretical results have been obtained. To mention a few, [13–16] investigated the deterministic case of $g_i = 0$. Specially when $d_i = p_i = q_i = 1$ and $f_i = 0$, the authors designed a continuous controller by proposing a sliding mode control approach in [13]. When systems contain nonlinear drifts and unknown time-varying coefficients, [15] presented an adaptive control approach to achieve sate-feedback stabilization. High-order nonholonomic systems, that is, $p_i \geq 1$, were introduced in [16]. The design procedure in [16] combined the idea of a discontinuous change of coordinate and adding a power integrator. However, it did not consider nonlinear parameterizations. Because many practical control systems such as biochemical processes and machines with friction often contain unknown parameters, [14] studied the problem of adaptive stabilization control design for a class of high-order nonholonomic systems with strong nonlinear drifts. Since stochastic noise frequently arises and is inevitable in practical control systems, how to extend these approaches to stochastic high-order nonholonomic systems is a very interesting problem. When $g_i \neq 0$, [17] studied the problem of

state feedback stabilization for a class of high-order stochastic nonholonomic systems with disturbed control directions and more general nonlinear drifts. However, nonlinear parameterization and nonlinear drift term in x_0-subsystem were not discussed in [17]. Unfortunately, all these mentioned results require that the nonlinearities $f_i(\cdot)$ are dependent on $(x_1, \ldots, x_i)$. Naturally, an interesting problem is put forward: *For system (1), under weaker assumptions, can a stabilizable state feedback controller be designed?*

In this paper, we will provide a satisfactory answer to this problem. By constructing the appropriate Lyapunov function, skillfully combining parameter separation, sign function, and backstepping design approach, an adaptive state feedback controller is designed to guarantee global asymptotic stability in probability of the closed-loop system. Finally, a simulation example is used to demonstrate the effectiveness of this approach.

The contributions and difficulties of this paper are highlighted from three aspects.

(i) The system under consideration is more general than those investigated in [13–17]. System (1) has unknown control coefficients and permits unknown parameters to enter nonlinear equations. Nonlinear functions $f_i(\cdot)$ are dependent on $(x_1, \ldots, x_{i+1})$, which makes discontinuous change of coordinates be inapplicable to the adaptive state feedback control of systems (1). In this paper, we propose a novel design approach to solve this obstacle.

(ii) The unknown growth rates of the upper bounds of $f_i(\cdot)$ and $g_i(\cdot)$ are extended; see Remark 13. Therefore, some new mathematical tools, such as sign function and transformation technique, are introduced to simplify the construction of Lyapunov function.

(iii) An practical example for mobile robot with small angle measurement error is modeled and solved by the proposed approach.

This paper is organized as follows. Section 2 gives some preliminaries. Section 3 presents the design and analysis of the adaptive controller, following a practical example in Section 4. Section 5 concludes this paper. The proofs of Propositions 16–19 are given in Appendix.

2. Mathematical Preliminaries

Some notations and definitions will be used throughout this paper. R^+ stands for the set of all nonnegative real numbers and R^n denotes real n-dimensional space. For vector $\chi = (\chi_1, \ldots, \chi_n)^\top$, $\overline{\chi}_i$ represents $(\chi_1, \ldots, \chi_i)^\top$. For a given vector or matrix x, $x^\top$ denotes its transpose, $|x|$ is the Euclidean norm of a vector x, and $\mathrm{Tr}\{x\}$ denotes its trace when x is square. $\mathscr{C}^i$ denotes the set of all functions with continuous ith partial derivatives. $\mathscr{K}$ denotes the set of all functions: $R^+ \longrightarrow R^+$ that are continuous, strictly increasing, and vanishing at zero; $\mathscr{K}_\infty$ denotes the set of all functions that are of class $\mathscr{K}$ and unbounded. For any $y \in R$, sign function $\mathrm{sgn}(y)$ is defined as $\mathrm{sgn}(y) = 1$ if $y > 0$; $\mathrm{sgn}(y) = 0$ if $y = 0$ and $\mathrm{sgn}(y) = -1$ if $y < 0$. For simplicity, we denote $\lceil y \rceil^p = |y|^p \mathrm{sgn}(y)$. The arguments of functions are sometimes omitted or simplified, for example, a function $f(x(t))$ is denoted by $f(x)$ or f.

In what follows, we present some definitions and lemmas which will be frequently used in the design and analysis of controller. Consider stochastic nonlinear system

$$
\begin{aligned}
dx &= f(x, \theta)\, dt + g(x, \theta)\, d\omega, \\
x(t_0) &= x_0 \in R^n,
\end{aligned}
\tag{2}
$$

where ω is a r-dimensional standard Wiener process defined on a probability space $(\Omega, \mathscr{F}, P)$, $x \in R^n$ is system state, $\theta \in R^m$ is an unknown constant vector and the functions $f : R^n \times R^m \longrightarrow R^n$, and $g : R^n \times R^m \longrightarrow R^{n \times r}$ are locally bounded and locally Lipschitz with respect to x.

Definition 1 (see [1]). The equilibrium $x(t) = 0$ of system (2) with $f(0, \theta) = 0$, $g(0, \theta) = 0$ is

(i) globally stable in probability if for any $\varepsilon > 0$, there exists a class $\mathscr{K}$ function $r(\cdot)$ such that $P\{|x(t)| < r(x_0)\} \geq 1 - \varepsilon$, for any $t \geq 0$, $x_0 \in R^n \setminus \{0\}$

(ii) globally asymptotically stable in probability if it is globally stable in probability and $P\{\lim_{t \to \infty} |x(t) = 0|\} = 1$, for any $x_0 \in R^n$

Lemma 2 (see [1]). *For system (2), if there exists a $\mathscr{C}^2$ function $V(x) : R^n \longrightarrow R^+$ and class $\mathscr{K}_\infty$ functions $\alpha_1(\cdot)$ and $\alpha_2(\cdot)$, such that*

$$
\begin{aligned}
\alpha_1(|x|) &\leq V(x) \leq \alpha_2(|x|), \\
\mathscr{L}V(x) &= \frac{\partial V(x)}{\partial x} f(x, \theta) \\
&\quad + \frac{1}{2}\mathrm{Tr}\left\{ g^T(x, \theta) \frac{\partial^2 V(x)}{\partial x^2} g(x, \theta) \right\} \\
&\leq -W(x),
\end{aligned}
\tag{3}
$$

where $W(x)$ is a nonnegative continuous function, then, for any $x(t_0) \in R^n$,

(i) there exists an almost surely unique solution on $[t_0, \infty)$

(ii) the equilibrium $x = 0$ is globally stable in probability, and the solution $x(t)$ satisfies $P\{\lim_{t \to \infty} W(x(t)) = 0\} = 1$

Lemma 3 (see [18]). *For any real-valued continuous function $h(x, y)$, where $x \in R^m$, $y \in R^n$, there are smooth scalar functions $c(x) \geq 1$ and $d(y) \geq 1$, such that $|h(x, y)| \leq c(x)d(y)$.*

Lemma 4 (see [19]). *Let x, y be real variables, then for any positive real numbers b, c, d and continuous function $a(\cdot) \geq 0$, one has $a(\cdot)|x|^c|y|^d \leq b|x|^{c+d} + (d/(c+d))((c+d)/c)^{-c/d}a(\cdot)^{(c+d)/d}b^{-c/d}|y|^{c+d}$.*

Lemma 5 (see [20]). *$f(x) = \lceil x \rceil^a$ is continuously differentiable and $\dot{f}(x) = a|x|^{a-1}$, where $a \geq 1$, $x \in R$. Moreover, $df(x(t))/dt = a|x(t)|^{a-1}\dot{x}(t)$.*

Lemma 6 (see [20]). *If $p = a/b \in R_{odd}^{\geq 1}$, $b \geq 1$, then $|x^p - y^p| \leq 2^{1-1/b}|\lceil x \rceil^a - \lceil y \rceil^a|^{1/b}$ for any $x, y \in R$.*

Lemma 7 (see [20]). *For given $r \geq 0$ and every $x \in R$, $y \in R$, $|x + y|^r \leq C_r(|x|^r + |y|^r)$ holds, where $C_r = 2^{r-1}$ if $r \geq 1$ and $C_r = 1$ if $0 \leq r < 1$.*

Lemma 8 (see [20]). *Let $f : [a,b] \longrightarrow R$ be a continuous monotone function with $f(a) = 0$; then $|\int_a^b f(x)dx| \leq |f(b)||b-a|$.*

3. Design and Analysis of Adaptive Controller

3.1. Problem Formulation and Assumptions. The aim of this paper is to design an adaptive state feedback controller for system (1) to guarantee global asymptotic stability in probability of the closed-loop system. We need the following assumptions to achieve this aim.

Assumption 9. The sign of $d_i(t,\theta)$, $i = 0,\ldots,n$, is assumed to be positive, and there exist smooth functions $0 < \lambda_{i,1}(t) \leq \lambda_{i,2}(t,\theta)$, such that

$$0 < \lambda_{i,1}(t) \leq d_i(t,\theta) \leq \lambda_{i,2}(t,\theta). \tag{4}$$

Assumption 10. For smooth functions $f_0(x_0,\theta)$ and $g_0(x_0,\theta)$, there exist bounded smooth functions $\alpha_0(x_0,\theta)$, $\beta_0(x_0,\theta)$ such that

$$\begin{aligned} f_0(x_0,\theta) &= x_0\alpha_0(x_0,\theta), \\ g_0(x_0,\theta) &= x_0\beta_0(x_0,\theta). \end{aligned} \tag{5}$$

Assumption 11. For $i = 1,\ldots,n$, there exist nonnegative continuous functions $\alpha_i(x_0,\overline{x}_i,\theta)$, $\beta_i(x_0,\overline{x}_i,\theta)$ and a constant τ satisfying $-r_n < \tau < 0$ such that

$$\begin{aligned} &\left|f_i(x_0,x,\theta)\right| \\ &\leq \left(|x_1|^{(r_i+\tau)/r_1} + \cdots + |x_i|^{(r_i+\tau)/r_i}\right)\alpha_i(x_0,\overline{x}_i,\theta) \\ &\quad + c_i u_0^{q_i} d_i(t,\theta)|x_{i+1}|^{(r_i+\tau)/r_{i+1}}, \\ &\left|g_i(x_0,x,\theta)\right| \\ &\leq \left(|x_1|^{(2r_i+\tau)/2r_1} + \cdots + |x_i|^{(2r_i+\tau)/2r_i}\right)\beta_i(x_0,\overline{x}_i,\theta), \end{aligned} \tag{6}$$

where r_{n+1} is a given positive constant and $r_1, \cdots, r_n$ are recursively defined by $r_i = p_i r_{i+1} - \tau$ and nonnegative constants $c_i < 1/2^{p_i-1}$.

Remark 12. Assumptions 10-11 imply that $f_0(0,\theta) = 0$, $g_0(0,\theta) = 0$, $f_i(0,0,\theta) = 0$, $g_i(0,0,\theta) = 0$, $i = 1,\ldots,n$; that is, origin is the equilibrium of system (1).

Remark 13. Assumption 11 enlarges the scope of nonholonomic systems. Specifically, if state x_{i+1} does not appear in the nonlinear function $f_i(\cdot)$, Assumption 11 is degenerated into the following form:

$$\begin{aligned} &\left|f_i(x_0,x,\theta)\right| \\ &\leq \left(|x_1|^{(r_i+\tau)/r_1} + \cdots + |x_i|^{(r_i+\tau)/r_i}\right)\alpha_i(x_0,\overline{x}_i,\theta), \\ &\left|g_i(x_0,x,\theta)\right| \\ &\leq \left(|x_1|^{(2r_i+\tau)/2r_1} + \cdots + |x_i|^{(2r_i+\tau)/2r_i}\right)\beta_i(x_0,\overline{x}_i,\theta), \end{aligned} \tag{7}$$

where $r_1 = 1$, $r_i = 1 + (i-1)\tau$ with τ being a constant for each $i = 1,\ldots,n$. Particularly, when $\tau = 0$, (7) becomes the following growth condition:

$$\begin{aligned} \left|f_i(x_0,x,\theta)\right| &\leq (|x_1| + \cdots + |x_i|)\alpha_i(x_0,\overline{x}_i,\theta), \\ \left|g_i(x_0,x,\theta)\right| &\leq (|x_1| + \cdots + |x_i|)\beta_i(x_0,\overline{x}_i,\theta). \end{aligned} \tag{8}$$

If $\alpha_i(x_0,\overline{x}_i,\theta)$ and $\beta_i(x_0,\overline{x}_i,\theta)$ are constants, (8) becomes the linear growth condition in [21]. If $\alpha_i(x_0,\overline{x}_i,\theta)$ and $\beta_i(x_0,\overline{x}_i,\theta)$ are smooth nonnegative functions, (8) becomes the linear growth condition in [22, 23]. When $\tau = -p/q \in (-2/(4n+1),0)$ with p being an even integer and q being an odd integer, condition (7) becomes the low-order growth condition in [24, 25]. Assumption 11 extends the value of τ to an explicit interval rather than a ratio of an even integer over an odd integer.

Remark 14. By Lemma 3, there are smooth scalar functions $\widetilde{\lambda}_{i,2}(t) \geq 1$, $\overline{\alpha}_i(x_0,\overline{x}_i) \geq 1$, $\overline{\beta}_i(x_0,\overline{x}_i) \geq 1$, $i = 0,\ldots,n$, and $\Theta(\theta) \geq 1$ such that

$$\begin{aligned} &\lambda_{i,1}(t) \leq d_i(t,\theta) \leq \widetilde{\lambda}_{i,2}(t)\Theta, \\ &\left|f_0(x_0,\theta)\right| \leq |x_0|\overline{\alpha}_0(x_0)\Theta, \\ &\left|g_0(x_0,\theta)\right| \leq |x_0|\overline{\beta}_0(x_0)\Theta, \\ &\left|f_i(x_0,x,\theta)\right| \\ &\leq \left(|x_1|^{(r_i+\tau)/r_1} + \cdots + |x_i|^{(r_i+\tau)/r_i}\right)\overline{\alpha}_i(x_0,\overline{x}_i)\Theta \\ &\quad + c_i u_0^{q_i} d_i(t,\theta)|x_{i+1}|^{(r_i+\tau)/r_{i+1}}, \\ &\left|g_i(x_0,x,\theta)\right| \\ &\leq \left(|x_1|^{(2r_i+\tau)/2r_1} + \cdots + |x_i|^{(2r_i+\tau)/2r_i}\right)\overline{\beta}_i(x_0,\overline{x}_i)\Theta. \end{aligned} \tag{9}$$

3.2. Stability and Convergence Analysis. We state the main result in this paper.

Theorem 15. *If Assumptions 9–11 hold for system (1), under an appropriate controller, the origin of the closed-loop system is globally asymptotically stable in probability for any initial condition.*

Proof. The proof is based on inductive argument. Firstly, we design an adaptive controller by considering two cases: $x_0(t_0) \neq 0$ and $x_0(t_0) = 0$.

Case I ($x_0(t_0) \neq 0$). The structure of system (1) means that the design procedure is divided into two separate parts.

Part I (design of controller u_0). Let us consider x_0-subsystem in system (1), define $\Theta_0 = \lambda_{0,1}^{-1}\max\{\lambda_{0,1}^{-1}, \Theta, \Theta^2\}$, and choose $V_0 = x_0^4/4 + (\lambda_{0,1}/2)\widetilde{\Theta}_0^2$, where $\widetilde{\Theta}_0 = \Theta_0 - \widehat{\Theta}_0$. From Remark 14, it follows that

$$\begin{aligned} \mathscr{L}V_0 &= x_0^3\left(d_0 u_0^{p_0} + f_0\right) + \frac{3}{2}\text{Tr}\left\{g_0 x_0^2 g_0^\top\right\} - \lambda_{0,1}\widetilde{\Theta}_0\dot{\widehat{\Theta}}_0 \\ &\leq -\frac{a}{\lambda_{0,1}}x_0^4 + \frac{a}{\lambda_{0,1}}x_0^4 + x_0^3 d_0 u_0^{p_0} + \Theta\overline{\alpha}_0(x_0)x_0^4 \end{aligned}$$

$$+ \frac{3}{2}\Theta^2\overline{\beta}_0^2(x_0)x_0^4 - \lambda_{0,1}\widetilde{\Theta}_0\dot{\widehat{\Theta}}_0$$

$$\leq -\frac{a}{\lambda_{0,1}}x_0^4 + \lambda_{0,1}x_0^3\left(\frac{d_0}{\lambda_{0,1}}u_0^{p_0} + h(x_0)\widehat{\Theta}_0 x_0\right)$$

$$- \lambda_{0,1}\widetilde{\Theta}_0\left(\dot{\widehat{\Theta}}_0 - x_0^4 h(x_0)\right),$$

$$\tag{10}$$

where a is a positive constant and $h(x_0) = a + \overline{\alpha}_0(x_0) + (3/2)\overline{\beta}_0^2(x_0)$ is a positive smooth function. Substituting the adaptive controller

$$u_0 = -\left(x_0 h(x_0)\sqrt{\widehat{\Theta}_0^2 + 1}\right)^{1/p_0},$$

$$\dot{\widehat{\Theta}}_0 = x_0^4 h(x_0)$$

$$\tag{11}$$

into (10) leads to

$$\mathscr{L}V_0 \leq -\frac{a}{\lambda_{0,1}}x_0^4.$$

$$\tag{12}$$

The succeeding proposition characterizes the features of x_0-subsystem.

Proposition 16. *If Assumptions 9-10 hold for x_0-subsystem, controller (11) guarantees that*

(i) x_0-subsystem has an almost surely unique solution on $[t_0, \infty)$ for any $x_0(t_0) \neq 0$

(ii) the equilibrium $x_0 = 0$ of x_0-subsystem is globally asymptotically stable in probability

(iii) the solution of x_0-subsystem and u_0 does not cross zero for any $x_0(t_0) \neq 0$

Proof. See Appendix.   □

Part II (design of controller u). In this part, we need to consider x_i-subsystem in system (1). With the help of Proposition 16, controller u will be recursively constructed by applying the adding a power integrator approach. Before the beginning of recursive design, we define $\Theta_1 = \max\{1, \Theta, \Theta^2\}$ and state transformation

$$\xi_1 = \lceil x_1 \rceil^{\sigma/r_1} - \lceil x_1^* \rceil^{\sigma/r_1}, \quad x_1^* = 0,$$

$$\xi_2 = \lceil x_2 \rceil^{\sigma/r_2} - \lceil x_2^* \rceil^{\sigma/r_2},$$

$$x_2^* = -\phi_1\left(x_0, x_1, \widehat{\Theta}_1\right)\lceil \xi_1 \rceil^{r_2/\sigma},$$

$$\vdots$$

$$\xi_k = \lceil x_k \rceil^{\sigma/r_k} - \lceil x_k^* \rceil^{\sigma/r_k},$$

$$x_k^* = -\phi_{k-1}\left(x_0, \overline{x}_{k-1}, \widehat{\Theta}_1\right)\lceil \xi_{k-1} \rceil^{r_k/\sigma},$$

$$\tag{13}$$

where σ is a positive constant satisfying the relationship $\sigma \geq \max\{p_1 r_2, \ldots, p_n r_{n+1}\}$, $\widehat{\Theta}_1$ is the estimate of Θ_1, and functions $\phi_k(\cdot)$ satisfying $\mathrm{sgn}(\phi_k) = \mathrm{sgn}(u_0^{q_k})$ will be determined later.

To solve the problem caused by sign function, the definition of $W_k(x_0, \overline{x}_k, \widehat{\Theta}_1)$ is given by

$$W_k\left(x_0, \overline{x}_k, \widehat{\Theta}_1\right)$$

$$= \int_{x_k^*}^{x_k}\left[\lceil s \rceil^{\sigma/r_k} - \lceil x_k^* \rceil^{\sigma/r_k}\right]^{(4l\sigma - \tau - r_k)/\sigma}\,\mathrm{d}s,$$

$$\tag{14}$$

where $k = 1, \ldots, n$ and $4l$ is an even number and satisfies $(4l - 2)\sigma \geq \tau + r_k$.

Proposition 17. *W_k is $\mathscr{C}^2$ function and satisfies $C_{k1}|x_k - x_k^*|^{(4l\sigma-\tau)/r_k} \leq W_k \leq C_{k2}|\xi_k|^{(4l\sigma-\tau)/\sigma}$, where $C_{k1} = 2^{(\sigma-r_k)(4l\sigma-\tau-r_k)/\sigma r_k}(r_k/(4l\sigma - \tau))$, $C_{k2} = 2^{1-r_k/\sigma}$, $k = 1, \ldots, n$.*

Proof. See Appendix.   □

To obtain the expression of u, we determine $\phi_1, \cdots, \phi_n$ by induction, the design procedure is implemented as follows.

Step 1. Let $V_1 = W_1 + (1/2)\widetilde{\Theta}_1^2$, where $\widetilde{\Theta}_1 = \Theta_1 - \widehat{\Theta}_1$. By using (13), (A.3), (A.7), and Lemma 7, we get

$$\mathscr{L}V_1 \leq \lceil \xi_1 \rceil^{(4l\sigma-\tau-r_1)/\sigma}\left(d_1 u_0^{q_1} x_2^{p_1} + f_1\right)$$

$$+ \frac{4l\sigma - \tau - r_1}{2r_1}|\xi_1|^{((4l-1)\sigma-\tau-r_1)/\sigma}|x_1|^{(\sigma-r_1)/r_1}|g_1|^2$$

$$- \widetilde{\Theta}_1\dot{\widehat{\Theta}}_1 \leq -n\xi_1^{4l} + n\xi_1^{4l}$$

$$+ \lceil \xi_1 \rceil^{(4l\sigma-\tau-r_1)/\sigma}d_1 u_0^{q_1}\left(x_2^{p_1} - x_2^{*p_1}\right)$$

$$+ \lceil \xi_1 \rceil^{(4l\sigma-\tau-r_1)/\sigma}d_1 u_0^{q_1} x_2^{*p_1} - \widetilde{\Theta}_1\dot{\widehat{\Theta}}_1$$

$$+ \frac{4l\sigma - \tau - r_1}{2r_1}|\xi_1|^{((4l-1)\sigma-\tau-r_1)/\sigma}|x_1|^{(\sigma+r_1+\tau)/r_1}\overline{\beta}_1^2\Theta^2$$

$$+ \lceil \xi_1 \rceil^{(4l\sigma-\tau-r_1)/\sigma}$$

$$\cdot\left(|x_1|^{(r_1+\tau)/r_1}\overline{\alpha}_1\Theta + c_1 d_1 u_0^{q_1}|x_2|^{p_1}\right).$$

$$\tag{15}$$

Clearly, the application of the fact $\lceil \xi_1 \rceil^{(4l\sigma-\tau-r_1)/\sigma}d_1 u_0^{q_1} x_2^{*p_1} \leq 0$ implies

$$\lceil \xi_1 \rceil^{(4l\sigma-\tau-r_1)/\sigma}u_0^{q_1}d_1 x_2^{*p_1}$$

$$+ \lceil \xi_1 \rceil^{(4l\sigma-\tau-r_1)/\sigma}c_1 d_1 u_0^{q_1}|x_2|^{p_1}$$

$$\leq \lceil \xi_1 \rceil^{(4l\sigma-\tau-r_1)/\sigma}u_0^{q_1}d_1 x_2^{*p_1}$$

$$+ |\xi_1|^{(4l\sigma-\tau-r_1)/\sigma}\overline{c}_1 d_1|u_0|^{q_1}|x_2^*|^{p_1}$$

$$+ |\xi_1|^{(4l\sigma-\tau-r_1)/\sigma}\overline{c}_1 d_1|u_0|^{q_1}|\xi_2|^{(r_1+\tau)/\sigma}$$

$$\leq \lceil \xi_1 \rceil^{(4l\sigma-\tau-r_1)/\sigma}d_1 u_0^{q_1}(1 - \overline{c}_1)x_2^{*p_1}$$

$$+ |\xi_1|^{(4l\sigma-\tau-r_1)/\sigma}\overline{c}_1 d_1|u_0|^{q_1}|\xi_2|^{(r_1+\tau)/\sigma},$$

$$\tag{16}$$

where $\bar{c}_1 = 2^{p_1-1}c_1$ is a positive constant. Substituting (16) into (15) and choosing

$$x_2^* = -\left(\frac{1}{\lambda_{1,1} u_0^{q_1} (1 - \bar{c}_1)} \right.$$

$$\left. \cdot \widehat{\Theta}_1 \left(\bar{\alpha}_1 + n + \frac{4l\sigma - \tau - r_1}{2r_1} \bar{\beta}_1^2 \right) \right)^{1/p_1} \lceil \xi_1 \rceil^{r_2/\sigma} \quad (17)$$

$$\triangleq -\phi_1 \left(x_0, x_1, \widehat{\Theta}_1 \right) \lceil \xi_1 \rceil^{r_2/\sigma},$$

one has

$$\mathcal{L}V_1 \le -n\xi_1^{4l} + \lceil \xi_1 \rceil^{(4l\sigma-\tau-r_1)/\sigma} d_1 u_0^{q_1} \left(x_2^{p_1} - x_2^{*\,p_1} \right)$$

$$+ \left(\Psi_1 - \dot{\widehat{\Theta}}_1 \right) \left(\widetilde{\Theta}_1 + \eta_1 \right) \quad (18)$$

$$+ \bar{c}_1 d_1 |u_0|^{q_1} |\xi_1|^{(4l\sigma-\tau-r_1)/\sigma} |\xi_2|^{(r_1+\tau)/\sigma},$$

where $\Psi_1 = \xi_1^{4l}(\bar{\alpha}_1 + n + ((4l\sigma - \tau - r_1)/2r_1)\bar{\beta}_1^2) \ge 0$ and $\eta_1 = 0$.

Step k ($k = 2, \ldots, n$). At Step $k - 1$, assume that there exists a $\mathscr{C}^2$ function V_{k-1} and virtual controllers $x_2^*, \cdots, x_k^*$ such that

$$\mathcal{L}V_{k-1}$$

$$\le - (n - k + 2) \sum_{i=1}^{k-1} \xi_i^{4l}$$

$$+ \lceil \xi_{k-1} \rceil^{(4l\sigma-\tau-r_{k-1})/\sigma} d_{k-1} u_0^{q_{k-1}} \left(x_k^{p_{k-1}} - x_k^{*\,p_{k-1}} \right) \quad (19)$$

$$+ \left(\Psi_{k-1} - \dot{\widehat{\Theta}}_1 \right) \left(\widetilde{\Theta}_1 + \eta_{k-1} \right)$$

$$+ \bar{c}_{k-1} d_{k-1} |u_0|^{q_{k-1}} |\xi_{k-1}|^{(4l\sigma-\tau-r_{k-1})/\sigma} |\xi_k|^{(r_{k-1}+\tau)/\sigma},$$

where $\bar{c}_{k-1} = 2^{p_{k-1}-1}c_{k-1}$ is a nonnegative constant and

$$0 \le \Psi_{k-1} \le \left(\xi_1^{4l} + \cdots + \xi_{k-1}^{4l} \right) \varphi_{k-1}, \quad (20)$$

for a $\mathscr{C}^\infty$ function $\varphi_{k-1} \ge 0$. In what follows, we prove that (19) also holds at Step k. To prove this point, considering $V_k = V_{k-1} + W_k$, we deduce from (19) and (A.3)-(A.13) that

$$\mathcal{L}V_k$$

$$\le - (n - k + 2) \sum_{i=1}^{k-1} \xi_i^{4l}$$

$$+ \lceil \xi_{k-1} \rceil^{(4l\sigma-\tau-r_{k-1})/\sigma} d_{k-1} u_0^{q_{k-1}} \left(x_k^{p_{k-1}} - x_k^{*\,p_{k-1}} \right)$$

$$+ \frac{\partial W_k}{\partial \widehat{\Theta}_1} \dot{\widehat{\Theta}}_1 + \left(\Psi_{k-1} - \dot{\widehat{\Theta}}_1 \right) \left(\widetilde{\Theta}_1 + \eta_{k-1} \right)$$

$$+ \bar{c}_{k-1} d_{k-1} |u_0|^{q_{k-1}} |\xi_{k-1}|^{(4l\sigma-\tau-r_{k-1})/\sigma} |\xi_k|^{(r_{k-1}+\tau)/\sigma}$$

$$+ \frac{\partial W_k}{\partial x_0} \left(d_0 u_0^{p_0} + f_0 \right) + \frac{\partial W_k}{\partial x_k} \left(d_k u_0^{q_k} x_{k+1}^{p_k} + f_k \right)$$

$$+ \frac{1}{2} \left| \frac{\partial^2 W_k}{\partial x_0^2} \right| |g_0|^2 + \sum_{i=1}^{k-1} \frac{\partial W_k}{\partial x_i} \left(d_i u_0^{q_i} x_{i+1}^{p_i} + f_i \right)$$

$$+ \frac{1}{2} \left| \frac{\partial^2 W_k}{\partial x_k^2} \right| |g_k|^2 + \frac{1}{2} \sum_{i=1}^{k-1} \left| \frac{\partial^2 W_k}{\partial x_i^2} \right| |g_i|^2$$

$$+ \frac{1}{2} \left| \frac{\partial^2 W_k}{\partial x_k \partial x_0} \right| |g_k| \left| g_0^\top \right| + \frac{1}{2} \sum_{i=1}^{k-1} \left| \frac{\partial^2 W_k}{\partial x_i \partial x_0} \right| |g_i| \left| g_0^\top \right|$$

$$+ \frac{1}{2} \sum_{i=1}^{k-1} \left| \frac{\partial^2 W_k}{\partial x_i \partial x_k} \right| |g_i| \left| g_k^\top \right|$$

$$+ \frac{1}{2} \sum_{i,j=1, i \ne j}^{k-1} \left| \frac{\partial^2 W_k}{\partial x_i \partial x_j} \right| |g_i| \left| g_j^\top \right|.$$

$$(21)$$

We give the following proposition, whose proof is placed in the Appendix.

Proposition 18. *There is a smooth function σ_k such that*

$$\lceil \xi_{k-1} \rceil^{(4l\sigma-\tau-r_{k-1})/\sigma} d_{k-1} u_0^{q_{k-1}} \left(x_k^{p_{k-1}} - x_k^{*\,p_{k-1}} \right)$$

$$+ \frac{\partial W_k}{\partial x_0} \left(d_0 u_0^{q_0} + f_0 \right) + \sum_{i=1}^{k-1} \frac{\partial W_k}{\partial x_i} \left(d_i u_0^{q_i} x_{i+1}^{p_i} + f_i \right)$$

$$+ \frac{\partial W_k}{\partial x_k} f_k + \frac{1}{2} \left| \frac{\partial^2 W_k}{\partial x_0^2} \right| |g_0|^2 + \frac{1}{2} \left| \frac{\partial^2 W_k}{\partial x_k^2} \right| |g_k|^2$$

$$+ \frac{1}{2} \sum_{i=1}^{k-1} \left| \frac{\partial^2 W_k}{\partial x_i \partial x_k} \right| |g_i| \left| g_k^\top \right| + \frac{1}{2} \sum_{i=1}^{k-1} \left| \frac{\partial^2 W_k}{\partial x_i^2} \right| |g_i|^2$$

$$+ \frac{1}{2} \left| \frac{\partial^2 W_k}{\partial x_k \partial x_0} \right| |g_k| \left| g_0^\top \right| + \frac{1}{2} \sum_{i=1}^{k-1} \left| \frac{\partial^2 W_k}{\partial x_i \partial x_0} \right| |g_i| \left| g_0^\top \right| \quad (22)$$

$$+ \frac{1}{2} \sum_{i,j=1, i \ne j}^{k-1} \left| \frac{\partial^2 W_k}{\partial x_i \partial x_j} \right| |g_i| \left| g_j^\top \right|$$

$$+ \bar{c}_{k-1} d_{k-1} |u_0|^{q_{k-1}} |\xi_{k-1}|^{(4l\sigma-\tau-r_{k-1})/\sigma} |\xi_k|^{(r_{k-1}+\tau)/\sigma}$$

$$\le \frac{12}{13} \sum_{i=1}^{k-1} \xi_i^{4l} + \sigma_k \Theta_1 \xi_k^{4l}$$

$$+ \bar{c}_k d_k |u_0|^{q_k} |\xi_k|^{(4l\sigma-\tau-r_k)/\sigma} |\xi_{k+1}|^{(r_k+\tau)/\sigma}$$

$$+ |\xi_k|^{(4l\sigma-\tau-r_k)/\sigma} d_k \bar{c}_k |u_0|^{q_k} |x_{k+1}^*|^{p_k},$$

where $\bar{c}_k = 2^{p_k-1}c_k$ is a nonnegative constant. Substituting (22) into (21), we arrive at

$$\mathcal{L}V_k \le -\left(n - k + \frac{14}{13} \right) \sum_{i=1}^{k-1} \xi_i^{4l}$$

$$+ \left(\Psi_{k-1} - \dot{\widehat{\Theta}}_1 \right) \left(\widetilde{\Theta}_1 + \eta_{k-1} \right) + \frac{\partial W_k}{\partial x_k} d_k u_0^{q_k} x_{k+1}^{p_k}$$

$$+ \sigma_k \xi_k^{4l} \Theta_1 + \frac{\partial W_k}{\partial \widehat{\Theta}_1} \dot{\widehat{\Theta}}_1$$

$$+ \bar{c}_k d_k \left| u_0 \right|^{q_k} \left| \xi_k \right|^{(4l\sigma - \tau - r_k)/\sigma} \left| \xi_{k+1} \right|^{(r_k + \tau)/\sigma}$$

$$+ \left| \xi_k \right|^{(4l\sigma - \tau - r_k)/\sigma} d_k \bar{c}_k \left| u_0 \right|^{q_k} \left| x_{k+1}^* \right|^{p_k}.$$

$$(23)$$

Obviously, defining $\Psi_k = \Psi_{k-1} + \xi_k^{4l}(\sigma_k + n - k + 14/13)$ *and* $\eta_k = \eta_{k-1} - \partial W_k / \partial \widehat{\Theta}_1$, *it follows from (20) that* $0 \leq \Psi_k \leq (\xi_1^{4l} + \cdots + \xi_k^{4l}) \varphi_k$, *where* $\varphi_k = \max\{\varphi_{k-1}, \sigma_k + n - k + 14/13\}$ *is a nonnegative smooth function. With this in mind, (23) can be rewritten as*

$$\mathcal{L}V_k \leq - \left(n - k + \frac{14}{13} \right) \sum_{i=1}^{k} \xi_i^{4l} + \left(\Psi_k - \dot{\widehat{\Theta}}_1 \right) \left(\widetilde{\Theta}_1 + \eta_k \right)$$

$$+ \frac{\partial W_k}{\partial x_k} d_k u_0^{q_k} x_{k+1}^{p_k}$$

$$+ \bar{c}_k d_k \left| u_0 \right|^{q_k} \left| \xi_k \right|^{(4l\sigma - \tau - r_k)/\sigma} \left| \xi_{k+1} \right|^{(r_k + \tau)/\sigma}$$

$$+ \left| \xi_k \right|^{(4l\sigma - \tau - r_k)/\sigma} d_k \bar{c}_k \left| u_0 \right|^{q_k} \left| x_{k+1}^* \right|^{p_k} \qquad (24)$$

$$+ \xi_k^{4l} \widehat{\Theta}_1 \left(\sigma_k + n - k + \frac{14}{13} \right) + \eta_{k-1} \Psi_{k-1}$$

$$- \eta_k \Psi_k.$$

With the help of Lemma 4, one has

$$\left| \eta_{k-1} \Psi_{k-1} - \eta_k \Psi_k \right| \leq \left| \eta_{k-1} \xi_k^{4l} \widehat{\Theta}_1 \left(\sigma_k + n - k + \frac{14}{13} \right) \right|$$

$$+ \left| \frac{\partial W_k}{\partial \widehat{\Theta}_1} \Psi_k \right| \qquad (25)$$

$$\leq \frac{1}{13} \sum_{i=1}^{k} \xi_i^{4l} + \overline{\varphi}_k \widehat{\Theta}_1 \xi_k^{4l},$$

which, together with (24), implies that

$$\mathcal{L}V_k \leq - (n - k + 1) \sum_{i=1}^{k} \xi_i^{4l} + \left(\Psi_k - \dot{\widehat{\Theta}}_1 \right) \left(\widetilde{\Theta}_1 + \eta_k \right)$$

$$+ \lceil \xi_k \rceil^{(4l\sigma - \tau - r_k)/\sigma} d_k u_0^{q_k} \left(x_{k+1}^{p_k} - x_{k+1}^{*p_k} \right)$$

$$+ \lceil \xi_k \rceil^{(4l\sigma - \tau - r_k)/\sigma} \left(d_k u_0^{q_k} \left(1 - \bar{c}_k \right) x_{k+1}^{*p_k} + \lceil \xi_k \rceil^{(r_k + \tau)/\sigma} \right) \quad (26)$$

$$\cdot \widehat{\Theta}_1 \left(\overline{\varphi}_k + \sigma_k + n - k + \frac{14}{13} \right) \right) + \bar{c}_k d_k \left| u_0 \right|^{q_k}$$

$$\cdot \left| \xi_k \right|^{(4l\sigma - \tau - r_k)/\sigma} \left| \xi_{k+1} \right|^{(r_k + \tau)/\sigma},$$

where $\overline{\varphi}_k \geq 0$ *is a smooth function. It is easy to see that the virtual controller*

$$x_{k+1}^* = - \left(\frac{1}{\lambda_{k,1} u_0^{q_k} \left(1 - \bar{c}_k \right)} \right.$$

$$\cdot \widehat{\Theta}_1 \left(\overline{\varphi}_k + \sigma_k + n - k + \frac{14}{13} \right) \right)^{1/p_k} \lceil \xi_k \rceil^{r_{k+1}/\sigma}$$

$$\triangleq - \phi_k \left(x_0, \overline{x}_k, \widehat{\Theta}_1 \right) \lceil \xi_k \rceil^{r_{k+1}/\sigma},$$

$$(27)$$

renders

$$\mathcal{L}V_k \leq - (n - k + 1) \sum_{i=1}^{k} \xi_i^{4l}$$

$$+ \lceil \xi_k \rceil^{(4l\sigma - \tau - r_k)/\sigma} d_k u_0^{q_k} \left(x_{k+1}^{p_k} - x_{k+1}^{*p_k} \right)$$

$$+ \left(\Psi_k - \dot{\widehat{\Theta}}_1 \right) \left(\widetilde{\Theta}_1 + \eta_k \right) \qquad (28)$$

$$+ \bar{c}_k d_k \left| u_0 \right|^{q_k} \left| \xi_k \right|^{(4l\sigma - \tau - r_k)/\sigma} \left| \xi_{k+1} \right|^{(r_k + \tau)/\sigma},$$

which still holds for $k = n$. *Hence at last step, we can explicitly construct a positive-definite and proper Lyapunov function* $V_n = V_{n-1} + W_n$ *and a smooth controller* x_{n+1}^* *with form (27) such that*

$$\mathcal{L}V_n \leq - \sum_{i=1}^{n} \xi_i^{4l} + \left(\Psi_n - \dot{\widehat{\Theta}}_1 \right) \left(\widetilde{\Theta}_1 + \eta_n \right)$$

$$+ \lceil \xi_n \rceil^{(4l\sigma - \tau - r_n)/\sigma} d_n u_0^{q_n} \left(u^{p_n} - x_{n+1}^{*p_n} \right) \qquad (29)$$

$$+ \bar{c}_n d_n \left| u_0 \right|^{q_n} \left| \xi_n \right|^{(4l\sigma - \tau - r_n)/\sigma} \left| \xi_{n+1} \right|^{(r_n + \tau)/\sigma}.$$

Noting $\xi_{n+1} = 0$, *by choosing the smooth actual controller* u *and the adaptive law for* $\widehat{\Theta}_1$,

$$u = x_{n+1}^* = - \phi_n \left(x_0, \overline{x}_n, \widehat{\Theta}_1 \right) \lceil \xi_n \rceil^{r_{n+1}/\sigma}, \quad \dot{\widehat{\Theta}}_1 = \Psi_n, \quad (30)$$

we get

$$\mathcal{L}V_n \leq - \sum_{i=1}^{n} \xi_i^{4l}. \qquad (31)$$

The succeeding proposition characterizes the features of x_i-*subsystem.*

Proposition 19. *If Assumptions 9–11 hold for system (1), controllers (11) and (30) guarantee that*

(i) the closed-loop system has an almost surely unique solution on $[t_0, \infty)$ *for each* $(x_1(t_0), \ldots, x_n(t_0), \widehat{\Theta}_1(t_0))$

(ii) the equilibrium of the closed-loop system is globally asymptotically stable in probability, $P\{\lim_{t \to \infty} |x(t)| = 0\} = 1$, $P\{\lim_{t \to \infty} \widehat{\Theta}_1(t)$ *exist and is finite*$\} = 1$ *for each* $(x_1(t_0), \ldots, x_n(t_0), \widehat{\Theta}_1(t_0))$

Proof. See Appendix. □

Case II $(x_0(t_0) = 0)$. When $x_0(t_0) = 0$, how to select the controllers u_0 and u is an interesting topic. If the initial state is zero, one chooses an open loop controller $u_0 = u_0^* \neq 0$ to drive the state x_0 away from zero; a novel controller $u = u^*$ can be obtained by the above procedure of original x_i-subsystem in (1). So there exists $t_s > 0$ such that $x_0(t_s) \neq 0$; after that, controllers u_0 in (11) and u in (30) can be used.

In view of the argument above, there exists an adaptive controller, such that the equilibrium of the closed-loop system is globally asymptotically stable in probability for any initial condition. The proof of Theorem 15 is completed.

4. Simulation Example

Consider the bilinear model of a mobile robot with small angle measurement error [26], which is described by

$$
\begin{aligned}
\dot{x}_c &= \left(1 - \frac{\varepsilon^2}{2}\right)v, \\
\dot{y}_c &= \theta v + \varepsilon v, \\
\dot{\theta} &= \omega,
\end{aligned}
\tag{32}
$$

where ε is a small bias in orientation. v and ω are two control inputs to denote the linear velocity and angular velocity, respectively. Since stochastic disturbance frequently arises in practical control systems, when the angular velocity ω is subject to some stochastic disturbances, ω can be expressed as $\omega(x_c, y_c, \theta) = \omega_1(x_c, y_c, \theta) + \omega_2(x_c, y_c, \theta)\dot{B}(t)$, where $B(t)$ is the so-called white noise [2]. Then system (32) is transformed into

$$
\begin{aligned}
dx_c &= \left(1 - \frac{\varepsilon^2}{2}\right)v\,dt, \\
dy_c &= (\theta v + \varepsilon v)\,dt, \\
d\theta &= \omega_1 dt + \omega_2 dB.
\end{aligned}
\tag{33}
$$

For system (33), by taking the state and input transformation $x_0 = x_c, x_1 = y_c, x_2 = \theta + \varepsilon, u_0 = v, u_1 = \omega_1$, one has

$$
\begin{aligned}
dx_0 &= \left(1 - \frac{\varepsilon^2}{2}\right)u_0 dt, \\
dx_1 &= x_2 u_0 dt, \\
dx_2 &= u_1 dt + \omega_2 dB.
\end{aligned}
\tag{34}
$$

System (34) is a special case of system (1). For simplicity, we assume $1 - \varepsilon^2/2 > 0$, $\omega_2 = x_2 \sin x_1$, and $x_0(0) \neq 0$.

Following the above design process, an adaptive controller can be explicitly given

$$
\begin{aligned}
u_0 &= -x_0, \\
u_1 &= -\widehat{\Theta}_1 \left(\overline{\varphi}_2 + \sigma + \frac{14}{13}\right)\lceil \xi_1 \rceil, \\
\dot{\widehat{\Theta}}_1 &= 2\xi_1^4 + \xi_2^4 \left(\sigma + \frac{14}{13}\right),
\end{aligned}
\tag{35}
$$

where $\xi_1 = \lceil x_1 \rceil$, $\xi_2 = \lceil x_2 \rceil - 2\widehat{\Theta}_1 u_0^{-1}\lceil \xi_1 \rceil$, $\overline{\varphi}_2 = 12|u_0|^{-4/3}(\sigma + 14/13)^{-4/3}(\xi_1^4 + \xi_2^4)$, $\sigma = 231.7 u_0^4 \widehat{\Theta}_1^3 + 6\widehat{\Theta}_1 + 12\widehat{\Theta}_1^{4/3} + 231.7$. By choosing $\varepsilon = 1$ and initial values $(x_0(0), x_1(0), x_2(0), \widehat{\Theta}_1(0)) = (2, -0.25, 2, 1)$, Figure 1 demonstrates the effectiveness of the control scheme.

5. Conclusions

This paper investigates adaptive state feedback stabilization for more general stochastic high-order nonholonomic systems. There still exist some problems to be investigated; for instance, we have the following. (1) The result in this paper can be applied to the case of $p_i \in R_{odd}^{\geq 1}$. However, if p_i is an even number or a ratio of odd integer and even integer, it is unclear whether the control strategy can be applied or not. (2) Recently, some results on stochastic nonlinear systems with SiISS dynamic uncertainty have been obtained [27–35]. An important problem is how to solve adaptive feedback control for stochastic nonholonomic nonlinear systems with SiISS dynamic and parametric uncertainties.

Appendix

Proof of Proposition 16 . It is not hard to prove that $\widetilde{\Theta}_0$ is bounded. By using (12) and Lemma 2, the equilibrium $x_0(t) = 0$ of x_0-subsystem is globally stable in probability and $P\{\lim_{t\to\infty}|x_0(t) = 0|\} = 1$ for any $x_0(t_0) \neq 0$. By Definition 1, the equilibrium $x_0(t) = 0$ of x_0-subsystem is globally asymptotically stable in probability. Substituting (11) into (1), it is easy to obtain
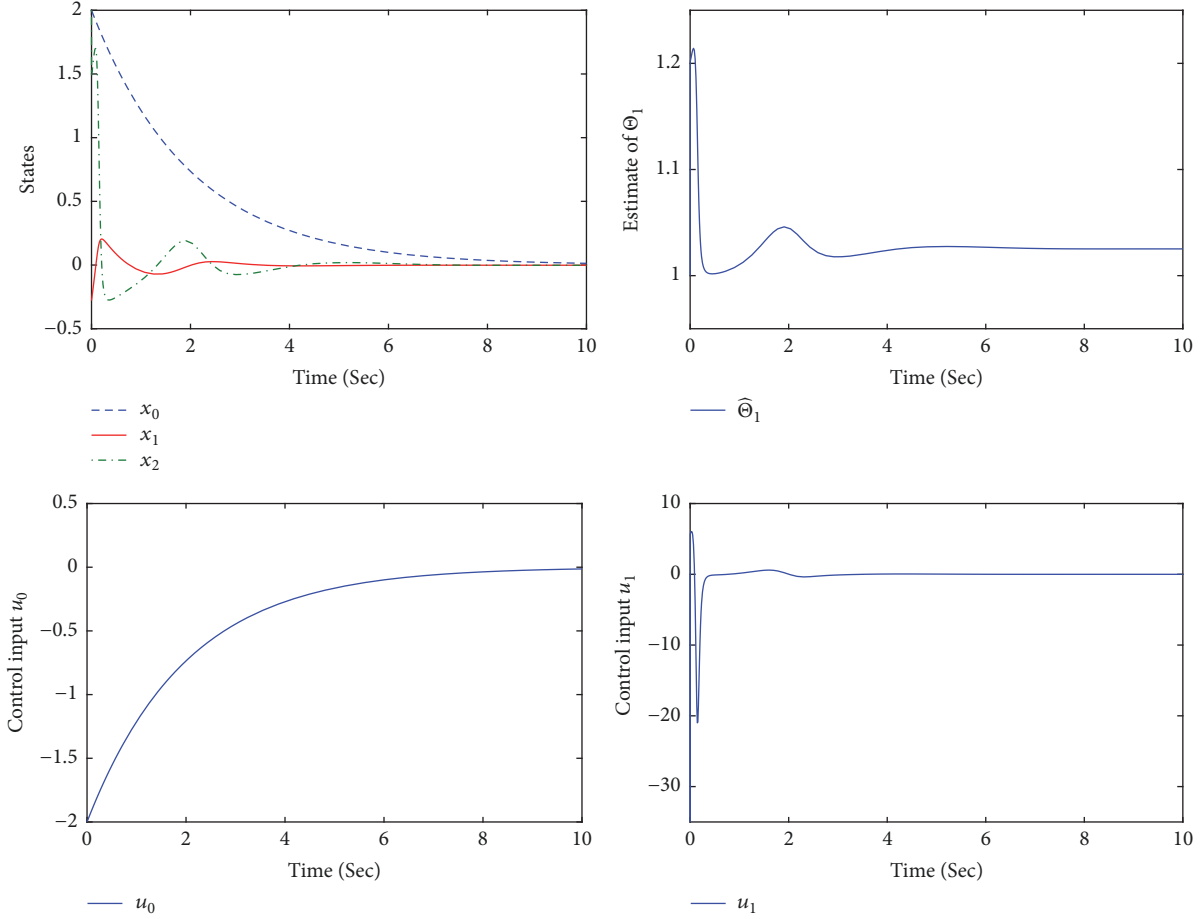
$$
\begin{aligned}
dx_0 &= \left(-d_0 h\left(x_0\right)\sqrt{\widehat{\Theta}_0^2 + 1} + \alpha_0\left(x_0, \theta\right)\right)x_0 dt \\
&\quad + \beta_0^\top\left(x_0, \theta\right)x_0 d\omega,
\end{aligned}
\tag{A.1}
$$

whose solution is

$$
\begin{aligned}
x_0(t) &= x_0(t_0)\exp\left(\int_{t_0}^t \left(-d_0(s, \theta)h(x_0(s))\sqrt{\widehat{\Theta}_0^2 + 1} + \alpha_0(x_0(s), \theta) - \frac{1}{2}\beta_0^\top(x_0(s), \theta)\beta_0(x_0(s), \theta)\right)ds \right. \\
&\quad \left. + \int_{t_0}^t \beta_0^\top(x_0(s), \theta)d\omega\right).
\end{aligned}
\tag{A.2}
$$

By using Lemma 2.3 in [2], for any $x_0(t_0) \neq 0$, there holds $x_0(t) \neq 0$ for any $t \geq t_0$, which implies that u_0 does not cross zero. The proof of Proposition 16 is completed. $\qquad\square$

FIGURE 1: Trajectories of x_0, x_1, x_2, $\widehat{\Theta}_1$, u_0 u_1.

Proof of Proposition 17 . From (14) and Lemma 5, we obtain

$$\frac{\partial W_k}{\partial x_k} = \lceil \xi_k \rceil^{(4l\sigma - \tau - r_k)/\sigma} , \tag{A.3}$$

$$\frac{\partial W_k}{\partial x_i} = -\frac{4l\sigma - \tau - r_k}{\sigma} \frac{\partial \left(\lceil x_k^* \rceil^{\sigma/r_k} \right)}{\partial x_i}$$
$$\cdot \int_{x_k^*}^{x_k} \left| \lceil s \rceil^{\sigma/r_k} - \lceil x_k^* \rceil^{\sigma/r_k} \right|^{((4l-1)\sigma - \tau - r_k)/\sigma} ds, \tag{A.4}$$

where $i = 1, \ldots, k - 1$. Exactly following the same procedure as in the proof of (A.4), we get

$$\frac{\partial W_k}{\partial \widehat{\Theta}_1} = -\frac{4l\sigma - \tau - r_k}{\sigma} \frac{\partial \left(\lceil x_k^* \rceil^{\sigma/r_k} \right)}{\partial \widehat{\Theta}_1}$$
$$\cdot \int_{x_k^*}^{x_k} \left| \lceil s \rceil^{\sigma/r_k} - \lceil x_k^* \rceil^{\sigma/r_k} \right|^{((4l-1)\sigma - \tau - r_k)/\sigma} ds, \tag{A.5}$$

$$\frac{\partial W_k}{\partial x_0} = -\frac{4l\sigma - \tau - r_k}{\sigma} \frac{\partial \left(\lceil x_k^* \rceil^{\sigma/r_k} \right)}{\partial x_0}$$
$$\cdot \int_{x_k^*}^{x_k} \left| \lceil s \rceil^{\sigma/r_k} - \lceil x_k^* \rceil^{\sigma/r_k} \right|^{((4l-1)\sigma - \tau - r_k)/\sigma} ds. \tag{A.6}$$

By using (A.3), (A.4), and Lemma 5,

$$\frac{\partial^2 W_k}{\partial x_k^2} = \frac{4l\sigma - \tau - r_k}{r_k} \left| \xi_k \right|^{((4l-1)\sigma - \tau - r_k)/\sigma} \left| x_k \right|^{(\sigma - r_k)/r_k} , \tag{A.7}$$

$$\frac{\partial^2 W_k}{\partial x_i^2} = \left(\frac{(4l-1)\sigma - \tau - r_k}{\sigma} \right.$$
$$\cdot \int_{x_k^*}^{x_k} \left| \lceil s \rceil^{\sigma/r_k} - \lceil x_k^* \rceil^{\sigma/r_k} \right|^{((4l-2)\sigma - \tau - r_k)/\sigma} ds \left(\frac{\partial \left(\lceil x_k^* \rceil^{\sigma/r_k} \right)}{\partial x_i} \right)^2$$
$$- \int_{x_k^*}^{x_k} \left| \lceil s \rceil^{\sigma/r_k} - \lceil x_k^* \rceil^{\sigma/r_k} \right|^{((4l-1)\sigma - \tau - r_k)/\sigma} ds \frac{\partial^2 \left(\lceil x_k^* \rceil^{\sigma/r_k} \right)}{\partial x_i^2} \right)$$
$$\cdot \frac{4l\sigma - \tau - r_k}{\sigma}. \tag{A.8}$$

Similar to (A.8), it follows that

$$\frac{\partial^2 W_k}{\partial x_0^2} = \left(\frac{(4l-1)\sigma - \tau - r_k}{\sigma} \right.$$

$$\cdot \int_{x_k^*}^{x_k} \left| \lceil s \rceil^{\sigma/r_k} - \lceil x_k^* \rceil^{\sigma/r_k} \right|^{((4l-2)\sigma-\tau-r_k)/\sigma} ds \left(\frac{\partial \left(\lceil x_k^* \rceil^{\sigma/r_k} \right)}{\partial x_0} \right)^2$$

$$- \int_{x_k^*}^{x_k} \left| \lceil s \rceil^{\sigma/r_k} - \lceil x_k^* \rceil^{\sigma/r_k} \right|^{((4l-1)\sigma-\tau-r_k)/\sigma} ds \frac{\partial^2 \left(\lceil x_k^* \rceil^{\sigma/r_k} \right)}{\partial x_0^2} \right) \qquad \cdot \frac{4l\sigma - \tau - r_k}{\sigma}. \tag{A.9}$$

Moreover, it is clear that

$$\frac{\partial^2 W_k}{\partial x_i \partial x_k} = \frac{\partial^2 W_k}{\partial x_k \partial x_i} = -\frac{4l\sigma - \tau - r_k}{\sigma} \left| \xi_k \right|^{\frac{(4l-1)\sigma - \tau - r_k}{\sigma}} \frac{\partial \left(\lceil x_k^* \rceil^{\sigma/r_k} \right)}{\partial x_i}, \tag{A.10}$$

$$\frac{\partial^2 W_k}{\partial x_0 \partial x_k} = \frac{\partial^2 W_k}{\partial x_k \partial x_0} = -\frac{4l\sigma - \tau - r_k}{\sigma} \left| \xi_k \right|^{((4l-1)\sigma-\tau-r_k)/\sigma} \frac{\partial \left(\lceil x_k^* \rceil^{\sigma/r_k} \right)}{\partial x_0}, \tag{A.11}$$

$$\frac{\partial^2 W_k}{\partial x_i \partial x_0} = \frac{\partial^2 W_k}{\partial x_0 \partial x_i} = \left(\frac{(4l-1)\sigma - \tau - r_k}{\sigma} \int_{x_k^*}^{x_k} \left| \lceil s \rceil^{\sigma/r_k} - \lceil x_k^* \rceil^{\sigma/r_k} \right|^{((4l-2)\sigma-\tau-r_k)/\sigma} ds \frac{\partial \left(\lceil x_k^* \rceil^{\sigma/r_k} \right)}{\partial x_i} \frac{\partial \left(\lceil x_k^* \rceil^{\sigma/r_k} \right)}{\partial x_0} \right.$$

$$\left. - \int_{x_k^*}^{x_k} \left| \lceil s \rceil^{\sigma/r_k} - \lceil x_k^* \rceil^{\sigma/r_k} \right|^{((4l-1)\sigma-\tau-r_k)/\sigma} ds \frac{\partial^2 \left(\lceil x_k^* \rceil^{(\sigma/r_k)} \right)}{\partial x_i \partial x_0} \right) \frac{4l\sigma - \tau - r_k}{\sigma}, \tag{A.12}$$

$$\frac{\partial^2 W_k}{\partial x_i \partial x_j} = \frac{\partial^2 W_k}{\partial x_j \partial x_i} = \left(\frac{(4l-1)\sigma - \tau - r_k}{\sigma} \int_{x_k^*}^{x_k} \left| \lceil s \rceil^{\sigma/r_k} - \lceil x_k^* \rceil^{\sigma/r_k} \right|^{((4l-2)\sigma-\tau-r_k)/\sigma} ds \frac{\partial \left(\lceil x_k^* \rceil^{\sigma/r_k} \right)}{\partial x_i} \frac{\partial \left(\lceil x_k^* \rceil^{\sigma/r_k} \right)}{\partial x_j} \right.$$

$$\left. - \int_{x_k^*}^{x_k} \left| \lceil s \rceil^{\sigma/r_k} - \lceil x_k^* \rceil^{\sigma/r_k} \right|^{((4l-1)\sigma-\tau-r_k)/\sigma} ds \frac{\partial^2 \left(\lceil x_k^* \rceil^{\sigma/r_k} \right)}{\partial x_i \partial x_j} \right) \frac{4l\sigma - \tau - r_k}{\sigma}, \tag{A.13}$$

where $j = 1, \ldots, k-1$ and $j \neq i$. From the expression of x_k^*, we know that $\partial(\lceil x_k^* \rceil^{\sigma/r_k})/\partial x_i$, $\partial^2(\lceil x_k^* \rceil^{\sigma/r_k})/\partial x_i^2$, $\partial^2(\lceil x_k^* \rceil^{\sigma/r_k})/\partial x_i \partial x_j$, $\partial^2(\lceil x_k^* \rceil^{\sigma/r_k})/\partial x_i \partial x_0$, $\partial(\lceil x_k^* \rceil^{\sigma/r_k})/\partial x_0$, $\partial(\lceil x_k^* \rceil^{\sigma/r_k})/\partial \widehat{\Theta}_1$ are continuous functions. It is straightforward to deduce the conclusion that $W_k(x_0, \overline{x}_k, \widehat{\Theta}_1)$ is $\mathscr{C}^2$ function.

The rest of proof is divided into two steps. Firstly, we prove the right side of inequality in Proposition 17. In view of Lemmas 6 and 8, one concludes that

$$W_k \left(x_0, \overline{x}_k, \widehat{\Theta}_1 \right)$$

$$\leq \left| \xi_k \right|^{(4l\sigma-\tau-r_k)/\sigma} 2^{1-r_k/\sigma} \left| \lceil x_k \rceil^{\sigma/r_k} - \lceil x_k^* \rceil^{\sigma/r_k} \right|^{r_k/\sigma} \tag{A.14}$$

$$= 2^{1-r_k/\sigma} \left| \xi_k \right|^{(4l\sigma-\tau)/\sigma}.$$

The next work is to prove the left side of inequality. The issue can be solved by considering two different cases. If $x_k \geq x_k^*$, then $\lceil s \rceil^{\sigma/r_k} - \lceil x_k^* \rceil^{\sigma/r_k} \geq 0$, where $x_k \geq s \geq x_k^*$. By Lemma 6, we obtain

$$W_k \left(x_0, \overline{x}_k, \widehat{\Theta}_1 \right)$$

$$\geq \int_{x_k^*}^{x_k} \left(2^{r_k/\sigma-1} (s - x_k^*) \right)^{(4l\sigma-\tau-r_k)/r_k} ds \tag{A.15}$$

$$= 2^{(r_k-\sigma)(4l\sigma-\tau-r_k)/\sigma r_k} \frac{r_k}{4l\sigma + q_k} (x_k - x_k^*)^{(4l\sigma-\tau)/r_k}.$$

If $x_k < x_k^*$, in a similar way,

$$W_k \left(x_0, \overline{x}_k, \widehat{\Theta}_1 \right)$$

$$\geq 2^{(r_k-\sigma)(4l\sigma-\tau-r_k)/\sigma r_k} \frac{r_k}{4l\sigma - \tau} (x_k^* - x_k)^{(4l\sigma-\tau)/r_k}. \tag{A.16}$$

Combining (A.15) with (A.16), the proof of Proposition 17 is completed. □

Proof of Proposition 18 . Since $p_{k-1}, k = 2, 3, \ldots, n+1$ are odd integers, by using Lemma 6,

$$\left| x_k^{p_{k-1}} - x_k^{* p_{k-1}} \right| \leq 2^{1-r_k p_{k-1}/\sigma} \left| \xi_k \right|^{r_k p_{k-1}/\sigma}. \tag{A.17}$$

According to (A.17), Remark 14, and Lemma 4, it follows that

$$\lceil \xi_{k-1} \rceil^{(4l\sigma-\tau-r_{k-1})/\sigma} d_{k-1} u_0^{q_{k-1}} \left(x_k^{p_{k-1}} - x_k^{* p_{k-1}} \right)$$

$$\leq \frac{1}{13} \xi_{k-1}^{4l} + \sigma_{k1} \Theta_1 \xi_k^{4l}, \tag{A.18}$$

where σ_{k1} is a positive constant. It is easy to deduce from (13) that

$$\left| \frac{\partial \left(\lceil x_k^* \rceil^{\sigma/r_k} \right)}{\partial x_0} \right| \leq \left| \xi_{k-1} \right| \left| \frac{\partial \phi_{k-1}^{\sigma/r_k}}{\partial x_0} \right|$$

$$+ \left|\phi_{k-1}\right|^{\sigma/r_k} \left|\frac{\partial\left(-\lceil x_{k-1}^*\rceil^{\sigma/r_{k-1}}\right)}{\partial x_0}\right|$$

$$\vdots$$

$$\leq \sum_{j=1}^{k-2}\prod_{l=j+1}^{k-1}\left|\phi_l\right|^{\sigma/r_{l+1}}\left|\frac{\partial\phi_j^{\sigma/r_{j+1}}}{\partial x_0}\right|\left|\xi_j\right| + \left|\xi_{k-1}\right|\left|\frac{\partial\phi_{k-1}^{\sigma/r_k}}{\partial x_0}\right|$$

$$\leq \sum_{j=1}^{k-1}\psi_{j,1}\left|\xi_j\right|,$$

$$(A.19)$$

where $\psi_{j,1} = \max\{(\prod_{l=j}^{k-1}|\phi_l|^{\sigma/r_{l+1}})|\partial\phi_j^{\sigma/r_{j+1}}/\partial x_0|, |\partial\phi_{k-1}^{\sigma/r_k}/\partial x_0|\}$, $j = 1,\ldots,k-1$, are continuous functions. It follows from (A.6), (A.19), Remark 14, and Lemmas 4 and 8 that

$$\frac{\partial W_k}{\partial x_0}\left(d_0 u_0^{p_0} + f_0\right) \leq \left(\left|d_0 u_0^{p_0}\right| + |f_0|\right)\left|\xi_k\right|^{-\tau/\sigma} 2^{1-r_k/\sigma}$$

$$\cdot \frac{4l\sigma - \tau - r_k}{\sigma}\left|\xi_k\right|^{(4l-1)\sigma/\sigma}\sum_{j=1}^{k-1}\psi_{j,1}\left|\xi_j\right| \leq \frac{1}{13}\xi_{k-1}^{4l} \quad (A.20)$$

$$+ \sigma_{k2}\Theta_1\xi_k^{4l},$$

where $\sigma_{k2} \geq 0$ is a smooth function. According to (13) and (A.19), we arrive at

$$\left|\frac{\partial^2\left(\lceil x_k^*\rceil^{\sigma/r_k}\right)}{\partial x_0^2}\right|$$

$$\leq \left|\frac{\partial^2\phi_{k-1}^{\sigma/r_k}}{\partial x_0^2}\right|\left|\xi_{k-1}\right| + \left|\frac{\partial\phi_{k-1}^{\sigma/r_k}}{\partial x_0}\right|\left|\frac{\partial\left(\lceil x_k^*\rceil^{\sigma/r_k}\right)}{\partial x_0}\right|$$

$$+ \sum_{j=1}^{k-2}\frac{\partial\left(\prod_{l=j+1}^{k-1}|\phi_l|^{\sigma/r_{l+1}}\left|\partial\phi_j^{\sigma/r_{j+1}}/\partial x_0\right|\right)}{\partial x_0}\left|\xi_j\right| \quad (A.21)$$

$$+ \sum_{j=1}^{k-2}\prod_{l=j+1}^{k-1}|\phi_l|^{\sigma/r_{l+1}}\left|\frac{\partial\phi_j^{\sigma/r_{j+1}}}{\partial x_0}\right|\left|\frac{\partial\left(\lceil x_j^*\rceil^{\sigma/r_j}\right)}{\partial x_0}\right|$$

$$\leq \sum_{j=1}^{k-1}\psi_{j,2}\left|\xi_j\right|,$$

where $\psi_{j,2} \geq 0$, $j = 1,\ldots,k-1$, are continuous functions. By using (A.9), (A.19), (A.21), Remark 14, and Lemmas 4 and 8, one has

$$\frac{1}{2}\left|\frac{\partial^2 W_k}{\partial x_0^2}\right|\left|g_0\right|^2 \leq 2^{-r_k/\sigma}\frac{4l\sigma - \tau - r_k}{\sigma}$$

$$\cdot \overline{\beta}_0^2\Theta_1 x_0^2\left(\left|\xi_k\right|^{-\tau/\sigma}\frac{(4l-1)\sigma - \tau - r_k}{\sigma}\left|\xi_k\right|^{(4l-2)\sigma/\sigma}\right.$$

$$\cdot \left(\sum_{j=1}^{k-1}\psi_{j,1}\left|\xi_j\right|\right)^2 + \left|\xi_k\right|^{-\tau/\sigma}\left|\xi_k\right|^{(4l-1)\sigma/\sigma}$$

$$\left.\cdot \sum_{j=1}^{k-1}\psi_{j,3}\left|\xi_j\right|\right) \leq \frac{1}{13}\xi_{k-1}^{4l} + \sigma_{k3}\Theta_1\xi_k^{4l},$$

$$(A.22)$$

where $\sigma_{k3} \geq 0$ is a smooth function. From (13) and Lemma 7, one arrives at

$$\sum_{i=1}^{k}\left|x_i\right|^{(r_k+\tau)/r_i}$$

$$\leq \sum_{i=1}^{k}\left(\left|\xi_i\right|^{(r_k+\tau)/\sigma} + \phi_{i-1}^{(r_k+\tau)/r_i}\left|\xi_{i-1}\right|^{(r_k+\tau)/\sigma}\right), \quad (A.23)$$

$$\sum_{i=1}^{k}\left|x_i\right|^{(2r_k+\tau)/2r_i}$$

$$\leq \sum_{i=1}^{k}\left(\left|\xi_i\right|^{(2r_k+\tau)/2\sigma} + \phi_{i-1}^{(2r_k+\tau)/2r_i}\left|\xi_{i-1}\right|^{(2r_k+\tau)/2\sigma}\right), \quad (A.24)$$

where $\phi_0 = 0$ and $\xi_0 = 0$. By (A.3), (A.23) and Lemma 4, it is easy to verify that

$$\frac{\partial W_k}{\partial x_k}f_k \leq \lceil\xi_k\rceil^{(4l\sigma-\tau-r_k)/\sigma}$$

$$\cdot \sum_{i=1}^{k}\left(\left|\xi_i\right|^{(r_k+\tau)/\sigma} + \phi_{i-1}^{(r_k+\tau)/r_i}\left|\xi_{i-1}\right|^{(r_k+\tau)/\sigma}\right)\Theta_1\overline{\alpha}_k$$

$$+ c_k d_k\left|u_0\right|^{q_k}\left|\xi_k\right|^{(4l\sigma-\tau-r_k)/\sigma}\left|x_{k+1}\right|^{p_k} \leq \frac{1}{13}\sum_{i=1}^{k-1}\xi_i^{4l} \quad (A.25)$$

$$+ \sigma_{k4}\Theta_1\xi_k^{4l} + \overline{c}_k d_k\left|u_0\right|^{q_k}\left|\xi_k\right|^{(4l\sigma-\tau-r_k)/\sigma}$$

$$\cdot \left|\xi_{k+1}\right|^{(r_k+\tau)/\sigma} + \overline{c}_k d_k\left|u_0\right|^{q_k}\left|\xi_k\right|^{(4l\sigma-\tau-r_k)/\sigma}$$

$$\cdot \left|x_{k+1}^*\right|^{p_k},$$

where $\sigma_{k4} \geq 0$ is a smooth function. Based on (13), one has

$$\left|\frac{\partial\left(\lceil x_k^*\rceil^{\sigma/r_k}\right)}{\partial x_i}\right| \leq \left|\frac{\partial\left(\sum_{l=1}^{k-1}\prod_{j=l}^{k-1}\phi_j^{\sigma/r_{j+1}}\lceil x_l\rceil^{\sigma/r_l}\right)}{\partial x_i}\right|$$

$$\leq \prod_{j=i}^{k-1}\phi_j^{\sigma/r_{j+1}}\frac{\sigma}{r_i}\left(\left|\xi_i\right|^{(\sigma-r_i)/\sigma} + \phi_{i-1}^{(\sigma-r_i)/r_i}\left|\xi_{i-1}\right|^{(\sigma-r_i)/\sigma}\right) \quad (A.26)$$

$$+ \sum_{l=1}^{k-1}\left|\frac{\partial\left(\prod_{j=i}^{k-1}\phi_j^{\sigma/r_{j+1}}\right)}{\partial x_i}\right|\left(\left|\xi_l\right| + \phi_{l-1}^{\sigma/r_l}\left|\xi_{l-1}\right|\right).$$

Similarly, by using (13), (A.4), (A.23), (A.26), Remark 14, and Lemma 4,

$$\sum_{i=1}^{k-1} \frac{\partial W_k}{\partial x_i} \left(d_i u_0^{q_i} x_{i+1}^{p_i} + f_i \right)$$

$$\leq \sum_{i=1}^{k-1} \sum_{j=1}^{i+1} \Theta_1 \left(|\xi_j|^{(r_i+\tau)/\sigma} + \phi_{j-1}^{(r_i+\tau)/r_i} |\xi_{j-1}|^{(r_i+\tau)/\sigma} \right)$$

$$\cdot F \times \frac{4l\sigma - \tau - r_k}{\sigma} 2^{1-r_k/\sigma} \left| \frac{\partial \left(\lceil x_k^* \rceil^{\sigma/r_k} \right)}{\partial x_i} \right|$$

$$\cdot |\xi_k|^{((4l-1)\sigma-\tau)/\sigma} \leq \frac{1}{13} \sum_{i=1}^{k-1} \xi_i^{4l} + \sigma_{k5} \Theta_1 \xi_k^{4l}, \tag{A.27}$$

where $\sigma_{k5} \geq 0$ is a smooth function and $F \triangleq \max\{u_0^{q_i}(1 - \bar{c}_i)\bar{\lambda}_{i,2}, \bar{\alpha}_i\}$. Because of (A.11), (A.19), (A.24), Remark 14, and Lemma 4, the following always holds:

$$\frac{1}{2} \left| \frac{\partial^2 W_k}{\partial x_k \partial x_0} \right| |g_k| |g_0^\top| \leq \sum_{i=1}^{k} \bar{\beta}_0 \bar{\beta}_k |x_0| \Theta_1$$

$$\cdot \frac{4l\sigma - \tau - r_k}{2\sigma} |\xi_k|^{((4l-1)\sigma-\tau-r_k)/\sigma} \left| \frac{\partial \left(\lceil x_k^* \rceil^{\sigma/r_k} \right)}{\partial x_0} \right|$$

$$\cdot \left(|\xi_i|^{(2r_k+\tau)/2\sigma} + \phi_{i-1}^{(2r_k+\tau)/2r_i} |\xi_{i-1}|^{(2r_k+\tau)/2\sigma} \right) \leq \frac{1}{13} \tag{A.28}$$

$$\cdot \sum_{i=1}^{k-1} \xi_i^{4l} + \sigma_{k6} \Theta_1 \xi_k^{4l},$$

where $\sigma_{k6} \geq 0$ is a smooth function. Using (A.7), (A.10), (A.24), (A.26), and Lemma 4, one has

$$\frac{1}{2} \sum_{i=1}^{k-1} \left| \frac{\partial^2 W_k}{\partial x_k \partial x_i} \right| |g_k| |g_i^\top| \leq \frac{1}{2}$$

$$\cdot \sum_{i=1}^{k-1} \sum_{l=1}^{k} \sum_{j=1}^{i} \bar{\beta}_i \bar{\beta}_k \Theta_1 \frac{4l\sigma - \tau - r_k}{2\sigma} |\xi_k|^{((4l-1)\sigma-\tau-r_k)/\sigma}$$

$$\times \left| \frac{\partial \left(\lceil x_k^* \rceil^{\sigma/r_k} \right)}{\partial x_i} \right| \left(|\xi_l|^{(2r_k+\tau)/2\sigma} \right. \tag{A.29}$$

$$+ \phi_{l-1}^{(2r_k+\tau)/2r_k} |\xi_{l-1}|^{(2r_k+\tau)/2\sigma} \right) \times \left(|\xi_j|^{(2r_i+\tau)/2\sigma} \right.$$

$$+ \phi_{j-1}^{(2r_i+\tau)/2r_i} |\xi_{j-1}|^{(2r_i+\tau)/2\sigma} \right) \leq \frac{1}{13} \sum_{i=1}^{k-1} \xi_i^{4l}$$

$$+ \sigma_{k7} \Theta_1 \xi_k^{4l},$$

$$\frac{1}{2} \left| \frac{\partial^2 W_k}{\partial x_k^2} \right| |g_k|^2 \leq \sum_{i=1}^{k} \bar{\beta}_k^2 \Theta_1$$

$$\cdot \frac{4l\sigma + q_k - r_k}{2\sigma} |\xi_k|^{((4l-1)\sigma-\tau-r_k)/\sigma} |x_k|^{(\sigma+\tau)/r_k}$$

$$\times \left(|\xi_i|^{(2r_k+\tau)/2\sigma} + \phi_{i-1}^{(2r_k+\tau)/2r_k} |\xi_{i-1}|^{(2r_k+\tau)/2\sigma} \right)^2$$

$$\leq \frac{1}{13} \sum_{i=1}^{k-1} \xi_i^{4l} + \sigma_{k8} \Theta_1 \xi_k^{4l}, \tag{A.30}$$

where $\sigma_{k7} \geq 0$ and $\sigma_{k8} \geq 0$ are smooth functions. In terms of (13) and (A.19), it follows that

$$\left| \frac{\partial^2 \left(\lceil x_k^* \rceil^{\sigma/r_k} \right)}{\partial x_i \partial x_0} \right|$$

$$\leq \left| \frac{\partial^2 \phi_{k-1}^{\sigma/r_k}}{\partial x_i \partial x_0} \right| |\xi_{k-1}|$$

$$+ \sum_{j=1}^{k-2} \frac{\partial \left(\prod_{l=j+1}^{k-1} |\phi_l|^{\sigma/r_{l+1}} \left| \partial \phi_j^{\sigma/r_{j+1}} / \partial x_0 \right| \right)}{\partial x_i} |\xi_j|$$

$$+ \sum_{j=1}^{k-2} \prod_{l=j+1}^{k-1} |\phi_l|^{\sigma/r_{l+1}} \left| \frac{\partial \phi_j^{\sigma/r_{j+1}}}{\partial x_0} \right| \left| \frac{\partial \xi_j}{\partial x_i} \right|$$

$$+ \left| \frac{\partial \xi_{k-1}}{\partial x_i} \right| \left| \frac{\partial \phi_{k-1}^{\sigma/r_k}}{\partial x_0} \right| \tag{A.31}$$

$$\leq \left| \frac{\partial^2 \phi_{k-1}^{\sigma/r_k}}{\partial x_i \partial x_0} \right| |\xi_{k-1}|$$

$$+ \sum_{j=1}^{k-2} \frac{\partial \left(\prod_{l=j+1}^{k-1} |\phi_l|^{\sigma/r_{l+1}} \left| \partial \phi_j^{\sigma/r_{j+1}} / \partial x_0 \right| \right)}{\partial x_i} |\xi_j|$$

$$+ \sum_{j=1}^{k-2} \prod_{l=j+1}^{k-1} |\phi_l|^{\sigma/r_{l+1}} \left| \frac{\partial \phi_j^{\sigma/r_{j+1}}}{\partial x_0} \right| \left| \frac{\partial \left(\lceil x_j^* \rceil^{\sigma/r_j} \right)}{\partial x_i} \right|$$

$$+ \left| \frac{\partial \left(\lceil x_{k-1}^* \rceil^{\sigma/r_{k-1}} \right)}{\partial x_i} \right| \left| \frac{\partial \phi_{k-1}^{\sigma/r_k}}{\partial x_0} \right|.$$

Combining (A.12), (A.24), (A.26), (A.31), Remark 14, and Lemma 4 yields

$$\frac{1}{2} \sum_{i=1}^{k-1} \left| \frac{\partial^2 W_k}{\partial x_i \partial x_0} \right| |g_i| |g_0^\top| \leq \sum_{i=1}^{k-1} \sum_{j=1}^{i} 2^{-r_k/\sigma} \bar{\beta}_0 \bar{\beta}_i |x_0| \Theta_1$$

$$\cdot \frac{4l\sigma - \tau - r_k}{\sigma} |\xi_k|^{((4l-2)\sigma-\tau)/\sigma}$$

$$\cdot \left(|\xi_j|^{(2r_i+\tau)/2\sigma} + \phi_{j-1}^{(2r_i+\tau)/2\sigma} |\xi_{j-1}|^{(2r_i+\tau)/2r_i} \right)$$

$$\times \left(\frac{(4l-1)\sigma - \tau - r_k}{\sigma} \left| \frac{\partial \left(\lceil x_k^* \rceil^{\sigma/r_k} \right)}{\partial x_i} \right| \sum_{l=1}^{k-1} \psi_{l,1} |\xi_j| \right.$$

$$+ \left| \frac{\partial^2 \left(\lceil x_k^* \rceil^{\sigma/r_k} \right)}{\partial x_i \partial x_0} \right| |\xi_k| \right) \le \frac{1}{13} \sum_{i=1}^{k-1} \xi_i^{4l} + \sigma_{k9} \Theta_1 \xi_k^{4l},$$

$$(A.32)$$

where $\sigma_{k9} \ge 0$ is a smooth function. Exactly following the same procedure, tedious calculations conclude that

$$\frac{1}{2} \sum_{i=1}^{k-1} \sum_{j=1}^{i} \left| \frac{\partial^2 W_k}{\partial x_i^2} \right| |g_i|^2$$

$$\le \sum_{i=1}^{k-1} \sum_{j=1}^{i} \overline{\beta}_i^2 \Theta_1 2^{-r_k/\sigma} |\xi_k|^{((4l-2)\sigma - \tau)/\sigma}$$

$$\cdot \frac{4l\sigma - \tau - r_k}{\sigma} \left(\left(\frac{\partial \left(\lceil x_k^* \rceil^{\sigma/r_k} \right)}{\partial x_i} \right)^2 \right.$$

$$(A.33)$$

$$\times \frac{(4l-1)\sigma - \tau - r_k}{\sigma} + \left| \frac{\partial^2 \left(\lceil x_k^* \rceil^{\sigma/r_k} \right)}{\partial x_i^2} \right| |\xi_k| \right)$$

$$\times \left(|\xi_j|^{(2r_i + \tau)/2\sigma} + \phi_{j-1}^{(2r_i + \tau)/2r_i} |\xi_{j-1}|^{(2r_i + \tau)/2\sigma} \right)^2$$

$$\le \frac{1}{13} \sum_{i=1}^{k-1} \xi_i^{4l} + \sigma_{k10} \Theta_1 \xi_k^{4l},$$

$$\sum_{i,j=1, i\ne j}^{k-1} \frac{1}{2} \left| \frac{\partial^2 W_k}{\partial x_i \partial x_j} \right| |g_i| |g_j^\top| \le \sum_{i,j=1, i\ne j}^{k-1} \sum_{l=1}^{i} \sum_{h=1}^{j} 2^{-r_k/\sigma}$$

$$\cdot \frac{4l\sigma - \tau - r_k}{\sigma} \overline{\beta}_i \overline{\beta}_j \Theta_1 |\xi_k|^{((4l-2)\sigma - \tau)/\sigma}$$

$$\times \left(\frac{(4l-1)\sigma - \tau - r_k}{\sigma} \left| \frac{\partial \left(\lceil x_k^* \rceil^{\sigma/r_k} \right)}{\partial x_i} \right| \right.$$

$$\cdot \left| \frac{\partial \left(\lceil x_k^* \rceil^{\sigma/r_k} \right)}{\partial x_j} \right| + \left| \frac{\partial^2 \left(\lceil x_k^* \rceil^{\sigma/r_k} \right)}{\partial x_i \partial x_j} \right| |\xi_k| \right)$$

$$(A.34)$$

$$\cdot \left(|\xi_l|^{(2r_i + \tau)/2\sigma} + \phi_{l-1}^{(2r_i + \tau)/2r_i} |\xi_{l-1}|^{(2r_i + \tau)/2\sigma} \right)$$

$$\times \left(|\xi_h|^{(2r_j + \tau)/2\sigma} + \phi_{h-1}^{(2r_j + \tau)/2r_j} |\xi_{h-1}|^{(2r_j + \tau)/2\sigma} \right)$$

$$\le \frac{1}{13} \sum_{i=1}^{k-1} \xi_i^{4l} + \sigma_{k11} \Theta_1 \xi_k^{4l},$$

where $\sigma_{k10} \ge 0$ and $\sigma_{k11} \ge 0$ are smooth functions. At last, by using Lemma 4,

$$\overline{c}_{k-1} d_{k-1} |u_0|^{q_{k-1}} |\xi_{k-1}|^{(4l\sigma - \tau - r_{k-1})/\sigma} |\xi_k|^{(r_{k-1} + \tau)/\sigma}$$

$$\le \frac{1}{13} \xi_{k-1}^{4l} + \sigma_{k12} \Theta_1 \xi_k^{4l},$$

$$(A.35)$$

where σ_{k12} is a position constant. Define the nonnegative smooth function $\sigma_k = \sum_{i=1}^{12} \sigma_{ki}$, and put (A.18), (A.20), (A.22), (A.25), (A.27)-(A.30), and (A.32)-(A.35) together, the proof of Proposition 18 is completed. □

Proof of Proposition 19. By (31), Proposition 17, and Lemma 2, Proposition 19 is proved. □

Data Availability

The data supporting the conclusions of our article are some open access articles that have been properly cited, and the readers can easily obtain these articles to verify the conclusions, replicate the analysis, and conduct secondary analyses. Therefore, we do not create a publicly available data repository.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

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