Research Article

Three-Dimensional Free Vibration Analysis of Gears with Variable Thickness Using the Chebyshev–Ritz Method

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Received 11 September 2018; Revised 18 October 2018; Accepted 22 October 2018; Published 5 November 2018

Academic Editor: Gaetano Giunta

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To reflect vibration more comprehensively and to satisfy the machining demand for high-order frequencies, we presented a three-dimensional free vibration analysis of gears with variable thickness using the Chebyshev–Ritz method based on three-dimensional elasticity theory. We derived the eigenvalue equations. We divided the gear model into three annular parts along the locations of the step variations, and the admissible function was a Ritz series that consisted of a Chebyshev polynomial multiplying boundary function. The convergence study demonstrated the high accuracy of the present method. We used a hammering method for a modal experiment to test two annular plates and one gear's eigenfrequencies in a completely free condition. We also applied the finite element method to solve the eigenfrequencies. Through a comparative analysis of the frequencies obtained by these three methods, we found that the results achieved by the Chebyshev–Ritz method were close to those obtained from the experiment and finite element method. The relative errors of four sets of data were greater than 4%, and the errors of the other 48 sets were less than 4%. Thus, it was feasible to use the Chebyshev–Ritz method to solve the eigenfrequencies of gears with variable thickness.

1. Introduction

Gears have a wide range of applications in industrial production and are essential for transmitting power [1, 2]. Ultrasonic machining has many advantages compared with traditional machining, such as reducing cutting force and surface roughness and improving machining accuracy [3–5]. Therefore, it is essential to introduce ultrasonic machining into gear machining to improve the gear's machining quality. The quality of ultrasonic gear machining mainly depends on the vibration characteristic of the ultrasonic machining system. During ultrasonic gear machining, the gear is not only a workpiece but also a part of the ultrasonic machining system [6, 7]. In past research, the gear was simplified as a uniform plate or an annular stepped plate, while regarding the gear's pitch diameter as the plate's diameter, to solve the gear's natural frequency [8, 9].

There are several plate theories according to different thickness-diameter ratios [10, 11]. Kirchhoff established the

foundation of a thin-plate theory with research on the vibration of a thin plate [12]. The theoretical results of thin-plate theory are close to accurate natural frequencies when the thickness-diameter ratio is less than 0.2, but the relative error increases rapidly when the thickness-diameter ratio is greater than 0.2 [13, 14]. Mindlin considered shear deformation to lay the foundation for a moderately thick-plate theory [15]. Some investigations of the vibration characteristics of a stepped plate based on Mindlin theory have been presented [16, 17]. Only the frequencies of transverse bending vibration mode have been given, however, as Mindlin theory assumes that the plate displacement varies linearly in thickness direction of plate with the plate thickness remaining unchanged and this theory also ignores the stress in the thickness direction of plate. Leissa summarized previous work to form an analytical system of different kinds of plates [18]. Thereafter, many researchers had deeper study of plate theory and established a vibration model, that is, the derivation of an analytical formula and the solving of a numerical solution under

FIGURE 1: Simplified model of gear Ω and the cylindrical coordinate system (r, θ, z) : (a) stereogram and (b) cross-section.

different conditions for engineering applications [19–21]. The finite element method and calculus method also have been used to analyze the plate's vibration characteristic [22–25]. These theories and methods have restrictions, however; for example, different kinds of plates require different theories for analysis, high-order frequencies cannot be obtained, and the finite element method cannot obtain a solution in theory.

Thus, some research has been conducted based on three-dimensional elasticity theory using the Ritz method for more accurate solutions. The Ritz method was established by Ritz in 1909 [26] to efficiently obtain accurate frequencies. Since then, many scholars have used this method to analyze different objects' vibration characteristics, including uniform and nonuniform beams, rectangular plates, and thin plates, according to one-dimensional or two-dimensional theory [27–30]. Then, Leissa and Kang used the Ritz method with an admissible function composed of algebraic polynomial multiplying boundary functions based on three-dimensional elasticity theory to analyze the vibration characteristic of thick, linearly tapered, annular plates [31, 32]. This method can be used to analyze the vibration characteristics of plates with arbitrary thickness. The numerical stability of an algebraic polynomial is bad, however, and high-order frequencies cannot be obtained. Because the Chebyshev polynomial possesses the simplicity of algebraic polynomial and avoids the numerical instability of high-order algebraic polynomial, Zhou applied the Chebyshev–Ritz method, which uses the Chebyshev polynomial to replace the algebraic polynomial in admissible functions to obtain more accurate eigenfrequencies of plates with constant thickness [33–35]. Although Zhou obtained more accurate eigenfrequencies, he did not apply this method to plates with variable thickness. In engineering, gears usually have variable thickness, so further research by the Chebyshev–Ritz method is needed.

To reflect the vibration more comprehensively and satisfy the machining demand for high-order frequencies, we presented a three-dimensional free vibration analysis of gears with variable thickness using the Chebyshev–Ritz method based on three-dimensional elasticity theory. We derived the eigenvalue equations. To achieve the eigenfrequencies, we

divided the gear model into three parts along the locations of the step variations and verified the convergence of this method. We used a hammering method for the modal experiment to test two annular plates and a gear's frequencies in a completely free condition. We also applied the finite element method to solve the eigenfrequencies. Through a comparative analysis of the frequencies obtained by the three methods discussed earlier, the results achieved according to the Chebyshev–Ritz method were close to those achieved by the experiment and finite element method. This finding indicated that it is feasible to use the Chebyshev–Ritz method to solve the eigenfrequencies of gears with variable thickness. Because three-dimensional elasticity theory can be used for any kinds of plates, the analysis method has a broad application in engineering.

2. Method of Analysis

To omit keyway, cast fillet, machining chamfer, and hole to reduce the weight of a straight-toothed spur gear and helical-spur gear, a gear with variable thickness can be simplified as an annular plate, as shown in Figure 1. A cylindrical coordinate (r, θ, z) is defined. The coordinate origin is coincident with the center of the plate's neutral surface. The radial (r) and axial (z) coordinates are measured from the central axis, and θ is the circumferential angle. The corresponding displacement components at a generic point are u , v , and w in the r , θ , and z direction, respectively. Thus, the ranges of the coordinate are given for the annular plate by the following: $0 \leq r \leq R_4$, $0 \leq \theta \leq 2\pi$, and $0 \leq z \leq t_1/2$.

To analyze vibration characteristics, the annular plate shown in Figure 1 can be regarded as a combination of three parts that are the annular plates shown in Figure 2. Therefore, the annular plate $\Omega = \Omega_1 - \Omega_2 - \Omega_3$.

According to the three-dimensional linear elasticity theory, the strain energy V of the annular plate Ω in Figure 1 is given in integral form as follows:

$$V = \frac{G}{2} \int_{\Omega} \left(\frac{2\nu}{1-2\nu} (\epsilon_{rr} + \epsilon_{\theta\theta} + \epsilon_{zz})^2 + 2 (\epsilon_{rr}^2 + \epsilon_{\theta\theta}^2 + \epsilon_{zz}^2) + \epsilon_{r\theta}^2 + \epsilon_{rz}^2 + \epsilon_{\theta z}^2 \right) r dr d\theta dz, \quad (1)$$

FIGURE 2: (a) Annular plate without web Ω_1 ; (b) annular plate of web Ω_2 and web Ω_3 .

where G is the shear modulus and ν is Poisson's ratio. The strain components ε_{ij} ($i, j = r, \theta, z$) are defined as follows:

$$\begin{aligned}\varepsilon_{rr} &= \frac{\partial u}{\partial r}; \\ \varepsilon_{\theta\theta} &= \frac{u}{r} + \frac{\partial v}{r \partial \theta}; \\ \varepsilon_{zz} &= \frac{\partial w}{\partial z}, \\ \varepsilon_{r\theta} &= \frac{\partial u}{r \partial \theta} + \frac{\partial v}{\partial r} - \frac{\nu}{r}; \\ \varepsilon_{rz} &= \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}; \\ \varepsilon_{\theta z} &= \frac{\partial v}{\partial z} + \frac{\partial w}{r \partial \theta}.\end{aligned}\quad (2)$$

The kinetic energy T of the plate is given as

$$T = \frac{\rho}{2} \int_{\Omega} \left(\left(\frac{\partial u}{\partial t} \right)^2 + \left(\frac{\partial v}{\partial t} \right)^2 + \left(\frac{\partial w}{\partial t} \right)^2 \right) r dr d\theta dz, \quad (3)$$

where ρ is the mass density per unit volume and t denotes time.

For simplicity and convenience in the mathematical formation, three dimensionless parameters are given as follows:

$$\begin{aligned}\bar{r} &= \frac{2r}{R_4 - R_1} - \frac{R_4 + R_1}{R_4 - R_1}; \\ \bar{\theta} &= \theta; \\ \bar{z} &= \frac{2z}{t_1}.\end{aligned}\quad (4)$$

For free vibration, the displacement function can be expressed as

$$\begin{aligned}u(r, \theta, z, t) &= U(\bar{r}, \bar{\theta}, \bar{z}) e^{i\omega t}; \\ v(r, \theta, z, t) &= V(\bar{r}, \bar{\theta}, \bar{z}) e^{i\omega t}; \\ w(r, \theta, z, t) &= W(\bar{r}, \bar{\theta}, \bar{z}) e^{i\omega t},\end{aligned}\quad (5)$$

where ω is the eigenfrequency of the annular plate in Figure 1 and $i = \sqrt{-1}$.

Considering the circumferential symmetry of the annular plate about coordinate θ , the displacement amplitude functions can be expressed by

$$\begin{aligned}U(\bar{r}, \bar{\theta}, \bar{z}) &= \bar{U}(\bar{r}, \bar{z}) \cos(s\bar{\theta}); \\ V(\bar{r}, \bar{\theta}, \bar{z}) &= \bar{V}(\bar{r}, \bar{z}) \sin(s\bar{\theta}); \\ W(\bar{r}, \bar{\theta}, \bar{z}) &= \bar{W}(\bar{r}, \bar{z}) \cos(s\bar{\theta}),\end{aligned}\quad (6)$$

where s is the circumferential wave number, and it is taken to an integer ($s = 0, 1, 2, 3, \dots, \infty$) to ensure periodicity in $\bar{\theta}$. It is obvious that $s = 0$ means the axisymmetric vibration. In this situation, $U(\bar{r}, \bar{\theta}, \bar{z}) = \bar{U}(\bar{r}, \bar{z})$, $V(\bar{r}, \bar{\theta}, \bar{z}) = 0$, and $W(\bar{r}, \bar{\theta}, \bar{z}) = \bar{W}(\bar{r}, \bar{z})$. Axisymmetric vibration only involves u and w . Rotating the symmetric axes by $\pi/2$, another set of free vibration modes can be obtained, corresponding to an interchange of $\cos(s\bar{\theta})$ and $\sin(s\bar{\theta})$ in (6), and $s = 0$ means $U(\bar{r}, \bar{\theta}, \bar{z}) = 0$; $V(\bar{r}, \bar{\theta}, \bar{z}) = \bar{V}(\bar{r}, \bar{z})$; and $W(\bar{r}, \bar{\theta}, \bar{z}) = 0$, representing torsional vibration. Torsional vibration involves only v , uncoupled from u and w . For $s > 0$ named bending vibration, the displacement functions are the same as (6), with the symmetric axes of the vibration mode shape being rotated. When $s = 1$, $V(\bar{r}, \bar{\theta}, \bar{z}) = V(\bar{r}, \bar{\theta} + 2\pi, \bar{z})$, every vibration mode shape occurs once in 2π in the $\bar{\theta}$ direction. When $s = 2$ and $s = 3$, $V(\bar{r}, \bar{\theta}, \bar{z}) = V(\bar{r}, \bar{\theta} + \pi, \bar{z})$ and $V(\bar{r}, \bar{\theta}, \bar{z}) = V(\bar{r}, \bar{\theta} + (2/3)\pi, \bar{z})$, every vibration mode shape occurs two or three times in 2π .

In consideration of the annular plate in Figure 1, which can be regarded as three parts, and substituting (4) and (5) into (1) and (3), the maximum strain energy $V_{\max}$ and kinetic energy $T_{\max}$ can be given as follows:

$$V_{\max} = \frac{Gt_1}{4} (\Delta_{V1} - \Delta_{V2} - \Delta_{V3}), \quad (7)$$

where

$$\begin{aligned}\Delta_{V1} &= \int_{-1}^1 \int_{-1}^1 \left(\frac{2\nu}{1-2\nu} \Gamma_1 (\bar{\varepsilon}_{rr} + \bar{\varepsilon}_{\theta\theta} + \bar{\varepsilon}_{zz})^2 \right. \\ &\quad + 2\Gamma_1 ((\bar{\varepsilon}_{r\theta})^2 + (\bar{\varepsilon}_{\theta z})^2 + (\bar{\varepsilon}_{rz})^2) + \Gamma_1 (\bar{\varepsilon}_{rz})^2 \\ &\quad \left. + \Gamma_2 ((\bar{\varepsilon}_{r\theta})^2 + (\bar{\varepsilon}_{\theta z})^2) \right) \left(\bar{r} + \frac{R_4 + R_1}{R_4 - R_1} \right) d\bar{z} d\bar{r},\end{aligned}$$

$$\Delta_{V2} = \int_{(2R_2-R_4-R_1)/(R_4-R_1)}^{(2R_3-R_4-R_1)/(R_4-R_1)} \int_{t_2/t_1}^1 \left(\frac{2\nu}{1-2\nu} \Gamma_1 (\bar{\varepsilon}_{rr} + \bar{\varepsilon}_{\theta\theta} + \bar{\varepsilon}_{zz})^2 \right. \\ \left. + 2\Gamma_1 ((\bar{\varepsilon}_{rr})^2 + (\bar{\varepsilon}_{\theta\theta})^2 + (\bar{\varepsilon}_{zz})^2) + \Gamma_1 (\bar{\varepsilon}_{rz})^2 \right. \\ \left. + \Gamma_2 ((\bar{\varepsilon}_{r\theta})^2 + (\bar{\varepsilon}_{\theta z})^2) \right) \left(\bar{r} + \frac{R_4 + R_1}{R_4 - R_1} \right) d\bar{z} d\bar{r},$$

$$\Delta_{V3} = \int_{(2R_2-R_4-R_1)/(R_4-R_1)}^{(2R_3-R_4-R_1)/(R_4-R_1)} \int_{-1}^{-t_2/t_1} \left(\frac{2\nu}{1-2\nu} \Gamma_1 (\bar{\varepsilon}_{rr} + \bar{\varepsilon}_{\theta\theta} + \bar{\varepsilon}_{zz})^2 \right. \\ \left. + 2\Gamma_1 ((\bar{\varepsilon}_{rr})^2 + (\bar{\varepsilon}_{\theta\theta})^2 + (\bar{\varepsilon}_{zz})^2) + \Gamma_1 (\bar{\varepsilon}_{rz})^2 \right. \\ \left. + \Gamma_2 ((\bar{\varepsilon}_{r\theta})^2 + (\bar{\varepsilon}_{\theta z})^2) \right) \left(\bar{r} + \frac{R_4 + R_1}{R_4 - R_1} \right) d\bar{z} d\bar{r},$$

$$T_{\max} = \frac{\rho(R_4 - R_1)^2 t_1}{16} \omega^2 (\Delta_{T1} - \Delta_{T2} - \Delta_{T3}),$$

where

$$\Delta_{T1} = \int_{-1}^1 \int_{-1}^1 \left(\Gamma_1 \bar{U}^2 + \Gamma_2 \bar{V}^2 + \Gamma_1 \bar{W}^2 \right) \left(\bar{r} + \frac{R_4 + R_1}{R_4 - R_1} \right) d\bar{z} d\bar{r},$$

$$\Delta_{T2} = \int_{(2R_2-R_4-R_1)/(R_4-R_1)}^{(2R_3-R_4-R_1)/(R_4-R_1)} \int_{t_2/t_1}^1 \left(\Gamma_1 \bar{U}^2 + \Gamma_2 \bar{V}^2 + \Gamma_1 \bar{W}^2 \right) \left(\bar{r} + \frac{R_4 + R_1}{R_4 - R_1} \right) d\bar{z} d\bar{r},$$

$$\Delta_{T3} = \int_{(2R_2-R_4-R_1)/(R_4-R_1)}^{(2R_3-R_4-R_1)/(R_4-R_1)} \int_{-1}^{-t_2/t_1} \left(\Gamma_1 \bar{U}^2 + \Gamma_2 \bar{V}^2 + \Gamma_1 \bar{W}^2 \right) \left(\bar{r} + \frac{R_4 + R_1}{R_4 - R_1} \right) d\bar{z} d\bar{r},$$

where

$$\bar{\varepsilon}_{rr} = \frac{\partial \bar{U}}{\partial \bar{r}};$$

$$\bar{\varepsilon}_{\theta\theta} = \frac{\bar{U}}{\bar{r} + (R_4 + R_1)/(R_4 - R_1)} + \frac{s\bar{V}}{\bar{r} + (R_4 + R_1)/(R_4 - R_1)};$$

$$\bar{\varepsilon}_{zz} = \frac{\partial \bar{W}}{(t_1/(R_4 - R_1)) \partial \bar{z}};$$

$$\bar{\varepsilon}_{r\theta} = -\frac{s\bar{U}}{\bar{r} + (R_4 + R_1)/(R_4 - R_1)} + \frac{\partial \bar{V}}{\partial \bar{r}} - \frac{\bar{V}}{\bar{r} + (R_4 + R_1)/(R_4 - R_1)};$$

$$\bar{\varepsilon}_{rz} = \frac{\partial \bar{U}}{(t_1/(R_4 - R_1)) \partial \bar{z}} + \frac{\partial \bar{W}}{\partial \bar{r}},$$

$$\bar{\varepsilon}_{\theta z} = \frac{\partial \bar{V}}{(t_1/(R_4 - R_1)) \partial \bar{z}} - \frac{s\bar{W}}{\bar{r} + (R_4 + R_1)/(R_4 - R_1)},$$

$$\Gamma_1 = \int_0^{2\pi} \cos^2(s\theta) d\theta = \begin{cases} 2\pi & s = 0, \\ \pi & s > 0, \end{cases}$$

$$\Gamma_2 = \int_0^{2\pi} \sin^2(s\theta) d\theta = \begin{cases} 0 & s = 0, \\ \pi & s > 0. \end{cases}$$

Each of the displacement amplitude functions is composed of a double series of Chebyshev polynomials multiplied by boundary functions as follows:

$$\bar{U}(\bar{r}, \bar{z}) = \eta_u \sum_{i=1}^I \sum_{j=1}^J A_{ij} P_i(\bar{r}) P_j(\bar{z}),$$

$$\bar{V}(\bar{r}, \bar{z}) = \eta_v \sum_{k=1}^K \sum_{l=1}^L B_{kl} P_k(\bar{r}) P_l(\bar{z}),$$

$$\bar{W}(\bar{r}, \bar{z}) = \eta_w \sum_{m=1}^M \sum_{n=1}^N C_{mn} P_m(\bar{r}) P_n(\bar{z}),$$

where I, J, K, L, M , and N are the truncation orders of the Chebyshev polynomial series and A_{ij}, B_{kl} , and C_{mn} are coefficients yet to be determined. $P_i(\chi)$ ($i = 1, 2, 3, \dots$; $\chi = \bar{r}, \bar{z}$) is the one-dimensional i th Chebyshev polynomial, which can be written as follows:

$$P_i(\chi) = \cos((i-1) \arccos(\chi)).$$

The boundary functions η_u, η_v , and η_w should enable the displacement components u, v , and w to satisfy the geometric boundary conditions of the rod. For example,

- (1) completely free: $\eta_u = \eta_v = \eta_w = 1$,
- (2) inner edge fixed, remaining boundaries free: $\eta_u = \eta_v = \eta_w = 1 + \bar{r}$,
- (3) outer edge fixed, remaining boundaries free: $\eta_u = \eta_v = \eta_w = 1 - \bar{r}$, and
- (4) both ends fixed: $\eta_u = \eta_v = \eta_w = (1 + \bar{r})(1 - \bar{r})$.

The Chebyshev polynomial series $P_i(\chi)$ ($i = 1, 2, 3, \dots$) is a set of complete and orthogonal series in the interval $[-1, 1]$. Thus, the double series $P_i(\chi)P_j(\chi)$ ($i, j = 1, 2, 3, \dots$) is also a complete and orthogonal set. The Chebyshev polynomial has excellent properties in the approximation of functions. Therefore, its numerical stability is better than the algebraic polynomial series.

The energy functional of the plate is defined as follows:

$$\Pi = V_{\max} - T_{\max}.$$

Minimizing the previous function with respect to the coefficient,

$$\begin{aligned}\frac{\partial \Pi}{\partial A_{ij}} &= 0; \\ \frac{\partial \Pi}{\partial B_{kl}} &= 0; \\ \frac{\partial \Pi}{\partial C_{mn}} &= 0,\end{aligned}\quad (16)$$

leads to the following eigenvalue equation:

$$(K - \Lambda^2 M) \mathbf{x} = 0, \quad (17)$$

in which $\Lambda = \omega(R_4 - R_1) \sqrt{\rho/G}$ and K and M are the stiffness and mass matrices resulting from the maximum strain energy ($V_{\max}$) and the maximum kinetic energy ($T_{\max}$). The vector $\mathbf{x}$ is given as

$$\mathbf{x} = (A_{11}, A_{12}, \dots, A_{IJ}; B_{11}, B_{12}, \dots, B_{KL}; C_{11}, C_{12}, \dots, C_{MN}). \quad (18)$$

3. Convergence Study

To verify the accuracy of frequencies obtained by the previous method, we conducted a convergence study. A convergence study is based on the fact that all of the frequencies obtained by the Ritz method should converge to their exact value in an upper bound manner when (13) and (14) are used. If the results did not converge properly, it would be likely that the assumed admissible function might not be the right choice.

Table 1 gives the results of the convergence study of the completely free annular plate in Figure 1, with $R_1 = 15$, $R_2 = 25$, $R_3 = 70$, $R_4 = 100$, $t_1 = 25$, and $t_2 = 15$ when $s = 1$. R_1, R_2, R_3, R_4, t_1 , and t_2 have same unit. Table 1 lists the first five eigenfrequencies in $\omega R_4 \sqrt{\rho/G}$ with Poisson's ratio $\nu = 0.3$ and retains four significant digits. To make the study of convergence less complicated, equal numbers of polynomial terms were taken in the r ($I = K = M$) and z ($J = L = N$) directions.

Table 1 shows the convergence of the first five eigenfrequencies. The values of I and J begin with six and eight, respectively, increasing I from 6 to 16 and J from 8 to 14. This result showed that the first and second eigenvalues were the easiest to converge to the accurate results. $I = 14$ and $J = 14$ were the smallest terms to obtain accurate values in the r and z directions. The Chebyshev polynomial series needed additional terms for higher order eigenfrequencies. When $I \times J = 16 \times 12$, the first five eigenfrequencies all converged to the accurate values. This result proved that the Chebyshev–Ritz method can be used to analyze the vibration characteristic of the annular plate in Figure 1 and to achieve good convergence.

In Table 2, we listed three sets of convergence comparison with the Chebyshev–Ritz method and the Ritz method using an algebraic polynomial as the admissible function from $I \times J = 10 \times 10$ to $I \times J = 12 \times 12$. The eigenfrequencies worked by the two methods in $I \times J = 10 \times 10$ were quite

close. The eigenfrequencies worked by the Chebyshev–Ritz method became increasingly accurate with an increase of I and J from $I \times J = 10 \times 10$ to $I \times J = 12 \times 12$. When $I \times J = 11 \times 11$, however, the eigenfrequencies worked by the Ritz method using an algebraic polynomial were ill-conditioned and the eigenfrequencies began to diverge. When $I \times J = 12 \times 12$, the eigenfrequencies all became complex numbers. This result indicated that the eigenfrequencies using an algebraic polynomial as the admissible function could not converge to the accurate values for gears with variable thickness. It is well known that ill-conditioned eigenfrequencies will occur with an increase in the number of terms. Sooner or later, the occurrence of ill-conditioned eigenfrequencies will depend on the admissible function adopted in the analysis. Because the Chebyshev polynomial possesses better numerical stability, the occurrence of ill-conditioned eigenfrequencies was delayed so that accurate eigenfrequencies could be obtained.

4. The Comparative Analysis of the Eigenfrequencies Using the Chebyshev–Ritz Method, Finite Element Method, and a Modal Experiment

To validate the accuracy of the eigenfrequencies worked out by the Chebyshev–Ritz method, we designed two annular plates and a gear for the modal experiment. The two annular plates are shown in Figure 3, and the gear is shown in Figure 4(a). The dimension parameters of the plates are listed in Table 3.

To test the eigenfrequencies, we adopted the hammering method and established the vibration measurement system to receive the hammering pulse of the annular plates. To simulate the completely free condition, we hung the annular plate with a rope. We used 13 measuring channels of the data acquisition instrument to test the frequencies. One channel was an incentive channel, and the others were parallel corresponding channel. The sensor on the hammer was connected to the data acquisition instrument and the data acquisition instrument was connected to the laptop with a program called Coinv DASP V 10.0. We placed three three-dimensional acceleration sensors on the plate in equal intervals in the r and θ directions. The three-dimensional acceleration sensors were connected to the data acquisition instrument and the frequencies collected by data acquisition instrument were limited between 1 kHz and 24 kHz. During the experiment, we established an analysis model by inputting the coordinate point of the annular plate to DASP V 10.0. We used the hammer beating the annular plate's surface and read the data from DASP V 10.0. The experimental platform was established as shown in Figure 4.

The eigenfrequencies of the two annular plates are listed in Tables 4 and 5 according to the Chebyshev–Ritz method and the experiment. CR represents the eigenfrequencies solved by Chebyshev–Ritz method, EX represents eigenfrequencies by experiment, and FE represents eigenfrequencies by finite element method. The finite element analysis type was a modal analysis with the Block Lanczos method (using Ansys 15.0). The unit type was SOLID 95 with 20 nodes, and

TABLE 1: Convergence of eigenvalues in $\Lambda = \omega R_4 \sqrt{\rho/G}$ of a completely free annular plate with $R_1 = 15, R_2 = 25, R_3 = 70, R_4 = 100, t_1 = 25, t_2 = 15$, and $s = 1$ for the first five eigenvalue parameters ($\nu = 0.3$).

$I \times J$	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5
6×8	1.477	2.672	3.886	5.643	7.001
6×10	1.475	2.671	3.830	5.639	6.978
6×12	1.475	2.671	3.826	5.636	6.967
6×14	1.474	2.671	3.801	5.634	6.953
10×8	1.446	2.669	3.575	5.598	6.502
10×10	1.442	2.669	3.568	5.591	6.462
10×12	1.441	2.668	3.546	5.586	6.451
10×14	1.439	2.668	3.542	5.582	6.430
14×8	1.438	2.668	3.529	5.585	6.425
14×10	1.437	2.668	3.508	5.579	6.414
14×12	1.434	2.668	3.504	5.574	6.388
14×14	1.433	2.667	3.488	5.571	6.383
15×8	1.438	2.668	3.526	5.582	6.417
15×10	1.435	2.667	3.505	5.573	6.390
15×12	1.433	2.667	3.500	5.571	6.369
15×14	1.433	2.667	3.487	5.565	6.362
16×8	1.437	2.668	3.510	5.580	6.408
16×10	1.433	2.667	3.503	5.570	6.387
16×12	1.433	2.667	3.487	5.565	6.362
16×14	1.433	2.667	3.487	5.565	6.362

TABLE 2: Comparison of the convergence study of the Ritz method using Chebyshev and algebraic polynomials in $\Lambda = \omega R_4 \sqrt{\rho/G}$ of a completely free annular plate with $R_1 = 15, R_2 = 25, R_3 = 70, R_4 = 100, t_1 = 25, t_2 = 15$, and $s = 1$ for the first five eigenvalue parameters ($\nu = 0.3$).

	$I \times J$	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5
CR	10×10	1.442	2.669	3.568	5.591	6.462
AR	10×10	1.442	2.669	3.556	5.585	6.467
CR	11×11	1.441	2.668	3.546	5.585	6.448
AR	11×11	2.051	2.182	5.828	6.419	6.786
						(ill-conditioned)
CR	12×12	1.437	2.668	3.522	5.579	6.415
AR	12×12	210.5i	162.9i	$1.957 + 118.5i$	$1.957 - 118.5i$	80.77i
						(ill-conditioned)

CR: results worked by the Chebyshev–Ritz method. AR: results worked by the Ritz method using algebraic polynomial as the admissible function. $i = \sqrt{-1}$.

FIGURE 3: (a) Brass annular plate and (b) grey cast iron annular plate.

TABLE 3: Dimension parameters of two annular plates.

Plate	Parameters (mm)					
	R_1	R_2	R_3	R_4	t_1	t_2
Brass annular plate	10	22	70	80	20	10
Grey iron annular plate	10	20	68	80	20	10

TABLE 4: Frequencies between 1 kHz and 24 kHz of the brass annular plate obtained by the Chebyshev–Ritz method and the experiment compared with those obtained by finite element method.

		Frequency (Hz)				
		f_1	f_2	f_3	f_4	
$s = 0^t$	CR	17867				
	EX	18981				
	FE	18410				
	CRE	2.95				
	FEX	3.10				
$s = 0^a$	CR	2355.2	8722.0	12790	17762	
	EX	2362.5	9096.9	13316	18244	
	FE	2319.0	8652.2	12822	17838	
	CRE	1.56	0.81	0.25	0.43	
	FEX	1.88	5.14	3.85	2.28	
$s = 1$	CR	4520.2	9986.4	11570	20546	
	EX	4496.9	9894.7	11644	20131	
	FE	4563.4	9982.0	11452	20404	
	CRE	0.95	0.04	1.03	0.70	
	FEX	1.46	0.87	1.68	1.34	
$s = 2$	CR	7910.2	15610	16118	23549	
	EX	8262.5	15313	16753	23353	
	FE	7898.6	15627	16540	23533	
	CRE	0.15	0.11	2.55	0.07	
	FEX	4.61	2.01	1.29	0.76	
$s = 3$	CR	3447.2	12263	22130		
	EX	3531.3	13175	21344		
	FE	3445.4	12822	22117		
	CRE	0.05	4.36	0.06		
	FEX	2.49	2.75	3.50		

0^a represents axisymmetric mode and 0^t represents torsional mode.

we adopted an intellectual finite element mesh division with four-stage precision for the mesh division. CRE represented relative error as $(|f_{CR} - f_{FE}|/f_{FE})\%$, and FEX was $(|f_{EX} - f_{FE}|/f_{FE})\%$. The brass annular plate's Poisson's ratio was $\nu = 0.36$, Young's modulus was $E = 88$ GPa, and density $\rho = 8557$ kg/m³. The grey cast iron annular plate's Poisson's ratio was $\nu = 0.3$, Young's modulus was $E = 130$ GPa, and density $\rho = 7410$ kg/m³.

We also designed the gear shown in Figure 4(a) to validate this method. Parameter R_4 is equal to the pitch radius. The gear was made of No. 45 steel with Poisson's ratio $\nu = 0.27$, Young's modulus $E = 210$ GPa, and density $\rho = 7800$ kg/m³. The dimension parameters are listed in Table 6.

In Table 4, 11 CREs are not greater than 1% and 1 CRE is greater than 4% in the 16 datasets. The CRE, in total, is not greater than 1% in Figure 5(a), which means $(11/16) \times 100\% = 68.8\%$. 2 FEXs are not greater than 1%, and 2 FEXs are

greater than 4% in the 16 datasets in Table 4. The FEX, in total, is not greater than 1% in Figure 5(a), which means $(2/16) \times 100\% = 12.5\%$. The CRE and FEX, in total, in Figures 5(b) and 5(c) are the same as that for Figure 5(a). In Table 5, 12 CREs and 2 FEXs are not greater than 1%, and 1 CRE and 4 FEXs are greater than 4%. In Table 7, 10 CREs and 3 FEXs are not greater than 1%, and 2 CREs and 3 FEXs are greater than 4%.

Figure 5 shows that the CREs in total are greater than 4% for each plate and the gear is very low. This low percentage indicated that it was feasible to use the Chebyshev–Ritz method based on three-dimensional elasticity theory to analyze the eigenfrequencies of gears with variable thickness. The low percentage of FEX greater than 4% indicated that the experimental scheme was reasonable and that the experimental results were reliable. The FEXs were mainly concentrated in the range of 1%–4% and rarely appeared in

TABLE 5: Frequencies between 1 kHz and 24 kHz of the grey cast iron annular plate obtained by the Chebyshev–Ritz method and the experiment compared with those obtained by finite element method.

		Frequency (Hz)			
		f_1	f_2	f_3	f_4
$s = 0^t$	CR	23892			
	EX	23293			
	FE	22575			
	CRE	5.83			
	FEX	3.18			
$s = 0^a$	CR	3023.5	11034	15837	22800
	EX	3131.3	11116	15869	22384
	FE	3007.7	10936	15948	22575
	CRE	0.53	0.90	0.70	1.00
	FEX	4.12	1.65	0.50	0.85
$s = 1$	CR	5882.7	13214	14808	
	EX	6021.9	12817	14288	
	FE	5866.1	13208	14657	
	CRE	0.28	0.05	1.03	
	FEX	2.66	2.96	2.52	
$s = 2$	CR	1911.9	10215	20563	
	EX	2002.5	10494	21072	
	FE	1906.8	10283	20422	
	CRE	0.27	0.66	0.69	
	FEX	5.02	2.05	3.18	
$s = 3$	CR	4704.3	14759	15977	
	EX	4953.1	13988	16314	
	FE	4703.0	14720	15948	
	CRE	0.03	0.26	0.18	
	FEX	5.32	4.97	2.29	

0^a represents axisymmetric mode and 0^t represents torsional mode.

TABLE 6

		Dimension parameters (mm)					
		Number of teeth	Module	R ₁	R ₂	R ₃	t ₁
Gear	100	2.25	19	28	96	20	12

(a)

(b)

FIGURE 4: (a) Location of three-dimensional acceleration sensors and (b) data acquisition instrument.

TABLE 7: Frequencies between 1 kHz and 24 kHz of the gear obtained by the Chebyshev–Ritz method and the experiment compared with those obtained by finite element method.

		Frequency (Hz)			
		f_1	f_2	f_3	f_4
$s = 0^t$	CR	22515			
	EX	23114			
	FE	21895			
	CRE	2.83			
	FEX	5.57			
$s = 0^a$	CR	2175.1	8695.0	13610	18864
	EX	2075.9	8685.7	13131	19094
	FE	2169.8	8675.3	13601	18776
	CRE	0.24	0.23	0.07	0.47
	FEX	4.33	0.12	3.46	1.69
$s = 1$	CR	4463.2	11742	21532	
	EX	4302.7	11874	21976	
	FE	4454.1	11512	20955	
	CRE	0.20	2.00	2.75	
	FEX	3.40	3.14	4.87	
$s = 2$	CR	1345.0	7751.4	8710.1	16698
	EX	1305.7	7779.2	8685.7	16312
	FE	1344.8	7733.1	8703.0	16011
	CRE	.0.01	0.24	0.08	4.29
	FEX	2.91	0.60	0.20	1.88
$s = 3$	CR	3327.5	14853	21965	
	EX	3397.0	15013	20314	
	FE	3326.2	14839	20955	
	CRE	0.04	0.09	4.82	
	FEX	2.13	1.17	3.06	

the range that was not greater than 1%. This phenomenon showed that it was difficult to achieve high precision using the experimental method because of the influence of the experimental platform's systematic error. In Figure 5, beyond 65% of the CREs were not greater than 1% for each plate and gear, and less than 13% of the CREs were greater than 4%. These values proved that the accuracy of results derived according to the Chebyshev–Ritz method and the finite element method was basically at the same level. We regarded the distribution of the percentage of CRE, in total, to be a linear distribution. This phenomenon provided additional guidance and a theoretical basis for the establishment of a system error compensation model for the experimental platform.

5. Conclusion

We should have repeated the finite element method's preprocessing steps to analyze the vibration characteristics, including modeling for different gears and different mesh division for accurate frequencies. The Chebyshev–Ritz method could be used to omit these preprocessing steps. By using the Chebyshev–Ritz method, we could substitute only the material property parameters and the dimension parameters into

the computing program to analyze the vibration characteristics of different gears with variable thickness. It was easy to achieve parametric programming and analysis. We were able to conduct three-dimensional free vibration analysis of gears with variable thickness using the Chebyshev–Ritz method to analyze the eigenfrequencies in different models with different s . This analysis method can be used for plates with any thickness–diameter ratios to solve their eigenfrequencies. The CREs, in total, that were greater than 4%, were low, which indicated that it was feasible to use the Chebyshev–Ritz method based on three-dimensional elasticity theory to analyze the eigenfrequencies of gears with variable thickness. The accuracy of results according to the Chebyshev–Ritz method and finite element method were basically at the same level because the CREs were mainly concentrated in a range not greater than 1%. The experimental method was reasonable, and the experimental results were reliable. This method can be used to analyze different types of stepped objects and also to omit the preprocessing steps for different analysis objects using the finite element method.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

FIGURE 5: Bar graphs of the percentage of FEX and CRE, in total, for each plate and gear according to Tables 4, 5, and 7.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

The authors acknowledge the funding support of the National Nature Science Fund of China (no. 51575375) and Shanxi Key Laboratory of Precision Machining.

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