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Research Article

Dominance Degree Multiple Attribute Decision Making Based on Z-Number Cognitive Information

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Cognitive information can be described by Z-number fully and effectively. However, many problems of Z-number need to be further studied. In this paper, two hidden probability models for calculating Z-number are established to provide more intuitive and abundant information. Next, the dominance degree relationship of Z-number is developed and subdivided. Furthermore, combined with the hidden probability of calculation, three different measurements of dominance degree are defined from three levels of geometry, algebra, and cross entropy based on the outranking relationship. The influencing factors are analyzed for different combinations of two probability models and three dominance degree measures. A multiattribute decision model is established on the basis of new grey association analysis and QUALIFLEX method. Finally, a decision example is given to verify the effectiveness and feasibility of the method. And sensitivity analysis is made to determine the impact of parameters and hidden probability on the decision model.

1. Introduction

Cognitive information for real-world decision making often has an element of uncertainty and is imprecise and only partially reliable. In 2011, Zadeh [1] put forward the Z-number theory to combine objective information with subjective understanding of cognitive information and enhance the understanding of natural language. A Z-number is expressed by a pair of ordered arrays A and B, where A represents the real value function of the uncertain variable X and B is the measure of reliability of A.

The current research on Z-numbers can be mainly divided into three categories. The first is theoretical basis study [2]. The concepts of Z-valuation [3], Z^+-number [4] and Z-information [5] are closely related to the concept of Z-numbers. The proposed four concepts are shown in Table 1, where u_A is the membership function of A and p_X is the probability distribution of X. The second aspect is language-type Z-number calculation and related extension [6, 7], for example, Zarrin and Azadeh [8] combined Z-number with fuzzy cognitive map (FCM) and proposed a novel approach named Z-number cognitive map. They evaluated and analyzed the impacts of resilience engineering (RE) principles on integrated health, safety, environment, and ergonomics (HSEE) management system. The main advantages of the proposed approach are determination of the weighted causality relations (for employing FCM) as well as handling uncertainty (for considering Z-number concept). Kant [9] extended the analysis of cognitive work as some requirements that gathered the framework of sociotechnical systems. It is helpful for the study of knowing and acting in technological contexts in the human. The third aspect is establishing a decision model based on Z-number [10–13]. For example, Li et al. [14] proposed a method to convert Z-numbers into fuzzy numbers. Aliev et al. [4] proposed some algorithms about Z-numbers. They considered two approaches for decision making with Z-information. The first approach is based on converting the Z-numbers to crisp numbers to determine the priority weight of each alternative, which would decrease some uncertain information during processing. The second approach is based on expected utility theory by using Z-numbers. This method of selecting

Table 1: Several concepts which are closely related to the concept of a Z-number.

Designation	Expression
Z-number	$Z = (A, B)$
Z-valuation	Z-valuation (X, A, B)
Z-information	A collection of Z-valuations
Z^+-number	$Z^+ = (A, R)$ or (A, p_X) or (μ_A, p_X)
Z^+-valuation	Z^+-valuation (X, A, p_X) or (X, μ_A, p_X)

expected utility is an uncertain factor, and it influences the effect of using Z-number. Kang et al. [15] proposed a utility function of Z-numbers. These decision methods, utility functions, or conversion methods may lose some information during the operation, and these shortcomings should be further studied.

At present, the motivation for using discrete Z-number in the study of Z-numbers is mainly divided into three levels [16]. Firstly, Z-numbers are used to describe relevant decision information in many decision problems by discrete language terms which are usually expressed as linguistic information. Secondly, the computational complexity of continuous fuzzy numbers [17, 18] and density functions is significantly higher than that of discrete fuzzy numbers [19–21] and discrete probability distributions. Thirdly, the universality of the multiattribute decision model is established according to the uncertain information. When the decision information is expressed as discrete Z-numbers, one does not need to assume a probability distribution to limit the model. When the probability of natural state and surrogate results are described by Z-numbers, it need transform numbers or sets according to some operators. Aliev and Zeinalova [22] performed decision analysis from two stages. Above all, the Z-number is converted to a fuzzy number; next, the value of the utility function is calculated and sorted, and then the scheme with the largest value of the utility function is selected as the optimal selection scheme. Z-information [16] belongs to the category of probability limits; thus, the probability distribution can be regarded as the probability limit. Li et al. [14] proposed a fuzzy expectation based on the Z-number that is a fuzzy set as well. In the fuzzy expectation, B is decomposed into a detailed numerical value α. The fuzzy number obtained the first element A; multiplying by α is considered as a transformation of the original Z-number. Chen [23] combined the method in the literature [24] to establish a multicriteria decision-making method based on Z-number. This method used the standard weight and standard value as Z-number. Obviously, the Z-number of this method is a set of real numbers. Its means that a lot of uncertain information contained in Z-number is neglected and lost, that is, decision makers cannot obtain an accurate optimal decision-making method. Farhadinia [25] proposed a new measure of information entropy, which is closely related to probability and comprehensively reflected more uncertain information.

Peng and Wang [26] defined a kind of outranking relationship between two Z-numbers, but it directly calculates the dominance degree of the first element A and the second element B in a Z-number by a simple comparison and accumulation, which makes no sense in some practical problems. For example, suppose that $Z_1 =$ (high, uncertainly) and $Z_2 =$ (low, absolutely); then, the "high" and "low" can be directly compared. It is obvious that "high" is better than "low" in evaluating the grade of a commodity, but "absolutely" is better than "uncertainly" according to intuitive literal sense. It is worth noting that "uncertainly" and "absolutely" are measurements of the possibility of "high" and "low," respectively. It is meaningless to compare them separately. That is, A and B in $Z = (A, B)$ should be tied together to evaluate the target.

Form the work reviewed, it can be concluded that ranking of Z-number cognitive information is a necessary operation, and the hidden probability and dominance degree of Z-numbers is a challenging practical issue.

The multiattribute decision-making problem is to give some alternative schemes, and each scheme needs to be comprehensively evaluated according to several attributes. The purpose of decision making is to find a scheme that makes the decision maker feel most satisfied from the given alternative schemes through the comprehensive evaluation sequence. Mao et al. [27] reported the interval-valued intuitionistic fuzzy entropy which reflects intuitionism and fuzziness of interval-valued intuitionistic fuzzy set (IvIFS) based on interval-valued intuitionistic fuzzy cross entropy. According to the compositive entropy use for multiple attributes decision making, they adopted the weighted correlation coefficient between IvIFSs and pattern recognition by a similarity measure transformed from the compositive entropy. However, in practical problems, there are often some individual subjective factors of decision makers, objective factors of the attributes, unpreventable error factors, and other uncertain factors, which are one of the characteristics of uncertain information itself. How to make better use of dubious, inaccurate, and uncertain information in multiattribute decision making is the main research problem in the field of multiattribute decision making [28]. Yang and Wang [29] established a linear programming model to solve Z-number probability, and made a multicriteria decision based on Z-number probability. Yang and Wang [29] combined the concept of reliability to judge the decision maker and established the multicriteria decision aiding model based on stochastic multicriteria acceptability analysis (SMAA).

Li et al. [30] developed a linear programming methodology for solving multiattribute group decision-making problems using intuitionistic fuzzy (IF) sets. Wan and Li [31] extended the linear programming technique for multidimensional analysis of preference (LINMAP) for solving heterogeneous MADM problems which involve intuitionistic fuzzy (IF) sets (IFSs), trapezoidal fuzzy numbers (TrFNs), intervals, and real numbers. They presented decision maker's preference given through pairwise comparisons of alternatives with hesitation degrees which are represented as IFSs. They constructed a new fuzzy mathematical programming model, obtained FIS and the attribute weights, and calculated the distances of all alternatives to the FIS, which are used to determine the ranking order of the alternatives. Yu et al. [32] developed a compromise-typed variable weight decision method for solving hybrid multiattribute decision-making problems with multiple types of attribute values, and variable weight synthesis and orness measures based on the coefficients of absolute risk aversion are analyzed in variable weight decision making. The comprehensive values of alternatives based on the compromise-typed variable weight decision method are calculated, and the decision-making results are determined according to the comprehensive values. Yu et al. [33] developed a novel method for HMAGDM problems based on the orness measures by analyzing the relative closeness of alternatives and preference deviation degrees of each decision maker (DM). The weights of the attributes of each alternative and weights preference of DMs are obtained using two linear programming models, and a ranking of alternative is determined according to the decision making preference of alternatives.

Shih and Chen [34] studied grey relational analysis of the series similarity and approximation. Liu et al. [35] defined the multiattribute and multistage decision-making problem, that is, the attribute weights and time weights in each decision stage are unknown and the attribute value is interval numbers; after dimensionless processing of attribute values, the grey correlation analysis method is used to determine the attribute weights of each attribute in different time periods to make decisions. Paelinck [36] proposed a simple and flexible outranking model. The QUALIFLEX method tested the binary relation of each possible ranking possibility and directed distance measure under different attributes and then calculated the comprehensive concordance/discordance index of each ranking to determine the optimal ranking. The cardinal and ordinal information can be correctly processed and the uncertain information can be fully considered by this method as well. The QUALIFLEX method has been studied and extended to various applications, such as investment risk assessment [37], supplier evaluation [38], and product design selection [39].

Based on the reviewed literature, the authors conclude that the little attention has been paid to the important issue of ranking Z-number and measuring uncertainty. And now many researchers do not consider the hidden probability and reliability of Z-number in the Z-number multiattribute decision problem. This paper will consider the important role of hidden probability and reliability in decision making.

Based on the previous discussion, the three primary motivations of this paper are as follows:

(1) The structure of Z-number expresses the subjectivity and objectivity of natural language, and the expression of its reliability cannot completely limit the subjective uncertainty information. Therefore, the hidden probability of Z-number is introduced to limit the reliability.

(2) The dominance degree accumulates A and B, respectively, in Z-number, which is unreasonable in practical problems. This paper distinguishes this from the geometric, algebraic, and cross-entropy levels to define the superiority of the three metrics. A multiattribute decision model with unknown attribute weights is established.

(3) Z-number is an important tool with which humans communicate with a computer. With effective use of Z-number to express natural language, humans will not spend a lot of time learning their own computer language. Natural language and Z-number combination can better reflect uncertain information.

This paper is divided into nine sections. Section 2 gives the basic definitions of Z-number, discrete Z-number, and Z^+-number. In Section 3, two probability models, MD I and MD II, are established. Section 4 is the main part of this paper: outranking relationships and three measurements of dominance degree of discrete Z-numbers. Section 5 is about two models as well: one is to establish a model for solving attribute weight by new grey relational analysis and the other is a multiattribute decision model based on three measurements of dominance degree, mainly using the QUALIFLEX method. Section 6 is about a multiattribute decision case of a venture capital company. Section 7 is a sensitivity analysis based on the decision cases. In Section 8, comparison and summary of the decision method of Section 6 are given, and Section 9 concludes this paper.

2. Preliminaries

In this section, we introduce some basic knowledge about fuzzy numbers, discrete Z-number, and Z^+-number in detail. Let X be a universal set, and a fuzzy set A in X is represented as

$$A = \{ <x, \mu_A(x)> \mid x \in X \}, \quad (1)$$

where $\mu_A(x) : X \longrightarrow [0,1]$ indicates the membership degree of the element $x \in X$ to A subset of X.

Definition 1 (discrete fuzzy number) [4, 26]. A fuzzy subset A of the real line R with membership function $\mu_A : R \longrightarrow [0,1]$ is a discrete fuzzy number if its support is finite, i.e., there exist $x_1, \ldots, x_n \in R$ with $x_1 < x_2 < \cdots < x_n$, such that $\text{supp}(A) = x_1, \ldots, x_n$ and there exist natural numbers s, t with $1 \le s \le t \le n$ satisfying the following conditions:

(1) $\mu_A(x_i) = 1$ for any natural number i with $s \le i \le t$

(2) $\mu_A(x_i) \le \mu_A(x_j)$ for each natural number i, j with $1 \le i \le j \le s$

(3) $\mu_A(x_i) \ge \mu_A(x_j)$ for each natural number i, j with $t \le i \le j \le n$

Definition 2 (Z-number) [1]. A Z-number is an ordered pair of fuzzy number, (A, B). It is associated with a real-valued uncertain variable, X. The first component, A, playing the role of a fuzzy restriction, $R(X)$, about the values which X can take, written as X is A, where A is a fuzzy set. The restriction

$$R(X) : X \quad \text{is } A \longrightarrow \text{Poss}(X = u) = \mu_A(u), \quad (2)$$

is referred to as a possibility restriction (constraint), where μ_A is the membership function of A and u is a generic value of X. The second component, B, is referred to as certainty. Closely related to certainty are the concepts of sureness, confidence, reliability, strength of belief, and probability. The underlying probability distribution, p_x, is not known. What is known is a restriction on p_x which may be expressed as

$$\int_R \mu_A p_x(u)\mathrm{d}u \quad \text{is } B. \tag{3}$$

Definition 3 (discrete Z-number) [4]. Let X be a random variable, A and B be two discrete fuzzy numbers, and $\mu_A : x_1, x_2, \ldots, x_n \longrightarrow [0,1]$ and $\mu_B : b_1, b_2, \ldots, b_n \longrightarrow [0,1]$. For the membership function of A and B, respectively, $x_1, x_2, \ldots, x_n \in R$ and $b_1, b_2, \ldots, b_n \in [0,1]$. A discrete Z-number is defined as an ordered pair of discrete fuzzy numbers $Z = (A, B)$ on X, where A is the fuzzy restriction of X and B is the fuzzy restriction of the probability measure of A.

Definition 4 (discrete Z^+-number) [4]. A discrete Z^+-number, denoted as $Z^+ = (A, R)$, where A is the fuzzy restriction and R is the probability distribution $p(x)$ of X, is expressed as

$$\begin{aligned} \mu &= \frac{\mu_1}{x_1} + \frac{\mu_2}{x_2} + \cdots + \frac{\mu_n}{x_n}, \\ p(x) &= \frac{p_1}{x_1} + \frac{p_2}{x_2} + \cdots + \frac{p_n}{x_n}, \end{aligned} \tag{4}$$

where μ_i/x_i means that μ_i, $i = 1, \ldots, n$, is the possibility that $X = x_i$. Similarly, p_i/x_i is the probability that $X = x_i$. And A plays the same role in Z^+-number as it does in Z-numbers, and R plays the role of the probability distribution.

3. Two Probability Models of Discrete Z-Numbers

The uncertainty of some information includes fuzziness and randomness. In order to obtain more uncertain information, we considered the hidden probability distribution of Z-numbers, comprehensively considered the relation between A and B in Z-numbers, and established two linear programming models to determine the hidden probability of discrete Z-numbers.

A discrete Z-number $Z = (A, B)$ described imperfect information on real-valued random variable X values and satisfies the following conditions:

(1) $\sum_i^n p_X(x_i) = \sum_i^n x_i \mu_A(x_i) / \sum_i^n \mu_A(x_i)$ (the slash symbol denotes division)

(2) $\sum_i^n p_X(x_i) = 1$ and $\forall p_X(x_i) \geq 0$

3.1. Probability Model I of Discrete Z-Numbers. Condition (1) ensures that these distributions are compatible when the centroid of μ_A and p_X is coincident. Condition (2) is the normalization and nonnegativity condition of the probability. We can define the mathematical model as follows:

$$\min \quad \sum_i^n \left(\mu_{A_{x_i}} p_j(x_i) - b_j\right)^2,$$

$$\begin{cases} \sum_i^n p_j(x_i)\, x_i = \dfrac{\sum_i^n x_i \mu_A(x_i)}{\sum_i^n \mu_A(x_i)}, \\ \sum_i^n p_j(x_i) = 1, \\ p_j(x_i) \geq 0, \end{cases} \tag{5}$$

and then the hidden probability matrix $\mathbf{P}$ of Z-number is calculated by MATLAB programming language, denoted as

$$\mathbf{P} = \left[p_j(x_i)\right]_{n\times n}. \tag{6}$$

Use $\mathbf{P}$ matrix to calculate the comprehensive probability of $p(x_i)$:

$$p(x_i) = \frac{\sum_j^n p_j(x_i)}{n}. \tag{7}$$

3.2. Probability Model II of Discrete Z-Numbers. On the basis of probability MD I of Z-number, MD II after simplification is shown as follows. Compared with MD I, MD II is relatively concise, but its response has less hidden uncertain information than MD I.

$$\min \quad \sum_i^n \left(\mu_{A_{x_i}} p(x_i) - b_{\max}\right)^2,$$

$$\begin{cases} \sum_i^n p(x_i)\, x_i = \dfrac{\sum_i^n x_i \mu_A(x_i)}{\sum_i^n \mu_A(x_i)}, \\ \sum_i^n p(x_i) = 1, \\ p(x_i) \geq 0, \end{cases} \tag{8}$$

where $\mu_B(b_{\max}) = \max\{\mu_B(b_1), \mu_B(b_2), \ldots, \mu_B(b_n)\}$; to make the distinction, the probability calculated under the $\mathfrak{R}\,(\mathfrak{R} \in \{\mathrm{I}, \mathrm{II}\})$ model is denoted as $p^{\mathfrak{R}}$. The hidden probability of the problem containing Z-number is solved, that is to say, the problem is resolved, and the optimization problem is solved by MATLAB language. Subsequently, the probability model presented in this paper can be used to evaluate the basic probability distribution.

Example 1. Let $Z_1 = (A_1, B_1)$ and $Z_2 = (A_2, B_2)$ be two discrete Z-numbers, where

$$A_1 = \frac{0}{1} + \frac{0.5}{2} + \frac{0.8}{3} + \frac{1}{4} + \frac{0.8}{5} + \frac{0.7}{6} + \frac{0.6}{7} + \frac{0.4}{8} + \frac{0.2}{9} + \frac{0}{10},$$

$$B_1 = \frac{0}{0.1} + \frac{0.3}{0.2} + \frac{0.6}{0.3} + \frac{0.7}{0.4} + \frac{0.8}{0.5} + \frac{0.9}{0.6} + \frac{1}{0.7} + \frac{0.8}{0.8} + \frac{0.5}{0.9} + \frac{0}{1},$$

$$A_2 = \frac{0}{1} + \frac{0.3}{2} + \frac{0.5}{3} + \frac{0.6}{4} + \frac{0.7}{5} + \frac{0.8}{6} + \frac{1}{7} + \frac{0.6}{8} + \frac{0.4}{9} + \frac{0}{10},$$

$$B_2 = \frac{0}{0.1} + \frac{0.5}{0.2} + \frac{1}{0.3} + \frac{0.8}{0.4} + \frac{0.7}{0.5} + \frac{0.6}{0.6} + \frac{0.4}{0.7} + \frac{0.2}{0.8} + \frac{0.1}{0.9} + \frac{0}{1}. \quad (9)$$

The hidden probabilities $p_1(x_{1i})$ and $p_2(x_{2i})$ of Z_1 and Z_2 are calculated, respectively, by using the above probabilistic MD I and MD II, which are called $Z_1^+ = (A_1, R_1)$ and $Z_2^+ = (A_2, R_2)$, where R_1 and R_2 are the following discrete probability distributions:

$$R_1 = \frac{p_1(x_{11})}{x_{11}} + \cdots + \frac{p_1(x_{1i})}{x_{1i}} + \cdots + \frac{p_1(x_{1n})}{x_{1n}},$$
$$R_2 = \frac{p_2(x_{21})}{x_{21}} + \cdots + \frac{p_2(x_{2i})}{x_{2i}} + \cdots + \frac{p_2(x_{2n})}{x_{2n}}, \quad (10)$$

where n is the number of discrete fuzzy number A in discrete Z-number. In MD I, first we need to figure out the hidden probability matrix P_1 and P_2 of Z_1 and Z_2 and then obtain the comprehensive probabilities $p_1^{\mathrm{I}}(x_i)$ and $p_2^{\mathrm{I}}(x_i)$ of Z-number from MD I.

$$P_1 = \begin{bmatrix} 0.4729 & 0.0253 & 0.0158 & 0.0129 & 0.0159 & 0.0182 & 0.0206 & 0.0305 & 0.0394 & 0.3485 \\ 0.3933 & 0.0510 & 0.0329 & 0.0268 & 0.0327 & 0.0369 & 0.0419 & 0.0568 & 0.0723 & 0.2556 \\ 0.3123 & 0.0764 & 0.0515 & 0.0425 & 0.0510 & 0.0563 & 0.0626 & 0.0811 & 0.0930 & 0.1733 \\ 0.2228 & 0.1085 & 0.0772 & 0.0632 & 0.0721 & 0.0756 & 0.0790 & 0.0886 & 0.0913 & 0.1218 \\ 0.1408 & 0.1216 & 0.1087 & 0.0989 & 0.0959 & 0.0919 & 0.0882 & 0.0860 & 0.0831 & 0.0849 \\ 0.0737 & 0.1065 & 0.1381 & 0.1580 & 0.1216 & 0.1053 & 0.0927 & 0.0768 & 0.0696 & 0.0576 \\ 0.0342 & 0.0622 & 0.1172 & 0.2952 & 0.1401 & 0.1056 & 0.0855 & 0.0635 & 0.0575 & 0.0391 \\ 0.0098 & 0.0181 & 0.0416 & 0.5238 & 0.1047 & 0.0895 & 0.0893 & 0.0476 & 0.0494 & 0.0262 \\ 0.0000 & 0.0000 & 0.0000 & 0.6536 & 0.0000 & 0.0000 & 0.3464 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.6536 & 0.0000 & 0.0000 & 0.3464 & 0.0000 & 0.0000 & 0.0000 \end{bmatrix},$$

$$p_1^{\mathrm{I}}(x_1) = \frac{0.4729 + 0.3933 + \cdots + 0.0098}{10} = 0.16598,$$

$$p_1^{\mathrm{I}} = \frac{0.16598}{1} + \frac{0.05696}{2} + \frac{0.0583}{3} + \frac{0.25285}{4} + \frac{0.0634}{5} + \frac{0.05793}{6} + \frac{0.12526}{7} + \frac{0.05309}{8} + \frac{0.05556}{9} + \frac{0.1107}{10}. \quad (11)$$

Then, the same logic applies to

$$p_2^{\mathrm{I}} = \frac{0.11998}{1} + \frac{0.05516}{2} + \frac{0.12109}{3} + \frac{0.05853}{4} + \frac{0.05956}{5} + \frac{0.06388}{6} + \frac{0.27207}{7} + \frac{0.05151}{8} + \frac{0.05218}{9} + \frac{0.14601}{10}. \quad (12)$$

Using MD II, the hidden probability $p(x_i)$ of Z-number is calculated as follows:

$$p_1^{\mathrm{II}} = \frac{0.3123}{1} + \frac{0.0764}{2} + \frac{0.0515}{3} + \frac{0.0425}{4} + \frac{0.0510}{5} + \frac{0.0563}{6} + \frac{0.0626}{7} + \frac{0.0811}{8} + \frac{0.0930}{9} + \frac{0.1733}{10},$$

$$p_2^{\mathrm{II}} = \frac{0.04}{1} + \frac{0.0585}{2} + \frac{0.0822}{3} + \frac{0.0896}{4} + \frac{0.1057}{5} + \frac{0.1436}{6} + \frac{0.3412}{7} + \frac{0.0659}{8} + \frac{0.0448}{9} + \frac{0.0284}{10}. \quad (13)$$

From the calculation results of Example 1, it can be seen that the calculation results of MD I reflect more hidden information, and the results of MD II are only part of the results of MD I. However, which model the dominance degree proposed in the following part applies to depends on the results of our sensitivity analysis.

4. Outranking Relationship and Dominance Degree of Discrete Z-Number

This section redefines a more detailed Z-number outranking relation for the outranking relation in article [26]. Combining the three aspects of geometry, algebra, and information entropy with Z-number and the hidden probability information of Z-number itself, three different computing methods of dominance degree are defined. And a model for calculating the hidden probability P of discrete Z-number is established.

A linguistic term set (LTS) noted $S = \{s_i \mid i = 0, 1, 2, \ldots, 2t \cdot t \in N\}$ is a finite and completely ordered discrete LTS with odd cardinality, where s_i represents the possible value for a linguistic variable, s_i and s_j satisfy $s_i \leq s_j$ if $i \leq j$, and $\mathrm{neg}(s_i) = s_j$ if $i + j = 2t$ for any $i, j \in N$.

Definition 5 (outranking relationship of discrete Z-number). Let two Z-numbers characterized by language variables, denoted as $Z_i = (s_i, s_i')$ and $Z_j = (s_j, s_j')$, where $s_i, s_j \in S$, $s_i' s_i' \in S'$ and $S = s_0, s_1, \ldots, s_{2l}, l \in N$ and $S' = s_0', s_1', \ldots, s_{2r}'$, $r \in N$, be two LTSs. Then, the outranking relationship of two Z-numbers can be defined as follows:

(1) Extremely strong dominance: when $s_i > s_j$ and $s_i' > s_j'$, Z_i extremely and strongly dominates Z_j; let us label this relationship as $Z_i >_e Z_j$ or $Z_j <_e Z_i$

(2) Strong dominance: when $s_i > s_j$ and $s_i' = s_j'$, Z_i strongly dominates Z_j; let us label this relationship as $Z_i >_s Z_j$ or $Z_j <_s Z_i$

(3) Weak dominance: when $s_i = s_j$ and $s_i' > s_j'$, Z_i weakly dominates Z_j; let us label this relationship as $Z_i >_w Z_j$ or $Z_j <_w Z_i$

(4) Equal dominance: when $s_i = s_j$ and $s_i' = s_j'$, Z_i equally dominates Z_j; let us label this relationship as $Z_i \cong Z_j$ or $Z_j <_s \cong Z_i$

(5) Incomparable relation: if neither Z_i nor Z_j satisfies the above conditions, then Z_i and Z_j are called incomparable; let us label this relationship as $Z_i \perp Z_j$ or $Z_j \perp Z_i$

Property 1. Let $Z_i = (s_i, s_i')$, $Z_j = (s_j, s_j')$, and $Z_l = (s_l, s_l')$ be three arbitrary Z-numbers. In this paper, we define the property of extremely strong dominant relation of Z-number as follows:

(1) Nonreflexivity: $Z_i \not>_e Z_i$, where $\not>_e$ indicates non-extremely strong dominance

(2) Asymmetry: $Z_i >_e Z_j \not\Longrightarrow Z_j >_e Z_i$

(3) Transitivity: $Z_i >_e Z_j$ and $Z_j >_e Z_l \Longrightarrow Z_i >_e Z_l$

Proof

(1) The nonreflexivity property uses the method of proof by contradiction; if $Z_i >_e Z_i$, then $s_i > s_i$ and $s_i' > s_i'$. It is obviously contradictory.

(2) The asymmetry property uses the method of proof by contradiction as well. Let us say that $Z_i >_e Z_j \Longrightarrow Z_j >_e Z_i$ is true. If $Z_i >_e Z_j$, then $s_i > s_j$ and $s_i' > s_j'$. If $Z_j >_e Z_i$, then $s_j > s_i$ and $s_j' > s_i'$. These two conclusions are obviously contradictory, and the asymmetric property is proved.

(3) If $Z_i >_e Z_j$, then $s_i > s_j$ and $s_i' > s_j'$; if $Z_j >_e Z_l$, then $s_j > s_l$ and $s_j' > s_l'$. According to the transitivity of inequality, we can draw a conclusion $s_i > s_l$ and $s_i' > s_l'$, that is, $Z_i >_e Z_l$. □

Definition 6 (three measurements of dominance degree for discrete Z-number). Let X_j and X_k $(j, k \in N)$ be two random variables, $Z_j = (A_j, B_j)$ and $Z_k = (A_k, B_k)$ be the discrete Z-numbers of X_j and X_k, and $\mu_{A_j} : \{x_{j1}, x_{j2}, \ldots, x_{jn}\} \longrightarrow [0, 1]$ and $\mu_{B_j} : \{b_{j1}, b_{j2}, \ldots, b_{jn}\} \longrightarrow [0, 1]$ be the membership function of A_j and B_j; Z_k is similar to Z_j. Firstly, Z-number is normalized:

$$Z_j = (A_j, B_j) = \left(\sum \frac{\mu_{A_j}(x_{ji})}{x_{ji}}, \sum \frac{\mu_{B_j}(b_{ji})}{b_{ji}}\right), \tag{14}$$

normalized:

$$Z_j' = (A_j', B_j') = \left(\sum \frac{\mu_{A_j}'(x_{ji}')}{x_{ji}'}, \sum \frac{\mu_{B_j}'(b_{ji}')}{b_{ji}'}\right), \tag{15}$$

where $x_{ji}' = ((a_{ji} - \mathrm{mim}\{a_{j1}, a_{k1}\}) / (\max\{a_{jn}, a_{kn}\} - \mathrm{mim}\{a_{j1}, a_{k1}\})) - (1/2)$, $b_{ji}' = b_{ji} - (1/2)$, $\mu_{A_j}(x_{ji}') = \mu_{A_j}(x_{ji})$, and $\mu_{B_j}(b_{ji}') = \mu_{B_j}(b_{ji})$; Z_k is similar to Z_j. After Z-number standardization, dimensional relationship between data is eliminated, making data comparable.

Then, three measurements of dominance degree for discrete Z-number as follows.

Definition 7 (the geometric measurements of dominance degree for discrete Z-number). The geometric measurements of dominance degree for discrete Z-number is Z-number after the number of standardized data one by one to quantify. The angle is measured as the degree of dominance between the two Z-numbers. The geometric measurements of dominance degree for discrete Z_j and Z_k are defined as follows:

$$P_{\mathfrak{R}}\left(Z_j \succ Z_k\right) = \alpha + \beta = \sum_i^n \alpha_i^{jk} + \sum_i^n \beta_i^{jk}$$
$$= \sum_i^n \operatorname{sgn} t \arccos\left(\frac{b'_{ki}\mu_{B_k}(b'_{ki})\cos\theta + x'_{ki}\mu_{A_k}(x'_{ki})\sin\theta}{\sqrt{\left(b'_{ki}\mu_{B_k}(b'_{ki})\cos\theta + x'_{ki}\mu_{A_k}(x'_{ki})\sin\theta\right)^2 + \left(b'_{ki}\mu_{B_k}(b'_{ki})\sin\theta - x'_{ki}\mu_{A_k}(x'_{ki})\cos\theta\right)^2}}\right)$$
$$+ \sum_i^n \operatorname{sgn} s \arccos\left(\frac{p(x'_{ki})\cos\gamma + x'_{ki}\sin\gamma}{\sqrt{\left(p(x'_{ki})\cos\gamma + x'_{ki}\sin\gamma\right)^2 + \left(p(x'_{ki})\sin\gamma - x'_{ki}\cos\gamma\right)^2}}\right), \quad (16)$$

where

$$t = b'_{ki}\mu_{B_k}(b'_{ki})\cos\theta - x'_{ki}\mu_{A_k}(x'_{ki})\sin\theta,$$
$$s = p(x'_{ki})\cos\gamma - x'_{ki}\sin\gamma,$$
$$\cos\gamma = \frac{x'_{ji}}{\sqrt{\left(x'_{ji}\right)^2 + \left(p\left(x'_{ji}\right)\right)^2}},$$
$$\sin\gamma = \frac{p\left(x'_{ji}\right)}{\sqrt{\left(x'_{ji}\right)^2 + \left(p\left(x'_{ji}\right)\right)^2}}, \quad (17)$$
$$\cos\theta = \frac{x'_{ji}\mu_{A_j}\left(x'_{ji}\right)}{\sqrt{\left[x'_{ji}\mu_{A_j}\left(x'_{ji}\right)\right]^2 + \left[b'_{ji}\mu_{B_j}\left(b'_{ji}\right)\right]^2}},$$
$$\sin\theta = \frac{b'_{ji}\mu_{B_j}\left(b'_{ji}\right)}{\sqrt{\left[x'_{ji}\mu_{A_j}\left(x'_{ji}\right)\right]^2 + \left[b'_{ji}\mu_{B_j}\left(b'_{ji}\right)\right]^2}},$$

and θ and γ are the positive angles between $\overrightarrow{Z_j} = (x'_{ji}\mu_{A_j}(x'_{ji}), b'_{ji}\mu_{B_j}(b'_{ji}))$ and $\overrightarrow{Z_j^+} = (x'_{ji}, p(x'_{ji}))$ vectors and the X-axis, respectively. $P_{\mathfrak{R}}(Z_j \succ Z_k)$ represents the degree of preference of Z_j to Z_k calculated with the hidden probability solved by probability MD $\mathfrak{R}$, $\mathfrak{R} \in \{\mathrm{I}, \mathrm{II}\}$.

Property 2. Let $P_{\mathfrak{R}}(Z_j \succ Z_k)$ be the geometric measurements of dominance degree for Z_j over Z_k; they have the following properties:

(1) $P_{\mathfrak{R}}(Z_j \succ Z_k) = 0$ if and only if $Z_j = Z_k$

(2) $-2n\pi \le P_{\mathfrak{R}}(Z_j \succ Z_k) \le 2n\pi$ or $-720^\circ n \le P_{\mathfrak{R}}(Z_j \succ Z_k) \le 720^\circ n$

(3) $P_{\mathfrak{R}}(Z_j \succ Z_k) = -P_{\mathfrak{R}}(Z_k \succ Z_j)$

Example 2. Use Z_1 and Z_2 in Example 1 and $Z_3 = (A_3, B_3)$ to calculate the dominance degree $P_{\mathfrak{R}}(Z_j \succ Z_k)$ of geometric measurements Z_j and Z_k $(j, k \in \{1, 2, 3\}, \mathfrak{R} \in \{\mathrm{I}, \mathrm{II}\})$.

$$A_3 = \frac{0}{1} + \frac{0.4}{2} + \frac{0.7}{3} + \frac{0.8}{4} + \frac{1}{5} + \frac{0.8}{6} + \frac{0.5}{7} + \frac{0.3}{8} + \frac{0.1}{9} + \frac{0}{10},$$
$$B_3 = \frac{0}{0.1} + \frac{0.4}{0.2} + \frac{0.8}{0.3} + \frac{0.9}{0.4} + \frac{1}{0.5} + \frac{0.7}{0.6} + \frac{0.6}{0.7} + \frac{0.5}{0.8} + \frac{0.3}{0.9} + \frac{0}{1}. \quad (18)$$

The first step is to calculate the hidden probability of Z-number with two models; then, we calculated the results as follows:

$$p_1^{\mathrm{I}} = \frac{0.43525}{1} + \frac{0.03902}{2} + \frac{0.03003}{3} + \frac{0.02589}{4} + \frac{0.02753}{5} + \frac{0.02809}{6} + \frac{0.02889}{7} + \frac{0.3281}{8} + \frac{0.04268}{9} + \frac{0.3098}{10},$$
$$p_2^{\mathrm{I}} = \frac{0.3437}{1} + \frac{0.03848}{2} + \frac{0.03187}{3} + \frac{0.03004}{4} + \frac{0.02882}{5} + \frac{0.02797}{6} + \frac{0.02618}{7} + \frac{0.3286}{8} + \frac{0.04045}{9} + \frac{0.3967}{10},$$
$$p_3^{\mathrm{I}} = \frac{0.41821}{1} + \frac{0.04209}{2} + \frac{0.03319}{3} + \frac{0.03051}{4} + \frac{0.02746}{5} + \frac{0.02825}{6} + \frac{0.03222}{7} + \frac{0.3708}{8} + \frac{0.04174}{9} + \frac{0.30927}{10},$$
$$p_1^{\mathrm{II}} = \frac{0.3970}{1} + \frac{0.0520}{2} + \frac{0.0340}{3} + \frac{0.0266}{4} + \frac{0.0335}{5} + \frac{0.0378}{6} + \frac{0.0427}{7} + \frac{0.0559}{8} + \frac{0.0943}{9} + \frac{0.2261}{10},$$
$$p_2^{\mathrm{II}} = \frac{0.4717}{1} + \frac{0}{2} + \frac{0}{3} + \frac{0}{4} + \frac{0}{5} + \frac{0}{6} + \frac{0}{7} + \frac{0}{8} + \frac{0}{9} + \frac{0.5283}{10},$$
$$p_3^{\mathrm{II}} = \frac{0.5522}{1} + \frac{0.0004}{2} + \frac{0.0002}{3} + \frac{0.0002}{4} + \frac{0.0001}{5} + \frac{0.0002}{6} + \frac{0.0003}{7} + \frac{0.0005}{8} + \frac{0.0007}{9} + \frac{0.4453}{10}. \quad (19)$$

Table 2: The dominance degree of Z_i and Z_j, $i, j = 1, 2, 3$.

	$P(Z_1 \succ Z_2)$	$P(Z_1 \succ Z_3)$	$(Z_2 \succ Z_3)$
MD I	$-499°34'$	$-102°11'38''$	$139°1'41''$
MD II	$-71°50'28''$	$21°37'44''$	$-98°1'6''$

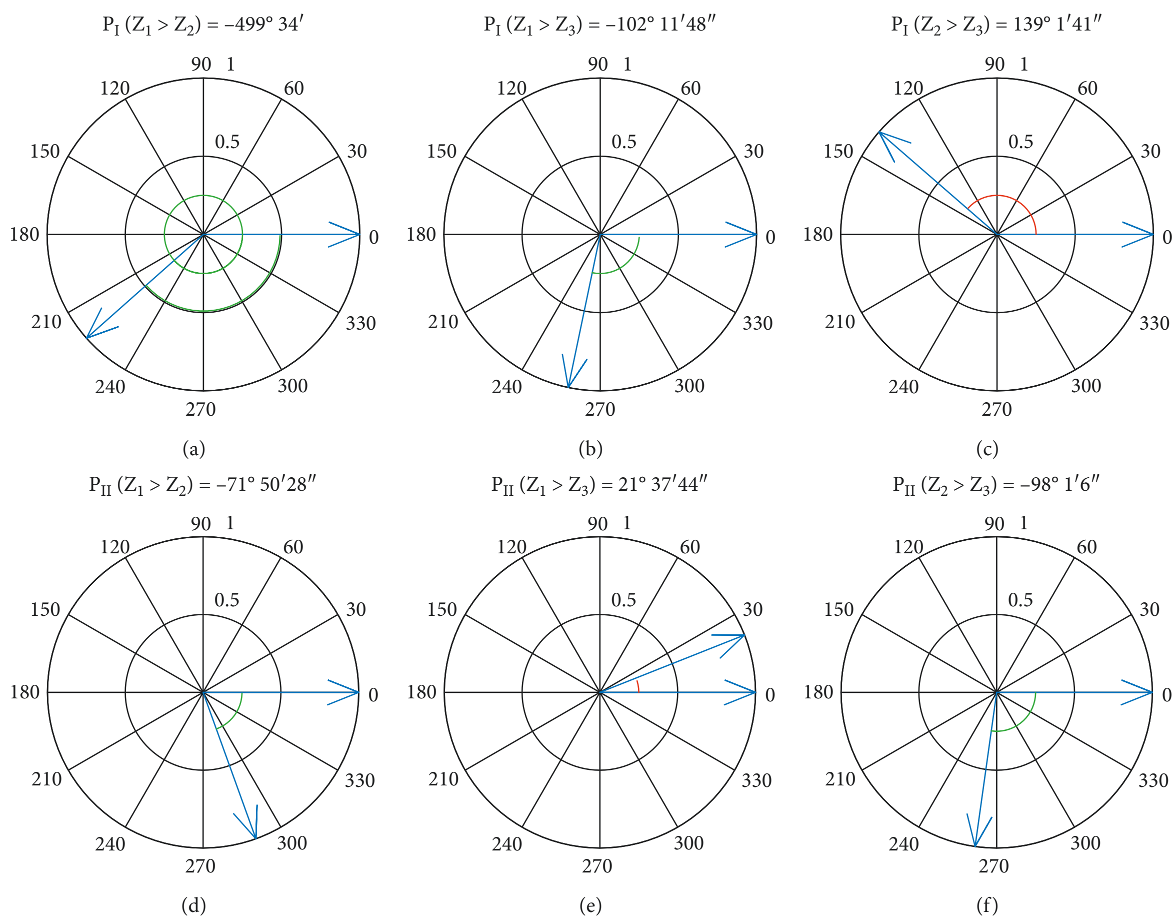

Figure 1: The geometric measurements of dominance degree for three discrete Z-numbers. (a) $P_I(Z_1 > Z_2) = -499°34'$. (b) $P_I(Z_1 > Z_3) = -102°11'48''$. (c) $P_I(Z_2 > Z_3) = 139°1'41''$. (d) $P_{II}(Z_1 > Z_2) = -71°50'28''$. (e) $P_{II}(Z_1 > Z_3) = 21°37'44''$. (f) $P_{II}(Z_2 > Z_3) = -98°1'6''$.

And then, the dominance degree is calculated by equation (16), as shown in Table 2.

In the polar coordinate system, the geometric measurement dominance degree of Z_1, Z_2, and Z_3 is shown in Figure 1. Figure 1 shows the dominance degree to which Z_j is partial to Z_k intuitively. In the figure, the green line represents the negative angle, the red line represents the positive angle, and a complete circle represents 360°. The superposition of the angles gives the final dominance degree.

Definition 8 (the algebra measurements of dominance degree for discrete Z-number). Similar to the geometric measurements of dominance degree for discrete Z-number, the algebra measurements of dominance degree for discrete Z-number are defined as follows after Z-number standardization:

$$
\begin{aligned}
P_{\mathfrak{R}}\left(Z_j \succ Z_k\right) = \operatorname{sgn} u \Bigg| & \sum_i^n \left[b'_{ji}\left(x'_{ji}\right)^{\lambda} - b'_{ki}\left(x'_{ki}\right)^{\lambda} \right] + \sum_i^n \left[\mu_{B_j}\left(b'_{ji}\right)\left(\mu_{A_j}\left(x'_{ji}\right)\right)^{\lambda} - \mu_{B_k}\left(b'_{ki}\right)\left(\mu_{A_k}\left(x'_{ki}\right)\right)^{\lambda} \right] \\
& + \sum_i^n \left[p\left(x'_{ji}\right)\left(x'_{ji}\right)^{\lambda} - p\left(x'_{ki}\right)\left(x'_{ki}\right)^{\lambda} \right] \Bigg|^{1/\lambda},
\end{aligned}
\tag{20}
$$

where $u=\sum_i^n[b'_{ji}(x'_{ji})^{\lambda}-b'_{ki}(x'_{ki})^{\lambda}]+\sum_i^n[\mu_{B_j}(b'_{ji})(\mu_{A_j}(x'_{ji}))^{\lambda}-\mu_{B_k}(b'_{ki})(\mu_{A_k}(x'_{ki}))^{\lambda}]+\sum_i^n[p(x'_{ji})(x'_{ji})^{\lambda}-p(x'_{ki})(x'_{ki})^{\lambda}]$, and $\lambda>0$; substituting λ into Z-numbers as the weight, the hidden information of Z-number is fully applied in combination with the hidden probability $p(x_i)$.

Property 3. Let $P_{\mathfrak{R}}(Z_j\succ Z_k)$ be the algebra measurements of dominance degree for Z_j over Z_k; they have the following properties:

(1) $P_{\mathfrak{R}}(Z_j\succ Z_k)=0$ if and only if $Z_j=Z_k$

(2) $-3n\le P_{\mathfrak{R}}(Z_j\succ Z_k)\le 3n$

(3) $P_{\mathfrak{R}}(Z_j\succ Z_k)=-P_{\mathfrak{R}}(Z_k\succ Z_j)$

Example 3. Use Z_1, Z_2, and Z_3 in Example 2 to calculate the dominance degree and algebra measurements Z_j and Z_k with different λ values; then, we use equation (20) to calculate the results as shown in Table 3.

Since there is a big difference in the change of algebraic measurement dominance degree when $\lambda\in[0,1]$ and $\lambda\in[1,10]$, we divide the algebraic measurement dominance degree into two cases and obtain Figures 2 and 3. When $\lambda\in[0,1]$, $P_{\mathfrak{R}}(Z_1\succ)Z_2\,\mathfrak{R}\in\{\text{I},\text{II}\}$ and $P_{\mathfrak{R}}(Z_2\succ)Z_3$ decrease with the increase of λ, while the monotonicity of $P_{\mathfrak{R}}(Z_1\succ)Z_3$ is the opposite. This is because there is a large gap between the value of p_3^{II} calculated by the probability model MD II and the value of p_3^{I} calculated by the probability model MD I, resulting in a large gap between the value of $P_{\mathfrak{R}}(Z_1\succ)Z_3$. When $\lambda\in[1,10]$, $P_{\mathfrak{R}}(Z_1\succ)Z_3$ and $P_{\mathfrak{R}}(Z_2\succ)Z_3$ decrease with the increase of λ, and the monotonicity of $P_{\mathfrak{R}}(Z_1\succ)Z_2$ basically increases monotonically.

Definition 9 (the cross-entropy measurements of dominance degree for discrete Z-number). Similar to the geometric and algebra measurements of dominance degree for discrete Z-number, the cross-entropy measurements of dominance degree for discrete Z-number are defined as follows after Z-number standardization. The cross-entropy measurement of dominance degree for discrete Z-number was defined as follows:

$$\begin{aligned}
\mathrm{CE}\left(Z_j,Z_k\right)&=\sum_i^n\left\{\left[\mu_{A_j}\left(x'_{ji}\right)\right]^{\mu_{B_j}\left(b'_{ji}\right)}\left|\ln\frac{2\left[\mu_{A_j}\left(x'_{ji}\right)\right]^{\mu_{B_j}\left(b'_{ji}\right)}}{\left[\mu_{A_j}\left(x'_{ji}\right)\right]^{\mu_{B_j}\left(b'_{ji}\right)}+\left[\mu_{A_k}\left(x'_{ki}\right)\right]^{\mu_{B_k}\left(b'_{ki}\right)}}\right|\right.\\
&\left.\quad+\left[p\left(x'_{ji}\right)\right]^{\mu_{A_j}\left(x'_{ji}\right)}\left|\ln\frac{2\left[p\left(x'_{ji}\right)\right]^{\mu_{A_j}\left(x'_{ji}\right)}}{\left[p\left(x'_{ji}\right)\right]^{\mu_{A_j}\left(x'_{ji}\right)}+\left[p\left(x'_{ki}\right)\right]^{\mu_{A_k}\left(x'_{ki}\right)}}\right|\right\},\\
\mathrm{CE}\left(Z_k,Z_j\right)&=\sum_i^n\left\{\left[\mu_{A_k}\left(x'_{ki}\right)\right]^{\mu_{B_k}\left(b'_{ki}\right)}\left|\ln\frac{2\left[\mu_{A_k}\left(x'_{ki}\right)\right]^{\mu_{B_k}\left(b'_{ki}\right)}}{\left[\mu_{A_k}\left(x'_{ki}\right)\right]^{\mu_{B_k}\left(b'_{ki}\right)}+\left[\mu_{A_j}\left(x'_{ji}\right)\right]^{\mu_{B_j}\left(b'_{ji}\right)}}\right|\right.\\
&\left.\quad+\left[p\left(x'_{ki}\right)\right]^{\mu_{A_k}\left(x'_{ki}\right)}\left|\ln\frac{2\left[p\left(x'_{ki}\right)\right]^{\mu_{A_k}\left(x'_{ki}\right)}}{\left[p\left(x'_{ki}\right)\right]^{\mu_{A_k}\left(x'_{ki}\right)}+\left[p\left(x'_{ji}\right)\right]^{\mu_{A_j}\left(x'_{ji}\right)}}\right|\right\}.
\end{aligned}\tag{21}$$

However, $\mathrm{CE}(Z_j,Z_k)$ is not symmetric, so in analogy with article [40], we proceed to the following definition of the Z-number symmetric cross-entropy measure uncertain information:

$$\mathrm{DE}_Z\left(Z_j,Z_k\right)=\mathrm{CE}\left(Z_j,Z_k\right)+\mathrm{CE}\left(Z_k,Z_j\right),\tag{22}$$

and then we define the cross-entropy measurements of dominance degree for discrete Z-number as follows:

$$P_{\mathfrak{R}}\left(Z_j\succ Z_k\right)=\operatorname{sgn}(k-j)\frac{1}{2n\ln 2}\left(\mathrm{CE}\left(Z_j,Z_k\right)+\mathrm{CE}\left(Z_k,Z_j\right)\right),\tag{23}$$

where n is the number of discrete fuzzy number A in discrete Z-number.

Property 4. Let $\mathrm{DE}_Z(Z_j,Z_k)$ be the Z-number symmetric cross entropy for Z_j and Z_k; they have the following properties:

(1) Symmetry: $\mathrm{DE}_Z(Z_j,Z_k)=\mathrm{DE}_Z(Z_k,Z_j)$

(2) Negative: $\mathrm{DE}_Z(Z_j,Z_k)\ge 0$

(3) Normative: $0\le\mathrm{DE}_Z(Z_j,Z_k)\le 2n\ln 2$

Table 3: Algebraic measurements of dominance degree of discrete Z-number under different λ values.

λ	0.1	0.5	1	1.5	2	5	10
$P_{\mathrm{I}}(Z_1 \succ Z_2)$	$8.4244 + 1.6991i$	$1.0351 + 0.1266i$	0.8717	$0.8979 - 0.0225i$	0.9070	0.8950	0.9104
$P_{\mathrm{II}}(Z_2 \succ Z_1)$	$6.8079 + 0.9580i$	$0.9627 + 0.0320i$	0.8717	$0.8685 + 0.0047i$	0.8712	0.8947	0.9103
$P_{\mathrm{I}}(Z_1 \succ Z_3)$	$3.4056e-05 - 2.5965e-06i$	$0.0511 - 0.0039i$	0.1032	−0.0313	−0.2862	−0.7746	−0.8903
$P_{\mathrm{II}}(Z_1 \succ Z_1)$	$1.8109e-05 - 3.5458e-06i$	$0.0362 - 0.0160i$	0.1032	−0.1316	−0.3600	−0.7747	−0.8904
$P_{\mathrm{I}}(Z_2 \succ Z_3)$	−0.2659	−0.6211	−0.7684	−0.8906	−0.9415	−0.9685	−0.9655
$P_{\mathrm{II}}(Z_2 \succ Z_3)$	−0.2682	−0.6222	−0.7685	−0.8917	−0.9427	−0.9685	−0.9655

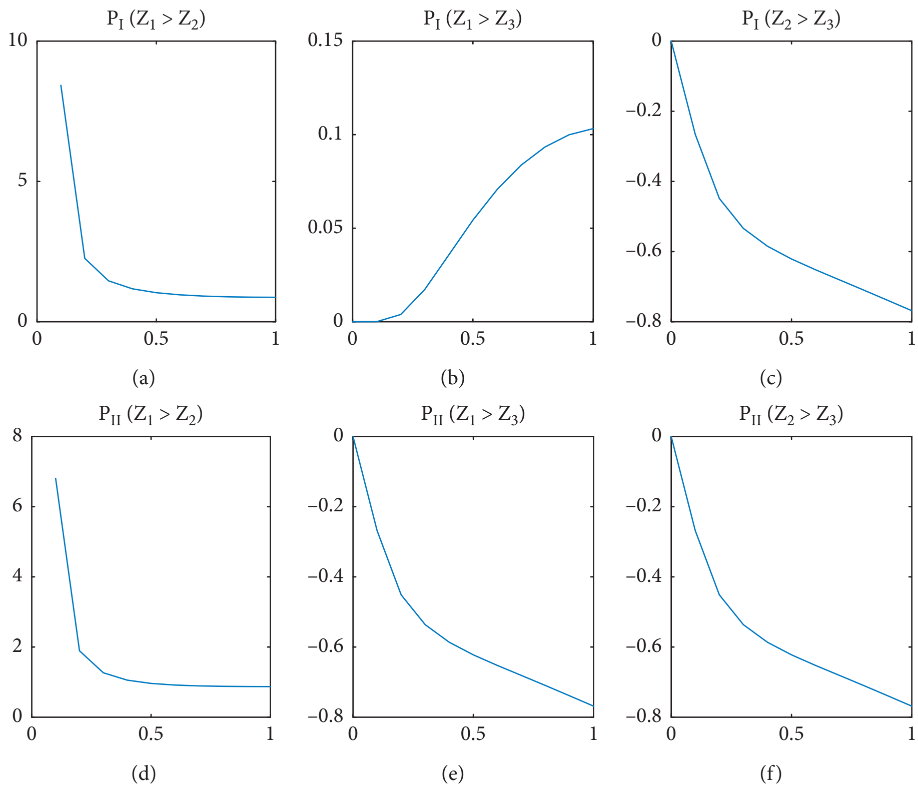

Figure 2: $P_{\mathfrak{R}}(Z_j \succ Z_k)(j,k = 1,2,3)$ change with respect to $\lambda(\lambda \in [0,1])$. (a) $P_{\mathrm{I}}(Z_1 > Z_2)$. (b) $P_{\mathrm{I}}(Z_1 > Z_3)$. (c) $P_{\mathrm{I}}(Z_2 > Z_3)$. (d) $P_{\mathrm{II}}(Z_1 > Z_2)$. (e) $P_{\mathrm{II}}(Z_1 > Z_3)$. (f) $P_{\mathrm{II}}(Z_2 > Z_3)$.

Proof

(1) The proof is trivial

(2) Because $\mu_{A_j}(x'_{ji})$, $\mu_{B_j}(b'_{ji})$, $\mu_{A_k}(x'_{ki})$, $\mu_{B_k}(b'_{ki})$, $p(x'_{ji})$, and $p(x'_{ki})$ are all members of $[0,1]$, $\mathrm{DE}_Z(Z_j, Z_k) \geq 0$ obviously

(3) $[\mu_{A_j}]^{\mu_{B_j}} \in [0,1]$ when $\mu_{A_j} \in [0,1], \mu_{B_j} \in [0,1]$, $[\mu_{A_k}]^{\mu_{B_k}} \in [0,1]$ can be obtained similarly:

$$\left| \ln \frac{2\left[\mu_{A_j}\left(x'_{ji}\right)\right]^{\mu_{B_j}\left(b'_{ji}\right)}}{\left[\mu_{A_j}\left(x'_{ji}\right)\right]^{\mu_{B_j}\left(b'_{ji}\right)} + \left[\mu_{A_k}\left(x'_{ki}\right)\right]^{\mu_{B_k}\left(b'_{ki}\right)}} \right| \in [0, \ln 2], \tag{24}$$

so

$$\left| \ln \frac{2\left[\mu_{A_j}\right]^{\mu_{B_j}}}{\left[\mu_{A_j}\right]^{\mu_{B_j}} + \left[\mu_{A_k}\right]^{\mu_{B_k}}} \right| \in [0, n\ln 2]. \tag{25}$$

Then,

$$\mu_{A_j}^{\mu_{B_j}} \left| \ln \frac{2\left[\mu_{A_j}\right]^{\mu_{B_j}}}{\left[\mu_{A_j}\right]^{\mu_{B_j}} + \left[\mu_{A_k}\right]^{\mu_{B_k}}} \right| \in [0, n\ln 2]. \tag{26}$$

And the following equation can be obtained similarly:

$$p_j^{\mu_{A_j}} \left| \ln \frac{2\left[p_j\right]^{\mu_{A_j}}}{\left[p_j\right]^{\mu_{A_j}} + \left[p_k\right]^{\mu_{A_k}}} \right| \in [0, n\ln 2]. \tag{27}$$

So, we can figure out

$$\mathrm{CE}\left(Z_j, Z_k\right), \mathrm{CE}\left(Z_k, Z_j\right) \in [0, n\ln 2]. \tag{28}$$

Therefore,

$$0 \leq \mathrm{DE}_Z = \mathrm{CE}\left(Z_j, Z_k\right) + \mathrm{CE}\left(Z_k, Z_j\right) \leq 2n\ln 2. \tag{29}$$

□

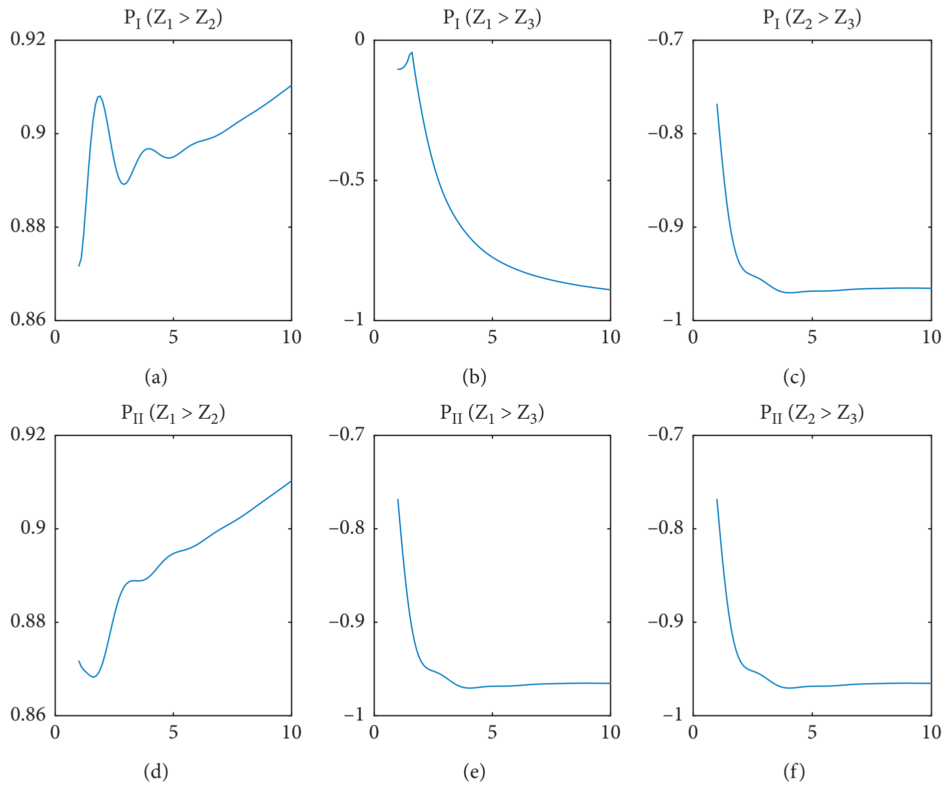

FIGURE 3: $P_{\Re}(Z_j \succ Z_k)(j,k=1,2,3)$ change with respect to $\lambda(\lambda \in [1,10])$. (a) $P_{\mathrm{I}}(Z_1 > Z_2)$. (b) $P_{\mathrm{I}}(Z_1 > Z_3)$. (c) $P_{\mathrm{I}}(Z_2 > Z_3)$. (d) $P_{\mathrm{II}}(Z_1 > Z_2)$. (e) $P_{\mathrm{II}}(Z_1 > Z_3)$. (f) $P_{\mathrm{II}}(Z_2 > Z_3)$.

Property 5. Let $P_{\Re}(Z_j \succ Z_k)$ be the cross-entropy measurements of dominance degree for Z_j over Z_k; they have the following properties:

(1) $P_{\Re}(Z_j \succ Z_k) = 0$ if and only if $Z_j = Z_k$

(2) $-1 \le P_{\Re}(Z_j \succ Z_k) \le 1$

(3) $P_{\Re}(Z_j \succ Z_k) = -P_{\Re}(Z_k \succ Z_j)$

Example 4. Use Z_1, Z_2, and Z_3 in Example 2 to calculate the cross-entropy measurements of dominance degree of Z_j and Z_k; the results are as follows:

$$\begin{aligned}
P_{\mathrm{I}}(Z_1 \succ Z_2) &= \operatorname{sgn}(2-1)\frac{1}{20\ln 2}(1.6189 + 1.7094) = 0.2401,\\
P_{\mathrm{II}}(Z_1 \succ Z_2) &= \operatorname{sgn}(2-1)\frac{1}{20\ln 2}(2.2681 + 0) = 0.1636,\\
P_{\mathrm{I}}(Z_1 \succ Z_3) &= \operatorname{sgn}(3-1)\frac{1}{20\ln 2}(0.6771 + 0.7164) = 0.1005,\\
P_{\mathrm{II}}(Z_1 \succ Z_3) &= \operatorname{sgn}(3-1)\frac{1}{20\ln 2}(1.2463 + 0.8180) = 0.1489,\\
P_{\mathrm{I}}(Z_2 \succ Z_3) &= \operatorname{sgn}(3-2)\frac{1}{20\ln 2}(1.3064 + 1.3744) = 0.1934,\\
P_{\mathrm{II}}(Z_2 \succ Z_3) &= \operatorname{sgn}(3-2)\frac{1}{20\ln 2}(1.0251 + 1.0699) = 0.1511.
\end{aligned} \tag{30}$$

The dominance degree of Z_j and Z_k is measured from three perspectives of geometry, algebra, and information entropy, respectively. Compared with the degree of superiority defined in article [26], the three measurements in this paper consider part A and part B of Z-number together. It is unreasonable to calculate part B separately in the literature. In the calculation of practical problems, it is meaningless to separately calculate the limit of A and B. For example, $Z_1 = (A_1, B_1) = (\text{good}, \text{likely})$ and $Z_2 = (A_2, B_2) = (\text{bad}, \text{likely})$, where both B_1 and B_2 are "likely," but B_1 is a restriction and constraint on the "good" of A_1, which has nothing to do with A_2, while B_2 is a restriction and constraint on the "bad" of A_2, which has nothing to do with A_1, so B cannot be calculated separately.

In order to explore the relationship between different probability models and different measurements of dominance degree, we consider it from two levels. The influence of different probability models on the dominance results of the same measurement and the relationship between the dominance results of different measurements under the same probability model are analyzed. With the help of MATLAB, the final result is shown in Table 4, and the following information can be obtained.

When MD I was selected, Z_2 was the most superior degree to Z_1, Z_3 was the least dominance degree to Z_1, and Z_2 was the middle superior to Z_3. When MD II is selected, the degree of superiority of Z_3 over Z_2 is the largest, and the degree of superiority of Z_2 over Z_1 is the middle. Obviously, the value of the degree of superiority of geometric forms

Table 4: Different probability models and different measurements of dominance degree.

Dominance degree	Geometric	Algebra ($\lambda = 1$)	Cross entropy
$P_{\mathrm{I}}(Z_1 \succ Z_2)$	$-499°34'$	0.8717	0.2401
$P_{\mathrm{II}}(Z_2 \succ Z_1)$	$-71°50'28''$	0.8717	0.1636
$P_{\mathrm{I}}(Z_1 \succ Z_3)$	$-102°11'48''$	0.1032	0.1005
$P_{\mathrm{II}}(Z_1 \succ Z_1)$	$21°37'44''$	0.1032	0.1489
$P_{\mathrm{I}}(Z_2 \succ Z_3)$	$139°1'41''$	−0.7684	0.1934
$P_{\mathrm{II}}(Z_2 \succ Z_3)$	$-98°1'6''$	−0.7685	0.1511

under different models has obvious changes, and the influence of model selection on its results is obvious and has a great influence on it.

The algebra measurements of dominance degree contain parameter λ. We first analyze the influence of different probability models on the degree of dominance when $\lambda = 1$ is fixed. According to the results in Table 3, the selection of probability models has little influence on the results. From Example 3, it can be seen that when λ selects a sufficiently large value, the change of the probability model does not affect the degree of superiority of the algebraic form. From this, we can conclude that the algebraic form of the dominance degree is the most stable one, not affected by the probability of the model changes.

In the calculation of MD I and MD II, the dominance degree of the form of information entropy is the highest in Z_1 over Z_2, while Z_2 over Z_3 and Z_1 over Z_3 are the second, respectively. Therefore, it can be seen that the influence of the change of probability model on the dominance degree of the form of information entropy is less than that of the geometric form, which has little influence on the algebra measurements of dominance degree, and the information entropy is relatively stable.

Under the hidden probability calculated by the MD I model, the dominance degree of the geometric form was compared, the numerical values of the dominance degree were all larger, the numerical values of the algebraic form were all smaller than the geometric form when $\lambda = 1$, and the dominance degree of the information entropy form was the smallest. Because the three forms are considered from different levels, including angle and distance, there is no direct connection between the values of different forms, and they only reflect the degree of preference between two Z-numbers.

Under the hidden probability p calculated by the MD II model, the geometric superiority values are less than those calculated by the MD I model because the hidden probability of MD II is a part of the MD I model, which reduces the degree that Z_j is better than Z_k, and the MD I response is a comprehensive hidden probability, so it is more real and effective. The algebraic superiority is almost unchanged when $\lambda = 1$. The superiority of cross entropy is very unstable and varies greatly. The information entropy reflects the uncertain information of data, so the change is greatly influenced by uncertain factors.

5. Decision Model Based on Dominance Degree of Z-Number

In this section, we mainly introduce the grey correlation and QUALIFLEX method model based on Z-number. The unknown attribute weight is calculated by grey relational analysis, and the ranking of the optimal scheme is calculated by the QUALIFLEX method.

5.1. Grey Relation Analysis to Determine Attribute Weight. In the general multiattribute decision making, the attribute weight of each scheme is unknown, and many multiattribute decision models will calculate the attribute weight without the given attribute weight artificially. There will be too much subjectivity in this method, and the attribute weight given may not be accurate, so there is a large degree of uncertainty.

The degree of correlation is a measure of the degree of correlation between the factors of two systems, which vary from time to time or from different objects. In the process of system development, if the trend of the two factors is consistent, that is, the degree of synchronous change is high, in other words, the degree of correlation between the two factors is high. On the contrary, the degree of correlation between the two factors is lower. The grey correlation analysis method, therefore, is based on the development trend of the degree of similarity or dissimilarity between factors, namely, "grey correlation", as a measure of the degree of correlation between a way in order to better reflect the various properties of each scheme, an improved method to calculate the unknown attribute weight is distrcued based on the grey relation analysis method. The change of the time is ignored, and we only consider the same time of the relationship between the properties of the combined with dominance degree of Z-number to calculate the location of the attribute weights.

There are n alternatives under m criteria; denote $A = \{a_1, \ldots, a_i, \ldots, a_n\}$ and $C = \{c_1, \ldots, c_j, \ldots, c_m\}$ as a decision matrix $(Z_{ij})_{n\times m}$, using Z_{ij} to represent the decision value of the i^{th} scheme and the j^{th} attribute. Formula (16) is used to calculate the dominance degree of Z-number of each group of schemes under the same attribute, which is represented by $P_{\mathfrak{R}}(Z_{ij} \succ Z_{kj})\,(\mathfrak{R} \in \{\mathrm{I}, \mathrm{II}\})$.

$$\begin{bmatrix} P_{\mathfrak{R}}(Z_{11} \succ Z_{11}) & P_{\mathfrak{R}}(Z_{21} \succ Z_{11}) & \cdots & P_{\mathfrak{R}}(Z_{m1} \succ Z_{11}) \\ P_{\mathfrak{R}}(Z_{11} \succ Z_{21}) & P_{\mathfrak{R}}(Z_{21} \succ Z_{21}) & \cdots & P_{\mathfrak{R}}(Z_{m1} \succ Z_{21}) \\ \vdots & \vdots & \vdots & \vdots \\ P_{\mathfrak{R}}(Z_{11} \succ Z_{n1}) & P_{\mathfrak{R}}(Z_{21} \succ Z_{n1}) & \cdots & P_{\mathfrak{R}}(Z_{m1} \succ Z_{n1}) \end{bmatrix}. \tag{31}$$

Establish the following model to solve attribute weight.

Step 1. Calculate the dominance degree of Z-number for each group of schemes under the same attribute $P_{\mathfrak{R}}(Z_{ij} \succ Z_{kj})$. For the convenience of later calculation, $P_{\mathfrak{R}j} = (P_{\mathfrak{R}}(Z_{ij} \succ Z_{kj}))_{n\times m}$ is denoted as the scheme combination under attribute j which is superior to the degree matrix.

Step 2. In the scheme combination superiority matrix of attribute j, select the maximum value as its positive ideal solution, denoted as $p^{+}_{\mathfrak{R}j}$. The dominance degree decision matrices are obtained as $P_{\mathfrak{R}j} = [P_{\mathfrak{R}}(Z_{ij} \succ Z_{kj})]_{n\times m}$.

Step 3. Calculate the correlation coefficient $\xi^{+}_{\mathfrak{R}}(Z_{ij}, Z_{kj})$ of the i^{th} and k^{th} schemes with positive ideal $p^{+}_{\mathfrak{R}j}$ under the attribute c_j, where $i, k = 1, 2, \ldots, n$.

$$\xi^{+}_{\mathfrak{R}}\left(Z_{ij}, Z_{kj}\right) = \frac{\min_i \min_k\left(P_{\mathfrak{R}}\left(Z_{ij}, Z_{kj}\right) - p^{+}_{\mathfrak{R}j}\right) + \rho \max_i \max_k\left(P_{\mathfrak{R}}\left(Z_{ij}, Z_{kj}\right) - p^{+}_{\mathfrak{R}j}\right)}{\left(P_{\mathfrak{R}}\left(Z_{ij}, Z_{kj}\right) - p^{+}_{\mathfrak{R}j}\right) + \rho \max_i \max_k\left(P_{\mathfrak{R}}\left(Z_{ij}, Z_{kj}\right) - p^{+}_{\mathfrak{R}j}\right)}. \tag{32}$$

Usually $\rho = 0.5$.

Step 4. Calculate attribute weight $\omega_{\mathfrak{R}j}$:

$$\omega_{\mathfrak{R}j} = \frac{\sum_i^n \sum_k^n \xi^{+}_{\mathfrak{R}}\left(Z_{ij}, Z_{kj}\right)}{\sum_i^n \sum_i^n \sum_i^m \xi^{+}_{\mathfrak{R}}\left(Z_{ij}, Z_{kj}\right)}, \tag{33}$$

where $\omega_{\mathfrak{R}j}$ satisfies the following conditions:

$$\begin{aligned} &\sum_j^n \omega_{\mathfrak{R}j} = 1,\\ &\omega_{\mathfrak{R}j} \geq 0. \end{aligned} \tag{34}$$

5.2. Decision Model of QUALIFLEX Method. The QUALIFLEX method is to assume that there are n choices, and then there are $n!$ possible sorting results, for example, there are three alternatives $A = \{a_1, a_2, a_3\}$, and we can get six choices: $G_1 = (a_1, a_2, a_3)$, $G_2 = (a_1, a_3, a_2)$, $G_3 = (a_2, a_1, a_3)$, $G_4 = (a_2, a_3, a_1)$, $G_5 = (a_3, a_1, a_2)$, and $G_6 = (a_3, a_2, a_1)$; there are $6 = 3!$ sorts.

The consistency/inconsistency index is defined based on the dominance degree of Z_j superior to Z_k, and the comprehensive consistency index/inconsistency index of each sort is calculated by combining the weight, in which the order corresponding to the maximum value is the optimal sorting scheme. Next, we will specify the specific steps of the multiattribute decision model based on the QUALIFLEX method:

Step 1. Normalize the Z-numbers in the decision matrix and obtain the standardized decision matrix $E' = (Z'_{ij})_{n\times m}$.

Step 2. Calculate the dominance degree of $Z'-$numbers; $P_{\mathfrak{R}}(Z'_{ik} \succ Z'_{kj})$ $\mathfrak{R} \in \{\text{I}, \text{II}\}$ represents the dominance degree of the i^{th} scheme and the k^{th} scheme under the j attribute with the $\mathfrak{R}$ probability model.

Step 3. Sort the options according to the given solutions and write out all possible sorting results: $G_l = (\ldots, a_\alpha, a_\beta, \ldots)$, $l = 1, 2, \ldots, n!$ where $a_\alpha, a_\beta \in A$ and satisfies a_α superior to a_β.

Step 4. Calculate the concordance/discordance index of schemes a_α and a_β under attribute j in the result of i possible sort $I^{l}_{\mathfrak{R}j}(a_\alpha, a_\beta)$:

$$I^{l}_{\mathfrak{R}j}\left(a_\alpha, a_\beta\right) = P_{\mathfrak{R}j}\left(a_\alpha, a_\beta\right) - \overline{P_{\mathfrak{R}j}}, \tag{35}$$

where $\overline{P_{\mathfrak{R}j}} = \max_i \max_k (P_{\mathfrak{R}}(Z_{ij} \succ Z_{kj}))$,

$$I^{l}_{\mathfrak{R}j}\left(a_\alpha, a_\beta\right) = \begin{cases} P_{\mathfrak{R}j}\left(a_\alpha, a_\beta\right) - \overline{P_{\mathfrak{R}j}} > 0, & \text{concordance exists},\\ P_{\mathfrak{R}j}\left(a_\alpha, a_\beta\right) - \overline{P_{\mathfrak{R}j}} = 0, & \text{ex aequo exists},\\ P_{\mathfrak{R}j}\left(a_\alpha, a_\beta\right) - \overline{P_{\mathfrak{R}j}} < 0, & \text{discordance exists}. \end{cases} \tag{36}$$

Step 5. The attribute weight is determined by grey relational analysis:

$$W_{\mathfrak{R}} = \left[w_{\mathfrak{R}1}, w_{\mathfrak{R}2}, \ldots, w_{\mathfrak{R}m}\right]. \tag{37}$$

Step 6. Calculate the weighting concordance/discordance index $I^{l}_{\mathfrak{R}}(a_\alpha, a_\beta)$:

$$I^{l}_{\mathfrak{R}}\left(a_\alpha, a_\beta\right) = \sum_j^m I^{l}_{\mathfrak{R}j}\left(a_\alpha, a_\beta\right) \cdot w_{\mathfrak{R}j}. \tag{38}$$

Step 7. Calculate the comprehensive concordance/discordance index $I^{l}_{\mathfrak{R}}$:

$$I^{l}_{\mathfrak{R}} = \sum_{a_\alpha, \ldots, a_\beta \in A} I^{l}_{\mathfrak{R}j}\left(a_\alpha, a_\beta\right) \cdot w_{\mathfrak{R}j}. \tag{39}$$

Step 8. Sort the comprehensive concordance/discordance index calculated in Step 5, and the maximum value corresponds to the optimal sorting scheme.

6. Case Study: Multiple Attribute Decision Based on Z-Number Dominance Degree

In this section, a mall service satisfaction assessment as an example to illustrate the feasibility and effectiveness of the proposed method is given. Because customer service satisfaction and store staff service awareness are the characteristics of psychological that Z-number can express better. State mall service satisfaction directly affects the store customer traffic and sales performance of goods until the store service customers feel good and return to the store to shopping again. And to a certain extent can recommend other customers to come to buy, which to a great extent bring benefits to the mall.

In this case, the service satisfaction evaluation is very important. However, there are many factors about mall service satisfaction, such as the words of the staff behavior ability, the quality of goods, and the mood of the shopper. These factors contain a lot of uncertain and inaccurate information. It makes quantitative characterization more suitable for evaluating qualitative tools than shopping service satisfaction. In this background, we use the multiple attribute decision making based on Z-number advantage degree model to solve the problem of market service satisfaction assessment.

In order to decide market efficiency, a company to evaluate four stores service satisfaction and four stores are expressed as (alternatives) $a_i\,(i = 1, 2, 3, 4)$. There are many factors that can influence,which is expressed as c_1, behavior c_2, selection c_3, and promotion c_4 as the standards (criterion) $c_j\,(j = 1, 2, 3, 4)$. The weights of the four criteria are completely unknown. First of all, experts went to four shopping malls to experience the service of shopping malls with their identities hidden and then evaluated the evaluation information of the four shopping malls after some discussion, as shown in Table 5 (the evaluation criteria are linguistic scale ambiguities in Tables 6 and 7). $Z_{11} = (A_{11}, B_{11}) = (s_5, s_4') = (\text{Good}, \text{Somewhat Certain})$ represents the evaluation of the first alternative a_1 in the first criterion c_1.

Table 5: Evaluation information on job satisfaction.

	c_1	c_2	c_3	c_4
a_1	(s_5, s_4')	(s_5, s_5')	(s_5, s_3')	(s_6, s_6')
a_2	(s_3, s_5')	(s_6, s_5')	(s_3, s_5')	(s_4, s_5')
a_3	(s_5, s_6')	(s_4, s_3')	(s_4, s_2')	(s_3, s_4')
a_4	(s_6, s_5')	(s_4, s_6')	(s_6, s_3')	(s_5, s_6')

$$A_{11} = \frac{0}{5} + \frac{0.25}{5.25} + \frac{0.5}{5.5} + \frac{0.75}{5.75} + \frac{1}{6} + \frac{0.75}{6.25} + \frac{0.5}{6.5} + \frac{0.25}{6.75} + \frac{0}{7},$$

$$B_{11} = \frac{0}{0.48} + \frac{0.3}{0.51} + \frac{0.5}{0.54} + \frac{0.8}{0.57} + \frac{1}{0.6} + \frac{0.8}{0.63} + \frac{0.5}{0.66} + \frac{0.3}{0.69} + \frac{0}{0.72}. \tag{40}$$

Step 1. Normalize the Z-numbers in the decision matrix and get the normalized matrix $E' = (Z'_{ij})_{4\times 4}$ $(i, j = 1, 2, 3, 4)$.

Step 2. Calculate the dominance degree decision matrices to obtain $P_{\mathfrak{R}j} = [P_{\mathfrak{R}}(Z'_{ij} \succ Z'_{kj})]_{n\times m}$. The following four matrices are the first geometric priority matrix that measures Z-number priority degree in this paper:

$$\begin{aligned}
P_{I1} &= \begin{bmatrix} 0^\circ & -942^\circ 32' 18'' & 46^\circ 51'' & 1087^\circ 55'' \\ 942^\circ 32' 18'' & 0^\circ & 735^\circ 59' 13'' & 775^\circ 20' 57'' \\ -46^\circ 51'' & -735^\circ 59' 13'' & 0^\circ & 933^\circ 6' 48'' \\ -1087^\circ 55'' & -775^\circ 20' 57'' & -933^\circ 6' 48'' & 0^\circ \end{bmatrix}, \\
P_{I2} &= \begin{bmatrix} 0^\circ & 981^\circ 12' 26'' & -898^\circ 51' 22'' & -932^\circ 55' 47'' \\ -981^\circ 12' 26'' & 0^\circ & -1065^\circ 50' 41'' & -741^\circ 36' 49'' \\ 898^\circ 51' 22'' & 1065^\circ 50' 41'' & 0^\circ & 58^\circ 59' 9'' \\ 932^\circ 55' 47'' & 741^\circ 36' 49'' & -58^\circ 59' 9'' & 0^\circ \end{bmatrix}, \\
P_{I3} &= \begin{bmatrix} 0^\circ & -467^\circ 56' 45'' & 290^\circ 29' 22'' & 539^\circ 6' 47'' \\ 467^\circ 56' 45'' & 0^\circ & 404^\circ 10' 53'' & 397^\circ 35' 53'' \\ -290^\circ 29' 22'' & -404^\circ 10' 53'' & 0^\circ & -842^\circ 20' 22'' \\ -539^\circ 6' 47'' & -397^\circ 35' 53'' & 842^\circ 20' 22'' & 0^\circ \end{bmatrix}, \\
P_{I4} &= \begin{bmatrix} 0^\circ & -715^\circ 39' 10'' & -799^\circ 56' 26'' & -905^\circ 35'' \\ 715^\circ 39' 10'' & 0^\circ & -1097^\circ 56' 51'' & 951^\circ 50' 15'' \\ 799^\circ 56' 26'' & 1097^\circ 56' 51'' & 0^\circ & 808^\circ 55' 8'' \\ 905^\circ 35'' & -951^\circ 50' 15'' & -808^\circ 55' 8'' & 0 \end{bmatrix}.
\end{aligned} \tag{41}$$

Step 3. In the case of multiattribute decision, there are four alternative schemes. We can list all the possible scenarios as follows:

Table 6: Linguistic terms in S and corresponding discrete fuzzy numbers.

Linguistic terms in S	Discrete fuzzy numbers
Very bad	0/0 + 0.25/0.25 + 0.5/0.5 + 0.75/0.75 + 1/1 + 0.75/1.25 + 0.5/1.5 + 0.25/1.75 + 0/2
Bad	0/1 + 0.25/1.25 + 0.5/1.5 + 0.75/1.75 + 1/2 + 0.75/2.25 + 0.5/2.5 + 0.25/2.75 + 0/3
Slightly bad	0/2 + 0.25/2.25 + 0.5/2.5 + 0.75/2.75 + 1/3 + 0.75/3.25 + 0.5/3.5 + 0.25/3.75 + 0/4
Middle	0/3 + 0.25/3.25 + 0.5/3.5 + 0.75/3.75 + 1/4 + 0.75/4.25 + 0.5/4.5 + 0.25/4.75 + 0/5
Slightly good	0/4 + 0.25/4.25 + 0.5/4.5 + 0.75/4.75 + 1/5 + 0.75/5.25 + 0.5/5.5 + 0.25/5.75 + 0/6
Good	0/5 + 0.25/5.25 + 0.5/5.5 + 0.75/5.75 + 1/6 + 0.75/6.25 + 0.5/6.5 + 0.25/6.75 + 0/7
Very good	0/6 + 0.25/6.25 + 0.5/6.5 + 0.75/6.75 + 1/7 + 0.75/7.25 + 0.5/7.5 + 0.25/7.75 + 0/8

Table 7: Linguistic terms in S' and corresponding discrete fuzzy numbers.

Linguistic terms in S'	Discrete fuzzy numbers
Strongly uncertain	0/0 + 0.3/0.03 + 0.5/0.06 + 0.8/0.09 + 1/0.12 + 0.8/0.15 + 0.5/0.18 + 0.3/0.21 + 0/0.24
Uncertain	0/0.12 + 0.3/0.15 + 0.5/0.18 + 0.8/0.21 + 1/0.24 + 0.8/0.27 + 0.5/0.3 + 0.3/0.33 + 0/0.36
Somewhat uncertain	0/0.24 + 0.3/0.27 + 0.5/0.3 + 0.8/0.33 + 1/0.36 + 0.8/0.39 + 0.5/0.42 + 0.3/0.45 + 0/0.48
Neutral	0/0.36 + 0.3/0.39 + 0.5/0.42 + 0.8/0.45 + 1/0.48 + 0.8/0.51 + 0.5/0.54 + 0.3/0.57 + 0/0.6
Somewhat certain	0/0.48 + 0.3/0.51 + 0.5/0.54 + 0.8/0.57 + 1/0.6 + 0.8/0.63 + 0.5/0.66 + 0.3/0.69 + 0/0.72
Certain	0/0.6 + 0.3/0.63 + 0.5/0.66 + 0.8/0.69 + 1/0.72 + 0.8/0.75 + 0.5/0.78 + 0.3/0.81 + 0/0.84
Strongly certain	0/0.72 + 0.3/0.75 + 0.5/0.78 + 0.8/0.81 + 1/0.84 + 0.8/0.87 + 0.5/0.9 + 0.3/0.93 + 0/0.96

$$
\begin{aligned}
G_1 &= (a_1, a_2, a_3, a_4),\\
G_2 &= (a_1, a_2, a_4, a_3),\\
G_3 &= (a_1, a_3, a_2, a_4),\\
G_4 &= (a_1, a_3, a_4, a_2),\\
G_5 &= (a_1, a_4, a_2, a_3),\\
G_6 &= (a_1, a_4, a_3, a_2),\\
G_7 &= (a_2, a_1, a_3, a_4),\\
G_8 &= (a_2, a_1, a_4, a_3),\\
G_9 &= (a_2, a_3, a_1, a_4),\\
G_{10} &= (a_2, a_3, a_4, a_1),\\
G_{11} &= (a_2, a_4, a_1, a_3),\\
G_{12} &= (a_2, a_4, a_3, a_1),\\
G_{13} &= (a_3, a_1, a_2, a_4),\\
G_{14} &= (a_3, a_1, a_4, a_2),\\
G_{15} &= (a_3, a_2, a_1, a_4),\\
G_{16} &= (a_3, a_2, a_4, a_1),\\
G_{17} &= (a_3, a_4, a_1, a_2),\\
G_{18} &= (a_3, a_4, a_2, a_1),\\
G_{19} &= (a_4, a_1, a_2, a_3),\\
G_{20} &= (a_4, a_1, a_3, a_2),\\
G_{21} &= (a_4, a_2, a_1, a_3),\\
G_{22} &= (a_4, a_2, a_3, a_1),\\
G_{23} &= (a_4, a_3, a_1, a_2),\\
G_{24} &= (a_4, a_3, a_2, a_1).
\end{aligned}
\tag{42}
$$

Step 4. Calculate the concordance/discordance index $I^l_{\mathfrak{R}j}(a_\alpha, a_\beta)$. The concordance/discordance index result of the first order is as follows (take the first attribute corresponding to schemes a_1 and a_2 in G_1 combination as an example):

$$
\begin{aligned}
I^1_{\mathrm{I}1}(a_1, a_2) &= P_{\mathrm{I}1}(Z_{11}, Z_{21}) - \overline{P_{\mathrm{I}1}}\\
&= 942^\circ 32' 18'' - 1087^\circ 55''\\
&= -144^\circ 28' 37''.
\end{aligned}
\tag{43}
$$

Step 5. Calculate the attribute weight by the grey relation analysis model (Table 8).

Step 6. Calculate weighting concordance/discordance index $I^l(a_\alpha, a_\beta)$.

Step 7. Calculate comprehensive concordance/discordance index I^l_i; the results are as follows:

$$
I^l_i = \begin{bmatrix}
-6009^\circ 1' 56'' & -6140^\circ 36' 23'' & -8133^\circ 42' 31'' & -5557^\circ 58' 58''\\
-5272^\circ 21' 27'' & -5406^\circ 30' 15'' & -7769^\circ 21' 33'' & -6604^\circ 23' 34''\\
-7055^\circ 26' 32'' & -7218^\circ 52' 1'' & -8265^\circ 16' 37'' & -5689^\circ 33' 24''\\
-6691^\circ 5' 34'' & -7868^\circ 50' 51'' & -8915^\circ 15' 27'' & -6767^\circ 49' 23''\\
-4908^\circ 28'' & -6053^\circ 54' 23'' & -8419^\circ 20' 2'' & -7682^\circ 39' 33''\\
-5954^\circ 24' 44'' & -7132^\circ 10' 22'' & -8550^\circ 54' 29'' & -7814^\circ 13' 39''
\end{bmatrix}.
\tag{44}
$$

Step 8. The sorting of the maximum value corresponding to the comprehensive consistency index in Step 7 is determined as the optimal sorting scheme. In this example, the maximum value is $-4908^\circ 28''$, and the corresponding optimal sorting scheme is $G_5 = (a_1, a_4, a_2, a_3)$. We can get the scheme selection sort as $a_1 \succ a_4 \succ a_2 \succ a_3$.

In Step 1 to Step 8, we used the hidden probability calculated by probability model MD II to calculate the dominance degree and obtain the attribute weight W_{II} and the comprehensive concordance/discordance index I^l_{II}.

Table 8: Attribute weight.

j	1	2	3	4
$\sum_i^n \sum_k^n \xi_i^+ (Z_{ij}, Z_{kj})$	21.4025	38.3146	9.9166	26.7168
w_{ij}	0.2221	0.3977	0.1029	0.2773

$$\begin{aligned} W_{\text{II}} &= [w_{\text{II}1}, w_{\text{II}2}, w_{\text{II}3}, w_{\text{II}4}] \\ &= [0.3561, 0.3205, 0.0906, 0.2328], \\ I_{\text{II}}^{l} &= \begin{bmatrix} -6307^\circ 9' 13'' & -6830^\circ 52' 17'' & -7902^\circ 35' 41'' & -5146^\circ 20' 49'' \\ -5481^\circ 59' 49'' & -6005^\circ 43' 14'' & -7346^\circ 23' 45'' & -5811^\circ 41' 18'' \\ -6972^\circ 29' 22'' & -7760^\circ 58' 36'' & -8426^\circ 19' 6'' & -5670^\circ 4' 13'' \\ -6416^\circ 17' 26'' & -7981^\circ 31' 52'' & -8646^\circ 52' 22'' & -6600^\circ 10' 33'' \\ -4925^\circ 47' 33'' & -6226^\circ 16' 9'' & -7566^\circ 56' 41'' & -6710^\circ 16' 52'' \\ -5591^\circ 8' 2'' & -7156^\circ 22' 28'' & -8090^\circ 40' 5'' & -7265^\circ 30' 41'' \end{bmatrix}. \end{aligned} \tag{45}$$

The maximum value is $-4925^\circ 47' 33''$, and the corresponding optimal sorting scheme is $G_5 = (a_1, a_4, a_2, a_3)$. We can get the scheme selection sort as $a_1 \succ a_4 \succ a_2 \succ a_3$. The result of probability model MD II is the same as that of model MD I. Then, we calculate the optimal ranking scheme under two probability models and three superior measures, respectively, and the results are shown in Table 9.

7. Sensitivity Analysis

In order to discuss the influence of probability model and the method of dominance degree measure on the optimal ranking, we considered different combinations of probability models (MD I and MD II) and different dominance degree measures (geometric measure, algebraic measure, and cross-entropy measure), and the final results are shown in Table 9 with the help of MATLAB calculation tool. In this paper, we consider the optimal ordering of two probability models and three superior degree measures and three cases in which the parameter $\lambda = 0.5, 1, 2$ of the dominance degree of the algebraic measure. By analyzing the optimal sorting results in Table 9, the following information can be obtained:

(1) The selection of probability model MD I or MD II has no influence on the optimal sorting scheme. Although the hidden probability calculated by MD II is part of the hidden probability calculated by MD I, from the decision result, the hidden probability of MD II is enough to replace the hidden probability of MD I, so the probability model in this paper has no influence on the optimal sequencing result. When making decisions, we choose the probability model MD II over the probability model MD I, and the calculation of MD II is simpler and easier than that of MD I.

(2) The order of the optimal sorting scheme does not change when the parameter $\lambda = 0.5$ and 1, which is the same as that of the geometric metric superior degree. When the parameter $\lambda = 2$, the optimal ordering changes.

(3) The results of the optimal ranking scheme with the superiority of geometric measures and the algebraic measures with the parameter $\lambda = 0.5$ and 1 are consistent, and the optimal schemes are all a_1.

(4) The optimal ranking result of the superior degree of cross-entropy measure is different from the ranking result of the superior degree of other measures. It is $a_1 \succ a_2 \succ a_3 \succ a_4$, and the best scheme is still a_1. This is because both geometric and algebraic measures are linear, and the degree of cross entropy is better than nonlinear, which can reflect more uncertain information.

8. Comparison and Summary

The ranking results of the existing five methods are showed in Table 10. The fourth ranking result [39] is the same as this paper and the other results are inconsistent with the results of this paper [12, 41]. The two components of the Z-number are converted into triangular blur sands and trapezoidal blurs, respectively, because there will be some loss of information during the conversion of Z-number, thus destroying the uncertain structure of Z-number. Z-number must use the membership and probability distribution function to reflect the twin peak information to express the probability characteristics and probability information. The disadvantage is that it is not possible to fully express the indeterminate information contained in Z-number.

However, it is ignored that A and B elements in Z-number cannot be calculated separately. Summarized the above analysis, the impacts of the existence of the hidden probability P of Z-numbers on uncertain information are considered, which are in line with Zadeh's definition of Z-number to a large extent, and from a geometric point of view, the algebraic angle and the cross-entropy angle consider the superiority between the two Z-numbers; the

Table 9: Ranking of schemes with different measures of dominance degree.

Method	MD I	MD II
Geometric	$a_1 \succ a_4 \succ a_2 \succ a_3$	$a_1 \succ a_4 \succ a_2 \succ a_3$
Algebra ($\lambda = 0.5$)	$a_1 \succ a_4 \succ a_2 \succ a_3$	$a_1 \succ a_4 \succ a_2 \succ a_3$
Algebra ($\lambda = 1$)	$a_1 \succ a_4 \succ a_2 \succ a_3$	$a_1 \succ a_4 \succ a_2 \succ a_3$
Algebra ($\lambda = 2$)	$a_2 \succ a_4 \succ a_1 \succ a_3$	$a_2 \succ a_4 \succ a_1 \succ a_3$
Cross entropy	$a_1 \succ a_2 \succ a_3 \succ a_4$	$a_1 \succ a_2 \succ a_3 \succ a_4$

Table 10: Optimal scheme sorting results for different methods.

Method	Ranking
Improved TOPSIS approach [12]	$a_4 \succ a_3 \succ a_1 \succ a_2$
Extended VIKOR method [13]	$a_1 \succ a_4 \succ a_2 \succ a_3$
Extended TODIM approach [11]	$a_4 \succ a_2 \succ a_3 \succ a_1$
Uncertain environment Z-number decision method [41]	$a_1 \succ a_4 \succ a_2 \succ a_3$
Excellent relationship ELECTRE III method [26]	$a_4 \succ a_1 \succ a_2 \succ a_3$

uncertainty and randomness of the Z-number also ignore the limitation of element A by element B. Combined with the QUALIFLEX method, the scheme is sorted, and then its comprehensive consistency index is calculated to get the optimal scheme order, which makes the method proposed in this paper more applicable than the previous method.

This paper mainly studies the advantages of discrete Z-number, taking into account the advantages of Z-number itself to express uncertain information, establishes two hidden probability models to solve Z-number, and defines the superiority between the two Z-numbers from three mathematical angles: geometry, algebra, and cross entropy. Many studies of Z-number cannot be directly compared. In the case study, this paper uses the problem of service satisfaction in shopping malls as a case study and analyzes the validity and feasibility of the proposed method with the existing method. Sensitivity analysis fully explains the shortcomings of this paper.

The main significance of this article is as follows. First, this paper put forward a two-solution hidden probability model, Z^+-number can be obtained by Z-number. However, Z^+-number obtained by different probability models is not uniform. In this regard to the subsequent calculation will bring a certain small error, but the error produced here is within the permissible range, and it does not affect the overall calculation results. Secondly, this paper perfectly combines the characteristics of Z-number and outranking relationship to define the superiority of the three forms. It is a great innovation to demonstrate the superiority between two discrete Z-numbers in many ways. Positive superiority represents the former is better than the latter, the latter is better than the former, the positive and negative sign expresses a priority relationship, and the number is the difference between the two Z-numbers. Finally, the decision model of this paper uses the improved grey association model to solve the attribute weight, using the QUALIFLEX method to sort the selection, which is an innovative idea.

In addition, the research method of this paper promotes the management application of Z-number in human cognition to a great extent.

9. Conclusion

Zadeh introduced the Z-number in 2011, and it became a powerful tool for describing human knowledge. In this paper, we have established two probability models to solve the hidden probability of Z-numbers and redefine the outranking relationship. In general, the new ranking relationship is more detailed. Then, we presented the dominance degree of different discrete Z-numbers of geometry, algebra, and cross entropy according to the ranking relationship and the hidden probability. This is useful for determining the outranking relationship of Z-numbers and simplifying applications based on fuzzy decision making. The dominance of the Z-number is calculated according to the format of the Z-number, and there is no subjective judgment. The proposed dominance of the Z-number is a general framework for dealing with discrete Z-numbers. Using the method proposed in this paper, three measurements of dominance degree of two discrete Z-numbers are obtained. The two probabilistic models and the different measurements of dominance degree were compared and analyzed in detail, and the sensitivity analysis was carried out through the case study in this paper to better illustrate the advantages of the dominance degree. The results of the proposed method were compared with the previous work results to verify the effectiveness of the dominance degree of the Z-number.

On the other hand, there exist shortcomings in these different measurements of dominance degree as well. The disadvantages of cross-entropy measurements are nonlinear, and the dominance degree of change is large and not very stable, but this is enough to explain this instability. The same is true for cross entropy. Finally, we built a model of the multiattribute decision making in uncertain environments with dominance degree of Z-number and the mall service satisfaction as the practical application of the dominance degree of Z-number. Based on the application results in the case, the proposed method is considered to help ranking the mall with the best service. The results of this method are meaningful for choosing the mall for shopping.

In future work, the authors will extend the hidden probability models and the dominance degree of continuous Z-numbers.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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