Hindawi
Mathematical Problems in Engineering
Volume 2019, Article ID 1813594, 9 pages
https://doi.org/10.1155/2019/1813594

Research Article

Robust Controllability and Observability of Boolean Control Networks under Different Disturbances

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Received 9 October 2018; Revised 3 December 2018; Accepted 20 December 2018; Published 2 January 2019

Academic Editor: Carlos-Renato Vázquez

This paper investigates robust controllability and observability of Boolean control networks under disturbances. Firstly, under unobservable disturbances, some sufficient conditions are obtained for robust controllability of BCNs. Then an algorithm is proposed to construct the least control sequences which drive the trajectory from a state to a given reachable state. If the disturbances are observable, by defining the order-preserving system, an efficient sufficient condition is obtained for robust controllability of BCNs. Finally, the robust observability problem is converted into an equivalent robust controllability via set controllability and is solved by using the results obtained for set controllability. Some numerical examples are presented to illustrate the obtained results.

1. Introduction

Boolean network (BN) was first proposed by Kauffman to model gene regularity networks [1]. It has been widely used in some other fields, such as biology and system science [2, 3]. To manipulate BN, external control inputs and outputs have been added to the logical dynamics, which yields Boolean control networks (BCNs) [4].

Recently, a new matrix product, called the semi-tensor product (STP) of matrices, has been proposed [5]. Using this method, an algebraic state space representation (ASSR) has been developed for dealing with some classical control problems of logical dynamic systems [6, 7], including controllability and observability [8–11], stability and stabilization [12–15], optimal control [16, 17], and disturbance decoupling [18, 19]. One may refer to [20–27] for some other applications.

It is known that the external disturbances are ubiquitous and may lead the network dynamics to some unexpected behaviour. Therefore, when modeling a gene regularity network, disturbances should be considered [28, 29]. In recent years, the disturbance decoupling problem of BCNs has been investigated in [18, 19]. Moreover, the global robust stability and stabilization of BCNs have been studied in [30]. However, to our best knowledge, there is few literatures devoted to the robust controllability and observability of BCNs, though they have obvious importance for BCNs with disturbances. Especially for robust observability problems, they are much harder to be understood and verified, compared with robust controllability problems. A recent work [31] proposes a new method to investigate observability of BCNs by converting it to a problem of set controllability. Under a fundamental framework of [31], we investigate robust controllability and observability problem in our earlier conference paper [32]. Regarding [32], the novelties of this paper consist of the following: (i) we discuss two kinds of control inputs, respectively, for robust controllability under unobservable disturbances: one is a free Boolean sequence, and the other inputs are logical variables satisfying certain input nteworks. For these two cases, we, respectively, give some sufficient conditions for robust controllability. But in [32], we only consider the case that control inputs are free; (ii) we give an actual example to illustrate our theoretical results; please see Example 22. But there is no such example in [32].

The main contributions of this paper include the following: (i) For unobservable disturbances, some sufficient conditions for robust controllability and observability of BCNs are obtained, respectively; (ii) based on the definition of order-preserving system, some easily verifiable sufficient conditions for robust controllability and observability under observable disturbances are presented, respectively; (iii) a general algorithm is proposed to design the least control sequences, which performs the required robust controllability or observability of BCNs.

The rest of this paper is organized as follows: Section 2 presents some preliminaries on the STP of matrices and set controllability of BCNs. Section 3 studies the robust controllability of BCNs under unobservable disturbances and gives an algorithm for designing the least control sequences. Section 4 discusses the robust controllability of BCNs under observable disturbances. The robust observability of BCNs is considered in Section 5. Section 6 is a brief conclusion.

Notation. The notations of this paper are fairly standard. $\mathscr{M}_{m\times n}$ denotes the set of $m \times n$ real matrices. $\mathrm{Col}(M)$ ($\mathrm{Row}(M)$) is the set of columns (rows) of M and $\mathrm{Col}_i(M)$ ($\mathrm{Row}_i(M)$) is the i-th column (row) of M. $\mathscr{D} := \{0,1\}$. $\Delta_n := \{\delta_n^i \mid i = 1,\dots,n\}$, where δ_n^i is the i-th column of the identity matrix I_n. $\mathbf{1}_\ell = (\underbrace{1,1,\dots,1}_{\ell})^T$. $\mathbf{1}_{p\times q}$ denotes a $p \times q$ matrix with all entries equal to 1. A matrix $L \in \mathscr{M}_{m\times n}$ is called a logical matrix if $\mathrm{Col}(L) \subset \Delta_m$. Denote by $\mathscr{L}_{m\times n}$ the set of $m \times n$ logical matrixes. If $L = [\delta_n^{i_1}, \delta_n^{i_2}, \dots, \delta_n^{i_r}]$, it is briefly denoted as $L = \delta_n[i_1, i_2, \dots, i_r]$. Denote by $\mathscr{B}_{m\times n}$ the set of $m \times n$ Boolean matrices. Let $M, N \in \mathscr{B}_{m\times n}$; then $M +_{\mathscr{B}} N$ is the Boolean addition (with respect to $+_{\mathscr{B}} = \vee$ and $\times_{\mathscr{B}} = \wedge$). A matrix $C > 0$ means all the entries are positive; that is, $c_{i,j} > 0$, $\forall i, j$.

2. Preliminaries

2.1. Semi-Tensor Product of Matrices. This subsection is a brief survey on STP of matrices [5].

Definition 1. Let $M \in \mathscr{M}_{m\times n}$, $N \in \mathscr{M}_{p\times q}$, and $t = \mathrm{lcm}\{n, p\}$ be the least common multiple of n and p. The STP of M and N is defined as

$$M \ltimes N := \left(M \otimes I_{t/n}\right)\left(N \otimes I_{t/p}\right) \in \mathscr{M}_{mt/n\times qt/p}, \tag{1}$$

where $\otimes$ is the Kronecker product.

When $n = p$, the STP becomes the conventional matrix product. Therefore, the STP is a generalization of the conventional matrix product. We can omit the symbol $\ltimes$ without confusion.

Proposition 2. *(1) Let $X \in \mathbb{R}^m$ be a column and M be a matrix. Then*

$$XM = \left(I_m \otimes M\right) X. \tag{2}$$

(2) Let $X \in \mathbb{R}^m$ and $Y \in \mathbb{R}^n$ be two columns. Then $W_{[m,n]} \ltimes X \ltimes Y = Y \ltimes X$, where $W_{[m,n]} \in \mathscr{M}_{mn\times mn}$ is called the swap matrix, which is defined by

$$\begin{aligned} W_{[m,n]} = \delta_{mn} [&1, m+1, \dots, (n-1)m \\ &+ 1; 2, \dots, (n-1)m + 2; \dots; m, 2m, \dots, nm]. \end{aligned} \tag{3}$$

By identifying $1 \sim \delta_2^1$ and $0 \sim \delta_2^2$, we have $\Delta \sim \mathscr{D}$. Using vector expression, a logical function can be expressed as its algebraic form.

Theorem 3. *Let $f : \underbrace{\mathscr{D} \times \cdots \times \mathscr{D}}_{n} \longrightarrow \mathscr{D}$ be a Boolean function. Then there exists a unique matrix $M_f \in \mathscr{L}_{2\times 2^n}$, called the structure matrix of f, such that in the vector form we have*

$$f\left(x_1, \dots, x_n\right) = M_f \ltimes_{i=1}^n x_i. \tag{4}$$

Theorem 4. *Let $u = M \ltimes_{i=1}^n x_i$ and $v = N \ltimes_{i=1}^n x_i$, where $x_i \in \Delta_2$, $i = 1,\dots,n$, are logical variables and $M \in \mathscr{L}_{p\times 2^n}$, $N \in \mathscr{L}_{q\times 2^n}$ are logical matrices. Then*

$$uv = (M * N) \ltimes_{i=1}^n x_i, \tag{5}$$

where $*$ is the Khatri-Lao product.

2.2. Set Controllability of BCNs. A BCN is described as

$$\begin{aligned} x_1(t+1) &= f_1\left(x_1(t), \dots, x_n(t); u_1(t), \dots, u_m(t)\right) \\ &\vdots \\ x_n(t+1) &= f_n\left(x_1(t), \dots, x_n(t); u_1(t), \dots, u_m(t)\right), \\ y_j(t) &= h_j\left(x_1(t), \dots, x_n(t)\right), \quad j = 1, \dots, p, \end{aligned} \tag{6}$$

where $x_i \in \mathscr{D}$, $i = 1,\dots,n$, are state variables, $u_j \in \mathscr{D}$, $j = 1,\dots,m$, are controls, and $f_i : \mathscr{D}^{m+n} \longrightarrow \mathscr{D}$, $i = 1,\dots,n$, and $h_j : \mathscr{D}^n \longrightarrow \mathscr{D}$, $j = 1,\dots,p$, are Boolean functions.

Definition 5 (see [5]). System (6) is

(1) controllable from x_0 to x_d, if there are $T > 0$ and a sequence of control $u(0), \dots, u(T-1)$, such that driven by these controls the trajectory can go from $x(0) = x_0$ to $x(T) = x_d$;

(2) controllable at x_0, if it is controllable from x_0 to any $x_d \in \mathscr{D}^n$;

(3) controllable, if it is controllable at any $x_0 \in \mathscr{D}^n$.

Using Theorems 3 and 4, (6) can be converted into its algebraic form as

$$\begin{aligned} x(t+1) &= Lu(t)x(t) \\ y(t) &= Hx(t), \end{aligned} \tag{7}$$

where $x(t) = \ltimes_{i=1}^n x_i(t)$, $u(t) = \ltimes_{j=1}^m u_j(t)$, $y(t) = \ltimes_{l=1}^p y_l(t)$, and $L \in \mathscr{L}_{2^n\times 2^{n+m}}$, $H \in \mathscr{L}_{2^p\times 2^n}$.

Define

$$M := \bigvee_{j=1}^{2^m} L\delta_{2^m}^j, \tag{8}$$

and set

$$\mathscr{C} := \bigvee_{i=1}^{2^n} M^{(i)}, \tag{9}$$

which is called the controllability matrix. Then we have the following result.

Theorem 6 (see [33]). *Consider system (6) (by free control sequence). Assume its controllability matrix is $\mathscr{C} = (c_{i,j})$; then system (6) is*

(1) *controllable from states $\delta_{2^n}^j$ to $\delta_{2^n}^i$, if and only if $c_{i,j} = 1$;*

(2) *controllable at $\delta_{2^n}^j$, if and only if $\mathrm{Col}_j(\mathscr{C}) = \mathbf{1}_{2^n}$;*

(3) *controllable, if and only if $\mathscr{C} = \mathbf{1}_{2^n \times 2^n}$.*

Denote by $N = \{1, 2, \ldots, 2^n\}$ the set of states for BCNs. Assuming $s \in 2^N$, the index column vector of s, denoted by $V(s) \in \mathbb{R}^{2^n}$, is defined as

$$(V(s))_i = \begin{cases} 1, & i \in s \\ 0, & i \notin s. \end{cases} \tag{10}$$

Define the set of initial sets P^0 and the set of destination sets P^d, respectively, as follows:

$$\begin{aligned} P^0 &:= \left\{s_1^0, s_2^0, \ldots, s_\alpha^0\right\} \subset 2^N, \\ P^d &:= \left\{s_1^d, s_2^d, \ldots, s_\beta^d\right\} \subset 2^N. \end{aligned} \tag{11}$$

Using initial sets and destination sets, the set controllability is defined as follows.

Definition 7. Consider system (6) with a set of initial sets P^0 and a set of destination sets P^d. System (6) is

(1) set controllable from $s_j^0 \in P^0$ to $s_i^d \in P^d$, if there exist $x_0 \in s_j^0$ and $x_d \in s_i^d$, such that x_d is controllable from x_0;

(2) set controllable at s_j^0, if, for any $s_i^d \in P^d$, the system is controllable from s_j^0 to s_i^d;

(3) set controllable, if it is set controllable at any $s_j^0 \in P^0$.

Using P^0 and P^d defined in (11), we define the initial index matrix J_0 and the destination index matrix J_d, respectively, as

$$\begin{aligned} J_0 &:= \left[V\left(s_1^0\right) \ V\left(s_2^0\right) \ \cdots \ V\left(s_\alpha^0\right)\right] \in \mathscr{B}_{2^n \times \alpha}; \\ J_d &:= \left[V\left(s_1^d\right) \ V\left(s_2^d\right) \ \cdots \ V\left(s_\beta^d\right)\right] \in \mathscr{B}_{2^n \times \beta}. \end{aligned} \tag{12}$$

Then we define a matrix, called the set controllability matrix, as

$$\mathscr{C}_S := J_d^T \times_{\mathscr{B}} \mathscr{C} \times_{\mathscr{B}} J_0 \in \mathscr{B}_{\beta \times \alpha}. \tag{13}$$

Note that hereafter all the matrix products are assumed to be Boolean product ($\times_{\mathscr{B}}$). Hence the symbol $\times_{\mathscr{B}}$ is omitted.

Definition 8 (see [31]). Consider system (6) with P^0 and P^d as defined in (11). Moreover, the corresponding set controllability matrix is $\mathscr{C}_S = (c_{i,j})$, which is defined by (13). Then system (6) is

(1) set controllable from s_j^0 to s_i^d, if and only if $c_{i,j} = 1$;

(2) controllable at s_j^0, if and only if $\mathrm{Col}_j(\mathscr{C}_S) = \mathbf{1}_\beta$;

(3) set controllable, if and only if $\mathscr{C}_S = \mathbf{1}_{\beta \times \alpha}$.

3. Robust Controllability of BCNs under Unobservable Disturbances

Consider the following disturbed BCN:

$$\begin{aligned} x_1(t+1) &= f_1\left(x_1(t), \ldots, x_n(t); u_1(t), \ldots, u_m(t); \xi_1(t), \ldots, \xi_q(t)\right) \\ &\vdots \\ x_n(t+1) &= f_n\left(x_1(t), \ldots, x_n(t); u_1(t), \ldots, u_m(t); \xi_1(t), \ldots, \xi_q(t)\right), \end{aligned} \tag{14}$$

with

$$y_j(t) = h_j\left(x_1(t), \ldots, x_n(t)\right), \quad j = 1, \ldots, p, \tag{15}$$

where $x_i(t), u_j(t), \xi_k(t), y_s(t) \in \mathscr{D}$ are states, controls, disturbances, and outputs, respectively, $i = 1, \ldots, n$, $j = 1, \ldots, m$, $k = 1, \ldots, q$, $s = 1, \ldots, p$. Using vector expression with $x(t) = \ltimes_{i=1}^n x_i(t)$, $u(t) = \ltimes_{j=1}^m u_j(t)$, $\xi(t) = \ltimes_{k=1}^q \xi_k(t)$, and $y(t) = \ltimes_{s=1}^p y_s(t)$, the algebraic forms of (14) and (15) are expressed as

$$x(t+1) = Fu(t)\xi(t)x(t) \tag{16}$$

with

$$y(t) = Hx(t), \tag{17}$$

where $F \in \mathscr{L}_{2^n \times 2^{n+m+q}}$, $H \in \mathscr{L}_{2^p \times 2^n}$.

First, the robust controllability of disturbed BCNs is defined as follows.

Definition 9. System (14) is

(1) robust controllable from x_0 to x_d, if there exist $T > 0$ and a control sequence $\{u(t) \mid t = 0, 1, \ldots, T-1\}$, such that, under any $\xi(t)$, $t = 0, 1, \ldots, T-1$, the state can be driven from $x(0) = x_0$ to $x(T) = x_d$;

(2) robust controllable at x_0, if it is robust controllable from x_0 to any $x_d \in \mathscr{D}^n$;

(3) robust controllable, if it is robust controllable at any $x_0 \in \mathscr{D}^n$.

In this paper, we consider two kinds of control inputs.

(i) The control is a free Boolean sequence.

(ii) The control inputs are logical variables satisfying certain input networks, such as

$$u_i(t+1) = g_i\left(u_1(t), \ldots, u_m(t)\right), \quad i = 1, \ldots, m. \tag{18}$$

3.1. Robust Controllability with a Free Control Sequence. For the algebraic form (16) of system (14), define

$$M := \bigvee_{j=1}^{2^m} \bigwedge_{k=1}^{2^q} F\delta_{2^m}^j \delta_{2^q}^k, \tag{19}$$

and

$$\mathscr{C}^R := \bigvee_{i=1}^{2^n} M^{(i)}, \tag{20}$$

which is called the robust controllability matrix; then we have the following result.

Theorem 10. *Consider system (14) under unobservable disturbances. Assume $\mathscr{C}^R = (c^R_{i,j})$; then system (14) is*

(1) *robust controllable from $\delta^j_{2^n}$ to $\delta^i_{2^n}$, if $c^R_{i,j} = 1$;*

(2) *robust controllable at $\delta^j_{2^n}$, if $\mathrm{Col}_j(\mathscr{C}^R) = \mathbf{1}_{2^n}$;*

(3) *robust controllable, if $\mathscr{C}^R = \mathbf{1}_{2^n \times 2^n}$.*

Proof. It is easy to see that the system is controllable from $\delta^j_{2^n}$ to $\delta^i_{2^n}$; then there exists a finite control sequence $u(0), u(1), \cdots, u(T-1)$ such that

$$\delta^{r_0}_{2^n} = \delta^j_{2^n} \xrightarrow{u(0)} \delta^{r_1}_{2^n} \xrightarrow{u(1)} \delta^{r_2}_{2^n} \xrightarrow{u(1)} \cdots \xrightarrow{u(T-1)} \delta^{r_T}_{2^n} = \delta^i_{2^n}. \tag{21}$$

Because of the construction of matrix M, particularly $(F\delta^j_{2^m}\delta^1_{2^q}) \bigwedge F\delta^j_{2^m}\delta^2_{2^q} \bigwedge \cdots \bigwedge F\delta^j_{2^m}\delta^{2^q}_{2^q}$, at any step t no matter which disturbance happens, we can always find the corresponding $u(t)$ to realize the corresponding one step transfer in (21). Hence, system (14) is controllable from $\delta^j_{2^n}$ to $\delta^i_{2^n}$. (The detailed argument is similar to the proof of Theorem 6.) □

Since robust controllability for any disturbance is a strong requirement, next we consider robust controllability under a constrained set of disturbance inputs. First, a constrained set for disturbance inputs is denoted by $Q_d = \{\delta^{\omega_1}_{2^q}, \delta^{\omega_2}_{2^q}, \ldots, \delta^{\omega_\tau}_{2^q}\}$, where τ is the cardinality of set Q_d. Define the robust controllability matrix under the constrained disturbances set Q_d as

$$\mathscr{C}^R_d := \bigvee_{i=1}^{2^n} {M_d}^{(i)}, \tag{22}$$

where

$$M_d := \bigvee_{j=1}^{2^m} \bigwedge_{k=1}^{\tau} F\delta^j_{2^m}\delta^{\omega_k}_{2^q}. \tag{23}$$

Then we have the following result.

Corollary 11. *Consider system (14) under the constrained disturbances set Q_d. Assume $\mathscr{C}^R_d = (c^R_{d\,(i,j)})$; then system (14) is*

(1) *robust controllable from $\delta^j_{2^n}$ to $\delta^i_{2^n}$, if $c^R_{d\,(i,j)} = 1$;*

(2) *robust controllable at $\delta^j_{2^n}$, if $\mathrm{Col}_j(\mathscr{C}^R_d) = \mathbf{1}_{2^n}$;*

(3) *robust controllable, if $\mathscr{C}^R_d = \mathbf{1}_{2^n \times 2^n}$.*

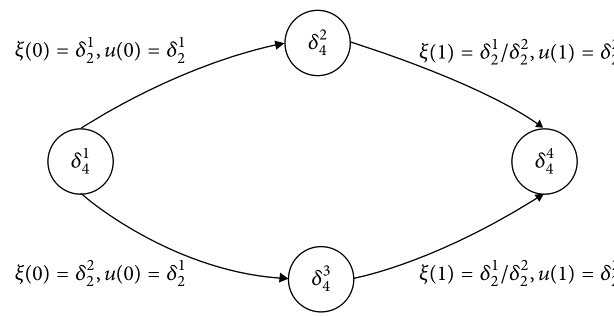

Figure 1: State trajectory under unobservable disturbance.

Remark 12. The main advantage of Theorem 10 is when the iterative steps increase, the corresponding matrices, which are defined in (19) and (20), do not increase their dimensions. Hence it is easily verifiable and computable. The main disadvantage is that this result is not necessary; refer to the following example.

Example 13. Consider the following BCN with unobservable disturbance:

$$x(t+1) = Fu(t)\,\xi(t)\,x(t), \tag{24}$$

where $x(t) = x_1(t) \ltimes x_2(t) \in \Delta_4$, $u(t), \xi(t) \in \Delta_2$, and $F = \delta_4[2, 3, 3, 1, 3, 2, 2, 4, 1, 4, 4, 4, 4, 4, 4, 2]$. Assuming $x_0 = \delta^1_4$ and $x_d = \delta^4_4$, then the state trajectory can be presented in Figure 1. It is easy to see that, under unobservable disturbance, system (24) is robust controllable from δ^1_4 to δ^4_4. However, by immediate calculation, we have

$$M = \bigvee_{j=1}^{2} \bigwedge_{k=1}^{2} F\delta^j_2\delta^k_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}. \tag{25}$$

It follows that

$$\mathscr{C}^R := \bigvee_{i=1}^{2^n} M^{(i)} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}. \tag{26}$$

Obviously, $(\mathscr{C}^R)_{4,1} = 0$. Hence we conclude that Theorem 10 is only sufficient but not necessary.

3.2. Robust Controllability with Input Networks. Using the STP method, the input networks (18) can be expressed in its algebraic form as

$$u(t+1) = Gu(t), \tag{27}$$

where $G \in \mathscr{L}_{2^m \times 2^m}$.

Putting (16) and (27) together, we have

$$\begin{aligned} x(t+1) &= Fu(t)\,\xi(t)\,x(t) \\ u(t+1) &= Gu(t). \end{aligned} \tag{28}$$

Set $w(t) = u(t)x(t)$; then (28) can be converted into

$$w(t+1) = \Phi\xi(t)w(t), \tag{29}$$

where $\Phi = [G(\mathbf{1}_{2^q}^T \otimes I_{2^m} \otimes \mathbf{1}_{2^n}^T)] * (FW_{[2^q,2^m]})$.

Similar to the argument in [34], we construct the set of initial sets P^0 and the set of destination sets P^d as

$$P^0 = P^d = \{s_i \mid i = 1,\dots,2^n\}, \tag{30}$$

where $s_i = \{u\delta_{2^n}^i \mid u \in \Delta_{2^m}\} = \{\delta_{2^m}^1\delta_{2^n}^i, \delta_{2^m}^2\delta_{2^n}^i, \dots, \delta_{2^m}^{2^m}\delta_{2^n}^i\}$.

Then it is easy to see that each s_i corresponds to a unique state $\delta_{2^n}^i$. Hence system (29) is set controllable from s_j to s_i, which is equivalent to the controllability from state $\delta_{2^n}^j$ to $\delta_{2^n}^i$ of system (14).

Theorem 14. *System (14) with inputs (18) is robust controllable, if and only if the extended system (29) is robust set controllable with respect to P^0 and P^d, which are defined in (30).*

Define $M_g := \bigwedge_{k=1}^{2^q}\Phi\delta_{2^q}^k$, and $\mathscr{C}_g^R := \bigvee_{i=1}^{2^{n+m}} M_g^{(i)}$. Using (12), we have the robust set controllable matrix as

$$\mathscr{C}_{sg}^R := J_d^T \times_{\mathscr{B}} \mathscr{C}_g^R \times_{\mathscr{B}} J_0 \in \mathscr{B}_{2^n\times 2^n}. \tag{31}$$

Similar to Theorem 10, we have an easily verifiable sufficient condition, which is shown as follows.

Theorem 15. *Consider system (14) with inputs (18). Assuming $\mathscr{C}_{sg}^R = (c_{i,j})$, then system (14) is*

(1) *robust controllable from $\delta_{2^n}^j$ to $\delta_{2^n}^i$, if $c_{i,j} = 1$;*

(2) *robust controllable at $\delta_{2^n}^j$, if $\mathrm{Col}_j(\mathscr{C}_{sg}^R) = \mathbf{1}_{2^n}$;*

(3) *robust controllable, if $\mathscr{C}_{sg}^R = \mathbf{1}_{2^n\times 2^n}$.*

3.3. Control Design. For system (14) with any disturbance, assume x_0 can be driven to x_d by a proper free control sequence. Since the trajectory from x_0 to x_d is in general not unique, in the following sequel, we try to find the shortest one. We propose the following "back stepping" algorithm.

Algorithm 16. Set $x_0 = \delta_{2^n}^j$ and $x_d = \delta_{2^n}^i$. Assume the (i, j)-th element of robust controllability matrix, $c_{i,j}^R = 1$.

(i) *Step 1.* Define (at most) k-step controllability matrix as $\mathscr{C}_k^R := \bigvee_{l=1}^k M^{(l)}$. Find smallest k, such that $[\mathscr{C}_k^R]_{i,j} = 1$.

(ii) *Step 2.* If $k = 1$, then $\mathscr{C}_1^R = M$. Find α_0 such that $[F\delta_{2^m}^{\alpha_0}\delta_{2^q}^r]_{i,j} = 1$, $\forall r$. Set $u(0) = \delta_{2^m}^{\alpha_0}$; stop. Else, find one s, such that $M_{i,s} = 1$ and $[\mathscr{C}_k^R]_{s,j} = 1$.

(iii) *Step 3.* Find α_{k-1} such that $[F\delta_{2^m}^{\alpha_{k-1}}\delta_{2^q}^r]_{i,s} = 1$ (note that α_{k-1} is not unique, but there is at least one such block). Set $u(k-1) = \delta_{2^m}^{\alpha_{k-1}}$.

(iv) *Step 4.* Set $k = k-1$ and replace i by s. Go back to Step 2.

The following proposition is an immediate consequence of Algorithm 16.

Proposition 17. *The control sequence generated by Algorithm 16 is a least sequence of controls, which can drive the trajectory from x_0 to x_d. Moreover, the corresponding trajectory is $\{x(0) = x_0, x(1), \dots, x(s) = x_d\}$, which is also produced by the algorithm.*

It is easily verified that Algorithm 16 can also be applied to the case with the constrained disturbances set.

Example 18. Consider the following BCN with unobservable disturbance:

$$\begin{aligned} x_1(t+1) &= (x_1(t) \longleftrightarrow x_2(t)) \vee u(t) \\ x_2(t+1) &= \neg x_1(t) \wedge \xi(t). \end{aligned} \tag{32}$$

Setting $x(t) = x_1(t) \ltimes x_2(t)$, we can get its algebraic form as

$$x(t+1) = Fu(t)\xi(t)x(t), \tag{33}$$

where $F = \delta_4[2, 2, 1, 1, 2, 2, 2, 2, 2, 4, 3, 1, 2, 4, 4, 2]$. It is easy to calculate that

$$M = \bigvee_{j=1}^{2}\bigwedge_{k=1}^{2} F\delta_2^j\delta_2^k = \begin{bmatrix} 0&0&0&0\\ 1&1&0&0\\ 0&0&0&0\\ 0&1&0&0 \end{bmatrix}. \tag{34}$$

It follows that

$$\mathscr{C}^R := \bigvee_{i=1}^{4} M^{(i)} = \begin{bmatrix} 0&0&0&0\\ 1&1&0&0\\ 0&0&0&0\\ 1&1&0&0 \end{bmatrix}. \tag{35}$$

According to Theorem 10, since $(\mathscr{C}^R)_{4,1} = 1$, we can see that system (32) is robust controllable from $x_0 = \delta_4^1$ to $x_d = \delta_4^4$. What is the least control sequence for driving δ_4^1 to δ_4^4?

Using Algorithm 16, we can find the smallest $k = 2$, such that $(\mathscr{C}_2^R)_{4,1} = 1$. It is easy to find that $M_{4,2} = 1$ and $(\mathscr{C}_1^R)_{2,1} = 1$. Choose $\alpha_1 = 2$, such that $[F\delta_2^{\alpha_1}\delta_2^r]_{4,2} = 1$, $r = 1, 2$. Set $u(1) = \delta_2^2$.

According to Step 4 in Algorithm 16, set $k = k-1 = 1$, and replace $i = 4$ by $s = 2$. Then we have $\mathscr{C}_1^R = M$. Find $\alpha_0 = 1$ (or 2), such that $[F\delta_2^{\alpha_0}\delta_2^r]_{2,1} = 1$, $r = 1, 2$. Set $u(0) = \delta_2^1(or\delta_2^2)$.

Hence the least control sequence is $\{u(0) = \delta_2^1$ (or δ_2^2), $u(1) = \delta_2^2\}$, which can drive δ_4^1 to δ_4^4, and the trajectory is $\{x(0) = \delta_4^1, x(1) = \delta_4^2, x(2) = \delta_4^4\}$.

For system (32) with constrained disturbance set $Q_d = \{\delta_2^1\}$, using (22) and (23), we have

$$M_d = \bigvee_{j=1}^{2} F\delta_2^j\delta_2^1 = \begin{bmatrix} 0&0&1&1\\ 1&1&0&0\\ 0&0&1&0\\ 0&1&0&0 \end{bmatrix}, \tag{36}$$

and

$$\mathscr{C}_d^R := \bigvee_{i=1}^{4} M_d^{(i)} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}. \qquad (37)$$

According to Corollary 11, we conclude that system (32) with $Q_d = \{\delta_2^1\}$ is robust controllable at $x_0 = \delta_4^3$.

4. Robust Controllability of BCNs under Observable Disturbances

Assume the disturbance $\xi(t) = \delta_{2^q}^r$ at time t; that is, the disturbance is dependent on time t, where $r \in \{1, 2, \ldots, 2^q\}$; then the dynamic model at time t is

$$x(t+1) = \widetilde{F}(r)\, u(t)\, x(t), \qquad (38)$$

where $\widetilde{F}(r) = FW_{[2^q,2^m]}\delta_{2^q}^r$.

Now we consider the overall robust controllability of system (14) by a free control sequence. It is easily extended to the case with networked inputs (18).

We give a necessary condition as follows, which is obvious, because the constant disturbance is legal.

Lemma 19. *Assuming system (14) is robust controllable, then each subsystem (38) is robust controllable.*

Then we consider the robust controllability problem under the following assumption:

(**A1**) Each subsystem (38) with $\widetilde{F}(r)$ is controllable, $r = 1, \ldots, 2^q$.

For each pair of states $x_d \in \Delta_{2^n}$ and $x \in \Delta_{2^n}$, we define the distance between them under model r.

$$d_{x_d}^r(x) = \min_k \{\text{state } x \text{ is driven to } x_d \text{ within } k \text{ steps}\}. \qquad (39)$$

It is noted that, under assumption (**A1**), the distance $d_{x_d}^r(x)$ is well defined for any x_d and any x. If we only consider the reachability to x_d, then assumption (**A1**) can be replaced by the following assumption.

(**A2**) Each subsystem with $\widetilde{F}(r)$ is reachable to the fixed x_d, $r = 1, \ldots, 2^q$.

Next we define the order-preserving of system (14).

Definition 20. System (14) is

(1) order-preserving at x_d, if for any $x, y \in \Delta_{2^n}$, and any $1 \le r, s \le 2^q$, we have

$$d_{x_d}^r(x) > d_{x_d}^r(y) \Longleftrightarrow d_{x_d}^s(x) > d_{x_d}^s(y). \qquad (40)$$

(2) order-preserving, if it is order-preserving at any $x_d \in \Delta_{2^n}$.

Theorem 21. *Consider system (14) with observable disturbances.*

(1) *Assume (**A2**) is satisfied and system (14) is order-preserving at the fixed x_d. Then it is robust reachable to x_d; i.e., it is robust controllable from any x_0 to x_d.*

(2) *Assume (**A1**) is satisfied and system (14) is order-preserving at any x_d. Then it is robust controllable.*

Proof. We only prove for the fixed state x_d case. The proof for general case is exactly the same. If we know the model at time t is r, then we can design the control $u(t)$ to drive $x(t)$ to $x(t+1)$, where

$$d_{x_d}^r(x(t+1)) = d_{x_d}^r(x(t)) - 1. \qquad (41)$$

Since the number of all possible models is $2^q < \infty$, after finite times, at least one model should be reached as many times as we wish. Because of the order-preserving, it is easy to see that at each time the distance is decreasing at least one. Hence the distance $d_{x_d}^r(x(t))$ will eventually reach zero. That is, system (14) will reach x_d. □

Example 22 (see [35]). Consider a lac operon regulatory network model

$$\begin{aligned} x_1(t+1) &= \neg u_1(t) \wedge (x_3(t) \vee u_2(t)), \\ x_2(t+1) &= x_1(t), \\ x_3(t+1) &= \neg u_1(t) \\ &\quad \wedge ((x_2(t) \wedge u_2(t)) \vee (x_3(t) \wedge \neg x_2(t))), \end{aligned} \qquad (42)$$

where x_1, x_2, and x_3 are state variables, which denote the mRNA, the LacZ polypeptide, and the internal lactose, respectively, and u_1, u_2 are control inputs, which represent the external lactose and the external glucose, respectively. The variable equal to 1 (or 0) means the corresponding molecular has a high (or low) concentration. In order to investigate the robust controllability of the model, we introduce an artificial disturbance ξ in $x_2(t+1) = x_1(t)$, which becomes $x_2(t+1) = x_1(t) \vee \xi(t)$.

Using the vector form of logical variables and setting $x(t) = x_1(t) \ltimes x_2(t) \ltimes x_3(t)$, $u(t) = u_1(t) \ltimes u_2(t)$, by using the STP, we have the following algebraic form:

$$x(t+1) = F\xi(t)\, u(t)\, x(t), \qquad (43)$$

where

$$\begin{aligned} F = \delta_8 [&6, 6, 6, 6, 6, 6, 6, 6, 6, 6, 6, 6, 8, 8, 8, 8, 1, \\ &1, 1, 2, 1, 1, 1, 2, 1, 1, 1, 2, 3, 3, 3, 4, 6, 6, 6, 6, \\ &6, 6, 6, 6, 6, 6, 6, 6, 8, 8, 8, 8, 2, 6, 1, 6, 2, 6, 1, \\ &6, 2, 6, 1, 6, 4, 8, 3, 8]. \end{aligned} \qquad (44)$$

Assume the disturbance is observable at time t; then we have

$$x(t+1) = F(\delta_2^r)\, u(t)\, x(t) = M_r u(t)\, x(t), \qquad (45)$$

where

$$\begin{aligned} M_1 &= \delta_8 [6, 6, 6, 6, 6, 6, 6, 6, 6, 6, 6, 6, 8, 8, 8, 8, \\ &\quad 1, 1, 1, 2, 1, 1, 1, 2, 1, 1, 1, 2, 3, 3, 3, 4], \\ M_2 &= \delta_8 [6, 6, 6, 6, 6, 6, 6, 6, 6, 6, 6, 6, 8, 8, 8, 8, \\ &\quad 2, 6, 1, 6, 2, 6, 1, 6, 2, 6, 1, 6, 4, 8, 3, 8]. \end{aligned} \qquad (46)$$

According to (8) and (9), we calculate that

$$\mathscr{C}_1 = \bigvee_{i=1}^{8} \widetilde{M_1}^{(i)} = \begin{bmatrix} 1&1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1&1\\ 0&0&0&0&0&0&0&0\\ 1&1&1&1&1&1&1&1\\ 0&0&0&0&0&0&0&0\\ 1&1&1&1&1&1&1&1 \end{bmatrix},$$
$$\mathscr{C}_2 = \bigvee_{i=1}^{8} \widetilde{M_2}^{(i)} = \begin{bmatrix} 0&0&1&0&0&0&1&0\\ 1&0&1&0&1&0&1&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&0\\ 1&1&1&1&1&1&1&1\\ 0&0&0&0&0&0&0&0\\ 1&1&1&1&1&1&1&1 \end{bmatrix}, \quad (47)$$

where $\widetilde{M_p} = \bigvee_{j=1}^{4} M_p \delta_4^j$, $p = 1, 2$.

It is obvious that both subsystems are not controllable. Hence system (43) does not satisfy Assumption (A1). From the above matrices $\mathscr{C}_1$ and $\mathscr{C}_2$, it is easy to see that both subsystems are all reachable to $x_d = \delta_8^8$. Using (39), for $r = 1$ and $r = 2$, respectively, we can figure out that $d_{x_d}^r(\delta_8^i) = 2$, $i = 1, 2, 3, 4$, and $d_{x_d}^r(\delta_8^j) = 1$, $j = 5, 6, 7$. Obviously, this system is order-preserving at $x_d = \delta_8^8$. According to Theorem 21, we conclude that system (43) is robust reachable to δ_8^8; i.e., it is robust controllable from any x_0 to δ_8^8.

5. Robust Observability of BCNs

First, we give the definition of robust observability for disturbed BCNs.

Definition 23. System (14)-(15) under unobservable disturbances is robust observable, if for any two initial states $x_0 \neq z_0$, and under any disturbance $\{\xi(t) \mid t = 0, 1, \cdots\}$, there exists a control sequence $\{u(t) \mid t = 0, 1, \cdots\}$, such that the corresponding output sequences are distinct; that is,

$$\{y(t) \mid u(t), x_0, t\} \neq \{\tilde{y}(t) \mid u(t), z_0, t\}. \quad (48)$$

As suggested in [31], we consider two kinds of state pairs. A pair $(x, z) \in \Delta_{2^n} \times \Delta_{2^n}$ is y-indistinguishable if $Hx = Hz$. Otherwise, (x, z) is called y-distinguishable.

Next, we split the product state space $\Delta_{2^n} \times \Delta_{2^n}$ into a partition of three components as

$$\begin{aligned} D &:= \{zx \mid z = x\}, \\ \Theta &:= \{zx \mid z \neq x \text{ and } Hz = Hx\}, \\ \Xi &:= \{zx \mid Hz \neq Hx\}. \end{aligned} \quad (49)$$

Using algebraic form (16), we construct a dual system as

$$\begin{aligned} z(t+1) &= Fu(t)\xi(t)z(t) \\ x(t+1) &= Fu(t)\xi(t)x(t). \end{aligned} \quad (50)$$

Set $w(t) = z(t)x(t)$; we have

$$\begin{aligned} z(t+1) &= F\left(I_{2^{m+q+n}} \otimes \mathbf{1}_{2^n}^T\right) u(t)\xi(t)z(t)x(t) \\ x(t+1) &= F\left(I_{2^{m+q}} \otimes \mathbf{1}_{2^n}^T\right) u(t)\xi(t)z(t)x(t). \end{aligned} \quad (51)$$

Then (51) can be expressed as a new system

$$w(t+1) = Lu(t)\xi(t)w(t), \quad (52)$$

where $L = [F(I_{2^{m+q+n}} \otimes \mathbf{1}_{2^n}^T)] * [F(I_{2^{m+q}} \otimes \mathbf{1}_{2^n}^T)] \in \mathscr{L}_{2^{2n} \times 2^{2n+m+q}}$.

Now the observability problem of system (14)-(15) can be converted into a set controllability problem of the extended system (52). Construct the set of initial sets and the set of destination sets as follows:

$$P^0 := \left\{ \bigcup_{zx \in \Theta} \{zx\} \right\} \quad (53)$$

and

$$P^d := \{\Xi\}. \quad (54)$$

Note that (53) means each state pair zx in Θ is an element of P^0, while (54) means P^d has only one element, which is Ξ. Then we have the following result.

Theorem 24. *Consider system (14)-(15) under unobservable disturbances with P^0 and P^d as defined in (53) and (54), respectively. The system is robust observable, if system (52) is robust set controllable. Precisely speaking, the conditions in Theorem 10 are satisfied.*

Example 25. Recall Example 18. Assuming that the output is

$$y(t) = x_1(t) \vee x_2(t), \quad (55)$$

we have its algebraic form as

$$y(t) = Hx(t), \quad (56)$$

where $H = \delta_2[1, 1, 1, 2]$.

It is easy to figure out that

$$\begin{aligned} \Theta &= \left\{ \left(\delta_4^1, \delta_4^2\right), \left(\delta_4^1, \delta_4^3\right), \left(\delta_4^2, \delta_4^3\right) \right\} \sim \left\{ \delta_{16}^2, \delta_{16}^3, \delta_{16}^7 \right\} \\ &= \{\theta_1, \theta_2, \theta_3\}, \end{aligned} \quad (57)$$

and

$$\Xi = \left\{ \left(\delta_4^1, \delta_4^4\right), \left(\delta_4^2, \delta_4^4\right), \left(\delta_4^3, \delta_4^4\right) \right\} \sim \left\{ \delta_{16}^4, \delta_{16}^8, \delta_{16}^{12} \right\}. \quad (58)$$

Set $w(t) = z(t)x(t)$; we have a new system as

$$w(t+1) = Lu(t)\xi(t)w(t), \tag{59}$$

where $L = \delta_{16}[6, 6, 5, 5, \cdots, 6, 8, 8, 6] \in \mathscr{L}_{16\times 64}$.

We calculate

$$C^R \coloneqq \bigvee_{i=1}^{16} M^{(i)} \in \mathscr{B}_{16\times 16}, \tag{60}$$

where $M = \bigvee_{j=1}^{2}\bigwedge_{k=1}^{2} L\delta_2^j\delta_2^k$.

Using $P^0 = \{\{\theta_1\}, \{\theta_2\}, \{\theta_3\}\}$ and $P^d = \{\Xi\}$, we have

$$J_d = \sum_{\delta_{16}^i\in\Xi}\delta_{16}^i, \quad J_0 = \delta_{16}[2, 3, 7]. \tag{61}$$

It follows that

$$\mathscr{C}_S^R \coloneqq J_d^T \times_{\mathscr{B}} \mathscr{C}^R \times_{\mathscr{B}} J_0 = [1, 0, 0]. \tag{62}$$

According to Theorem 24, since $\mathrm{Col}_j(\mathscr{C}_S^R) = 1$, we conclude that system (32) with output (55) is robust observable at $s_1^0 = \{\theta_1\}$.

Similarly, we can obtain an easily verifiable sufficient condition for the robust observability of BCNs under observable disturbances.

Theorem 26. *System (14)-(15) with observable disturbances is robust observable, if system (52) is robust set ontrollable from P^0 to P^d. Precisely speaking, the conditions in Theorem 21 are satisfied.*

6. Conclusion

Robust controllability and observability of disturbed BCNs have been investigated in this paper. Some sufficient conditions for robust controllability under unobservable and observable disturbances have been obtained, respectively. By constructing an extended system for a given BCN, an observability problem can be converted to a set controllability problem of its extended system. Then some corresponding conditions for robust observability are obtained for two cases, respectively. A universal algorithm is presented to construct the least control sequences to realize the required robust controllability.

The results in this paper can easily be extended to mix-valued logical systems.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

An earlier version of this paper was presented as a conference paper at the 37th Chinese Control Conference, 25-27, July, Wuhan, 2018. We would like to thank Prof. Daizhan Cheng for his valuable discussion. This work was supported by National Natural Science Foundation (NNSF) of China under Grants 61773371 and 71701115.

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