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Research Article

A Simultaneous Inversion Problem for the Variable-Order Time Fractional Differential Equation with Variable Coefficient

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This paper deals with an inverse problem of simultaneously determining the space-dependent diffusion coefficient and the fractional order in the variable-order time fractional diffusion equation by the measurements at one interior point. Numerical solution to the forward problem is given by the finite difference scheme, and the homotopy regularization algorithm is applied to solve the inverse problem utilizing Legendre polynomials as the basis functions of the approximate space. The inversion solutions with noisy data which give good approximations to the exact solution demonstrate effectiveness of the inversion algorithm for the simultaneous inversion problem.

1. Introduction

The fractional calculus and fractional differential equations and their applications have attracted much attention during the last two decades. If the usual integer-order derivative in a classical diffusion equation is replaced with a fractional derivative of any real number in time or space, we get a time or space fractional diffusion equation, which has more and more applications in the fields of physics, mechanics, viscoelastic materials, environmental science, hydrology, nanofluid, etc. On the attempting to describe some real processes and complex dynamical systems using the fractional diffusion models, several researches confronted with the situation that the fractional derivative did not remain constant and changed; say, in the interval from 0 to 1, from 1 to 2 or even from 0 to 2. To study such phenomena in mathematics, an effective way is to employ the fractional derivative of the variable order; i.e., the order of the fractional derivative can change with the time or/and space variables. The variable-order fractional models and the corresponding theory of variable-order operators give a new mathematical framework for dealing with complex dynamical systems, which becomes a new direction in the research of fractional differential equations (see, e.g., [1, 2]). For the concept of variable fractional differentials and integrals and the connection between the mathematical concepts of fractional order and physical process, see [3–5] and references therein.

There are quite a few of researches on numerical methods for solving the variable-order time/space fractional diffusion equation models. We refer to some work given by F. Liu and his group, see [6–9] for instance, also see the work given by Sun et al. [10], and the master degree thesis given by Zhang [11] for numerical solutions to the variable-order time fractional diffusion models with variable coefficients. Recently, we refer to the work by Bhrawy and Zaky et al. [12–16] for numerical methods for multidimensional space/time variable-order fractional differential equations, where approximate solutions of the forward variable-order fractional differential equations and spectral techniques are discussed based on orthogonal polynomials. On concrete applications of the fractional calculus models, we always encounter with inverse problems which need to determine some unknowns in the model using suitable additional data. As for inverse problems arising in the time/space fractional diffusion equations with constant fractional orders, we refer to [17–29], and recently see [30–32], etc. However, there are few studies on inverse problems associated with variable-order fractional differential equations to our knowledge. Jia and Peng gave a uniqueness result for the inverse source problem in the variable-order space fractional diffusion equation

using a Carleman type estimate [33], and Liu et al. [34] gave numerical inversions for determining the variable fractional order in the variable-order time fractional diffusion equation using the homotopy regularization algorithm.

Since the diffusion coefficient denotes the characteristics of the medium, the variable time fractional order is a key index denoting the correlations in the model, and they are always unknown in real-life problems, it is of much importance to determine these two variable-dependent parameters utilizing the inverse problem method. Although the uniqueness and stability of such a simultaneous inverse problem are very difficult to obtain at present, it is still meaningful to study its numerical inversions from numerics. In this paper, we deal with the simultaneous inversion problem of determining the space-dependent fractional order and diffusion coefficient in the variable-order time fractional diffusion equation utilizing the homotopy regularization algorithm. Such an inversion problem is more difficult than those discussed in the previous work by the following reasons:

(i) The model is complicated, which involves the variable-dependent fractional order and diffusion coefficient, and the computational complexity is high.

(ii) The inverse problem is to determine two kinds of parameters simultaneously, and both of the diffusion coefficient and the fractional order are spatially dependent functions such that the ill-posedness of the numerical inversion is much severe.

(iii) The approximate space for the unknowns is different. We employ the Legendre polynomials as the basis functions instead of the ordinary polynomials to perform the inversion algorithm.

Therefore it is much challenging to deal with the simultaneous inversion problem arising in the variable-order fractional diffusion equation. The most contribution of this paper is devoted to give effective numerical inversions for the simultaneous inversion problem utilizing the homotopy regularization in the framework of Legendre polynomials approximations.

The rest of the paper is organized as follows. In Section 2, a finite difference scheme for solving the forward problem is given. In Section 3, the homotopy regularization algorithm is introduced for solving the inverse problem of determining the fractional order and the diffusion coefficient simultaneously. In Section 4, numerical inversions with noisy data are performed in Legendre polynomial approximate space, and concluding remarks are given in Section 5.

2. The Forward Problem and Finite Difference Scheme

2.1. The Forward Problem. Denote $\Omega = (-L, L)$ and $\Omega_T = \Omega \times (0, T)$ for $L > 0$ and $T > 0$. Consider the variable-order time fractional diffusion equation with a space-dependent diffusion coefficient:

$$ {}_0D_t^{\gamma(x,t)} u(x,t) = \frac{\partial}{\partial x}\left(a(x)\frac{\partial u(x,t)}{\partial x}\right) + f(x,t), \quad (1)$$

where $u = u(x,t)$ is the state variable, $\gamma(x,t)$ is the fractional differential order depending upon the space and time variables $x \in \Omega$ and $t \in (0,T)$, $a(x) > 0$ is the diffusion coefficient, and $f(x,t)$ is a source term. Here $D_t^{\gamma(x,t)} u$ denotes the variable-order fractional derivative in the sense of Caputo, which is defined by

$$ {}_0D_t^{\gamma(x,t)} u(x,t) = \frac{1}{\Gamma(1-\gamma(x,t))}\int_0^t (t-\sigma)^{-\gamma(x,t)}\frac{\partial u(x,\sigma)}{\partial \sigma}d\sigma, \quad (2)$$

where $\Gamma(\cdot)$ is the Gamma function, and there exist two positive constants $\underline{\gamma}$ and $\overline{\gamma}$ such that

$$ 0 < \underline{\gamma} \le \gamma(x,t) \le \overline{\gamma} < 1, \quad (x,t) \in \Omega_T. \quad (3)$$

Given the initial value condition

$$ u(x,0) = u_0(x), \quad x \in \overline{\Omega}, \quad (4)$$

and the homogeneous boundary condition

$$ u(-L,t) = u(L,t) = 0, \quad t \in (0,T], \quad (5)$$

a forward problem is composed by (1) with the initial boundary conditions (4)-(5).

2.2. The Finite Difference Scheme. In this subsection, an implicit finite difference scheme is set forth to solve the forward problem (1), (4)-(5) numerically. Although a similar difference scheme has been discussed in [11], we still give it here for the completeness of this paper.

By discretizing the space domain by $x_i = -L + ih$ $(i = 0, 1, \ldots, M)$ and the time domain by $t_k = k\tau$ $(k = 0, 1, \ldots, N)$ and the variable-order fractional derivative by the difference discretization, we have

$$ {}_0D_t^{\gamma} u\big|_{(x_i,t_k)} = \frac{\tau^{-\gamma(x_i,t_k)}}{\Gamma(2-\gamma(x_i,t_k))}\sum_{j=0}^{k-1} b_{i,j}^k \left(u(x_i,t_{j+1}) - u(x_i,t_j)\right) + o(\tau), \quad (6)$$

where $b_{i,j}^k = (k-j)^{1-\gamma(x_i,t_k)} - (k-j-1)^{1-\gamma(x_i,t_k)}$ and $h = 2L/M$ is the space mesh step and $\tau = T/N$ is the time mesh step.

Next, denote $x_{i-1/2} = (1/2)(x_{i-1} + x_i)$ as the midpoint of the neighboring nodes x_{i-1} and x_i. The integer-order diffusion term in (1) is discretized by

$$ \frac{\partial}{\partial x}\left(a(x)\frac{\partial u}{\partial x}\right)\bigg|_{(x_i,t_k)} = \frac{1}{h}\left[a(x_{i+1/2})\frac{u(x_{i+1},t_k) - u(x_i,t_k)}{h} - a(x_{i-1/2})\frac{u(x_i,t_k) - u(x_{i-1},t_k)}{h}\right] + o(h^2). \quad (7)$$

Denoting $u_i^k \approx u(x_i,t_k)$, $\gamma_i^k = \gamma(x_i,t_k)$, $a_{i\pm1/2} = a(x_{i\pm1/2})$, and $f_i^k = f(x_i,t_k)$, substituting (6) and (7) into (1), and omitting high-order terms, we get

$$\frac{\tau^{-\gamma_i^k}}{\Gamma(2-\gamma_i^k)}\sum_{j=0}^{k-1} b_{i,j}^k \left(u_i^{j+1}-u_i^j\right) = \frac{1}{h}\left[a_{i+1/2}\frac{u_{i+1}^k-u_i^k}{h} - a_{i-1/2}\frac{u_i^k-u_{i-1}^k}{h}\right] + f_i^k. \tag{8}$$

The initial boundary value conditions are discretized as follows:

$$u_i^0 = u_0(x_i), \quad i = 0,1,\ldots,M, \tag{9}$$

and

$$u_0^k = U_M^k = 0, \quad k = 1,2,\ldots. \tag{10}$$

Finally, denote $p_i^k = \tau^{-\gamma_i^k}h^2/\Gamma(2-\gamma_i^k)$; (8) is arranged as

$$\begin{aligned} p_i^k &\left[u_i^k - \sum_{j=1}^{k-1}\left(b_{i,j}^k - b_{i,j-1}^k\right)u_i^j - b_{i,0}^k u_i^0\right] \\ &= a_{i+1/2}u_{i+1}^k - \left(a_{i+1/2}+a_{i-1/2}\right)u_i^k + a_{i-1/2}u_{i-1}^k \\ &\quad + h^2 f_i^k, \end{aligned} \tag{11}$$

for $k = 1,2,\ldots$. In the case of $k = 1$, there is

$$\begin{aligned} &-a_{i-1/2}u_{i-1}^1 + \left(p_i^1 + a_{i+1/2} + a_{i-1/2}\right)u_i^1 - a_{i+1/2}u_{i+1}^1 \\ &\quad = p_i^1 u_i^0 + h^2 f_i^1, \end{aligned} \tag{12}$$

and we have for $k > 1$

$$\begin{aligned} &-a_{i-1/2}u_{i-1}^k + \left(p_i^k + a_{i+1/2} + a_{i-1/2}\right)u_i^k - a_{i+1/2}u_{i+1}^k \\ &\quad = p_i^k\left[\sum_{j=1}^{k-1}\left(b_{i,j}^k - b_{i,j-1}^k\right)u_i^j + b_{i,0}^k u_i^0\right] + h^2 f_i^k. \end{aligned} \tag{13}$$

Denote $U^0 = (u_1^0, u_2^0, \ldots, u_{M-1}^0)^T$, and let

$$\begin{aligned} U^k &= \left(u_1^k, u_2^k, \ldots, u_{M-1}^k\right)^T, \\ F^k &= \left(f_1^k, f_2^k, \ldots, f_{M-1}^k\right)^T, \end{aligned} \tag{14}$$

for $k = 1,2,\ldots,N$. We rewrite (13) in a matrix form:

$$A_k U^k = \sum_{j=0}^{k-1} B_{j,k}U^j + h^2 F^k, \quad k = 1,2,\ldots, \tag{15}$$

where

$$A_k = \left(A_k^{i,j}\right)_{(M-1)\times(M-1)} \begin{pmatrix} p_1^k + a_{1+1/2} + a_{1-1/2} & -a_{1+1/2} & 0 & \cdots & 0 \\ -a_{2-1/2} & p_2^k + a_{2+1/2} + a_{2-1/2} & -a_{2+1/2} & \cdots & 0 \\ \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & -a_{M-1-1/2} & p_{M-1}^k + a_{M-1+1/2} + a_{M-1-1/2} \end{pmatrix} \tag{16}$$

for $k = 1,\ldots,N$, and

$$B_{j,k} = \begin{pmatrix} p_1^k\left(b_{1,j}^k - b_{1,j-1}^k\right) & 0 & \cdots & 0 \\ 0 & p_2^k\left(b_{2,j}^k - b_{2,j-1}^k\right) & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & p_{M-1}^k\left(b_{M-1,j}^k - b_{M-1,j-1}^k\right) \end{pmatrix} \tag{17}$$

for $k = 1,\ldots,N$ and $j = 0,1,\ldots,k-1$.

Lemma 1. *The coefficient matrix A_k ($k = 1,\ldots,N$) defined by (16) is strictly diagonally dominant, and the difference scheme (15) has only one solution.*

Proof. For any i,k, there is $p_i^k > 0$. Noting $a_i > 0$ there holds

$$\left|-a_{i+1/2} - a_{i-1/2}\right| < p_i^k + a_{i+1/2} + a_{i-1/2}, \tag{18}$$

which means that

$$\sum_{j=1, j\neq i}^{M-1}\left|A_k^{ij}\right| < A_k^{ii}. \tag{19}$$

This shows that the matrix A_k ($k = 1,\ldots,N$) is strictly diagonally dominant, and the difference scheme (15) has unique solution.

Table 1: The solutions error with time step ($h = 2\pi/100, t = 1$).

τ	1/50	1/150	1/300
Err	1.5149e-2	1.5147e-2	1.5144e-2
τ	1/400	1/500	1/800
Err	1.5139e-2	1.5118e-2	1.5015e-2

Table 2: The solutions error with space step ($\tau = 1/100, t = 1$).

h	$2\pi/50$	$2\pi/100$	$2\pi/150$
Err	3.0206e-2	1.5094e-2	1.0088e-2
h	$2\pi/200$	$2\pi/300$	$2\pi/400$
Err	7.6023e-3	5.1475e-3	3.9498e-3

Furthermore, it is not difficult to prove that the scheme (15) is of unconditional stability and convergence with a similar method as used in [34]. □

Lemma 2 (see [11, 34]). *The difference scheme (15) is unconditional stable and convergent to the exact solution of the forward problem for any finite time $T < \infty$.*

2.3. Numerical Testification. In this subsection, we give a numerical computation for the forward problem utilizing the difference scheme (15). The analytic solution is chosen as

$$u(x,t) = \sin(x)(1+t)^2, \tag{20}$$

and the model parameters are taken as follows:

$$\begin{aligned} L &= \pi, \\ \gamma(x,t) &= 0.5 + 0.15x, \\ a(x) &= 1 + x^2, \\ u_0(x) &= \sin(x), \end{aligned} \tag{21}$$

and the source term

$$\begin{aligned} f(x,t) &= \frac{2t^{1-\gamma(x,t)}\sin(x)}{\Gamma(2-\gamma(x,t))} + \frac{2t^{2-\gamma(x,t)}\sin(x)}{\Gamma(3-\gamma(x,t))} \\ &\quad - (t+1)^2\left[2x\cos(x) - \sin(x) - x^2\sin(x)\right]. \end{aligned} \tag{22}$$

The numerical results with different time step and space step are listed in Tables 1 and 2 at given $t = 1$, respectively. The relative error in the solutions is expressed by

$$Err = \frac{\left\|u(\cdot,1) - u^*(\cdot,1)\right\|_2}{\left\|u(\cdot,1)\right\|_2} \tag{23}$$

where $u(\cdot,1)$ and $u^*(\cdot,1)$ denote the exact and numerical solutions, respectively. Moreover, the numerical and exact solutions are plotted in Figure 1.

From Tables 1 and 2 and Figure 1, it can be seen that the error in the solutions gradually decreases as the grid step goes to small demonstrating that the numerical scheme is convergent, and the space step has a significant impact on the numerical scheme for given time step.

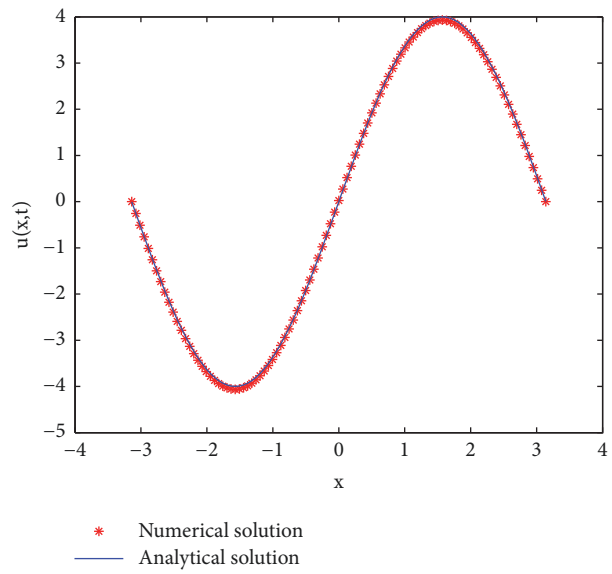

Figure 1: The numerical and analytical solutions at $t = 1$.

3. The Inverse Problem and Inversion Algorithm

3.1. The Inverse Problem. For the variable-order fractional diffusion equation (1), suppose that the variable fractional order $\gamma(x,t)$ and the space-dependent diffusion coefficient $a(x)$ are both unknown. Thus we encounter with an inverse problem that is to determine the variable diffusion coefficient and the fractional order simultaneously based on the forward problem and some additional measurements on the solution. Let $x_0 \in (-L, L)$ be the fixed measured point and t_j, $j = 1, \ldots, J$, be the measured time. The additional data are given as

$$u(x_0, t_j) = \theta_j, \quad t_j \in (0, T], \quad j = 1, \ldots, J, \tag{24}$$

and the vector

$$\Theta = (\theta_1, \theta_2, \ldots, \theta_J)^T \tag{25}$$

is called the measured data vector.

For simplicity we assume that the fractional order function is time independent on concrete inversions; i.e., the simultaneous inversion problem here is to determine the space-dependent functions $a(x)$ and $\gamma(x)$ based on the forward problem together with the overposed condition (24).

Together with the forward problem and a priori conditions for the diffusion coefficient and the fractional order, we give the following assumptions:

(A1) $a(x) \in C^1(\Omega)$ and $a(x) > 0$ for $x \in \Omega$;

(A2) $\gamma(x) \in C(\Omega)$, and $0 < \underline{\gamma} \le \gamma(x) \le \overline{\gamma} < 1$ for $x \in \Omega$.

Under the above assumptions for the unknowns we give the inversion algorithm in the following.

3.2. The Homotopy Regularization Algorithm. Let S denote the admissible space for the unknowns satisfying conditions (A1) and (A2), and

$$S_E = \left\{\langle a(x), \gamma(x)\rangle \in S : \|a\|_{C^1} + \|\gamma\|_\infty \le E\right\} \tag{26}$$

for some constant $E > 0$. For $\langle a(x), \gamma(x)\rangle \in S_E$ further assume that they have the generalized Fourier expansions

$$a(x) = \sum_{s=1}^{\infty} a_s \varphi_s(x) \tag{27}$$

and

$$\gamma(x) = \sum_{k=1}^{\infty} \gamma_k \varphi_k(x) \tag{28}$$

respectively, where $\{\varphi_i(x),\ i = 1, 2, \ldots\}$ is a group of basis functions in the continuous space and a_s $(s = 1, 2, \ldots)$ and γ_k $(k = 1, 2, \ldots)$ are the expansion coefficients of $a(x)$ and $\gamma(x)$, respectively. In numerical approximations, let

$$a^S(x) = \sum_{s=1}^{S} a_s \varphi_s(x) \tag{29}$$

and

$$\gamma^K(x) = \sum_{k=1}^{K} \gamma_k \varphi_k(x) \tag{30}$$

be the approximations to the exact functions $a(x)$ and $\gamma(x)$ and $S \geq 1$ and $K \geq 1$ be the dimensions of the approximate spaces, respectively. On concrete computations, it is convenient to set a S-dimensional vector $\mathbf{a} = (a_1, a_2, \ldots, a_S) \in \mathbf{R}^S$ corresponding to $a^S(x)$ and a K-dimensional vector $\mathbf{b} = (\gamma_1, \gamma_2, \ldots, \gamma_K) \in \mathbf{R}^K$ corresponding to $\gamma^K(x)$, respectively. Therefore, to get an approximate solution $\langle a^S(x), \gamma^K(x)\rangle$ is equivalent to finding a vector $(\mathbf{a}, \mathbf{b}) \in \mathbf{R}^S \times \mathbf{R}^K$. In what follows, we set $z := (\mathbf{a}, \mathbf{b}) = (a_1, a_2, \ldots, a_S, \gamma_1, \gamma_2, \ldots, \gamma_K)$ and equip with Euclid norm.

For given $z \in \mathbf{R}^S \times \mathbf{R}^K$, the forward problem is solved numerically whose solution is denoted by $u(z)(x, t)$. Then we get $u(z)(x_0, t_j)$ for $j = 1, 2, \ldots, J$ correspondingly to the measured time and position, which is denoted by a parameter-dependent vector

$$\begin{aligned} &V(z) \\ &\quad := \left(u(z)(x_0, t_1), u(z)(x_0, t_2), \ldots, u(z)(x_0, t_J)\right)^T. \end{aligned} \tag{31}$$

Combining with the additional conditions (24)-(25), we are to look for an optimal solution to the inverse problem from numerics by minimizing a cost functional between the measured data and the computational output data:

$$\min_{z \in \mathbf{R}^S \times \mathbf{R}^K} \quad \|V(z) - \Theta\|_2^2; \tag{32}$$

here and in the follows $\|\cdot\|_2$ denotes the Euclid norm. Furthermore by the homotopy idea and regularization method, we get the following minimization problem:

$$\min_{z \in \mathbf{R}^S \times \mathbf{R}^K} \quad \left\{(1-\lambda)\|V(z) - \Theta\|_2^2 + \lambda \|z\|_2^2\right\}, \tag{33}$$

where $\lambda \in (0, 1)$ is the homotopy parameter. The inversion algorithm is proceeded based on the minimization problem (33) by iteration, linearization, differentiation, and homotopy approximations as done in [27, 34, 35].

In fact, by linearization and numerical differentiation approximations, solving (33) is transformed to solve a normal equation combining with an iteration process:

$$\begin{aligned} &\left(\lambda I + (1-\lambda)\mathbf{G}^T\mathbf{G}\right)\delta z_n = (1-\lambda)\mathbf{G}^T\left(\Theta - V(z_n)\right), \\ &z_n + \delta z_n \longrightarrow z_{n+1}, \end{aligned} \tag{34}$$

where $\lambda \in (0, 1)$ is just the homotopy parameter which taking values from near 1 decreasingly approximating to 0, δz_n is a perturbation vector for any given $z_n \in \mathbf{R}^{S+K}$, n denotes the iterative number and z_0 is an initial iteration, and

$$\begin{aligned} &\mathbf{G} = \left(g_{ji}\right)_{J*(S+K)}, \\ &g_{ji} = \frac{u(z_n + \tau\mathbf{e}_i)(x_0, t_j) - u(z_n)(x_0, t_j)}{\tau}, \end{aligned} \tag{35}$$

where τ is the numerical differential step, $\mathbf{e}_i$ $(i = 1, 2, \ldots, S + K)$ is the unit basis vector in $\mathbf{R}^{S+K}$, and $V(z_n)$, Θ are the computational output and the measured data vectors, respectively. The algorithm can be terminated as long as an optimal perturbation satisfying the condition: $\|\delta z_n\|_2 \leq$ eps, where eps is the given convergent precision. On concrete inversions we employ the homotopy parameter by

$$\lambda = \lambda(n) = \frac{1}{1 + \exp\left(\beta(n - n_0)\right)}, \tag{36}$$

where n is the number of iterations, n_0 is the preestimated number, and $\beta > 0$ is the adjust parameter.

In the follows, we utilize the above inversion algorithm to perform numerical inversions for the simultaneous inverse problem. We always set the initial iteration $z_0 = \mathbf{0}$, the differential step $\tau = 1e-2$, the preestimated number $n_0 = 5$, and the adjust parameter $\beta = 0.5$ in the homotopy parameter given by (36) if there is no specification. Moreover, we choose $M = 100$ and $N = 100$ in computation of the solution to the forward problem using the finite scheme (15). The step-by-step procedures of the inversion algorithm are given as follows.

Step 1. Give an approximate space composed by orthogonal polynomials on $x \in [-L, L]$, choose $K \geq 1$, $S \geq 1$ as the dimensions of the approximate spaces for $a(x)$ and $\gamma(x)$, respectively, choose an initial iteration vector $z_n \in \mathbf{R}^{S+K}$ $(n = 0, 1, \ldots)$, numerical differentiation step τ, and convergent precision *eps*, and give the additional data vector Θ.

Step 2. Solve the forward problem with the difference scheme (15) to get $u(z_n)(x, t)$ and the vector $V(z_n)$ and then to obtain the matrix G by formula (35).

Step 3. Choose the homotopy parameter via (36) to work out an optimal perturbation δz_n by formula (34).

Step 4. If there holds $\|\delta z_n\|_2 \leq eps$, then the inversion algorithm can be terminated, and $z_{n+1} = z_n + \delta z_n$ is taken

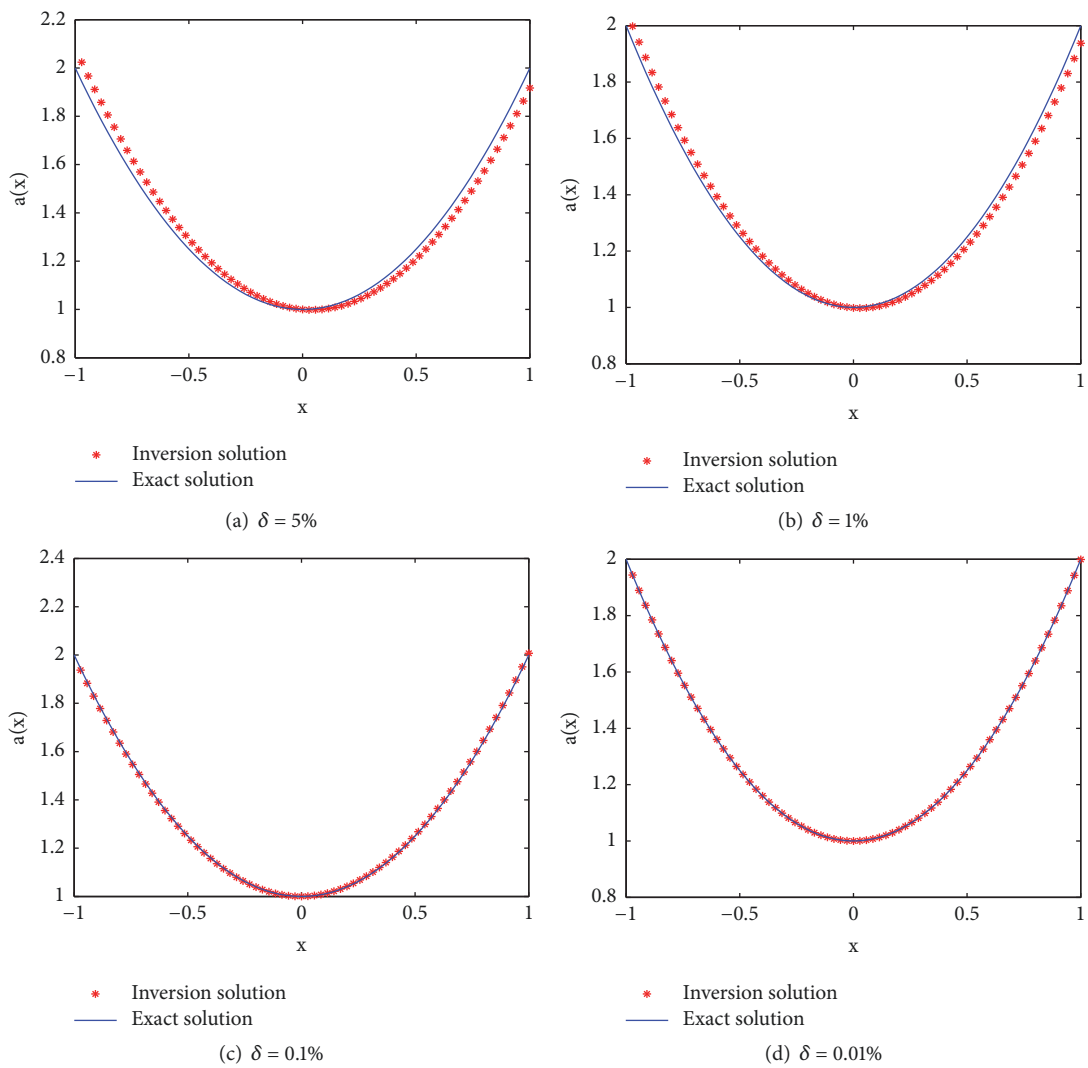

Figure 2: The inversion and exact diffusion coefficients in Section 4.2.1.

as the inversion solution vector. Otherwise, go to Step 2 by replacing z_n with z_{n+1}.

4. Numerical Inversion

Consider the forward problem for $(x,t) \in (-1,1)\times(0,T)$ and $T > 0$

$$\begin{aligned} {}_0D_t^{\gamma(x)}u &= \frac{\partial}{\partial x}\left(a(x)\frac{\partial u)}{\partial x}\right) + f(x,t), \\ u(x,0) &= u_0(x), \\ u(-1,t) &= u(1,t) = 0. \end{aligned} \tag{37}$$

We will perform numerical inversions for determining $a(x)$ and $\gamma(x)$ on $x \in [-1,1]$ simultaneously by the additional data $\{u(0,t)\}$ utilizing the homotopy regularization algorithm introduced in Section 3.2.

4.1. Inversion with Accurate Data. Let the exact solution of the forward problem be $u(x,t) = 10(1-x^2)(1+t)^2$, and the exact diffusion coefficient and the fractional order be $a(x) = 1 + x^2$ and $\gamma(x) = 1/2 + (3/20)x$ on $x \in [-1,1]$, respectively. The additional data are obtained by computing the forward problem using the exact diffusion coefficient and the fractional order, and we utilize Legendre polynomials as the basis functions in the approximate spaces. In the process of numerical implementation, let the approximate diffusion coefficient $a^S(x) \in \operatorname{span}\{1, x, (3x^2-1)/2, (5x^3-3x)/2\}$, and the approximate fractional order $\gamma^K(x) \in \operatorname{span}\{1, x, (3x^2-1)/2\}$. In other words, the exact solution of the simultaneous inversion problem in the finite-dimensional approximate space is expressed via

$$z = \left(\frac{4}{3}, 0, \frac{2}{3}, 0, \frac{1}{2}, \frac{3}{20}, 0\right). \tag{38}$$

By utilizing the inversion algorithm with accurate data, the inversion results are obtained which listed in Table 3, where J denotes the dimension of the space of the additional data, which is the number of the additional data measured at $x = 0$ and t_j $(j = 1, 2, \ldots, J)$, Err denotes the relative error in

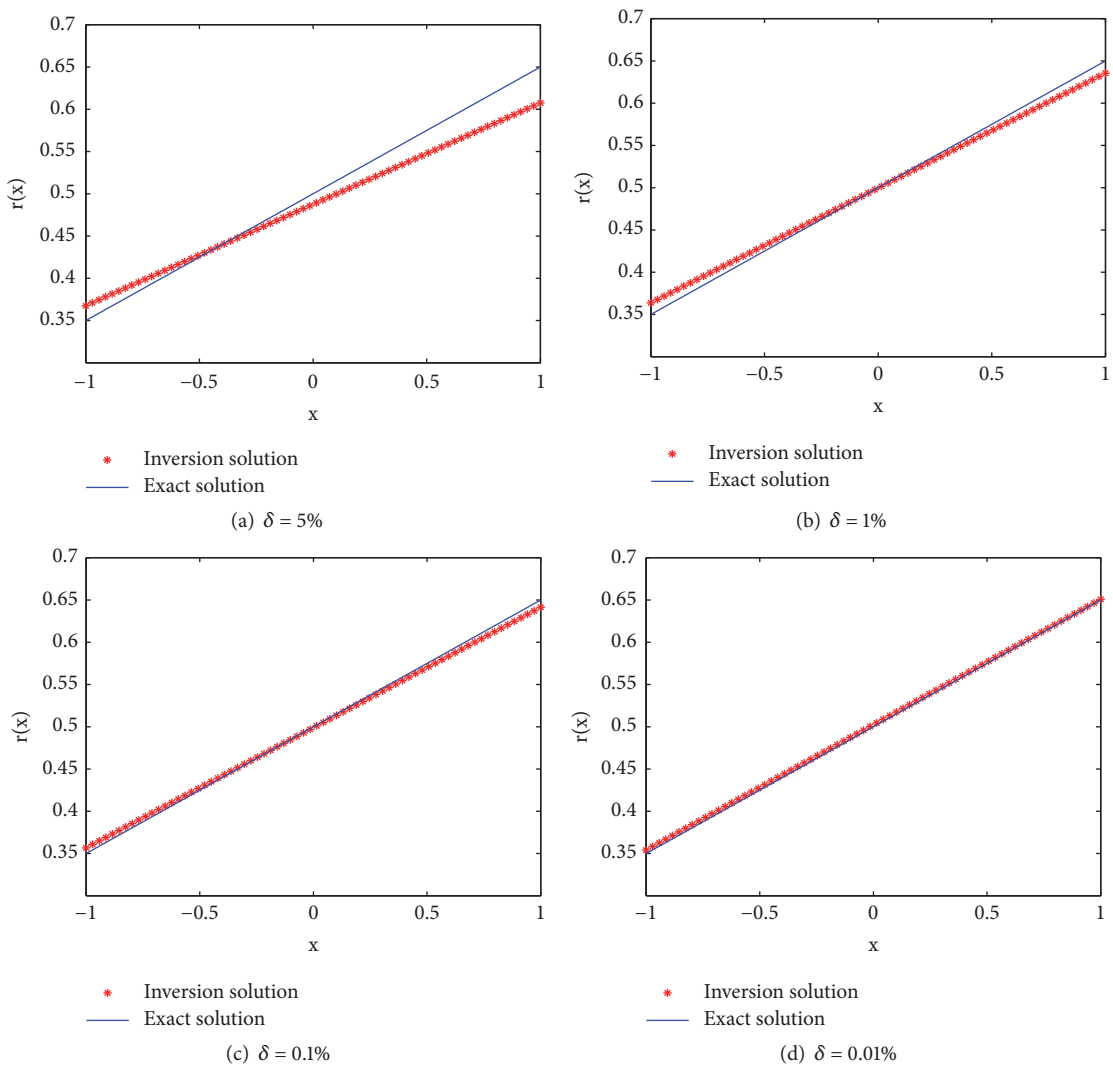

Figure 3: The inversion and exact fractional orders in Section 4.2.1.

Table 3: Impact of the number of additional data on the inversion algorithm.

J	Err	$\overline{z}$
4	8.89e-1	(1.5712, -0.0203, 0.7298, 0.0006, 0.6015, 0.1382, 0.0001)
8	1.20e-5	(1.3335, -0.0001, 0.6679, 0.0002, 0.5039, 0.1496, -0.0013)
20	1.66e-9	(1.3334, -0.0000, 0.6667, 0.0000, 0.5001, 0.1499, -0.0001)
30	1.76e-10	(1.3333, -0.0000, 0.6666, 0.0000, 0.5000, 0.1500, 0.0000)

the exact and inversion solutions, and $\overline{z}$ denotes the vector of the inverse solution.

It can be seen from Table 3 that the inversion results give good approximations to the exact solutions even using a few of measurements, and the solutions error goes to small rapidly as the number of the additional data increasing.

4.2. Inversion with Noisy Data. In this subsection, we implement the inversion algorithm using noisy data. Suppose the disturbed additional data vector is expressed by

$$\Theta^{\delta} = \Theta\left(1 + \xi\delta\right), \tag{39}$$

where $\delta > 0$ is the noise level and ξ is an random number ranged in $[-1, 1]$.

4.2.1. Inversion for Polynomial Coefficient. Take the model and parameters as utilized in Section 4.1, i.e., the exact solution to the inverse problem is also set to be the vector given by (38), and the inversion is performed using noisy data given by (39). Noting that the computational result for each iteration is different due to the random noise in the data, we repeat the inversion processes ten times with each noise level and take an average of gained inversion results. The average inversion results are listed in Table 4, where $\overline{z}^{ave}$

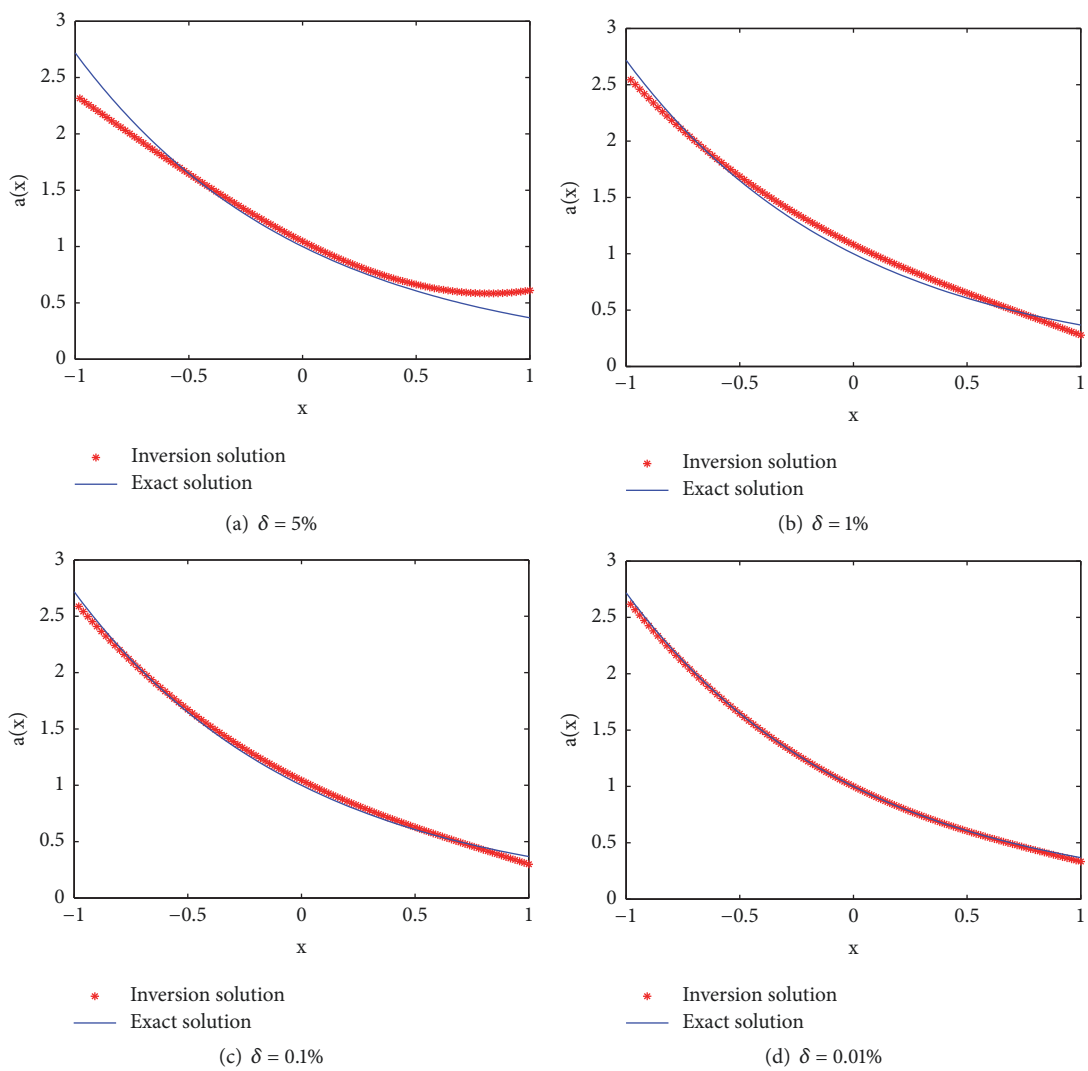

Figure 4: The inversion and exact diffusion coefficients in Section 4.2.2.

Table 4: The average inversion results with noise data in Section 4.2.1.

δ	Err	$\overline{z}^{ave}$
5%	5.26e-1	(1.2415, -0.1093, 0.5897, 0.0093, 0.4774, 0.1234, 0.0213)
1%	4.64e-2	(1.3132, 0.0091, 0.6647, 0.0012, 0.5167, 0.1611, 0.0026)
0.1%	3.70e-3	(1.3229, -0.0026,0.6659, 0.0007, 0.5098, 0.1543, 0.0009)
0.01%	5.87e-4	(1.3330, 0.0001, 0.6660, 0.0003, 0.5039, 0.1497, 0.0001)

denotes the average of the inversion solutions and $Err = \|\overline{z}^{ave} - z\|_2 / \|z\|$ is the relative error in the solutions. Moreover, the average inversion results and the exact solutions for the diffusion coefficient and the fractional order with different noise levels are plotted in Figures 2(a)–2(d) and Figures 3(a)–3(d), respectively.

It can be seen from Table 4 and Figures 2 and 3 that the inversion algorithm is stable from numerics for the simultaneous inversion problem of determining the space-dependence diffusion coefficient and variable fractional order; however, the inversion becomes difficult when the noise level is greater than 5% showing that the joint inversion here is of high ill-posedness.

Remark 3. If utilizing the ordinary polynomials as the basic functions of the approximate space to perform the inversion algorithm with noisy data, the average inversion results are listed in Table 5.

It can be seen from Table 5 as compared with Table 4 that the inversion results with ordinary polynomials in the approximate spaces are worse than those of using Legendre orthogonal polynomials.

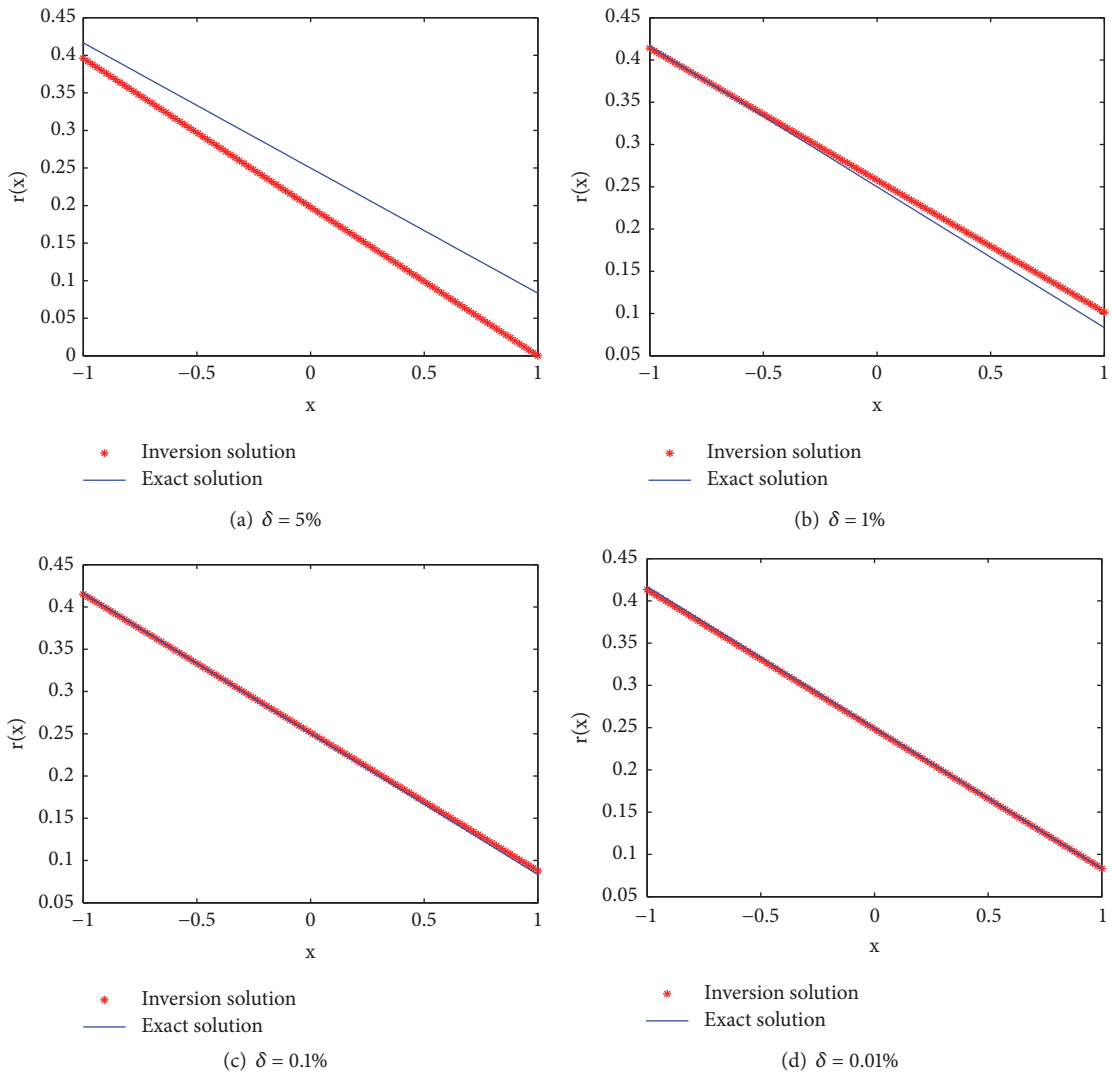

Figure 5: The inversion and exact fractional orders in Section 4.2.2.

Table 5: The inversion results with ordinary polynomials as basic functions in Section 4.2.1.

δ	Err	$\overline{z}^{ave}$
5%	5.270e-1	(0.9992, 0.0231, 0.9964, 0.0024, 0.4703, 0.0881, -0.0313)
1%	1.097e-1	(1.0002, 0.0044, 1.0018, 0.0072, 0.5039, 0.1648, 0.0031)
0.1%	7.680e-2	(1.0001, 0.0020, 0.9999, -0.0003, 0.4992, 0.1484, 0.0020)
0.01%	6.318e-3	(1.0000, -0.0005, 1.0001, -0.0004, 0.5002, 0.1504, 0.0003)

4.2.2. Inversion for Exponential Coefficient. Also let the exact solution to the forward problem (37) be $u(x,t) = 10(1 - x^2)(1+t)^2$, but let the exact diffusion coefficient be $a(x) = e^{-x}$ and the exact fraction order be $\gamma(x) = 1/4 - (1/6)x$. Since the exponential function has the truncated error in the polynomials space, we choose $S = 5$ to approximate the diffusion coefficient and also $K = 3$ to approximate fractional order on the concrete inversions. Therefore, the exact solution vector to the inverse problem here in the approximate space is expressed by

$$z = \left(\frac{47}{40}, -\frac{11}{10}, \frac{5}{14}, -\frac{1}{15}, \frac{1}{105}, \frac{1}{4}, -\frac{1}{6}, 0\right). \tag{40}$$

Like done in Section 4.2.1, the inversion is performed utilizing the same parameters with noisy data. The average inversion results are listed in Table 6, and the exact and inversion solutions with different noise levels are plotted in Figures 4(a)–4(d) and Figures 5(a)–5(d), respectively. From Table 6 and Figures 4 and 5, we can be see that the inversion has a similar result as shown in Section 4.2.1, and the inversion algorithm is testified again to be efficiency for the simultaneous inversion problem.

4.2.3. Inversion for Nonlinear Order Function. In this subsection we perform numerical inversions for the fractional

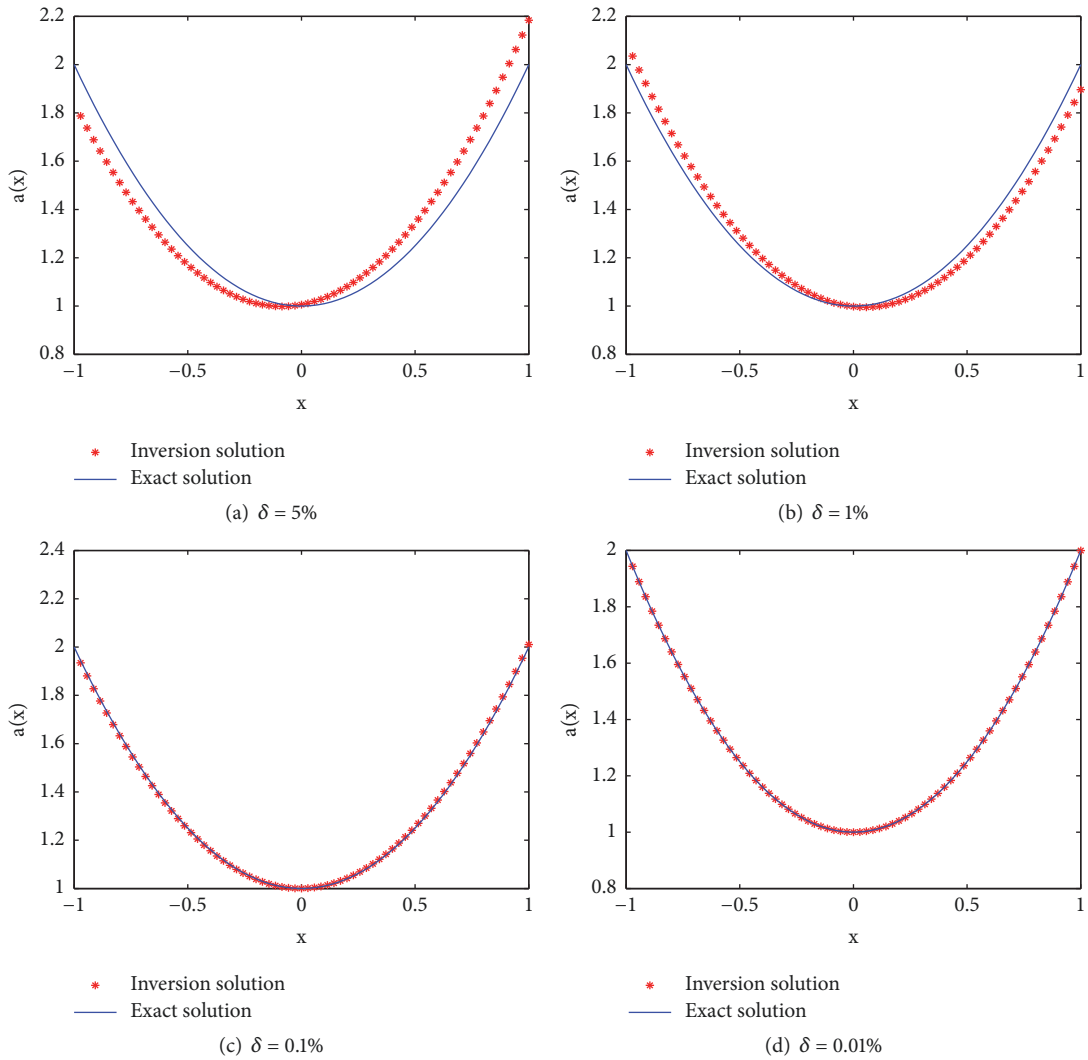

Figure 6: The inversion and exact diffusion coefficients in Section 4.2.3.

Table 6: The average inversion results with noise data in Section 4.2.2.

δ	Err	$\overline{z}^{ave}$
5%	8.76e-1	(1.1902, -1.2157, 0.3707, -0.07414, -0.0097, 0.2166, -0.1649, 0.0098)
1%	1.33e-1	(1.1825, -1.2670, 0.3312, -0.0698, 0.0123, 0.2621, -0.1746, 0.0213)
0.1%	2.35e-2	(1.1801, -1.1101, 0.3395, -0.0701, 0.0113, 0.2509, -0.1667, -0.0001)
0.01%	9.60e-3	(1.1787, -1.1000, 0.3486, -0.0669, 0.0100, 0.2498, -0.1662, 0.0001)

order taking nonlinear functions. Let the exact solution of the forward problem be $u(x,t) = 10(1-x^2)(1+t)^2$ and the exact diffusion coefficient also be $a(x) = 1 + x^2$ but the fractional order be $\gamma(x) = 1/2 + (\sin(x) + x^2)/4$ for $x \in [-1,1]$. On the finite-dimensional Legendre polynomials approximation we choose $S = 4$ to approximate the diffusion coefficient and $K = 5$ to approximate the fractional order, and the exact solution vector of the inverse problem is expressed via

$$z = \left(\frac{4}{3}, 0, \frac{2}{3}, 0, \frac{7}{12}, \frac{9}{40}, \frac{1}{6}, -\frac{1}{60}, 0\right). \tag{41}$$

Like done in the above, the inversion is performed utilizing the same parameters with noisy data. The average inversion results are listed in Table 7, and the exact and inversion solutions with different noise levels are plotted in Figures 6(a)–6(d) and Figures 7(a)–7(d), respectively. By Table 7 and Figures 6 and 7, we can see again that the inversion solutions approximate to the exact solutions as the noise level goes to small demonstrating the efficiency of the simultaneous inversion algorithm.

5. Conclusion

The simultaneous inverse problem of determining space-dependent diffusion coefficient and fractional order in the

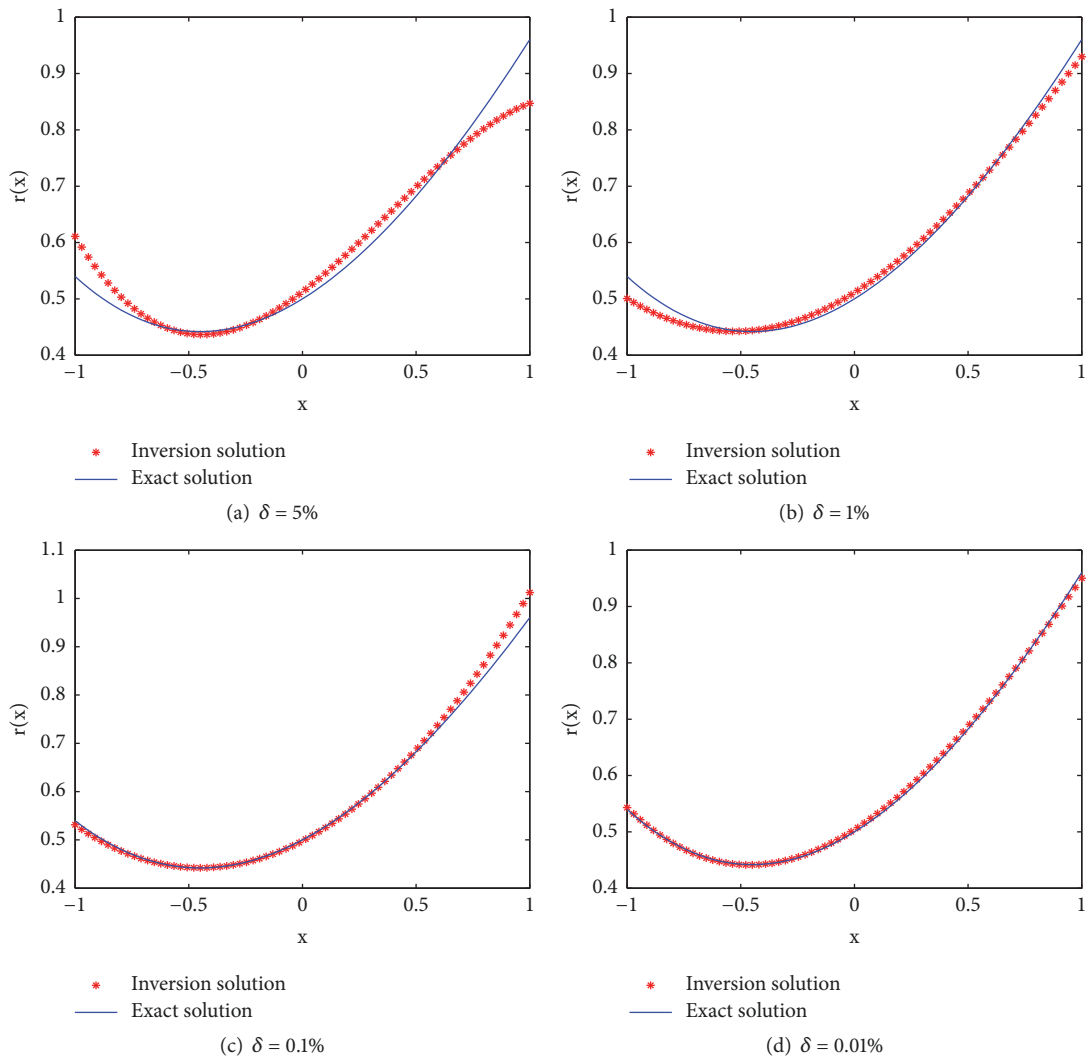

Figure 7: The inversion and exact fractional orders in Section 4.2.3.

Table 7: The average inversion results with noisy data in Section 4.2.3.

δ	Err	$\overline{z}^{ave}$
5%	2.8e-1	(1.3340, 0.0010, 0.6789, 0.0015, 0.5730, 0.2301, 0.1591, -0.0159, -0.0009)
1%	5.4e-2	(1.3334, -0.0019, 0.6671, 0.0015, 0.5830, 0.2245, 0.1667, -0.0167, 0.0012)
0.1%	6.1e-3	(1.3331, 0.0007, 0.6663, 0.0008, 0.5325, 0.2253, 0.1663, -0.0164, 0.0008)
0.01%	1.2e-3	(1.3332, -0.0002, 0.6667, 0.0001, 0.5832, 0.2249, 0.1665, -0.0165, -0.0000)

variable-order time fractional diffusion equation is investigated by using measurements at one interior point. Generally speaking, an inverse problem of determining multiple types of coefficients in a PDE is more difficult than that of determining a single type of coefficient, especially for the variable-order fractional differential equations. However, numerical inversions for such kinds of inverse problems can be performed successfully by the homotopy regularization algorithm using Legendre polynomials as the basis functions of the approximate space. The inversion becomes very difficult when the fractional order is time-space-dependent and the noise level is greater than 1%. We will focus our attention on the uniqueness of such inverse problems and theoretical analysis for the inversion algorithm utilizing limited additional data in the near future.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Acknowledgments

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