Hindawi
Mathematical Problems in Engineering
Volume 2019, Article ID 2912301, 8 pages
https://doi.org/10.1155/2019/2912301

Research Article

The Primal-Dual Active Set Method for a Class of Nonlinear Problems with T-Monotone Operators

Xiahui He and Peng Yang

Department of Business Administration, Hunan University of Finance and Economics, Changsha, Hunan 410205, China

Correspondence should be addressed to Peng Yang; pengyang947@gmail.com

Received 6 December 2018; Accepted 25 February 2019; Published 17 March 2019

Academic Editor: Vyacheslav Kalashnikov

The family of primal-dual active set methods is drawing more attention in scientific and engineering applications due to its effectiveness and robustness for variational inequality problems. In this work, we introduce and study a primal-dual active set method for the solution of the variational inequality problems with T-monotone operators. We show that the sequence generated by the proposed method globally and monotonously converges to the unique solution of the variational inequality problem. Moreover, the convergence rate of the proposed scheme is analyzed under the framework of the algebraic setting; i.e., the established convergence results show that the iteration number of the methods is bounded by the number of the unknowns. Finally, numerical results show that the efficiency can be achieved by the primal-dual active set method.

1. Introduction

The variational inequality problem associated with T-monotone operators has many applications, e.g., the diffusion problem involving Michaelis-Menten or second-order irreversible reactions; see, for example, [1–7] and the references therein for details. Hence, there is a growing interest in finding robust and efficient methods for solving this kind of complementarity problems, which reflects in an increasing number of proposals of numerical methods for its solution in recent years. It is well known that the projected relaxation method [8–10] is a popular solution technique for this class of complementarity problems. A great advantage of this approach is that it is easy to implement and can be convergent for both problems with symmetric and nonsymmetric operator. However, the convergence of this kind of methods depends crucially on the choice of the relaxation parameter. Another popular approach for the solution of variational inequality problems is the Schwarz algorithm [1, 7, 11–14], which is based on the framework of domain decomposition methods [15–17]. A major advantage of the Schwarz methods is amenable to implement and its convergence rate will not be deteriorated with the refinement of the mesh size when it is applied to the system arising from the discretization of partial differential equations (PDEs). The theory of monotone and global convergences for the classical Schwarz algorithms is also obtained. However, this kind of methods depends on the shape of the computational domain. To fix these issues, the family of active set strategies [18–24] can be used to solve variational inequality problems in an efficient way, which is the focus of this work.

The active set method consists of two major steps: in the first phase, an index set is decomposed into active and inactive parts with respect to the solution vector, based on a criterion specifying a certain active set method; and then in the second phase, a reduced system associated with the active and inactive sets is solved. We briefly mention a few related publications that partially motivated our current work. In [22], Kanzow shows that the primal-dual active set algorithm is an efficient and accurate method for large-scale linear complementarity problems. In [25], Puterman and Brumelle analyze the convergence properties for the primal-dual active set method with continuous state and control spaces. They showed that the algorithm is equivalent to a Newton's method, but under very restrictive assumptions which were not easily verifiable. In [24, 26], an active-set method with nonlinear elimination is proposed for the fully implicit simulation of two-phase flows. In the proposed algorithm, a variational inequality formulation of two-phase flow problems is used to avoid nonphysical undershoot or overshoot of the saturation fractions, and employ a class of active-set reduced-space algorithms to solve the resultant nonlinear complementarity system arising at each implicit time step. In particular, [18] builds the relationship between the primal-dual active set method and the semismooth Newton method and shows that the proposed method is a specific case of the Newton-type method under reasonable assumptions.

In general, there are two ways to study convergence of primal-dual active set methods for solving variational inequality problems [18, 27]. The first one is to prove that the method produces a monotone sequence, in which the initial guess often starts from a super-solution or a lower-solution of the problem. Convergence theorems established in this way are often based on the assumption that the matrix from the linear system is an M-matrix, and the corresponding variational inequality is usually restricted to the unilateral case. The other way is to prove that the active set method generates a minimizing iterative sequence under some objective function, which is usually used to the variational inequality with the bilateral case. In the later case, the matrix is often supposed to be symmetric and positive definite. However, when the primal-dual active set method is used to solve the variational inequality problem with T-monotone operators, there is not a general convergence theorem, owing to the fact that it is difficult to find such a merit function. In this paper, we get the equivalent relation between the primal-dual active set method and Howard's algorithm introduced in [9, 28, 29] for the variational inequality problem with T-monotone operators, then show the convergence theorem of Howard's algorithm, and thus obtain the convergence theorem of the primal-dual active set method. Using the equivalence of the proposed algorithms for variational inequality problems with T-monotone operators, we give a simple proof for the global monotone convergence of the primal-dual active set method and conclude that the primal-dual active set method converges in no more than $n+1$ iterations, where n is the size of the solution vector. To the best of the authors' knowledge, this is the first attempt to apply the primal-dual active set method for the variational inequality problem associated with T-monotone operators.

The rest of the paper is organized as follows. In Section 2, we present some notations and model problem and give some preliminaries which are used throughout the paper. In Section 3, the primal-dual active set method for the bilateral variational inequality problem is proposed. We establish the equivalence between Howard's algorithm and the primal-dual active set method in Section 4. Finally, in Section 5 we report some numerical results for the proposed methods, and the paper is concluded in Section 6.

2. Preliminaries

In this section, we present some notations and the model problem and give some preliminaries that are used throughout the paper. First of all, we introduce some notations. Let $N = \{1, 2, \ldots, n\}$ denote an index set. For any index sets $I, J \subseteq N$, A_{IJ} is denoted as the submatrix of a matrix $A \in \mathbb{R}^{n\times n}$ that consists of $a_{ij} (i \in I, j \in J)$, and r_I is defined as the subvector of a vector $r \in \mathbb{R}^n$ consisting of $r_i (i \in I)$. Let K be a subset of $\mathbb{R}^n$ and f an operator from K to $\mathbb{R}^n$. Let any element $v \in K$ be expressed by $v = v^+ + v^-$ with $v^+ = \max\{v, 0\}$, $v^- = \min\{v, 0\}$. Then the notion of T-monotone is defined as follows.

Definition 1. A function f is called T-monotone over the subspce $K \subseteq \mathbb{R}^n$, if it satisfies

$$\left\langle f(v) - f(w), (v - w)^+ \right\rangle \geq 0, \quad \forall v, w \in K, \tag{1}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product.

Definition 2. A function f is called strictly T-monotone over the subspce $K \subseteq \mathbb{R}^n$, if, for all $v, w \in K$, $\langle f(v) - f(w), (v - w)^+ \rangle = 0$ is equivalent to $(v - w)^+ = 0$.

Based on this definition, in this study we consider variational inequality problems associated with T-monotone operators as follows.

Problem 3. Let $K = \{v \in \mathbb{R}^n : \phi \leq v \leq \psi\}$, ϕ, ψ be given vectors in $\mathbb{R}^n$ satisfying $\phi \leq \psi$, and f is a continuous and strictly T-monotone operator. Then the variational inequality problem associated with T-monotone operators is defined by

$$\begin{aligned} &\text{find} \quad x \in K, \\ &\text{such that} \quad \left\langle f(x), v - x \right\rangle \geq 0, \quad \forall v \in K. \end{aligned} \tag{2}$$

T-monotone operators are a kind of important operators, which include many linear and nonlinear elliptic operators; see, e.g., [1, 3, 7]. One of the advantages of T-monotone operators is that the relevant algorithms often possess monotone convergence property. We would like to point out that problem (2) has a unique solution; see, e.g., [27, 30]. Moreover, if all components of vector ϕ become $-\infty$ and ψ become $+\infty$, then problem (2) reduces to the system of nonlinear equations

$$f(x) = 0. \tag{3}$$

Hence, system (3) also has a unique solution when f is a continuous and strictly T-monotone operator. Moreover, the T-monotone operator has the following properties.

Lemma 4. *Let I, J be the subsets of the index set $N = \{1, 2, \ldots, n\}$ that satisfies $J = N \setminus I$. For any vectors $y, z \in K$, if the subvector $y_I = z_I$ with $y_J \geq z_J$, and the function f is a continuous T-monotone operator over the subspace K, then $f_I(y) \leq f_I(z)$.*

Proof. Let the index set $\widehat{I}$ be defined as

$$\widehat{I} = \{i \in I : f_i(y) > f_i(z)\} \tag{4}$$

and $\widehat{J} = N \setminus \widehat{I}$. Without loss of generality, the set $\widehat{I}$ is denoted as $\{1, 2, \ldots, k\}$ and $\widehat{J} = \{k + 1, \ldots, n\}$.

In the following, we prove it by contradiction based on the assumption that $\widehat{I}$ is not empty. The set w_1 is defined as

$$w_1 = \{z_1 + \delta_1, \ldots, z_k + \delta_k, z_{k+1}, \ldots, z_n\} \tag{5}$$

and $w_2 = y$ with δ_i being a positive integer for all $i \in \widehat{I}$. If δ_i is small enough, then we have $f_i(w_1) < f_i(w_2)$ for all $i \in \widehat{I}$ by the continuity of the operator f. In addition to that, since we have $(w_1 - w_2)^+_{\widehat{J}} = 0$, then

$$\begin{aligned} 0 &\leq \left\langle f(w_1) - f(w_2), (w_1 - w_2)^+ \right\rangle \\ &= \sum_{i=1}^{k} \delta_i \left(f_i(w_1) - f_i(w_2)\right) < 0, \end{aligned} \tag{6}$$

which is a contradiction. Therefore, $\widehat{I} = \emptyset$ and we can get the conclusion $f_I(y) \leq f_I(z)$. □

Lemma 5. *Let the function f be a continuous strictly T-monotone operator over K and I, J be the subsets of the index set $N = \{1, 2, \ldots, n\}$ that satisfies $J = N \setminus I$. For any vectors $y, z \in K$, if $y_I \leq z_I$ and $f_J(y) \leq f_J(z)$, then $y \leq z$.*

Proof. Let $\widehat{I} = \{i \in N : y_i \leq z_i\}$ and $\widehat{J} = N \setminus \widehat{I}$. Similarly, without loss of generality, let $\widehat{I} = \{1, 2, \ldots, k\}$ and $\widehat{J} = N \setminus \widehat{I}$.

In the following, we also prove it by contradiction that $\widehat{J}$ is not empty. By the definition of $\widehat{I}$, we have that $I \subset \widehat{I}$ and $\widehat{J} \subset J$. Moreover, we have

$$\begin{aligned} f_{\widehat{J}}(y) &\leq f_{\widehat{J}}(z), \\ y_{\widehat{J}} &> z_{\widehat{J}}, \\ y_{\widehat{I}} &\leq z_{\widehat{I}}. \end{aligned} \tag{7}$$

Moreover, we know that

$$\begin{aligned} f_{\widehat{J}}\left(z_{\widehat{I}}, y_{\widehat{J}}\right) &\leq f_{\widehat{J}}\left(y_{\widehat{I}}, y_{\widehat{J}}\right) = f_{\widehat{J}}(y) \leq f_{\widehat{J}}(z) \\ &= f_{\widehat{J}}\left(z_{\widehat{I}}, z_{\widehat{J}}\right), \end{aligned} \tag{8}$$

where the first inequality comes from (7) and Lemma 4.

Let $w_1 = (z_{\widehat{I}}, y_{\widehat{J}})$ and $w_2 = z$, by the definition of the T-monotone operator; then we obtain that

$$\begin{aligned} 0 &\leq \left\langle f(w_1) - f(w_2), (w_1 - w_2)^+ \right\rangle \\ &= \left\langle f_{\widehat{J}}(w_1) - f_{\widehat{J}}(w_2), \left(y_{\widehat{J}} - z_{\widehat{J}}\right)^+ \right\rangle \leq 0, \end{aligned} \tag{9}$$

where the second inequality comes from $f_{\widehat{J}}(w_1) \leq f_{\widehat{J}}(w_2)$ and $(y_{\widehat{J}} - z_{\widehat{J}})^+ > 0$. Hence

$$\left\langle f_{\widehat{J}}(w_1) - f_{\widehat{J}}(w_2), \left(y_{\widehat{J}} - z_{\widehat{J}}\right)^+ \right\rangle = 0, \tag{10}$$

which means $y_{\widehat{J}} - z_{\widehat{J}} \leq 0$, since f is a strictly T-monotone operator. This is a contradiction to (7), and it means that $y \leq z$. □

3. Primal-Dual Active Set Method for the Variational Inequality Problem

In this section, we use the family of primal-dual active set methods [18–23] for solving the variational inequality problem (2) and then establish the equivalence between the primal-dual active set method and Howard's algorithm [9, 28, 29]. The focus of this study is on the following equivalent problem of the model problem (2):

$$\begin{aligned} f(x) - \lambda &= 0, \\ \mathscr{B}(x, \lambda) &= 0, \end{aligned} \tag{11}$$

where

$$\mathscr{B}(x, \lambda) \coloneqq \max\{\min\{\lambda, c(x - \phi)\}, c(x - \psi)\}. \tag{12}$$

Here, the max-operation or min-operation is understood componentwise; $c > 0$ is a constant. The primal-dual active set method is based on using (12) as a prediction strategy. In the method, an index set is partitioned into active and inactive parts, based on a criterion specifying a certain active set method; i.e., given a current primal-dual pair (x, λ), the choice for the next active and inactive sets is given by

$$\begin{aligned} \mathscr{J}_\phi &= \{i \in N : \lambda_i - c(x - \phi)_i > 0\}, \\ \mathscr{J}_\psi &= \{i \in N : \lambda_i - c(x - \psi)_i < 0\}, \end{aligned} \tag{13}$$

and $\mathscr{J} = N \setminus (\mathscr{J}_\phi \cup \mathscr{J}_\psi)$. Below, we present a high level description of the basic algorithm for a general problem in Algorithm 1.

Remark 6. The convergence theorem of the primal-dual active set method for the variational inequality problem with linear operator is based on a merit function, which is to prove that the proposed method generates a minimizing sequence for the merit function [23]. In this case, the matrix from the discretization of the linear operator is often supposed to be symmetric and positive definite. Theoretically, this condition number estimate cannot be applied immediately to the family of variational inequality problems with T-monotone operators, since these operators do not have these properties and we cannot use the technique of the merit function to get the convergence of Algorithm 1.

In this study, we establish the equivalence between and the primal-dual active set method and Howard's algorithm [9, 28, 29] for the solution of (2), and then we obtain the convergence theorem of Howard's algorithm. The use of Howard's algorithm is based on the following equivalent problem of (2):

$$\begin{aligned} &\text{find} \quad x \in \mathbb{R}^n, \\ &\text{such that} \quad \max\{\min\{f(x), x - \phi\}, x - \psi\} = 0. \end{aligned} \tag{14}$$

And then we define the following two functions $F, G \in \mathbb{R}^n$ by

$$\begin{aligned} F(x) &\coloneqq \min\{f(x), x - \phi\}, \\ G(x) &\coloneqq \max\{F(x), x - \psi\}, \end{aligned} \tag{15}$$

Step 1. Initialize x^0 and λ^0. Set $k := 0$.
Step 2. Determine the active and inactive sets by

$$\mathcal{J}_{\phi}^{k} = \left\{i \in N : \lambda_i^k - c\left(x^k - \phi\right)_i > 0\right\},$$
$$\mathcal{J}_{\psi}^{k} = \left\{i \in N : \lambda_i^k - c\left(x^k - \psi\right)_i < 0\right\}.$$

and $\mathcal{J}^k = N \setminus (\mathcal{J}_{\phi}^k \cup \mathcal{J}_{\psi}^k)$.
Step 3. Let x^{k+1} and λ^{k+1} be the solution of the nonlinear system

$$\begin{aligned} f(x^{k+1}) - \lambda^{k+1} &= 0, \\ x^{k+1} &= \phi \quad \text{on } \mathcal{J}_{\phi}^k, \\ x^{k+1} &= \psi \quad \text{on } \mathcal{J}_{\psi}^k, \\ \lambda^{k+1} &= 0 \quad \text{on } \mathcal{J}^k. \end{aligned}$$

Step 4. If $k \geq 1$ and $x^k = x^{k-1}$, then stop. Otherwise set $k := k + 1$ and return to Step 2.

Algorithm 1: The primal-dual active set method.

Step 1. Initialize β^0 in $\mathcal{A} = \{0, 1\}^n$, and set $k := 0$.
Step 2. Find $x^k \in \mathbb{R}^n$ such that $F^{\beta^k}(x^k) = 0$. If $k \geq 1$ and $x^k = x^{k-1}$, then stop. Otherwise go to Step 3.
Step 3. For every $i \in N$, take

$$\beta^{k+1} := \begin{cases} 0 & \text{if } F_i^{\beta}\left(x^k\right) \geq \left(x^k - \psi\right)_i, \\ 1 & \text{if } F_i^{\beta}\left(x^k\right) < \left(x^k - \psi\right)_i. \end{cases}$$

Set $k := k + 1$ and return to Step 2.

Algorithm 2: Howard's algorithm.

respectively. Based on above notations, for $\forall x \in \mathbb{R}^n$ (14) is equivalent to

$$\begin{aligned} &\text{find} \quad x \in \mathbb{R}^n, \\ &\text{such that} \quad G(x) = 0. \end{aligned} \tag{16}$$

In the following, we introduce the other equivalent formation of (2). Let $\mathcal{A} = \{0, 1\}^n$ with $\alpha = (\alpha_i)_{i \in N} \in \mathcal{A}$; then we can also define the function

$$F(x) := \min_{\alpha \in \mathcal{A}} f^{\alpha}(x) = 0, \tag{17}$$

where

$$f^{\alpha}(x) := \begin{cases} f_i(x) & \text{if } \alpha_i = 0, \\ (x - \phi)_i & \text{if } \alpha_i = 1. \end{cases} \tag{18}$$

Moreover, for every $\beta \in \mathcal{A}$, the function $F^{\beta}(x) : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is defined by

$$F^{\beta}(x) := \begin{cases} F_i(x) & \text{if } \beta_i = 0, \\ (x - \psi)_i & \text{if } \beta_i = 1. \end{cases} \tag{19}$$

In a similar way, the function $G(x) = \max_{\beta \in \mathcal{A}} F^{\beta}(x)$, and then (14) is equivalent to the following problem:

$$\begin{aligned} &\text{find} \quad x \in \mathbb{R}^n, \\ &\text{such that} \quad \max_{\beta \in \mathcal{A}} F^{\beta}(x) = 0. \end{aligned} \tag{20}$$

Based on the above notations, we also present a high level description of Howard's algorithm for solving (20) in Algorithm 2.

4. Convergence Results

In the section, we show that the primal-dual active set method is equivalent to Howard's algorithm for the model problem (2), and then we obtain the convergence theorem of the primal-dual active set method.

To begin, let us focus on Howard's algorithm (i.e., Algorithm 2). Note that, at each step, Howard's algorithm satisfies

$$\begin{aligned} \lambda^{k+1} &= f\left(x^{k+1}\right), \\ \lambda_i^{k+1} &= 0, \quad i \in \mathcal{J}^k, \\ \left(x^{k+1} - \phi\right)_i &= 0, \quad i \in \mathcal{J}_{\phi}^k, \\ \left(x^{k+1} - \psi\right)_i &= 0, \quad i \in \mathcal{J}_{\psi}^k. \end{aligned} \tag{21}$$

Then we have the following notation:

$$\widetilde{F}\left(x^{k+1}\right) = 0$$
$$:= \begin{cases} f_i\left(x^{k+1}\right) & \text{if } c\left(x^k - \psi\right)_i \leq f_i\left(x^k\right) \leq c\left(x^k - \phi\right)_i, \\ c\left(x^{k+1} - \phi\right)_i & \text{if } f_i\left(x^k\right) > c\left(x^k - \phi\right)_i, \\ c\left(x^{k+1} - \psi\right)_i & \text{if } f_i\left(x^k\right) < c\left(x^k - \psi\right)_i. \end{cases} \tag{22}$$

In the following, Howard's algorithm is used to solve the equivalent problem

$$\max\left\{\min\left\{f(x), c(x - \phi)\right\}, c(x - \psi)\right\} = 0. \tag{23}$$

For $k \geq 0$, if we set

$$\begin{aligned} &\beta_i^{k+1} = 0, \\ &\alpha_i^{k+1} = 0 \\ &\qquad \text{for } i \in \mathcal{I}^k, \\ &\beta_i^{k+1} = 0, \\ &\alpha_i^{k+1} = 1 \\ &\qquad \text{for } i \in \mathcal{J}_\phi^k, \\ &\beta_i^{k+1} = 1, \\ &\alpha_i^{k+1} = 0 \\ &\qquad \text{for } i \in \mathcal{J}_\psi^k. \end{aligned} \tag{24}$$

Note that α^{k+1} and β^{k+1} are defined from the previous step x^k, as introduced in Howard's algorithm, and then we obtain $\widetilde{F} = F^{\beta^{k+1}}$. Therefore, (22) is equivalent to

$$F^{\beta^{k+1}}\left(x^{k+1}\right) = 0, \tag{25}$$

and x^{k+1} is defined as in Howard's algorithm applied to (23).

Remark 7. If we choose the initial guess with $\lambda^0 = f(x^0)$ in Algorithm 1, then the primal-dual active set method (i.e., Algorithm 1) and Howard's algorithm (i.e., Algorithm 2) for the variational inequality problem (2) are equivalent.

We now state the main convergence result for Howard's algorithm; i.e., the iteration number of the method is bounded by the number of the unknowns. We start with the following preliminary results to show the monotone convergence of Algorithm 2.

Lemma 8. *If f is a continuous strictly T-monotone operator, then the functions F, F^β, and G are monotone operators with each $\beta \in \mathcal{A} = \{0,1\}^n$.*

Proof. Let the vectors $x \in \mathbb{R}^n$, $y \in \mathbb{R}^n$ such that $F(x) \leq F(y)$, and let $\alpha^x \in \mathcal{A}$ be a minimizer that is associated with the function $F(x)$ defined as follows:

$$\alpha^x := \begin{cases} 0 & \text{if } f_i(x) \leq (x-\phi)_i \\ 1 & \text{if } f_i(x) > (x-\phi)_i \end{cases} \quad \text{for } i \in N. \tag{26}$$

Based on this notation, we obtain

$$f^{\alpha^x}(x) = F(x) \leq F(y) = \min_{\alpha \in \mathcal{A}} f^\alpha(y) \leq f^{\alpha^x}(y). \tag{27}$$

Hence, we conclude that the inequality $x \leq y$ holds, owing to the fact that f^{α^x} is a monotone operator.

In what follows, we show that the function F^β is a monotone operator. Let x, y be in $\mathbb{R}^n$ such that $F^\beta(x) \leq F^\beta(y)$. On the one hand, if $\beta_i = 0$ for some $i \in N$, then we have $F_i^\beta(x) \leq F_i^\beta(y)$ and therefore

$$f_i^{\alpha^x}(x) \leq f_i^{\alpha^y}(y) \leq f_i^{\alpha^x}(y). \tag{28}$$

On the other hand, if $\beta_i = 1$ for some $i \in N$, then we have $(x-\psi)_i \leq (y-\psi)_i$.

Let $\overline{\alpha} \in \mathcal{A}$ be defined as follows:

$$\overline{\alpha} := \begin{cases} 0 & \text{if } \beta_i = 0,\ \alpha_i^x = 0 \\ 1 & \text{if otherwise} \end{cases} \quad \text{for } i \in N. \tag{29}$$

Then we can obtain that if $\beta_i = 0$ and $\alpha_i^x = 0$ then

$$f_i^{\overline{\alpha}}(x) = f_i^{\alpha^x}(x) \leq f_i^{\alpha^x}(y) = f_i^{\overline{\alpha}}(y), \tag{30}$$

and otherwise,

$$f_i^{\overline{\alpha}}(x) = (x-\phi)_i \leq (y-\phi)_i = f_i^{\overline{\alpha}}(y). \tag{31}$$

Hence, $f_i^{\overline{\alpha}}(x) \leq f_i^{\overline{\alpha}}(y)$, and we also conclude that the inequality $x \leq y$ holds by using the monotonicity of the function $f^{\overline{\alpha}}$.

In a similar way, we show that the function G is a monotone operator. We assume $G(x) \leq G(y)$. By (20), let β^y be the index such that $G(y) = F^{\beta^y}(y)$. Then we have

$$F^{\beta^y}(x) \leq G(y) = F^{\beta^y}(y). \tag{32}$$

Hence, the inequality $x \leq y$ is obtained, based on the fact that the function F^{β^y} is a monotone operator. ☐

Theorem 9. *Let f be a continuous strictly T-monotone operator and $x^* \in \mathbb{R}^n$ be the unique solution of (2). Then the sequence $\{x^k\}$ given by Algorithm 2 satisfies*

(a) $x^k \geq x^{k+1}$ *for all* $k \geq 0$ *and* $x^k \leq \psi$ *for all* $k \geq 1$;

(b) $x^k \longrightarrow x^*$ *at most* $n+1$ *iterations.*

Proof. The uniqueness of the solution is obtained by the monotonocity of the function G.

(a) Note that

$$F^{\beta^{k+1}}\left(x^{k+1}\right) = 0 = F^{\beta^k}\left(x^k\right) \leq G\left(x^k\right) = F^{\beta^{k+1}}\left(x^k\right). \tag{33}$$

Then, we have $x^k \geq x^{k+1}$, since the function $F^{\beta^{k+1}}$ is monotone by Lemma 8. In the following we prove that

$$x^k \leq \psi \quad \text{for all } k \geq 1. \tag{34}$$

In fact, we need only to prove that $x^1 - \psi \leq 0$. If $\beta_i^1 = 0$, then we have $(x^1 - \psi)_i = 0$ by definition of x^1. On the other hand if $\beta_i^1 = 1$, then we obtain $F_i(x^0) \geq (x-\psi)_i$. Moreover, either $F_i(x^0)$ or $(x^0-\psi)_i$ is zero by definition of x^0. Hence $(x^0-\psi)_i \leq 0$ and $(x^1-\psi)_i \leq 0$.

(b) We first show that the sequence $\{\beta^k\}_{k\geq 1}$ is decreasing in $\mathscr{A} = \{0,1\}^n$. In fact, if $\beta_i^k = 0$ for some $k \geq 1$, then $F_i(x^k) = 0$. Using $(x^k - \psi)_i \leq 0$ we have that $\beta_i^{k+1} = 0$. Since $\{\beta^k\}_{k\geq 1}$ is decreasing, the set $I^k := \{i \in N : F_i(x^k) \geq (x^k - \psi)_i\}$ is thus increasing for $k \geq 0$. On the other hand, since $\mathrm{card}(I^k) \leq n$, there exists a first index $k \in N$ such that $I^k = I^{k+1}$, and we have $\alpha^k = \alpha^{k+1}$. Moreover, $G(x^{k+1}) = F^{\alpha^{k+2}}(x^{k+1}) = F^{\alpha^{k+1}}(x^{k+1}) = 0$, and thus x^{k+1} is the solution for some $k \leq n$. This makes at most $n+1$ iterations.

□

5. Numerical Experiments

In the numerical experiments, we focus on investigating the qualitative properties of the solution algorithms and comparing them to some other known algorithms. In this test, we consider a problem defined in the unit square $\Omega = (0,1) \times (0,1)$ as follows: find $u^* \in K$ such that

$$\begin{aligned} -\triangle u + g(u,x,y) &> 0, \quad u = \phi, \\ -\triangle u + g(u,x,y) &< 0, \quad u = \psi, \\ -\triangle u + g(u,x,y) &= 0, \quad \phi \leq u \leq \psi, \end{aligned} \tag{35}$$

where $K = \{u \in H_0^1(\Omega) : \phi \leq u \leq \psi \text{ a.e. in } \Omega\}$, ϕ, and ψ are given functions and $g(u,x,y) = u/(1+u) + 10x + y - 8$. We discretize the model problem by using the standard five-point difference scheme with a constant mesh step size: $h_x = h_y = 1/(m+1)$, where m denotes the number of mesh sizes in x- and y-directions ($N = m^2$ is the total number of unknowns). Then the discretized problem (35) belongs to a variational inequality problem with a T-monotone operator [7].

In the following tests, we choose the bound obstacles $\phi = 0$ and $\psi = 1$ for the variational inequality problem and conduct the following experiments:

(1) Compare the primal-dual active set method with project successive overrelaxation (PSOR) method [8–10] and the classical additive Schwarz method [7, 12, 13].

(2) Compare different solution algorithms from the point of view of iteration numbers and the total computing time for the variational inequality problem with T-monotone operators.

Throughout this section, for the construction of the classical overlapping additive Schwarz method as introduced in [7, 31], denoted by *Schwarz*, we partition the computational domian $N = N_1 \cup N_2$ into two equal parts with the overlapping size $O(1/10)$, and the corresponding subproblems are solved by PSOR with the relaxation parameter $\omega = 1.8$. In the proposed primal-dual active set method, i.e., Algorithm 1, the corresponding nonlinear systems are solved by nonlinear Gauss-Seidel method. We mainly consider the effect of dimension (denoted by N) on the performance of each algorithm. The initial guess for all the methods is chosen as $u^0 = 8A^{-1}(1,\ldots,1)^T$ with A being the coefficient matrix of $-\triangle$. The tolerance of the three methods is chosen to be equal to 10^{-6} in the $\|\cdot\|_2$-norm for both inner and outer iterations.

Table 1: Comparison of iteration numbers.

N	PSOR	Schwarz	Algorithm 1
100	35	16	8
400	42	23	12
900	63	39	15
1600	120	60	19
2500	187	85	23
3600	267	116	27

Table 2: Comparison of the computing time in seconds.

N	PSOR	Schwarz	Algorithm 1
100	0.015	0.031	0.015
400	0.5	0.703	0.765
900	4.281	5.796	6.484
1600	23.984	29.312	33.312
2500	89.5	106.89	126.359
3600	289.046	334.64	406.734

In the experiment, we consider PSOR, the classical additive Schwarz method, and Algorithm 1 for the numerical solution of (35). Algorithm 1 for $c = 1$ is not convergent for (35), where similar results were discussed in [23]. Hence, we choose $c = 1000$ in Algorithm 1. In the test, we mainly focus on the effect of the computational mesh size (denoted by N) to the performance of each algorithm, as listed in Tables 1 and 2, respectively. Below we list the observations made from the results.

(1) From Table 1, we can see that the number of iterations for Schwarz and PSOR grows much with the increase of the number of the computational mesh sizes, while the number of iterations for Algorithm 1 does not grow much with the increase of the number of mesh sizes, which implies that Algorithm 1 is insensitive to the number of mesh sizes. Hence, we can conclude that the performance of the primal-dual active set method is better than the Schwarz or PSOR method in terms of the iteration number.

(2) On the other hand, if we focus on the total computing time, the performance of PSOR is better than that of the Schwarz method or the primal-dual active set method, as shown in Table 2. Since the operator in the test problem (35) is semilinear, i.e., almost linear, the PSOR method behaves much better than it usually does, while the Schwarz method and the primal-dual active set method spend much time to solve the related subproblem at each outer iteration step. As a result, the Schwarz method and the primal-dual active set method performed not so good as one expected in terms of computing time, as shown in our numerical results. It is consonant with [31], in which the PSOR and Schwarz methods are used to solve linear complementarity problem.

6. Concluding Remarks

In this work, we have developed the family of primal-dual active set methods to solve the discrete nonlinear systems, arising from the variational inequality problems with T-monotone operators. We build the equivalent relation between the primal-dual active set method and Howard's algorithm, then show the convergence theorem of Howard's algorithm, and thus obtain the convergence theorem of the primal-dual active set method. The numerical experiments confirm the advantage of the primal-dual active set method, when compared to the traditional PSOR and Schwarz methods.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Acknowledgments

This research was supported by the Natural Science Foundation of China under Grant No. 71502053 and the Natural Science Foundation of Hunan Province under Grant No. 2018JJ2012.

References

[1] L. Badea, "On the Schwarz alternating method with more than two subdomains for nonlinear monotone problems," *SIAM Journal on Numerical Analysis*, vol. 28, no. 1, pp. 179–204, 1991.

[2] C. M. Elliott and J. R. Ockendon, *Weak and Variational Methods for Moving Boundary Problems*, vol. 59 of *Research Notes in Mathematics*, Pitman (Advanced Publishing Program), London, UK, 1982.

[3] R. H. Hoppe, "Multigrid algorithms for variational inequalities," *SIAM Journal on Numerical Analysis*, vol. 24, no. 5, pp. 1046–1065, 1987.

[4] K.-H. Hoffmann and J. Zou, "Parallel solution of variational inequality problems with nonlinear source terms," *IMA Journal of Numerical Analysis (IMAJNA)*, vol. 16, no. 1, pp. 31–45, 1996.

[5] G. H. Meyer, "Free boundary problems with nonlinear source terms," *Numerische Mathematik*, vol. 43, no. 3, pp. 463–483, 1984.

[6] H. Yang, S. Sun, Y. Li, and C. Yang, "A scalable fully implicit framework for reservoir simulation on parallel computers," *Computer Methods Applied Mechanics and Engineering*, vol. 330, pp. 334–350, 2018.

[7] J.-P. Zeng and S. Z. Zhou, "A domain decomposition method for a kind of optimization problems," *Journal of Computational and Applied Mathematics*, vol. 146, no. 1, pp. 127–139, 2002.

[8] Y. Achdou and O. Pironneau, *Computational Methods for Option Pricing*, Frontiers in Applied Mathematics, SIAM, 2005.

[9] O. Bokanowski, S. Maroso, and H. Zidani, "Some convergence results for Howard's algorithm," *SIAM Journal on Numerical Analysis*, vol. 47, no. 4, pp. 3001–3026, 2009.

[10] R. Glowinski, J. L. Lions, and R. Tremolieres, *Numerical Analysis of Variational Inequalities*, North-Holland, Amsterdam, The Netherlands, 1981.

[11] X.-C. Tai and J. Xu, "Global and uniform convergence of subspace correction methods for some convex optimization problems," *Mathematics of Computation*, vol. 71, no. 237, pp. 105–124, 2002.

[12] H. Xu, J. Zeng, and Z. Sun, "Two-level additive Schwarz algorithms for nonlinear complementarity problem with an M-function," *Numerical Linear Algebra with Applications*, vol. 17, no. 4, pp. 599–613, 2010.

[13] H. Yang, F.-N. Hwang, and X.-C. Cai, "Nonlinear preconditioning techniques for full-space Lagrange-Newton solution of PDE-constrained optimization problems," *SIAM Journal on Scientific Computing*, vol. 38, no. 5, pp. A2756–A2778, 2016.

[14] H. Yang, C. Yang, and X.-C. Cai, "Mixed order discretization based two-level Schwarz preconditioners for a tracer transport problem on the cubed-sphere," *Computers & Fluids. An International Journal*, vol. 110, pp. 88–95, 2015.

[15] B. Smith, P. Björstad, and W. Gropp, *Domain Decomposition: Parallel Multilevel Methods for Elliptic Partial Differential Equations*, Cambridge University Press, Cambridge, UK, 1996.

[16] A. Toselli and O. B. Widlund, *Domain Decomposition Methods-Algorithms and Theory*, Springer, Berlin, Germany, 2005.

[17] H. Yang, C. Yang, and X.-C. Cai, "Parallel domain decomposition methods with mixed order discretization for fully implicit solution of tracer transport problems on the cubed-sphere," *Journal of Scientific Computing*, 2014.

[18] M. Hintermuller, K. Ito, and K. Kunisch, "The primal-dual active set strategy as a semismooth Newton method," *SIAM Journal on Optimization*, vol. 13, no. 3, pp. 865–888, 2003.

[19] K. Ito and K. Kunisch, "An augmented Lagrangian technique for variational inequalities," *Applied Mathematics & Optimization*, vol. 21, no. 3, pp. 223–241, 1990.

[20] K. Ito and K. Kunisch, "An active set strategy based on the augmented Lagrangian formulation for image restoration," *M2AN: Mathematical Modelling and Numerical Analysis*, vol. 33, no. 1, pp. 1–21, 1999.

[21] K. Ito and K. Kunisch, "The primal-dual active set method for nonlinear optimal control problems with bilateral constraints," *SIAM Journal on Control and Optimization*, vol. 43, no. 1, pp. 357–376, 2004.

[22] C. Kanzow, "Inexact semismooth Newton methods for large-scale complementarity problems," *Optimization Methods & Software*, vol. 19, no. 3-4, pp. 309–325, 2004.

[23] T. Karkkainen, K. Kunisch, and P. Tarvainen, "Augmented Lagrangian active set methods for obstacle problems," *Journal of Optimization Theory and Applications*, vol. 119, no. 3, pp. 499–533, 2003.

[24] H. Yang, C. Yang, and S. Sun, "Active-set reduced-space methods with nonlinear elimination for two-phase flow problems in porous media," *SIAM Journal on Scientific Computing*, vol. 38, no. 4, pp. B593–B618, 2016.

[25] M. L. Puterman and S. L. Brumelle, "On the convergence of policy iteration in stationary dynamic programming," *Mathematics of Operations Research*, vol. 4, no. 1, pp. 60–69, 1979.

[26] H. Yang, S. Sun, and C. Yang, "Nonlinearly preconditioned semismooth Newton methods for variational inequality solution of two-phase flow in porous media," *Journal of Computational Physics*, vol. 332, pp. 1–20, 2017.

[27] P. T. Harker and J.-S. Pang, "Finite-dimensional variational inequality and nonlinear complementarity problems: a survey of theory, algorithms and applications," *Mathematical Programming*, vol. 48, no. 2, pp. 161–220, 1990.

[28] M. S. Santos, "Accuracy of numerical solutions using the Euler equation residuals," *Econometrica*, vol. 68, no. 6, pp. 1377–1402, 2000.

[29] M. S. Santos and J. Rust, "Convergence properties of policy iteration," *SIAM Journal on Control and Optimization*, vol. 42, no. 6, pp. 2094–2115, 2004.

[30] U. Mosco, *An Introduction to Approximate Solution of Variational Inequalities*, Cremonese, Roma, 1973.

[31] H. Yang, Q. Li, and H. Xu, "A multiplicative Schwarz iteration scheme for solving the linear complementarity problem with an H-matrix," *Linear Algebra and its Applications*, vol. 430, no. 4, pp. 1085–1098, 2009.