

## Research Article

# An Application of the Newton-Homotopy Continuation Method for Solving the Forward Kinematic Problem of the 3-RRS Parallel Manipulator

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The forward kinematic problem (FKP) of the 3-RRS parallel manipulator is solved by means of the Newton-homotopy continuation method. The closure equations are formulated in the three-dimensional Euclidean spaces considering the coordinates of the centers of the spherical joints as unknown variables. The method is easy to follow and unlike the classical Newton-Raphson method it allows finding all the solutions of the FKP. A case study is included in the contribution in order to confirm the correctness of the method. In that concern, the numerical results obtained by means of the proposed method are verified with the aid of two different approaches such as the application of commercially available software like Maple16™ and the application of the PHCpack, a general purpose solver for polynomial systems based on homotopy continuation.

## 1. Introduction

A parallel manipulator is a mechanical device composed of a moving platform and a fixed platform connected each other by means of at least two kinematic chains or legs. Compared with their serial counterparts, parallel manipulators bring us interesting advantages like higher accuracy and stiffness as well as the possibility to place the servo motors near to the fixed platform. Owing these features, parallel manipulators allow for potential applications such as flight simulators, automobile simulators, mining machines, medical fields, machine tools with multiaxis, cable-actuated cameras, and haptic devices, in work processes. However, the complex forward kinematics of parallel manipulators is one of their major drawbacks.

This work deals with the forward kinematic problem of the 3-RRS parallel manipulator where R and S stand for revolute and spherical joint, respectively, which is a robot able to perform the 2RIT motion. In order to perform the kinematic analysis of parallel manipulators, a comprehension of their mobility is a crucial issue in order to select the motive joints. According to Liu and Bonev [1], the three degrees of freedom

of the robot at hand may be understood as two independent freedoms: two rotations and one translation. As occurs for the tangential 3-RPS parallel manipulator [2], the axes of the two rotations belong to the plane formed by the centers of the spherical mounted on the moving platform. The performance of the 3-RRS parallel manipulator, as well as its applications, has been well studied; consider, for instance, that Liu [3] investigated the dynamics of the 3-RRS parallel manipulator considering flexible links. Khajepour et al. [4] developed a 3-RRS parallel manipulator whose main advantage is that the torques produced by the servomotors are decreased by using a mass compensator. Zhao et al. [5] developed a family of metamorphic parallel manipulators based on the topology of the 3-RRS parallel manipulator in which the substitution of spherical joints by different kinematic devices plays a central role in order to modify their mobility. Tetik and Kiper [6] reported the inverse and forward singularity manifolds of the 3-RRS parallel manipulator. In order to overcome the limitations of the 3-RRS parallel manipulator, it can be used as a module of a serial-parallel manipulator [7] or provided with a reconfigurable fixed platform [8]. On the other hand it is interesting to note that if the topology of the 3-RRS parallel

manipulator is modified in such a way that its revolute joints intersect at a common point, and then it is transformed into a spherical parallel manipulator [9–11].

This work is focused on the FKP of the 3-RRS parallel manipulator generator of the 2RIT motion. The rest of the contribution is organized as follows. In Section 2 the mathematical resource employed in the contribution, namely, the Newton-homotopy method, is explained. In Section 3, the architecture of the 3-RRS parallel manipulator is briefly outlined. In Section 4 the FKM is formally presented. With the purpose to exemplify the method, in Section 5 a numerical example is solved and verified by means of two different approaches: (i) the application of the library Homotopy of Maple16™ and (ii) the application of the free available software PHCpack [12]. Finally, some conclusions are given at the end of the contribution.

## 2. Mathematical Background

There are many problems in kinematics involving interrogates dealing with mappings between algebraic spaces. Prominent amongst them the analytic solution of the forward kinematics of parallel manipulators is one of the most challengers problems in modern kinematics. In that concern, elegant methods to solve the forward kinematics of the Gough-Stewart platform [13] have been developed by resorting to the so-called Algebraic geometry [14–18] as a mean to overcome heavy manipulations of complex algebraic expressions. Naturally trend can be also implemented for solving the FKP of parallel manipulators having limited mobility [19] and the 3-RRS robot is not the exception. The mathematical tool chosen for approaching the issue at hand is the Newton-homotopy continuation method [20], a kind of perturbation technique, and in what follows the method is briefly explained. The method is focused on solving a square system of  $n$  polynomials in the unknowns  $w = (w_1, w_n)$  given by

$$f(w) := [f_1(w), \dots, f_n(w)] \quad (1)$$

In the method, the equations are modified by omitting some terms and adding new ones in such a way that a new system of equations emerges with the characteristic that the start system may be guessed avoiding divergence. Afterwards, the coefficients are deformed/perturbed into coefficients of the original system by a series of small increments; e.g., assuming that we need to solve  $f(w) = 0$ , then the auxiliary homotopy function may be chosen as follows:

$$g(w) := [g_1(w), \dots, g_n(w)] \quad (2)$$

The functions  $g(w)$  may be freely selected; e.g., an option would be simple linear combinations of the involved variables:

$$g(w) = \begin{bmatrix} g_1(w) \\ \vdots \\ g_n(w) \end{bmatrix} = \begin{bmatrix} \lambda_1^1 w_1 + \dots + \lambda_n^1 w_n + \lambda^1 \\ \vdots \\ \lambda_1^n w_1 + \dots + \lambda_n^n w_n + \lambda^n \end{bmatrix} \quad (3)$$

$$\forall \lambda \in \mathfrak{R}$$

Then, one defines  $H(w, t) := [h_1(w), \dots, h_n(w)]$  as the multiple homotopy continuation function given by

$$H(w, t) := t \begin{bmatrix} f_1(w) \\ \vdots \\ f_n(w) \end{bmatrix} + (1-t) \begin{bmatrix} g_1(w) \\ \vdots \\ g_n(w) \end{bmatrix} \quad (4)$$

where  $t$  is an arbitrary parameter given in the interval  $[0, 1]$  confined to  $t \in [0, 1]$ . Hence, it is evident that we have two boundary conditions as follows:

$$\begin{aligned} H(w, 0) &= g(w) \\ H(w, 1) &= f(w) \end{aligned} \quad (5)$$

These conditions are the basis of the homotopy continuation method.

The idea fundamental consists of solving  $H(w, t) = 0$  instead of  $f(w) = 0$  considering small variations of parameter  $t$  and resorting to the usual Newton-Raphson method. The iterations are performed according to

$$w_{iter+1} = w_{iter} - J(w_{iter})^{-1} H(w_{iter}, t) \quad (6)$$

where

$$J(w_{iter}) = \begin{bmatrix} \frac{\partial h_1(w)}{\partial w_1} & \frac{\partial h_1(w)}{\partial w_2} & \dots & \frac{\partial h_1(w)}{\partial w_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial h_n(w)}{\partial w_1} & \frac{\partial h_n(w)}{\partial w_2} & \dots & \frac{\partial h_n(w)}{\partial w_n} \end{bmatrix} \quad (7)$$

is the Jacobian matrix of the modified/perturbed system. Of course, in order to guarantee convergence of the algorithm avoiding divergence,  $g(w)$  is selected in such a way that  $\det(J(w_{iter})) \neq 0$ .

Finally, it is interesting to quote that the method explained in this section to solve higher nonlinear equations although requires proper guess start values; the solution, if any, usually is found in a few iterations. In that regard, other effective methods like the Sylvester dialytic method of elimination could produce spurious solutions or heavy polynomial equations depending on the expected degree of the involved polynomials.

## 3. Geometry of the Parallel Manipulator

In this section the topology and geometry of the parallel manipulator are outlined.

With reference to Figure 1, the 3-RRS parallel manipulator consists of two triangular platforms, one fixed and the other movable, connected by means of three identical limbs. The  $i$ th limb is composed of a lower link having a length  $a$  and an upper link having a length  $b$ , and it is articulated with three joints: (i) an active revolute joint R located at point  $A_i$  connecting the lower link to the fixed platform which is related with the  $i$ th generalized coordinate  $q_i$ , (ii) a passive revolute joint R located at point  $B_i$  articulating the two links

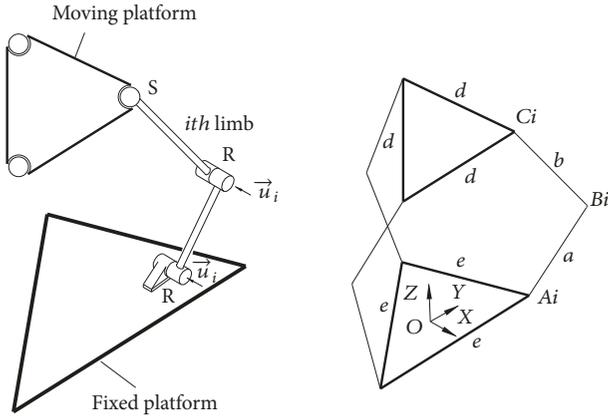


FIGURE 1: The 3-RRS parallel manipulator.

in the same limb, and (iii) a spherical joint  $S$  located at point  $C_i$  connecting the upper link to the moving platform. The revolute joints in the same limb have parallel axes along the unit vector  $\vec{u}_i$ . Furthermore, the moving and fixed platforms are in the shape of equilateral triangles  $C_1C_2C_3$  of side length  $d$  and  $A_1A_2A_3$  of side length  $e$ , respectively. With this topology the moving platform is able to perform the 2RIT motion. Finally, with the purpose to formulate the FKP, let us consider that  $O\_XYZ$  is a reference frame with associated unit vectors  $\hat{i}\hat{j}\hat{k}$  attached to the moving platform with the origin  $O$  located at the center of the triangle  $A_1A_2A_3$  and the  $Z$ -axis normal to the plane of the fixed platform.

#### 4. Formulation of the Forward Kinematic Problem

The FKP consists of finding the reachable poses of the moving platform meeting a prescribed set of generalized coordinates  $q_i$  ( $i = 1, 2, 3$ ). To this end, consider that points  $C_i$  are composed of unknown coordinates as follows:  $C_1 = (w_1, w_2, w_3)$ ,  $C_2 = (w_4, w_5, w_6)$ , and  $C_3 = (w_7, w_8, w_9)$ .

Owing the parallelism between the axes of the two revolute joints in the same limb, three closure equations may be written as follows:

$$(\vec{c}_i - \vec{a}_i) \cdot \vec{u}_i = 0 \quad (8)$$

(8) where  $\vec{a}_i$  and  $\vec{c}_i$  are the position vectors of points  $A_i$  and  $C_i$  while the dot ( $\cdot$ ) denotes the usual inner product of three-dimensional vectorial algebra. On the other hand, the Euclidean distance between points  $B_i$  and  $C_i$  gives us three quadratic equations as follows:

$$(\vec{c}_i - \vec{b}_i) \cdot (\vec{c}_i - \vec{b}_i) = b^2 \quad (9)$$

where  $\vec{b}_i$  denotes the position vector of the  $i$ th point  $B_i$ . In that regard it is straightforward to show that points  $B_i$  are obtained directly upon the generalized coordinates. Finally, other three quadratic equations are generated taking into

account the Euclidean norm between points  $C_i$ . In fact from the geometry of the moving platform it follows that

$$\begin{aligned} (\vec{c}_1 - \vec{c}_2) \cdot (\vec{c}_1 - \vec{c}_2) &= d^2 \\ (\vec{c}_2 - \vec{c}_3) \cdot (\vec{c}_2 - \vec{c}_3) &= d^2 \\ (\vec{c}_1 - \vec{c}_3) \cdot (\vec{c}_1 - \vec{c}_3) &= d^2 \end{aligned} \quad (10)$$

After a few computations, a system of nine equations in the unknown coordinates of points  $C_i$  is generated. This kind of equations may be reduced into a univariate 16<sup>th</sup> order polynomial [21]. In the contribution, taking into account that (3) are linear equations then expressions (8)-(10) are reduced into a nonlinear system of six quadratic equations that are solved by means of the Newton-homotopy continuation method. Once the coordinates of points  $C_i$  are calculated, the position of the moving platform is characterized by the coordinates of the center  $C$  of the moving platform. Indeed

$$C = \frac{(C_1 + C_2 + C_3)}{3} \quad (11)$$

Meanwhile, the rotation matrix  $Rot$  between the moving and the fixed platforms may be obtained as follows:

$$Rot = [\hat{w} \quad \hat{v} \quad \hat{w} \times \hat{v}] \quad (12)$$

where  $\hat{w}$  is a unit vector pointed from  $C$  to  $C_1$  and  $\hat{v}$  is a unit vector located in the plane of the moving platform perpendicular to  $\hat{w}$ . Furthermore, with the purpose to determine the orientation angles  $\theta$  and  $\psi$ , the rotation matrix may be also written as follows:

$$Rot = \begin{bmatrix} \cos(\psi) & 0 & \sin(\psi) \\ \sin(\theta)\sin(\psi) & \cos(\theta) & -\sin(\theta)\cos(\psi) \\ -\cos(\theta)\sin(\psi) & \sin(\theta) & \cos(\theta)\cos(\psi) \end{bmatrix} \quad (13)$$

Equating (12) and (13) it follows that

$$\cos(\theta) = \hat{v} \cdot \hat{j}, \quad (14)$$

$$\cos(\psi) = \hat{w} \cdot \hat{i} \quad (15)$$

#### 5. Case Study

With the purpose to illustrate the method, in this section a numerical example is provided. To this end the parameters of the parallel manipulator, using SI units, are given in Table 1.

The numerical example consists of finding all the poses that the moving platform can reach given the generalized coordinates  $q_1 = 120^\circ$ ,  $q_2 = 135^\circ$ , and  $q_3 = 95^\circ$ . Firstly, the closure equations are obtained according to the method introduced in Section 4. The application of (8) leads to

$$\begin{aligned} f_1 &= w_2, \\ f_2 &= -0.866w_4 - 0.5w_5, \\ f_3 &= 0.866w_7 - 0.5w_8 \end{aligned} \quad (16)$$

TABLE 1: Parameters of the numerical example.

Dimensions of the platforms	$e = 0.582\text{m}, d = 0.5\text{m}$
Lengths of the links	$a = 0.7\text{m}, b = 0.65\text{m}$
Orientation of the revolute joints	$\hat{u}_1 = \hat{j}, u_2 = -0.866\hat{i} - 0.5\hat{j} + \hat{k}, \hat{u}_3 = 0.866\hat{i} - 0.5\hat{j} + \hat{k}$

TABLE 2: One solution of the numerical example.

Initial values	$w_1 = 0.1, w_2 = 0.0, w_3 = 2.5,$ $w_4 = 0.5, w_5 = 1.5, w_6 = 1.0,$ $w_7 = 0.5, w_8 = -0.1, w_9 = 1.5$
Newton-homotopy method result	$w_1 = 0.1897, w_2 = -0.4e - 06, w_3 = 1.0256,$ $w_4 = -0.1860, w_5 = 0.3222, w_6 = 0.9550,$ $w_7 = 0.1206, w_8 = 0.2090, w_9 = 0.5766$
Equations (8)-(10)	$f_1 = -4.21e - 06, f_2 = -5.23e - 06, f_3 = -4.16e - 06,$ $f_4 = -7.58e - 06, f_5 = -1.56e - 06, f_6 = 6.96e - 07,$ $f_7 = 2.82e - 08, f_8 = 1.20e - 06, f_9 = -3.55e - 06$

Meanwhile based on (9) we have

$$\begin{aligned}
 f_4 &= w_1^2 - 1.372w_1 + w_2^2 + w_3^2 - 1.212w_3 + 0.416 \\
 f_5 &= w_4^2 + 0.831w_4 + w_5^2 - 1.439w_5 + w_6^2 - 0.989w_6 \\
 &\quad + 0.513 \\
 f_6 &= w_7^2 + 0.397w_7 + w_8^2 + 0.688w_8 + w_9^2 - 1.394w_9 \\
 &\quad + 0.221
 \end{aligned} \tag{17}$$

On the other hand, from (10) one obtains

$$\begin{aligned}
 f_7 &= w_1^2 - 2w_1w_4 + w_4^2 + w_2^2 - 2w_2w_5 + w_5^2 + w_3^2 \\
 &\quad - 2w_3w_6 + w_6^2 - 0.25 \\
 f_8 &= w_4^2 - 2w_4w_7 + w_7^2 + w_5^2 - 2w_5w_8 + w_8^2 + w_6^2 \\
 &\quad - 2w_6w_9 + w_9^2 - 0.25 \\
 f_9 &= w_1^2 - 2w_1w_7 + w_7^2 + w_2^2 - 2w_2w_8 + w_8^2 + w_3^2 \\
 &\quad - 2w_3w_9 + w_9^2 - 0.25
 \end{aligned} \tag{18}$$

The application of the Newton-homotopy method allows finding six real solutions for the FKP, one of them is shown in Table 2. In that regard the auxiliary homotopy function  $g(w)$  was selected as suggested in expression (3).

Furthermore, the six reachable poses of the moving platform are provided in compact form, e.g., through the computations of the center C of the moving platform and their orientation angles, in Table 3.

The numerical results obtained by means of the Newton-homotopy method were verified with the aid of special software such as the Homotopy library of Maple16™ and the free available software for general purpose solver for polynomial systems by homotopy continuation of Prof. Verschelde [12]. For brevity, only the results obtained for the variable  $w_9$  are listed in Table 4.

TABLE 3: Reachable poses of the moving platform.

Sol.	Center C m	$\theta$ deg	$\psi$ deg
1	(-0.057,0.106,0.993)	38.589	111.577
2	(-0.078,0.014,0.210)	5.572	92.922
3	(0.079,0.154,0.251)	82.892	122.343
4	(0.002,0,0.097)	10.240	89.867
5	(0.018,-0.022,0.154)	33.550	85.583
6	(0.041,0.177,0.852)	76.918	127.838

TABLE 4: Results of variable  $w_9$  using three methods.

Sol.	Maple16™	PHCpack [12]	Newton-homotopy
1	1.241462	1.241501	1.241466
2	0.061616	0.061612	0.061614
3	0.533831	0.533833	0.533832
4	0.057072	0.057072	0.057071
5	0.058178	0.058179	0.058174
6	0.576702	0.576701	0.576697

It is worth noting that the numerical results obtained by means of the Newton-homotopy continuation method are in excellent agreement with those generated with Maple16™ and the PHCpack solver [12].

Finally, it is opportune to mention that, for limited-dof parallel manipulators, the Sylvester dialytic method of elimination is an effective option to solve the forward kinematic problem, see for instance [21–23]. However, when the number of variables is increased, the possibility of spurious or strange solutions is increased, which does not occur with the homotopy continuation method.

## 6. Conclusions

This work is devoted to solve the forward kinematic problem of the 3-RRS parallel manipulator, a robot with the ability to perform the 2RIT motion. The closure equations of the

robot are formulated based on the unknown coordinates of the centers of the spherical joints. To the best knowledge of the author this procedure has not been employed in previous works for the robot at hand and is a genuine alternative to the classical method of kinematic analysis based on the rotation matrix, avoiding, among others drawbacks, the handling of orientation angles. Thereafter, three linear equations are generated taking into account the limited mobility imposed to the moving platform by the legs due to the parallel assembly of the revolute joints in the same limbs; e.g., the coordinates of the spherical joints are projected in the planes of the corresponding legs. On the other hand, six quadratic equations are formulated by computing the Euclidean norm between selected points of the robot manipulator; e.g., three quadratic equations are obtained upon the geometry of the moving platform while the remaining quadratic equations are obtained based on the link lengths. Afterwards, the system of equations is solved by means of the Newton-homotopy continuation method. The method is easy to follow and allows finding all the reachable poses of the moving platform. Furthermore the Newton-homotopy method guarantees convergence and the generation of spurious solutions can be avoided. A numerical example is provided with the purpose to illustrate the correctness and simplicity of the method. Finally, the numerical results are successfully compared with results generated with special software like Maple16™ and the free available software PHCpack of Prof. Verschelde [12].

## Nomenclature

$a, b$ :	Arm and forearm of a limb
$A_i$ :	Nominal coordinates of the lower $i$ th revolute joint
$\vec{a}_i$ :	Position vector of point $A_i$
$B_i$ :	Nominal coordinates of the intermediate $i$ th revolute joint
$\vec{b}_i$ :	Position vector of point $B_i$
$C$ :	Center of the moving platform
$C_i$ :	Nominal coordinates of the $i$ th spherical joint
$\vec{c}_i$ :	Position vector of point $C_i$
$d$ :	Side length of the moving triangular platform
dof:	Degrees-of-freedom
$e$ :	Side length of the fixed triangular platform
FKP:	Forward Kinematic Problem
$iter$ :	Iteration
$J$ :	Jacobian matrix
$P$ :	Prismatic joint
$q_i$ :	$i$ th generalized coordinate
$R$ :	Revolute joint
$Rot$ :	Rotation matrix
$S$ :	Spherical joint
SI:	International System of Units
$t$ :	Arbitrary parameter of homotopy
$T$ :	Translational motion
$\hat{u}_i$ :	Unit vector denoting the orientation of the $i$ th revolute joint

$\hat{v}, \hat{w}$ :	Unit vectors associated to the rotation matrix
$w = (w_1, \dots, w_n)$ :	Set of $n$ variables
$f(w)$ :	Square system of $n$ polynomials
$g(w)$ :	Auxiliary homotopy function
$H(w, t)$ :	Multiple homotopy continuation function
$O\_XYZ$ :	Fixed reference frame with associated unit vectors
$\theta, \psi$ :	Orientation angles of the moving platform.

## Data Availability

The Maplesheet data used to support the findings of this study are available from the corresponding author upon request.

## Conflicts of Interest

The author declares that they have no conflicts of interest.

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