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Research Article

Extreme Spectra Realization by Nonsymmetric Tridiagonal and Nonsymmetric Arrow Matrices

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We consider the following inverse extreme eigenvalue problem: given the real numbers $\{\lambda_1^{(j)}, \lambda_j^{(j)}\}_{j=1}^n$ and the real vector $\mathbf{x}^{(n)} = (x_1, x_2, \ldots, x_n)$, to construct a nonsymmetric tridiagonal matrix and a nonsymmetric arrow matrix such that $\{\lambda_1^{(j)}, \lambda_j^{(j)}\}_{j=1}^n$ are the minimal and the maximal eigenvalues of each one of their leading principal submatrices, and $(x^{(n)}, \lambda_n^{(n)})$ is an eigenpair of the matrix. We give sufficient conditions for the existence of such matrices. Moreover our results generate an algorithmic procedure to compute a unique solution matrix.

1. Introduction

We consider a particular inverse eigenvalue problem for real nonsymmetric tridiagonal matrices of the form

$$A = \begin{pmatrix} a_1 & b_1 & & & \\ c_1 & a_2 & b_2 & & \\ & c_2 & a_3 & \ddots & \\ & & \ddots & \ddots & b_{n-1} \\ & & & c_{n-1} & a_n \end{pmatrix}; \quad b_i c_i > 0, \; i = 1, 2, \ldots, n-1, \tag{1}$$

and for real nonsymmetric arrow matrices of the form

$$B = \begin{pmatrix} a_1 & b_1 & b_2 & \cdots & b_{n-1} \\ c_1 & a_2 & & & \\ c_2 & & a_3 & & \\ \vdots & & & \ddots & \\ c_{n-1} & & & & a_n \end{pmatrix}; \quad b_i c_i > 0, \; i = 1, 2, \ldots, n-1. \tag{2}$$

This kind of matrices appears in several areas of science and engineering, as in the Lanczos method for tridiagonalizing a nonsymmetric matrix or for computing the Gaussian quadrature [1–3]. The nonsymmetric arrow matrices used to be an important tool for computing eigenvalues via dividing and conquering approximations, in the study of nonsymmetric eigenvalue problem [4]. The symmetric inverse eigenvalue problem has attracted the attention of many authors. In contrast, the nonsymmetric case has been less studied [5, 6]. In this paper we discuss the inverse eigenvalues problem for matrices (1) and (2) considering the following spectral information: the set of minimal and maximal eigenvalues of all leading principal submatrices A_j, $j = 1, 2, \ldots, n$, of a matrix A of form (1) or (2), together with an eigenvector of A. This type of spectral information has been recently considered in the literature [7–10]. More precisely, we consider the following problem.

Problem 1. Given the list of real numbers

$$\left\{\lambda_1^{(n)}, \lambda_1^{(n-1)}, \ldots, \lambda_1^{(2)}, \lambda_1^{(1)}, \lambda_2^{(2)}, \ldots, \lambda_{n-1}^{(n-1)}, \lambda_n^{(n)}\right\}, \tag{3}$$

and the vector

$$\mathbf{x}^{(n)} = (x_1, x_2, \ldots, x_n), \tag{4}$$

construct a matrix A of form (1) or (2), such that $\lambda_1^{(j)}$ and $\lambda_j^{(j)}$ are, respectively, the minimal and the maximal eigenvalue of the leading principal submatrix A_j, $j = 1, 2, \ldots, n$ of A, and $(\mathbf{x}^{(n)}, \lambda_n^{(n)})$ is an eigenpair of A.

It is known that a matrix of form (1) is diagonally similar to the symmetric irreducible tridiagonal matrix

$$DAD^{-1} = \begin{pmatrix} a_1 & \sqrt{b_1c_1} & & & \\ \sqrt{b_1c_1} & a_2 & \sqrt{b_2c_2} & & \\ & \sqrt{b_2c_2} & \ddots & \ddots & \\ & & \ddots & a_{n-1} & \sqrt{b_{n-1}c_{n-1}} \\ & & & \sqrt{b_{n-1}c_{n-1}} & a_n \end{pmatrix}, \tag{5}$$

where $D = \operatorname{diag}(\gamma_1, \gamma_2, \ldots, \gamma_n)$, with $\gamma_i = \sqrt{(c_ic_{i+1}\cdots c_{n-1})/(b_ib_{i+1}\cdots b_{n-1})}$, $i = 1, \ldots, n-1$, and $\gamma_n = 1$ (see [2]).

In the same way, it can be determined that a matrix of form (2) is diagonally similar to the symmetric irreducible arrow matrix:

$$DBD^{-1} = \begin{pmatrix} a_1 & \sqrt{b_1c_1} & \sqrt{b_2c_2} & \cdots & \sqrt{b_{n-1}c_{n-1}} \\ \sqrt{b_1c_1} & a_2 & & & \\ \sqrt{b_2c_2} & & a_3 & & \\ \vdots & & & \ddots & \\ \sqrt{b_{n-1}c_{n-1}} & & & & a_n \end{pmatrix}, \tag{6}$$

where $D = \operatorname{diag}(\delta_1, \delta_2, \ldots, \delta_n)$, with $\delta_1 = 1$ and $\delta_{i+1} = \sqrt{b_i/c_i}$, $i = 1, \ldots, n-1$.

An important fact related to the above similarities is that they leave invariant the eigenvalues of the leading principal submatrices A_j, $j = 1, 2, \ldots, n$. Then the following results in [7, 11, 12] hold for matrices of forms (1) and (2) as well.

Lemma 2 (see [11]). *A necessary and sufficient condition for the existence of an $n \times n$ symmetric tridiagonal matrix of form (1) ($b_i = c_i$), such that $\lambda_1^{(j)}$ and $\lambda_j^{(j)}$ are, respectively, the minimal and the maximal eigenvalues of the leading principal submatrix A_j of A, $j = 1, 2, \ldots, n$, is*

$$\lambda_1^{(n)} < \lambda_1^{(n-1)} < \cdots < \lambda_1^{(2)} < \lambda_1^{(1)} < \lambda_2^{(2)} < \cdots < \lambda_{n-1}^{(n-1)} < \lambda_n^{(n)}. \tag{7}$$

Lemma 3 (see [12]). *Let A be a matrix of form (2) with $b_i = c_i \neq 0$, $i = 1, \ldots, n-1$. Let $\lambda_1^{(j)}$ and $\lambda_j^{(j)}$, respectively, be the minimal and the maximal eigenvalue of the leading principal submatrix A_j of A, $j = 1, 2, \ldots, n$. Then*

$$\lambda_1^{(j)} < \cdots < \lambda_1^{(3)} < \lambda_1^{(2)} < \lambda_1^{(1)} < \lambda_2^{(2)} < \lambda_3^{(3)} < \cdots < \lambda_j^{(j)}, \tag{8}$$

and

$$\lambda_1^{(j)} < a_i < \lambda_j^{(j)}, \quad i = 2, 3, \ldots, j, \ \text{for each } j = 2, 3, \ldots, n. \tag{9}$$

Lemma 4 (see [7]). *Let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ be a set of orthonormal eigenvectors associated with the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of an $n \times n$ matrix A of form (2) with $b_i = c_i > 0$, $i = 1, \ldots, n-1$, and with all its diagonal entries a_j distinct, $j = 2, 3, \ldots, n$. Then $x_{\mu j} \neq 0$ for $\mu, j = 1, 2, \ldots, n$, where $x_{\mu j}$ denotes the μ^{th} entry of the vector $\mathbf{x}_j$.*

In [11, 12], the authors show how to construct symmetric tridiagonal and symmetric arrow matrices from the spectral information (3). Then, if the given spectral information is only (3), we may construct matrices of form (1) and (2), respectively, from the symmetric matrices (5) and (6), by similarity. However, these constructions are not unique. In order to obtain a unique solution, we consider the spectral information (3) and (4). We shall need the following known results.

Lemma 5 (see [12]). *Let $P(\lambda)$ be a monic polynomial of degree n with all zeroes being real. If λ_1 and λ_n are, respectively, the minimal and maximal zero of $P(\lambda)$, then*
(1) if $\mu < \lambda_1$, we have $(-1)^n P(\mu) > 0$,
(2) if $\mu > \lambda_n$, we have $P(\mu) > 0$.

Lemma 6 (see [13]). *Let A be an $n \times n$ nonsymmetric tridiagonal matrix of form (1), and let A_j be the $j \times j$ leading principal submatrix of A, with characteristic polynomial $P_j(\lambda) = \det(\lambda I_j - A_j)$, $j = 1, 2, \ldots, n$. Then the sequence $\{P_j(\lambda)\}_{j=1}^n$ satisfies the recurrence relation*

$$\begin{aligned} P_0(\lambda) &= 1, \\ P_1(\lambda) &= (\lambda - a_1), \\ P_j(\lambda) &= (\lambda - a_j) P_{j-1}(\lambda) - b_{j-1}c_{j-1}P_{j-2}(\lambda), \\ & \qquad j = 2, 3, \ldots, n, \end{aligned} \tag{10}$$

Lemma 7. *Let A be an $n \times n$ nonsymmetric arrow matrix of form (2), and let A_j be the $j \times j$ leading principal submatrix of A, with characteristic polynomial $P_j(\lambda) = \det(\lambda I_j - A_j)$, $j = 1, 2, \ldots, n$. Then, the sequence $\{P_j(\lambda)\}_{j=1}^n$ satisfies the recurrence relation*

$$\begin{aligned} P_1(\lambda) &= (\lambda - a_1) \\ P_2(\lambda) &= (\lambda - a_2) P_1(\lambda) - b_1c_1 \\ P_j(\lambda) &= (\lambda - a_j) P_{j-1}(\lambda) - b_{j-1}c_{j-1}\prod_{i=2}^{j-1}(\lambda - a_i), \\ & \qquad j = 3, 4, \ldots, n. \end{aligned} \tag{11}$$

Proof. The result follows by expanding the determinants $\det(\lambda I_j - A_j)$, $j = 1, 2, \ldots, n$. □

2. Main Results

In this section we give a unique solution to Problem 1 for the matrices of forms *(1) and (2)*. Conditions (12) and (13), as well as conditions (31) and (32) below, arise from Lemmas 2, 3, and 4 by the similarity of matrices (1) and (5), as well as matrices (2) and (6).

Theorem 8. *Let the real numbers* $\{\lambda_1^{(j)}, \lambda_j^{(j)}\}_{j=1}^n$ *and the vector* $\mathbf{x}^{(n)} = (x_1, x_2, \ldots, x_n)$ *be satisfying*

$$\lambda_1^{(n)} < \lambda_1^{(n-1)} < \cdots < \lambda_1^{(2)} < \lambda_1^{(1)} < \lambda_2^{(2)} < \cdots < \lambda_{n-1}^{(n-1)} < \lambda_n^{(n)} \tag{12}$$

and

$$x_i \neq 0, \quad i = 1, 2, \ldots, n. \tag{13}$$

Then, there exists a unique nonsymmetric tridiagonal matrix A of form (1), such that $\lambda_1^{(j)}$ *and* $\lambda_j^{(j)}$ *are, respectively, the minimal and the maximal eigenvalue of the leading principal submatrix* A_j, $j = 1, 2, \ldots, n$, *of A, and* $(x^{(n)}, \lambda_n^{(n)})$ *is an eigenpair of A.*

Proof. Suppose $\{\lambda_1^{(j)}, \lambda_j^{(j)}\}_{j=1}^n$ satisfy (12), and the vector $x^{(n)}$ satisfies (13). To show the existence of a nonsymmetric tridiagonal matrix A with the required properties is equivalent to show that the system of equations

$$\begin{gathered} P_j\left(\lambda_i^{(j)}\right) = 0; \quad j = 1, 2, \ldots, n, \ i = 1, j \\ A\mathbf{x}^{(n)} = \lambda_n^{(n)}\mathbf{x}^{(n)}, \end{gathered} \tag{14}$$

where $P_j(\lambda) = \det(\lambda I_j - A_j)$, $j = 1, 2, \ldots, n$ satisfies Lemma 6, has real solutions a_j, $j = 1, 2, \ldots, n$ and b_i, c_i, $i = 1, 2, \ldots, n-1$, with $c_i b_i > 0$.

From Lemma 6 for $j = 1$, it follows that $P_1(\lambda_1^{(1)}) = \lambda_1^{(1)} - a_1 = 0$. Then,

$$a_1 = \lambda_1^{(1)}. \tag{15}$$

Now, from Lemma 6 for $j = 2, 3, \ldots, n$, system (14) can be written as

$$P_j\left(\lambda_1^{(j)}\right) = \left(\lambda_1^{(j)} - a_j\right) P_{j-1}\left(\lambda_1^{(j)}\right) - b_{j-1}c_{j-1}P_{j-2}\left(\lambda_1^{(j)}\right) = 0, \tag{16}$$

$$P_j\left(\lambda_j^{(j)}\right) = \left(\lambda_j^{(j)} - a_j\right) P_{j-1}\left(\lambda_j^{(j)}\right) - b_{j-1}c_{j-1}P_{j-2}\left(\lambda_j^{(j)}\right) = 0, \tag{17}$$

$$j = 2, 3, \ldots, n. \tag{18}$$

$$a_1 x_1 + b_1 x_2 = \lambda_n^{(n)} x_1 \tag{19}$$

$$c_{j-1}x_{j-1} + a_j x_j + b_j x_{j+1} = \lambda_n^{(n)} x_j, \quad j = 2, \ldots, n-1 \tag{20}$$

$$c_{n-1}x_{n-1} + a_n x_n = \lambda_n^{(n)} x_n. \tag{21}$$

From (19) and condition (13), it follows that

$$b_1 = \left(\lambda_n^{(n)} - a_1\right)\frac{x_1}{x_2}. \tag{22}$$

From (16) and (17) for $j = 2$ we have

$$\begin{gathered} P_2\left(\lambda_1^{(2)}\right) = \left(\lambda_1^{(2)} - a_2\right) P_1\left(\lambda_1^{(2)}\right) - b_1 c_1 P_0\left(\lambda_1^{(2)}\right) = 0 \\ P_2\left(\lambda_2^{(2)}\right) = \left(\lambda_2^{(2)} - a_2\right) P_1\left(\lambda_2^{(2)}\right) - b_1 c_1 P_0\left(\lambda_2^{(2)}\right) = 0, \end{gathered} \tag{23}$$

from which

$$c_1 = \frac{1}{b_1}\frac{\left(\lambda_2^{(2)} - \lambda_1^{(2)}\right) P_1\left(\lambda_1^{(2)}\right) P_1\left(\lambda_2^{(2)}\right)}{P_1\left(\lambda_1^{(2)}\right) - P_1\left(\lambda_2^{(2)}\right)}. \tag{24}$$

and

$$a_2 = \frac{\lambda_1^{(2)} P_1\left(\lambda_1^{(2)}\right) - \lambda_2^{(2)} P_1\left(\lambda_2^{(2)}\right)}{P_1\left(\lambda_1^{(2)}\right) - P_1\left(\lambda_2^{(2)}\right)}. \tag{25}$$

Moreover, from Lemma 5 and condition (12) we have

$$b_1 c_1 = \frac{(-1)\left(\lambda_2^{(2)} - \lambda_1^{(2)}\right) P_1\left(\lambda_1^{(2)}\right) P_1\left(\lambda_2^{(2)}\right)}{(-1)\left[P_1\left(\lambda_1^{(2)}\right) - P_1\left(\lambda_2^{(2)}\right)\right]} > 0. \tag{26}$$

Now, from (20) and condition (13), we obtain

$$b_j = \left(\lambda_n^{(n)} - a_j\right)\frac{x_j}{x_{j+1}} - c_{j-1}\frac{x_{j-1}}{x_{j+1}}; \quad j = 2, 3, \ldots, n-1. \tag{27}$$

From (16) and (17) it follows that

$$c_{j-1} = \frac{1}{b_{j-1}} \cdot \frac{\left(\lambda_j^{(j)} - \lambda_1^{(j)}\right) P_{j-1}\left(\lambda_1^{(j)}\right) P_{j-1}\left(\lambda_j^{(j)}\right)}{P_{j-1}\left(\lambda_1^{(j)}\right) P_{j-2}\left(\lambda_j^{(j)}\right) - P_{j-1}\left(\lambda_j^{(j)}\right) P_{j-2}\left(\lambda_1^{(j)}\right)}, \quad j = 3, \ldots, n. \tag{28}$$

and

$$a_j = \frac{\lambda_1^{(j)} P_{j-1}\left(\lambda_1^{(j)}\right) P_{j-2}\left(\lambda_j^{(j)}\right) - \lambda_j^{(j)} P_{j-1}\left(\lambda_j^{(j)}\right) P_{j-2}\left(\lambda_1^{(j)}\right)}{P_{j-1}\left(\lambda_1^{(j)}\right) P_{j-2}\left(\lambda_j^{(j)}\right) - P_{j-1}\left(\lambda_j^{(j)}\right) P_{j-2}\left(\lambda_1^{(j)}\right)}, \quad j = 3, \ldots, n. \tag{29}$$

Finally, from Lemma 5 and condition (12)

$$b_{j-1}c_{j-1} = \frac{(-1)^{j-1}\left(\lambda_j^{(j)} - \lambda_1^{(j)}\right) P_{j-1}\left(\lambda_1^{(j)}\right) P_{j-1}\left(\lambda_j^{(j)}\right)}{(-1)^{j-1}\left[P_{j-1}\left(\lambda_1^{(j)}\right) P_{j-2}\left(\lambda_j^{(j)}\right) - P_{j-1}\left(\lambda_j^{(j)}\right) P_{j-2}\left(\lambda_1^{(j)}\right)\right]} > 0, \tag{30}$$

for $j = 3, 4, \ldots, n$. Thus, we obtain a unique nonsymmetric tridiagonal matrix of form *(1)*. □

Theorem 9. *Let the real numbers* $\{\lambda_1^{(j)}, \lambda_j^{(j)}\}_{j=1}^n$ *and the vector* $\mathbf{x}^{(n)} = (x_1, x_2, \ldots, x_n)$ *be satisfying*

$$\lambda_1^{(n)} < \lambda_1^{(n-1)} < \cdots < \lambda_1^{(2)} < \lambda_1^{(1)} < \lambda_2^{(2)} < \cdots < \lambda_{n-1}^{(n-1)} < \lambda_n^{(n)} \quad (31)$$

and

$$x_i \neq 0, \quad i = 1, 2, \ldots, n. \quad (32)$$

Then there exists a unique nonsymmetric arrow matrix A *of form (2), such that* $\lambda_1^{(j)}$ *and* $\lambda_j^{(j)}$ *are, respectively, the minimal and the maximal eigenvalue of the leading principal submatrix* A_j, $j = 1, 2, \ldots, n$, *of* A, *and* $(x^{(n)}, \lambda_n^{(n)})$ *is an eigenpair of* A.

Proof. Suppose $\{\lambda_1^{(j)}, \lambda_j^{(j)}\}_{j=1}^n$ and $x^{(n)}$ satisfy conditions (31) and (32), respectively. To show the existence of a nonsymmetric arrow matrix A with the required properties is equivalent to show that the system of equations

$$P_j\left(\lambda_i^{(j)}\right) = 0, \quad j = 1, 2, \ldots, n, \ i = 1, j \quad (33)$$

$$A\mathbf{x}^{(n)} = \lambda_n^{(n)}\mathbf{x}^{(n)}, \quad (34)$$

where $P_j(\lambda) = \det(\lambda I_j - A_j)$, $j = 1, 2, \ldots, n$ satisfies Lemma 7, has real solutions a_j, $j = 1, 2, \ldots, n$ and b_i, c_i, $i = 1, 2, \ldots, n-1$, with $b_i c_i > 0$.

From Lemma 7 for $j = 1$ it follows that $P_1(\lambda_1^{(1)}) = \lambda_1^{(1)} - a_1 = 0$. Then

$$a_1 = \lambda_1^{(1)} \quad (35)$$

From (41) and (42) for $j = 2$, it follows that

$$\begin{aligned} P_2\left(\lambda_1^{(2)}\right) &= \left(\lambda_1^{(2)} - a_2\right) P_1\left(\lambda_1^{(2)}\right) - b_1 c_1 = 0 \\ P_2\left(\lambda_2^{(2)}\right) &= \left(\lambda_2^{(2)} - a_2\right) P_1\left(\lambda_2^{(2)}\right) - b_1 c_1 = 0 \end{aligned} \quad (36)$$

and

$$a_2 = \frac{\lambda_1^{(2)} P_1\left(\lambda_1^{(2)}\right) - \lambda_2^{(2)} P_1\left(\lambda_2^{(2)}\right)}{P_1\left(\lambda_1^{(2)}\right) - P_1\left(\lambda_2^{(2)}\right)}. \quad (37)$$

Now, from conditions (32) and (44), we have

$$c_1 = \left(\lambda_n^{(n)} - a_2\right)\left(\frac{x_2}{x_1}\right). \quad (38)$$

Thus,

$$b_1 = \frac{1}{c_1} \frac{\left(\lambda_2^{(2)} - \lambda_1^{(2)}\right) P_1\left(\lambda_1^{(2)}\right) P_1\left(\lambda_2^{(2)}\right)}{P_1\left(\lambda_1^{(2)}\right) - P_1\left(\lambda_2^{(2)}\right)}. \quad (39)$$

Moreover, from Lemma 5 and condition (31),

$$b_1 c_1 = \frac{(-1)\left(\lambda_2^{(2)} - \lambda_1^{(2)}\right) P_1\left(\lambda_1^{(2)}\right) P_1\left(\lambda_2^{(2)}\right)}{(-1)\left[P_1\left(\lambda_1^{(2)}\right) - P_1\left(\lambda_2^{(2)}\right)\right]} > 0 \quad (40)$$

From Lemma 7, for $j = 3, 4, \ldots, n$, system (33) can be written as

$$P_j\left(\lambda_1^{(j)}\right) = \left(\lambda_1^{(j)} - a_j\right) P_{j-1}\left(\lambda_1^{(j)}\right) - b_{j-1} c_{j-1} \prod_{i=2}^{j-1}\left(\lambda_1^{(j)} - a_i\right) = 0, \quad (41)$$

$$P_j\left(\lambda_j^{(j)}\right) = \left(\lambda_j^{(j)} - a_j\right) P_{j-1}\left(\lambda_j^{(j)}\right) - b_{j-1} c_{j-1} \prod_{i=2}^{j-1}\left(\lambda_j^{(j)} - a_i\right) = 0, \quad (42)$$

$$a_1 x_1 + \sum_{k=1}^{n-1} b_k x_{k+1} = \lambda_n^{(n)} x_1 \quad (43)$$

$$c_{j-1} x_1 + a_j x_j = \lambda_n^{(n)} x_j \quad j = 2, 3, \ldots, n. \quad (44)$$

Now, from (41) and (42), we have

$$a_j = \frac{\lambda_1^{(j)} P_{j-1}\left(\lambda_1^{(j)}\right) \prod_{i=2}^{j-1}\left(\lambda_j^{(j)} - a_i\right) - \lambda_j^{(j)} P_{j-1}\left(\lambda_j^{(j)}\right) \prod_{i=2}^{j-1}\left(\lambda_1^{(j)} - a_i\right)}{P_{j-1}\left(\lambda_1^{(j)}\right) \prod_{i=2}^{j-1}\left(\lambda_j^{(j)} - a_i\right) - P_{j-1}\left(\lambda_j^{(j)}\right) \prod_{i=2}^{j-1}\left(\lambda_1^{(j)} - a_i\right)}. \quad (45)$$

And from (44) and condition (32),

$$c_{j-1} = \left(\lambda_n^{(n)} - a_j\right)\left(\frac{x_j}{x_1}\right). \quad (46)$$

Then,

$$b_{j-1} = \frac{1}{c_{j-1}} \cdot \frac{\left(\lambda_j^{(j)} - \lambda_1^{(j)}\right) P_{j-1}\left(\lambda_1^{(j)}\right) P_{j-1}\left(\lambda_j^{(j)}\right)}{P_{j-1}\left(\lambda_1^{(j)}\right) \prod_{i=2}^{j-1}\left(\lambda_j^{(j)} - a_i\right) - P_{j-1}\left(\lambda_j^{(j)}\right) \prod_{i=2}^{j-1}\left(\lambda_1^{(j)} - a_i\right)}. \quad (47)$$

Finally, from Lemmas 3 and 5 and condition (31), we have

$$b_{j-1} c_{j-1} = \frac{(-1)^{j-1}\left(\lambda_j^{(j)} - \lambda_1^{(j)}\right) P_{j-1}\left(\lambda_1^{(j)}\right) P_{j-1}\left(\lambda_j^{(j)}\right)}{(-1)^{j-1}\left[P_{j-1}\left(\lambda_1^{(j)}\right) \prod_{i=2}^{j-1}\left(\lambda_j^{(j)} - a_i\right) - P_{j-1}\left(\lambda_j^{(j)}\right) \prod_{i=2}^{j-1}\left(\lambda_1^{(j)} - a_i\right)\right]} > 0. \quad (48)$$

Thus, we obtain a unique nonsymmetric arrow matrix of form *(2)*. □

We observe that the construction given by Theorems 8 and 9 generalizes the procedures given in [11, 12], in which the authors only consider the extremal eigenvalues as initial spectral information.

3. Numerical Examples

Example 1. The real numbers

$$\begin{matrix} \lambda_1^{(6)} & \lambda_1^{(5)} & \lambda_1^{(4)} & \lambda_1^{(3)} & \lambda_1^{(2)} & \lambda_1^{(1)} & \lambda_2^{(2)} & \lambda_3^{(3)} & \lambda_4^{(4)} & \lambda_5^{(5)} & \lambda_6^{(6)} \\ -8 & -5 & -3 & -2 & -1 & 2 & 6 & 7 & 9 & 12 & 15 \end{matrix}, \quad (49)$$

and the vector

$$x = (0.1, 0.2, 0.3, 0.4, 0.5, 0.6) \quad (50)$$

satisfy conditions (12) and (13). Our procedure from Theorem 8 gives the matrix

$$A = \begin{pmatrix} 2.0000 & 6.5000 & 0 & 0 & 0 & 0 \\ 1.8462 & 3.0000 & 7.3846 & 0 & 0 & 0 \\ 0 & 1.0833 & 2.0000 & 9.2083 & 0 & 0 \\ 0 & 0 & 2.3489 & 4.7865 & 6.7615 & 0 \\ 0 & 0 & 0 & 6.7122 & 2.5515 & 5.8989 \\ 0 & 0 & 0 & 0 & 12.8876 & 4.2603 \end{pmatrix}, \quad (51)$$

with the required spectral properties.

Example 2. The eigenvalues of matrix

$$A = \begin{pmatrix} \alpha & \beta & 0 & \cdots & 0 \\ \gamma & \alpha & \beta & \cdots & \vdots \\ 0 & \gamma & \alpha & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \beta \\ 0 & \cdots & 0 & \gamma & \alpha \end{pmatrix}; \quad \beta\gamma > 0, \quad (52)$$

are given by

$$\lambda_1^{(j)} = \alpha + 2\sqrt{\beta\gamma}\cos\left(\frac{\pi}{j+1}\right), \quad j = 2, 3, \ldots, n \quad (53)$$

$$\lambda_j^{(j)} = \alpha + 2\sqrt{\beta\gamma}\cos\left(\frac{j\pi}{j+1}\right), \quad j = 1, 2, \ldots, n. \quad (54)$$

And the components of an eigenvector associated with $\lambda_n^{(n)}$ are given by the recurrence relation:

$$\begin{aligned} x_1 &= 1; \\ x_2 &= \frac{1}{\beta}\left(\lambda_n^{(n)} - \alpha\right); \\ x_i &= \frac{1}{\beta}\left[\left(\lambda_n^{(n)} - \alpha\right)x_{i-1} - \gamma x_{i-2}\right], \quad i = 3, \ldots, n \end{aligned} \quad (55)$$

In Table 1 we consider A (52) with $\alpha = 4$, $\beta = 1$, and $\gamma = 2$. λ is the vector with components and the eigenvalues given in (53) and (54) and x is the vector defined by (55). We denote by $\widetilde{A}$ the constructed matrix by the procedure from Theorem 8 and $\widetilde{\lambda}$ is the vector with the extreme eigenvalues of $\widetilde{A}$. We consider the following expressions: $e_A = \log(\|A - \widetilde{A}\|/\|A\|)$, $e_\lambda = \log(\|\lambda - \widetilde{\lambda}\|/\|\lambda\|)$, and $e_x = \log(\|\widetilde{A}x - \widetilde{\lambda}_n^{(n)}x\|/\|\lambda_n^{(n)}x\|)$.

Example 3. Consider the $(2n-1)$-dimensional vector $\boldsymbol{\lambda}$, whose entries are real numbers chosen randomly in nondecreasing order and a n-dimensional vector x, whose components are all nonzero. Let $\widetilde{A}$ be $n \times n$ nonsymmetric tridiagonal matrix constructed from the entries of $\boldsymbol{\lambda}$ in such a way that $\widetilde{\lambda}_1^{(j)}$ and $\widetilde{\lambda}_j^{(j)}$ are the minimal and maximal eigenvalues of the $j \times j$ leading principal matrix $\widetilde{A}_j$ of $\widetilde{A}$, $j = 1, 2, \ldots, n$. Let $\widetilde{\boldsymbol{\lambda}}$ be the vector whose entries are the numbers $\widetilde{\lambda}_1^{(j)}$ and $\widetilde{\lambda}_j^{(j)}$, $j = 1, 2, \ldots, n$, in nondecreasing order. Figure 1 shows the plot of e_λ and e_x defined in Example 2, with $n = 50$ and 100 reconstructions of the matrix $\widetilde{A}$.

Example 4. In this example we consider random real numbers generated from the random Matlab function:

$$\begin{matrix} \lambda_1^{(7)} & \lambda_1^{(6)} & \lambda_1^{(5)} & \lambda_1^{(4)} & \lambda_1^{(3)} & \lambda_1^{(2)} & \\ -5.6310 & -5.2374 & -4.6520 & -4.3671 & -4.3250 & -3.7990 & \\ \lambda_1^{(1)} & \lambda_2^{(2)} & \lambda_3^{(3)} & \lambda_4^{(4)} & \lambda_5^{(5)} & \lambda_6^{(6)} & \lambda_7^{(7)} \\ -1.6036 & 0.1400 & 4.5734 & 4.5770 & 5.9097 & 6.4780 & 6.6793 \end{matrix} \quad (56)$$

and

$$\begin{aligned} x = (&0.5886, 0.3527, -0.1856, 0.0370, \\ &-0.4738, 0.3565, -0.3765) \end{aligned} \quad (57)$$

satisfying conditions (31) and (32), respectively. From Theorem 9 procedure we obtain the matrix

$$A = \begin{pmatrix} -1.6036 & 0.7312 & -7.0534 & 0.2249 & -3.1132 & 3.8302 & -1.2197 \\ 5.2347 & -2.0554 & & & & & \\ -1.1019 & & 3.1855 & & & & \\ 0.5884 & & & -2.6848 & & & \\ -1.5557 & & & & 4.7466 & & \\ 2.1299 & & & & & 3.1628 & \\ -3.5368 & & & & & & 1.1509 \end{pmatrix} \quad (58)$$

with the required spectral properties.

Table 1

n	e_A	e_λ	e_x
5	−14.8638	−15.7126	−15.5670
10	−13.8739	−14.9518	−15.0035
15	−13.1475	−14.6297	−14.3310
20	−12.7062	−14.7679	−14.8618
25	−12.1855	−14.3766	−14.1447
30	−12.2131	−13.8168	−13.4682
40	−12.0392	−12.9129	−12.2304
50	−11.4974	−11.8000	−11.2788

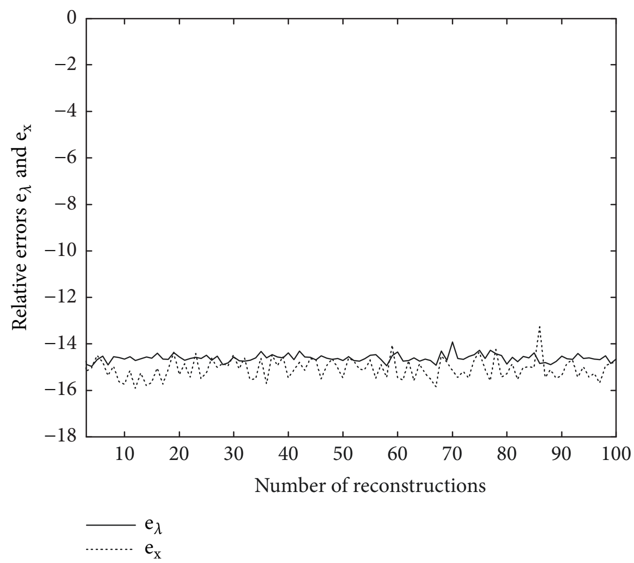

Figure 1

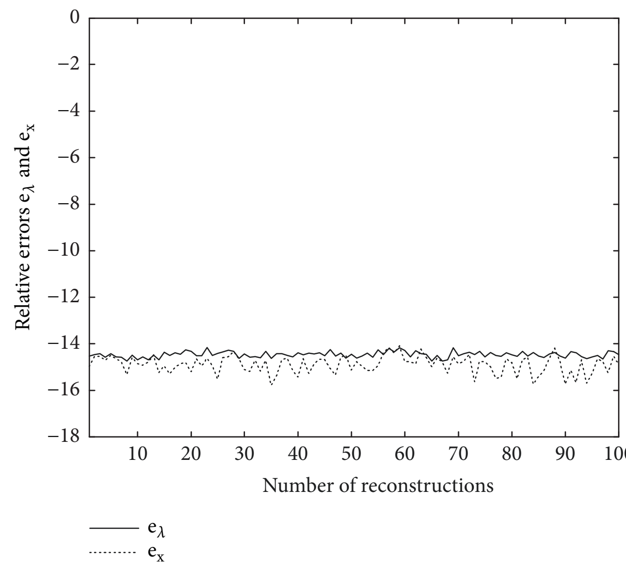

Figure 2

Example 5. In Table 2, we construct nonsymmetric arrow matrices of different orders, from data obtained arbitrarily by the random function in Matlab, which satisfy conditions (31) and (32) of Theorem 9. We denote by $\widetilde{A}$ the constructed matrix and $\widetilde{\lambda}$ is the vector with the extreme eigenvalues of $\widetilde{A}$. We consider the expressions, $e_\lambda = \log(\|\lambda - \widetilde{\lambda}\|/\|\lambda\|)$ and $e_x = \log(\|\widetilde{A}x - \widetilde{\lambda}_n^{(n)}x\|/\|\lambda_n^{(n)}x\|)$.

Example 6. Consider $(2n-1)$–dimensional vector $\boldsymbol{\lambda}$, whose entries are real numbers chosen randomly in nondecreasing order and a n-dimensional vector x, whose components are all nonzero. Let $\widetilde{A}$ be $n \times n$ nonsymmetric arrow matrix constructed from the entries of $\boldsymbol{\lambda}$ in such a way that $\widetilde{\lambda}_1^{(j)}$ and $\widetilde{\lambda}_j^{(j)}$ are the minimal and maximal eigenvalues of the $j \times j$ leading principal matrix $\widetilde{A}_j$ of $\widetilde{A}$, $j = 1, 2, \ldots, n$. Let $\widetilde{\boldsymbol{\lambda}}$ be the vector whose entries are numbers $\widetilde{\lambda}_1^{(j)}$ and $\widetilde{\lambda}_j^{(j)}$, $j = 1, 2, \ldots, n$, in nondecreasing order. Figure 2 shows the plot of e_λ and e_x defined in Example 5, with $n = 50$ and 100 reconstructions of the matrix $\widetilde{A}$.

Data Availability

No data were used to support this study.

Table 2

n	e_λ	e_x
5	−14.9841	−15.0703
10	−14.9891	−15.9881
15	−14.9286	−15.7874
20	−14.8021	−15.9188
25	−14.7349	−15.4561
30	−14.8234	−14.5340
40	−14.6183	−14.2118
50	−14.6217	−14.3512

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

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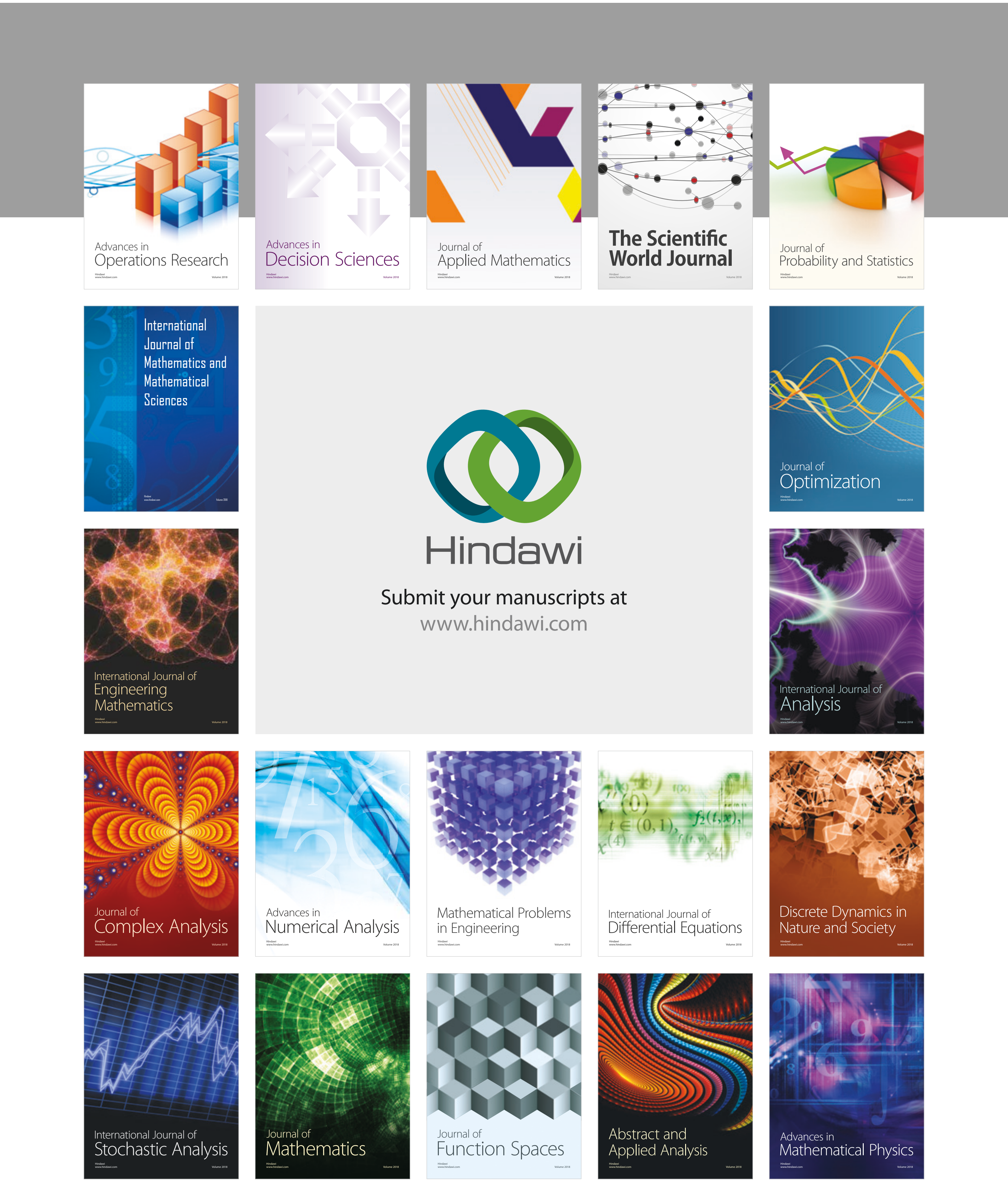
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