Research Article

Positive Solutions for a Fourth-Order Riemann–Stieltjes Integral Boundary Value Problem

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In this paper, we use the fixed point index to study the existence of positive solutions for the fourth-order Riemann–Stieltjes integral boundary value problem $\begin{cases} -x^{(4)}(t) = f(t, x(t), x'(t), x''(t), x'''(t)), & t \in (0, 1) \\ x(0) = x'(0) = x'''(1) = 0, & x''(0) = \alpha[x''(t)] \end{cases}$, where $f: [0, 1] \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a continuous function and $\alpha[x'']$ denotes a linear function. Two existence theorems are obtained with some appropriate inequality conditions on the nonlinearity f , which involve the spectral radius of related linear operators. These conditions allow $f(t, z_1, z_2, z_3, z_4)$ to have superlinear or sublinear growth in z_i , $i = 1, 2, 3, 4$.

1. Introduction

In this paper, we investigate the existence of positive solutions for the following fourth-order Riemann–Stieltjes integral boundary value problem:

$$\begin{cases} -x^{(4)}(t) = f(t, x(t), x'(t), x''(t), x'''(t)), & t \in (0, 1), \\ x(0) = x'(0) = x'''(1) = 0, & x''(0) = \alpha[x''(t)], \end{cases} \quad (1)$$

where $\alpha[x''(t)] = \int_0^1 x''(t) d\beta(t)$ denotes the Riemann–Stieltjes integral with a suitable function β of bounded variation and

$$(H0)\alpha[1] \in [0, 1]. \quad (2)$$

The deformation of an elastic beam in an equilibrium state can be described by a fourth-order ordinary equation boundary value problem [1], and there are a large number of papers in the literature in this direction; for example, see [1–30] and the references therein. In [1], the author used Krasnosel'skii's fixed point theorem to establish one or two positive solutions for the fourth-order boundary value problem:

$$\begin{cases} u^{(4)}(t) + \beta u''(t) = \lambda f(t, u(t), u''(t)), & t \in (0, 1), \\ u(0) = u(1) = \int_0^1 p(s)u(s)ds, \\ u''(0) = u''(1) = \int_0^1 q(s)u''(s)ds. \end{cases} \quad (3)$$

When $f \in C([0, 1] \times \mathbb{R}^+ \times \mathbb{R}^-, \mathbb{R}^+)$ and in [2], the authors studied the existence of positive solutions for the fourth-order m -point boundary value problem:

$$\begin{cases} u^{(4)} + \alpha u'' - \beta u = f(t, u), & t \in (0, 1), \\ u(0) = \sum_{i=1}^{m-2} a_i u(\xi_i), u(1) = \sum_{i=1}^{m-2} b_i u(\xi_i), \\ u''(0) = \sum_{i=1}^{m-2} a_i u''(\xi_i), \\ u''(1) = \sum_{i=1}^{m-2} b_i u''(\xi_i), \end{cases} \quad (4)$$

where $f \in C([0, 1] \times \mathbb{R}^+, \mathbb{R}^+)$ satisfies superlinear and sublinear growth conditions:

$$\begin{aligned} \text{superlinear: } \liminf_{u \rightarrow +\infty} \min_{t \in [0, 1]} \frac{f(t, u)}{u} &> \lambda^*, \\ \limsup_{u \rightarrow 0^+} \max_{t \in [0, 1]} \frac{f(t, u)}{u} &< \lambda^*, \\ \text{sublinear: } \liminf_{u \rightarrow 0^+} \min_{t \in [0, 1]} \frac{f(t, u)}{u} &> \lambda^*, \\ \limsup_{u \rightarrow +\infty} \max_{t \in [0, 1]} \frac{f(t, u)}{u} &< \lambda^*, \end{aligned} \quad (5)$$

where λ^* is the first eigenvalue of the relevant linear operator.

In [3], the authors studied the existence of an iterative solution for the fourth-order boundary value problem:

$$\begin{cases} u^{(4)}(t) = f(t, u(t), u'(t)), & t \in (0, 1), \\ u(0) = u'(0) = u'(1) = u''(1) = 0, \end{cases} \quad (6)$$

where $f: [0, 1] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ is continuous and satisfies some appropriate Lipschitz condition, and in [4], the authors used the method of upper and lower solution to establish existence results for the fourth-order four-point boundary value problem on time scales:

$$\begin{cases} u^{\Delta\Delta\Delta\Delta}(t) = f(t, u(\sigma(t)), u^{\Delta\Delta}(t)), & t \in [0, 1]_{\mathbb{T}}, \\ u(0) = u(\sigma^4(1)) = 0, \\ \alpha u^{\Delta\Delta}(\xi_1) - \beta u^{\Delta\Delta\Delta}(\xi_1) = 0, \\ \gamma u^{\Delta\Delta}(\xi_2) + \eta u^{\Delta\Delta\Delta}(\xi_2) = 0, \end{cases} \quad (7)$$

where $f: [0, 1] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function.

There are only a few papers in the literature which consider general nonlinearities for fourth-order boundary value problems. The difficulty lies in a priori estimates for third-order derivatives, so some authors adopted a Nagumo-type growth condition (see (H7) in Section 3) to overcome this difficulty; for example, see [15, 16, 31–34] and the references therein. In [15], the author studied the existence of positive solutions for the fourth-order boundary value problem:

$$\begin{cases} u^{(4)} = f(t, u, u', u'', u'''), & t \in (0, 1), \\ u(0) = u'(0) = u''(1) = u'''(1) = 0, \end{cases} \quad (8)$$

where $f \in C([0, 1] \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^-, \mathbb{R}^+)$ satisfies some inequality conditions, where f grows both superlinearly and sublinearly about its variables u, u', u'' , and u''' . When f is superlinear, a Nagumo-type condition is used to restrict the growth of f on u'' and u''' .

Integral boundary conditions arise in thermal conduction problems [35], semiconductor problems [36], and hydrodynamic problems [37], and there are some papers in the literature devoted to this direction (see [1, 9, 19, 34, 38–48]). In [19], the authors studied p -Laplacian fourth-order differential equations with Riemann–Stieltjes integral boundary conditions:

$$\begin{cases} (\phi_{p_1}(u''(t)))'' = \lambda^{p_1-1} a_1(t) f_1(t, u(t), v(t)), & 0 < t < 1, \\ (\phi_{p_1}(v''(t)))'' = \mu^{p_2-1} a_2(t) f_2(t, u(t), v(t)), \\ u(0) = u(1) = \int_0^1 u(s) d\xi_1(s), \\ \phi_{p_1}(u''(0)) = \phi_{p_1}(u''(1)) = \int_0^1 (u''(s)) d\eta_1(s) \phi_{p_1}, \\ v(0) = v(1) = \int_0^1 v(s) d\xi_2(s), \\ \phi_{p_2}(v''(0)) = \phi_{p_2}(v''(1)) = \int_0^1 \phi_{p_2}(v''(s)) d\eta_2(s). \end{cases} \quad (9)$$

The authors used fixed point theory in cones to obtain the existence of positive solutions for the above problem and provided the interval ranges of the parameters λ and μ for these solutions.

In [38], the authors studied the fractional differential equation with a singular decreasing nonlinearity and a p -Laplacian operator:

$$\begin{cases} -D_{0+}^\alpha (\phi_p(-D_{0+}^\gamma z))(x) = f(x, z(x)), & 0 < x < 1, \\ z(0) = 0, D_{0+}^\gamma z(0) = D_{0+}^\gamma z(1) = 0, \\ z(1) = \int_0^1 z(x) d\chi(x). \end{cases} \quad (10)$$

Using a double iterative technique, they showed that the above problem has a unique positive solution, and from an iterative technique, they established an appropriate sequence, which converges uniformly to the unique positive solution.

Motivated by the aforementioned works, the aim of this paper is to study the existence of positive solutions for the fourth-order Riemann–Stieltjes integral boundary value problem (1). The novelty is of two folds: (1) we provide some useful inequality conditions on f involving the first eigenvalue of the relevant linear operator (these conditions imply that f grows superlinearly and sublinearly) and (2) for the

superlinear case, an appropriate Nagumo-type condition is used to restrict the growth of f on x''' in (1).

2. Preliminaries

In this section, we first transform (1) into an equivalent Hammerstein-type integral equation. For this, let $x''(t) = y(t)$, for $t \in [0, 1]$. Then, from the conditions $x(0) = x'(0) = 0$, we have

$$\begin{aligned} x'(t) &= \int_0^t x''(s) ds = \int_0^t y(s) ds, \\ x(t) &= \int_0^t x'(s) ds = \int_0^t \int_0^s y(\tau) d\tau ds. \end{aligned} \quad (11)$$

Therefore, substituting (11) into (1) gives

$$\begin{cases} -y''(t) = f\left(t, \int_0^t \int_0^s y(\tau) d\tau ds, \int_0^t y(s) ds, y(t), y'(t)\right), & t \in (0, 1), \\ y(0) = \alpha[y(t)], \\ y'(1) = 0. \end{cases} \quad (12)$$

Lemma 1. *The problem (12) can be transformed into the Hammerstein-type integral equation:*

$$y(t) = \int_0^1 G_1(t, s) f\left(s, \int_0^s \int_0^r y(\tau) d\tau dr, \int_0^s y(\tau) d\tau, y(s), y'(s)\right) ds, \quad (13)$$

where $G_1(t, s) = (1/1 - \alpha[1]) \int_0^1 K_1(t, s) d\beta(t) + K_1(t, s)$ and $K_1(t, s) = \min\{t, s\}$, for $t, s \in [0, 1]$.

Proof. Using the function g on $[0, 1]$ to replace $f(t, \int_0^t \int_0^s y(\tau) d\tau ds, \int_0^t y(s) ds, y(t), y'(t))$ in (12), we consider the following problem:

$$\begin{cases} -y''(t) = g(t), & t \in (0, 1), \\ y(0) = \alpha[y(t)], \\ y'(1) = 0. \end{cases} \quad (14)$$

From the differential equation in (14), we obtain

$$y(t) = \int_0^t (s-t)g(s)ds + c_1 t + c_2, \quad \text{for some } c_i \in \mathbb{R}, i = 1, 2, \quad (15)$$

and then

$$\begin{aligned} y(t) &= \int_0^t (s-t)g(s)ds + \int_0^1 t g(s)ds + \frac{1}{1-\alpha[1]} \int_0^1 \int_0^1 K_1(t, s) g(s) ds d\beta(t) \\ &= \int_0^1 G_1(t, s) g(s) ds. \end{aligned} \quad (20)$$

This completes the proof.

Let $E = C^1[0, 1]$, $y = \max\{\|y\|_C, \|y'\|_C\}$ with $\|y\|_C = \max_{t \in [0, 1]} |y(t)|$, and $P = \{y \in E: y(t) \geq 0, y'(t) \geq 0, \forall t \in [0, 1]\}$. Then, $(E, \|\cdot\|)$ is a Banach space, and P is a cone on E . From Lemma 1, we can define an operator $A: P \rightarrow P$ as follows:

$$(Ay)(t) = \int_0^1 G_1(t, s) f\left(s, \int_0^s \int_0^r y(\tau) d\tau dr, \int_0^s y(\tau) d\tau, y(s), y'(s)\right) ds, \quad \text{for } y \in E. \quad (21)$$

Then, A is a completely continuous operator from the Arzelà–Ascoli theorem (this argument is standard). $\square$

Remark 1

- (i) In our work, we need the nonnegativity of Green's function G_1 , so we have the following assumption:

$$(H1) \int_0^1 K_1(t, s) d\beta(t) \geq 0, \quad \text{for } s \in [0, 1]. \quad (22)$$

- (ii) We need some inequality conditions on the non-linearity $f(t, z_1, z_2, z_3, z_4)$ with respect to the variables $z_i, i = 1, 2, 3, 4$. We consider some useful linear operators:

$$\begin{aligned} (L_1 y)(t) &= \int_0^1 G_1(t, s) \int_0^s \int_0^\tau y(r) dr d\tau ds := \int_0^1 G_2(t, s) y(s) ds, \\ (L_2 y)(t) &= \int_0^1 G_1(t, s) \int_0^1 \int_0^s y(\tau) d\tau ds := \int_0^1 G_3(t, s) y(s) ds, \quad \text{for } y \in E, t \in [0, 1]. \end{aligned} \quad (23)$$

If we know the function β , we can obtain the functions G_2 and G_3 .

Example 1. Let $\beta(t) = 0$ for $t \in [0, 1]$. Then, $G_1(t, s) = K_1(t, s)$, for $t, s \in [0, 1]$. Let

$$h(s, \tau) = \begin{cases} 1, & 0 \leq \tau \leq s \leq 1, \\ 0, & 0 \leq s \leq \tau \leq 1, \end{cases} \quad (24)$$

and then from (22), we find

$$G_3(t, s) = \int_0^1 G_1(t, \tau) h(\tau, s) d\tau, \quad \text{for } t, s \in [0, 1]. \quad (25)$$

We consider two cases:

- (i) Case 1: when $0 \leq t \leq s \leq 1$, we have

$$G_3(t, s) = \int_0^t 0 \cdot \tau d\tau + \int_t^s 0 \cdot t d\tau + \int_s^1 1 \cdot t d\tau = t(1 - s). \quad (26)$$

- (ii) Case 2: when $0 \leq s \leq t \leq 1$, we have

$$G_3(t, s) = \int_0^s 0 \cdot \tau d\tau + \int_s^t 1 \cdot \tau d\tau + \int_t^1 1 \cdot t d\tau = t - \frac{1}{2}t^2 - \frac{1}{2}s^2. \quad (27)$$

We now calculate G_2 . For this, let $\int_0^\tau y(r) dr = j(\tau)$ and $k(\tau, r) = \begin{cases} 0, & 0 \leq \tau \leq r \leq 1, \\ 1, & 0 \leq r \leq \tau \leq 1. \end{cases}$ Then, we have

$$\begin{aligned} \int_0^1 G_1(t, s) \int_0^s j(\tau) d\tau ds &= \int_0^1 G_3(t, \tau) j(\tau) d\tau \\ &= \int_0^1 G_3(t, \tau) \int_0^\tau y(r) dr d\tau \\ &= \int_0^1 G_3(t, \tau) \int_0^1 k(\tau, r) y(r) dr d\tau. \end{aligned} \quad (28)$$

Therefore, from (22), we have

$$G_2(t, s) = \int_0^1 G_3(t, \tau) k(\tau, s) d\tau, \quad \text{for } t, s \in [0, 1]. \quad (29)$$

We consider two cases:

- (i) Case 1: when $0 \leq t \leq s \leq 1$, we have

$$\begin{aligned} G_2(t, s) &= \int_0^t 0 \cdot \left(t - \frac{1}{2}t^2 - \frac{1}{2}\tau^2 \right) d\tau \\ &\quad + \int_t^s 0 \cdot t(1 - \tau) d\tau + \int_s^1 1 \cdot t(1 - \tau) d\tau \\ &= t \left(\frac{1}{2} + \frac{1}{2}s^2 - s \right). \end{aligned} \quad (30)$$

- (ii) Case 2: when $0 \leq s \leq t \leq 1$, we have

$$\begin{aligned} G_2(t, s) &= \int_0^s 0 \cdot \left(t - \frac{1}{2}t^2 - \frac{1}{2}\tau^2 \right) d\tau \\ &\quad + \int_s^t 1 \cdot \left(t - \frac{1}{2}t^2 - \frac{1}{2}\tau^2 \right) d\tau + \int_t^1 1 \cdot t(1 - \tau) d\tau \\ &= \frac{1}{2}t - ts + \frac{1}{2}t^2s - \frac{1}{6}t^3 + \frac{1}{6}s^3. \end{aligned} \quad (31)$$

Example 2. Let $\beta(t) = (t/2)$ for $t \in [0, 1]$. Then, we have

$$\begin{aligned} \frac{1}{1 - \alpha[1]} \int_0^1 K_1(t, s) d\beta(t) &= \int_0^1 K_1(t, s) dt = \int_0^s t dt \\ &\quad + \int_s^1 s dt = s - \frac{1}{2}s^2, \quad \text{for } s \in [0, 1]. \end{aligned} \quad (32)$$

Hence, $G_1(t, s) = s - (1/2)s^2 + K_1(t, s)$, for $t, s \in [0, 1]$. Note (22) and Example 1, so we only need to calculate

$$\begin{aligned} \int_0^1 \left(\tau - \frac{1}{2}\tau^2 \right) h(\tau, s) d\tau &= \int_s^1 \left(\tau - \frac{1}{2}\tau^2 \right) d\tau = \frac{1}{3} - \frac{1}{2}s^2 + \frac{1}{6}s^3, \\ \int_0^1 \left(\frac{1}{3} - \frac{1}{2}\tau^2 + \frac{1}{6}\tau^3 \right) k(\tau, s) d\tau &= \int_s^1 \left(\frac{1}{3} - \frac{1}{2}\tau^2 + \frac{1}{6}\tau^3 \right) d\tau \\ &= \frac{5}{24} - \frac{1}{3}s + \frac{1}{6}s^3 - \frac{1}{24}s^4, \quad \text{for } s \in [0, 1]. \end{aligned} \quad (33)$$

Therefore, we obtain

$$G_3(t, s) = \frac{1}{3} - \frac{1}{2}s^2 + \frac{1}{6}s^3 + \begin{cases} t(1-s), & 0 \leq t \leq s \leq 1, \\ t - \frac{1}{2}t^2 - \frac{1}{2}s^2, & 0 \leq s \leq t \leq 1, \end{cases}$$

$$G_2(t, s) = \frac{5}{24} - \frac{1}{3}s + \frac{1}{6}s^3 - \frac{1}{24}s^4 + \begin{cases} t\left(\frac{1}{2} + \frac{1}{2}s^2 - s\right), & 0 \leq t \leq s \leq 1, \\ \frac{1}{2}t - ts + \frac{1}{2}t^2s - \frac{1}{6}t^3 + \frac{1}{6}s^3, & 0 \leq s \leq t \leq 1. \end{cases} \quad (34)$$

Lemma 2. (Krein–Rutman, see([49] theorem 19.3)). Let P be a reproducing cone in a real Banach space E , and let $L: E \longrightarrow E$ be a compact linear operator with $L(P) \subset P$. If

$r(L) > 0$, then there exists $\varphi \in P, \{0\}$ such that $L\varphi = r(L)\varphi$, where $r(L)$ is the spectral radius of L .

Lemma 3. For not all zero numbers a, b, c , and $d \geq 0$, we let

$$(L_{a,b,c,d}y)(t) = a \int_0^1 G_2(t, s)y(s)ds + b \int_0^1 G_3(t, s)y(s)ds + c \int_0^1 G_1(t, s)y(s)ds + d \int_0^1 G_1(t, s)y'(s)ds,$$

$$(L_{a,b,c}y)(t) = a \int_0^1 G_2(t, s)y(s)ds + b \int_0^1 G_3(t, s)y(s)ds + c \int_0^1 G_1(t, s)y(s)ds, \quad (35)$$

$$(L_{a,b,c}^*y)(s) = a \int_0^1 G_2(t, s)y(t)dt + b \int_0^1 G_3(t, s)y(t)dt + c \int_0^1 G_1(t, s)y(t)dt, \quad \text{for } y \in P.$$

Then,

$$\left(\frac{1}{30}a + \frac{1}{8}b + \frac{1}{3}c + \frac{1}{3}d\right)\kappa_1 \leq r(L_{a,b,c,d}) \leq \left(\frac{1}{8}a + \frac{1}{3}b + \frac{1}{2}c + \frac{1}{2}d\right)\kappa_2,$$

$$\left(\frac{1}{30}a + \frac{1}{8}b + \frac{1}{3}c\right)\kappa_1 \leq r(L_{a,b,c}) \leq \left(\frac{1}{8}a + \frac{1}{3}b + \frac{1}{2}c\right)\kappa_2,$$

$$\left(\frac{1}{30}a + \frac{1}{8}b + \frac{1}{3}c\right)\kappa_1 \leq r(L_{a,b,c}^*) \leq \left(\frac{1}{8}a + \frac{1}{3}b + \frac{1}{2}c\right)\kappa_2, \quad (36)$$

Proof. We first give some inequalities for G_i , $i = 1, 2, 3$. Note that $ts \leq K_1(t, s) \leq s$, for $t, s \in [0, 1]$, and from the definition of G_1 , we see that

$$\kappa_1 ts = \frac{\alpha[t] - \alpha[1] + 1}{1 - \alpha[1]} ts \leq G_1(t, s) \leq \frac{1}{1 - \alpha[1]} s$$

$$= \kappa_2 s, \quad \text{for } t, s \in [0, 1]. \quad (37)$$

where $\kappa_1 := (\alpha[t] - \alpha[1] + 1)/(1 - \alpha[1])$ and $\kappa_2 := 1/(1 - \alpha[1])$.

With h and k as before, note

$$\frac{1}{2}\kappa_1 t(1 - s^2) = \int_s^1 \kappa_1 t \tau d\tau \leq G_3(t, s) = \int_0^1 G_1(t, \tau)h(\tau, s)d\tau \leq \int_s^1 \kappa_2 \tau d\tau = \frac{1}{2}\kappa_2(1 - s^2)$$

for $t, s \in [0, 1]$,

$$\frac{1}{6}\kappa_1 t(2 + s^3 - 3s) = \int_s^1 \frac{1}{2}\kappa_1 t(1 - \tau^2)d\tau \leq G_2(t, s) = \int_0^1 G_3(t, \tau)k(\tau, s)d\tau \leq \int_s^1 \frac{1}{2}\kappa_2(1 - \tau^2)d\tau,$$

$$= \frac{1}{6}\kappa_2(2 + s^3 - 3s), \quad \text{for } t, s \in [0, 1]. \quad (38)$$

For convenience, let $\tilde{\psi}_0(t) = t$, $\tilde{\psi}_1(t) = (1/6)(2 + t^3 - 3t)$, and $\tilde{\psi}_2(t) = (1/2)(1 - t^2)$, for $t \in [0, 1]$.

We only prove the inequalities in (35) about the spectral radius of $L_{a,b,c}$. For convenience, let $G_{a,b,c}(t, s) = aG_2(t, s) + bG_3(t, s) + cG_1(t, s)$ for $t, s \in [0, 1]$. Then, we have

$$(L_{a,b,c}y)(t) = \int_0^1 G_{a,b,c}(t, s)y(s)ds, \quad \text{for } y \in P. \quad (39)$$

Thus, we obtain

$$\begin{aligned} \|L_{a,b,c}\| &= \sup_{\|y\|=1} \|L_{a,b,c}y\| = \sup_{t \in [0,1], \|y\|=1} \int_0^1 G_{a,b,c}(t, s)y(s)ds \\ &\leq \max_{t \in [0,1]} \int_0^1 G_{a,b,c}(t, s)ds \leq \int_0^1 \kappa_2 [a\tilde{\psi}_1(s) \\ &\quad + b\tilde{\psi}_2(s) + c\tilde{\psi}_0(s)]ds \\ &\leq \left(\frac{1}{8}a + \frac{1}{3}b + \frac{1}{2}c\right)\kappa_2. \end{aligned} \quad (40)$$

For all $n \in \mathbb{N}_+$, we note that

$$\begin{aligned} (L_{a,b,c}^n y)(t) &= L(L_{a,b,c}^{n-1} y)(t) = \int_0^1 G_{a,b,c}(t, s_{n-1})(L_{a,b,c}^{n-1} y)(s_{n-1})ds_{n-1} \\ &= \dots \\ &= \int_0^1 \int_0^1 \dots \int_0^1 G_{a,b,c}(t, s_{n-1})G_{a,b,c}(s_{n-1}, s_{n-2}) \\ &\quad \dots G_{a,b,c}(s_1, s)y(s)ds_{n-1}ds_{n-2}\dots ds, \quad \text{for } y \in P. \end{aligned} \quad (41)$$

Hence, we can obtain

$$\begin{aligned} \|L_{a,b,c}^n\| &\leq \max_{t \in [0,1]} \underbrace{\int_0^1 \int_0^1 \dots \int_0^1}_{n} G_{a,b,c}(t, s_{n-1})G_{a,b,c}(s_{n-1}, s_{n-2}) \\ &\quad \dots G_{a,b,c}(s_1, s)ds_{n-1}ds_{n-2}\dots ds \\ &\leq \kappa_2^n \left[\int_0^1 (a\tilde{\psi}_1(s) + b\tilde{\psi}_2(s) + c\tilde{\psi}_0(s))ds \right]^n \\ &\leq \left(\frac{1}{8}a + \frac{1}{3}b + \frac{1}{2}c\right)^n \kappa_2^n. \end{aligned} \quad (42)$$

Gelfand's theorem implies that

$$r(L_{a,b,c}) = \lim_{n \rightarrow \infty} \sqrt[n]{\|L_{a,b,c}^n\|} \leq \left(\frac{1}{8}a + \frac{1}{3}b + \frac{1}{2}c\right)\kappa_2. \quad (43)$$

Next, we introduce a conclusion in ([50], problem 2.1.4). Let $y \in C[0, 1]$, and a functional J on $C[0, 1]$ be as

$$J(x) = \int_0^1 x(t)y(t)dt, \quad \forall x \in C[0, 1]. \quad (44)$$

Then, we have

$$\|J\| = \int_0^1 |y(t)|dt. \quad (45)$$

We note that there exists $t_0 \in [0, 1]$ such that $\max_{t \in [0,1]} \int_0^1 G_{a,b,c}(t, s)ds = \int_0^1 G_{a,b,c}(t_0, s)ds$. Then, in (38), for fixed t , we define a linear function:

$$L_{a,b,c,t}y = \int_0^1 G_{a,b,c}(t, s)y(s)ds, \quad \text{for } y \in C[0, 1], \quad (46)$$

and thus

$$\|L_{a,b,c,t}\| = \int_0^1 G_{a,b,c}(t, s)ds. \quad (47)$$

Then, by the definition of the norm of linear function, we know that for all $\varepsilon > 0$, there exists $y_{\varepsilon,t} \in C[0, 1]$ with $\|y_{\varepsilon,t}\| = 1$ such that

$$\begin{aligned} \int_0^1 G_{a,b,c}(t, s)ds &= \|L_{a,b,c,t}\| = \sup_{\|y\|=1} |L_{a,b,c,t}y| \geq |L_{a,b,c,t}y_{\varepsilon,t}| \\ &\geq \int_0^1 G_{a,b,c}(t, s)ds - \varepsilon. \end{aligned} \quad (48)$$

On the contrary, note from the definition of our norm we have

$$\|L_{a,b,c}y\| = \sup_{t \in [0,1]} |L_{a,b,c,t}y| \geq |L_{a,b,c,t_0}y|. \quad (49)$$

Consequently, we have

$$\|L_{a,b,c}\| = \sup_{\|y\|=1} \|L_{a,b,c}y\| \geq \|L_{a,b,c,t_0}y_{\varepsilon,t_0}\| \geq \int_0^1 G_{a,b,c}(t_0, s)ds - \varepsilon. \quad (50)$$

For the arbitrariness of ε , we have

$$\|L_{a,b,c}\| = \int_0^1 G_{a,b,c}(t_0, s)ds = \max_{t \in [0,1]} \int_0^1 G_{a,b,c}(t, s)ds. \quad (51)$$

Also, for all $n \in \mathbb{N}_+$, we obtain

$$\begin{aligned}
\|L_{a,b,c}^n\| &= \max_{t \in [0,1]} \int_0^1 \int_0^1 \cdots \int_0^1 G_{a,b,c}(t, s_{n-1}) G_{a,b,c}(s_{n-1}, s_{n-2}) \cdots G_{a,b,c}(s_1, s) ds_{n-1} ds_{n-2} \cdots ds \\
&\geq \kappa_1^n \left(\max_{t \in [0,1]} t \right) \int_0^1 \int_0^1 \cdots \int_0^1 [a\tilde{\psi}_1(s_{n-1}) + b\tilde{\psi}_2(s_{n-1}) + c\tilde{\psi}_0(s_{n-1})] \\
&\quad \cdot s_{n-1} [a\tilde{\psi}_1(s_{n-2}) + b\tilde{\psi}_2(s_{n-2}) + c\tilde{\psi}_0(s_{n-2})] \cdots s_1 [a\tilde{\psi}_1(s) + b\tilde{\psi}_2(s) + c\tilde{\psi}_0(s)] ds_{n-1} ds_{n-2} \cdots ds \\
&= \kappa_1^n \int_0^1 [a\tilde{\psi}_1(s) + b\tilde{\psi}_2(s) + c\tilde{\psi}_0(s)] ds \left(\int_0^1 [as\tilde{\psi}_1(s) + bs\tilde{\psi}_2(s) + cs\tilde{\psi}_0(s)] ds \right)^{n-1}.
\end{aligned} \tag{52}$$

From Gelfand's theorem, we have

$$\begin{aligned}
r(L_{a,b,c}) &= \lim_{n \rightarrow \infty} \sqrt[n]{\|L_{a,b,c}^n\|} \geq \kappa_1 \int_0^1 [as\tilde{\psi}_1(s) + bs\tilde{\psi}_2(s) + cs\tilde{\psi}_0(s)] ds \\
&\geq \left(\frac{1}{30}a + \frac{1}{8}b + \frac{1}{3}c \right) \kappa_1.
\end{aligned} \tag{53}$$

This completes the proof. $\square$

Lemma 4. (see [51]). Let E be a real Banach space and P be a cone on E . Suppose that $\Omega \subset E$ is a bounded open set and that $A: \overline{\Omega} \cap P \rightarrow P$ is a continuous compact operator. If there exists a $\omega_0 \in P, \{0\}$ such that

$$\omega - A\omega \neq \lambda\omega_0, \quad \forall \lambda \geq 0, \omega \in \partial\Omega \cap P, \tag{54}$$

then $i(A, \Omega \cap P, P) = 0$, where i denotes the fixed point index on P .

Lemma 5. (see [51]). Let E be a real Banach space and P be a cone on E . Suppose that $\Omega \subset E$ is a bounded open set with $0 \in \Omega$ and that $A: \overline{\Omega} \cap P \rightarrow P$ is a continuous compact operator. If

$$\omega - \lambda A\omega \neq 0, \quad \forall \lambda \in [0, 1], \omega \in \partial\Omega \cap P, \tag{55}$$

then $i(A, \Omega \cap P, P) = 1$.

3. Main Results

In our paper, we let $B_\rho = \{y \in P: \|y\| < \rho\}$ for $\rho > 0$. Now, $\partial B_\rho = \{y \in P: \|y\| = \rho\}$ and $\overline{B}_\rho = \{y \in P: \|y\| \leq \rho\}$. Now, we list our assumptions on the nonlinearity f .

$$(H2) f \in C([0, 1] \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+). \tag{56}$$

(H3) There exist not all zero numbers a_1, b_1, c_1 , and $d_1 \geq 0$ and $e_1 > 0$ such that $r(L_{a_1, b_1, c_1, d_1}) < 1$, and $f(t, z_1, z_2, z_3, z_4) \leq a_1 z_1 + b_1 z_2 + c_1 z_3 + d_1 z_4 + e_1$, for $(t, z_1, z_2, z_3, z_4) \in [0, 1] \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+$.

(H4) There exist not all zero numbers a_2, b_2, c_2 , and $d_2 \geq 0$ and $\rho_1 > 0$ such that $r(L_{a_2, b_2, c_2, d_2}) \geq 1$, and $f(t, z_1, z_2, z_3, z_4) \geq a_2 z_1 + b_2 z_2 + c_2 z_3 + d_2 z_4$, for $(t, z_1, z_2, z_3, z_4) \in [0, 1] \times [0, \rho_1] \times [0, \rho_1] \times [0, \rho_1] \times [0, \rho_1]$.

(H5) There exist not all zero numbers a_3, b_3, c_3 , and $d_3 \geq 0$ and $\rho_2 > 0$ such that $r(L_{a_3, b_3, c_3, d_3}) < 1$, and $f(t, z_1, z_2, z_3, z_4) \leq a_3 z_1 + b_3 z_2 + c_3 z_3 + d_3 z_4$, for $(t, z_1, z_2, z_3, z_4) \in [0, 1] \times [0, \rho_2] \times [0, \rho_2] \times [0, \rho_2] \times [0, \rho_2]$.

(H6) There exist not all zero numbers a_4, b_4 , and $c_4 \geq 0$ and $e_2 > 0$ such that $r(L_{a_4, b_4, c_4}^*) > 1$, and $f(t, z_1, z_2, z_3, z_4) \geq a_4 z_1 + b_4 z_2 + c_4 z_3 - e_2$, for $(t, z_1, z_2, z_3, z_4) \in [0, 1] \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+$.

(H7) For any $M > 0$, there exists a positive continuous function $H_M(\varrho)$ on $\mathbb{R}^+$ such that $\int_0^{+\infty} (\varrho d\varrho / H_M(\varrho) + \delta_0) = +\infty$, $\forall \delta_0 > 0$, and $f(t, z_1, z_2, z_3, z_4) \leq H_M(z_4)$, for $(t, z_1, z_2, z_3, z_4) \in [0, 1] \times [0, M] \times [0, M] \times [0, M] \times \mathbb{R}^+$.

Remark 2. Considering Lemma 3, one can adjust the coefficients a_i, b_i, c_i , and d_i ($i = 1, 2, 3, 4$) such that the spectral radii $r(L_{a_i, b_i, c_i, d_i}), r(L_{a_4, b_4, c_4}), r(L_{a_4, b_4, c_4}^*)$ ($i = 1, 2, 3$) satisfy their respective conditions in (H3)–(H6).

Theorem 1. Suppose that (H0–H4) hold. Then, (1) has at least one positive solution.

Proof. Let $W = \{y \in P: y = \lambda Ay, \lambda \in [0, 1]\}$. Now, we prove that W is a bounded set in P . If $y \in W$, then from (H3), we have

$$\begin{aligned}
y(t) &= \lambda (Ay)(t) \leq (Ay)(t) \leq \int_0^1 G_1(t, s) \left(a_1 \int_0^s \int_0^\tau y(r) dr d\tau + b_1 \int_0^s y(\tau) d\tau + c_1 y(s) + d_1 y'(s) + e_1 \right) ds \\
&= \int_0^1 (a_1 G_2(t, s) + b_1 G_3(t, s) + c_1 G_1(t, s)) y(s) ds + d_1 \int_0^1 G_1(t, s) y'(s) ds + e_1 \eta(t) \\
&= (L_{a_1, b_1, c_1, d_1} y)(t) + e_1 \eta(t),
\end{aligned} \tag{57}$$

where

$$\begin{aligned}\eta(t) &= \int_0^1 \left[\frac{1}{1-\alpha[1]} \int_0^1 K_1(t,s) d\beta(t) + K_1(t,s) \right] ds \\ &= \frac{1}{1-\alpha[1]} \int_0^1 \int_0^1 K_1(t,s) d\beta(t) ds + t - \frac{1}{2}t^2,\end{aligned}\quad (58)$$

for $t \in [0, 1]$. This implies that

$$\left((I - L_{a_1, b_1, c_1, d_1}) y \right)'(t) \leq e_1 \eta(t). \quad (59)$$

Note that $\eta \in P(\eta'(t) = 1 - t \geq 0, \forall t \in [0, 1])$, $y \in W$, and we obtain

$$\begin{aligned}y'(t) &= \lambda (Ay)'(t) = \lambda \int_t^1 f\left(s, \int_0^s \int_0^\tau y(r) dr d\tau, \int_0^s y(\tau) d\tau, y(s), y'(s)\right) ds \\ &\leq \int_0^1 \frac{\partial G_1}{\partial t}(t, s) \left(a_1 \int_0^s \int_0^\tau y(r) dr d\tau + b_1 \int_0^s y(\tau) d\tau + c_1 y(s) + d_1 y'(s) + e_1 \right) ds \\ &= \int_0^1 \frac{\partial}{\partial t} (a_1 G_2(t, s) + b_1 G_3(t, s) + c_1 G_1(t, s)) y(s) ds + d_1 \int_0^1 \frac{\partial}{\partial t} G_1(t, s) y'(s) ds + e_1 \eta'(t) \\ &= (L_{a_1, b_1, c_1, d_1} y)'(t) + e_1 \eta'(t).\end{aligned}\quad (60)$$

Thus,

$$\left((I - L_{a_1, b_1, c_1, d_1}) y \right)'(t) \leq e_1 \eta'(t). \quad (61)$$

Since $r(L_{a_1, b_1, c_1, d_1}) < 1$, we know that $I - L_{a_1, b_1, c_1, d_1}$ has a bounded inverse operator $(I - L_{a_1, b_1, c_1, d_1})^{-1}$, with

$$\begin{aligned}(I - L_{a_1, b_1, c_1, d_1})^{-1} &= I + L_{a_1, b_1, c_1, d_1} + L_{a_1, b_1, c_1, d_1}^2 + \dots \\ &\quad + L_{a_1, b_1, c_1, d_1}^n + \dots.\end{aligned}\quad (62)$$

Note that $L_{a_1, b_1, c_1, d_1}(P) \subset P$, and we obtain $(I - L_{a_1, b_1, c_1, d_1})^{-1}(P) \subset P$. Therefore,

$$\begin{aligned}y(t) &\leq \left((I - L_{a_1, b_1, c_1, d_1})^{-1} e_1 \eta \right)(t), \\ y'(t) &\leq \left((I - L_{a_1, b_1, c_1, d_1})^{-1} e_1 \eta \right)'(t), \\ &\text{for } t \in [0, 1].\end{aligned}\quad (63)$$

This implies that

$$\begin{aligned}\|y\|_C &\leq \left\| \left((I - L_{a_1, b_1, c_1, d_1})^{-1} e_1 \eta \right) \right\|_C, \\ \|y'\|_C &\leq \left\| \left((I - L_{a_1, b_1, c_1, d_1})^{-1} e_1 \eta \right)' \right\|_C.\end{aligned}\quad (64)$$

Therefore, we have

$$\|y\| \leq \left\| \left((I - L_{a_1, b_1, c_1, d_1})^{-1} e_1 \eta \right) \right\|. \quad (65)$$

That is, W is bounded. Now, we can select $R_1 > \max\{\sup W, \rho_1\}$ (ρ_1 is defined in (H4)) such that

$$y \neq \lambda Ay, \quad \text{for } y \in \partial B_{R_1} \cap P \text{ and } \lambda \in [0, 1]. \quad (66)$$

From Lemma 5, we have

$$i(A, B_{R_1} \cap P, P) = 1. \quad (67)$$

On the contrary, since $L_{a_2, b_2, c_2, d_2}(P) \subset P$ and $r(L_{a_2, b_2, c_2, d_2}) \geq 1$, it follows from Lemma 2 that there exists $\varphi_0 \in P \setminus \{0\}$ such that $L_{a_2, b_2, c_2, d_2} \varphi_0 = r(L_{a_2, b_2, c_2, d_2}) \varphi_0$ and $\varphi_0 = r(L_{a_2, b_2, c_2, d_2})^{-1} L_{a_2, b_2, c_2, d_2} \varphi_0 \in P$. Now, we show that

$$y - Ay \neq \lambda \varphi_0, \quad \text{for } y \in \partial B_{\rho_1} \cap P, \lambda \geq 0. \quad (68)$$

If this claim is false, then there exist $y_0 \in \partial B_{\rho_1} \cap P$ and $\lambda_0 \geq 0$ such that $y_0 - Ay_0 = \lambda_0 \varphi_0$. Note that $\lambda_0 > 0$ (otherwise, the theorem is proved). Then, from (H4), we have

$$\begin{aligned}(Ay_0)(t) &= \int_0^1 G_1(t, s) f\left(s, \int_0^s \int_0^\tau y_0(r) dr d\tau, \int_0^s y_0(\tau) d\tau, y_0(s), y_0'(s)\right) ds \\ &\geq \int_0^1 G_1(t, s) \left(a_2 \int_0^s \int_0^\tau y_0(r) dr d\tau + b_2 \int_0^s y_0(\tau) d\tau + c_2 y_0(s) + d_2 y_0'(s) \right) ds \\ &= \int_0^1 (a_2 G_2(t, s) + b_2 G_3(t, s) + c_2 G_1(t, s)) y_0(s) ds + d_2 \int_0^1 G_1(t, s) y_0'(s) ds \\ &= (L_{a_2, b_2, c_2, d_2} y_0)(t),\end{aligned}\quad (69)$$

which implies that

$$y_0 = Ay_0 + \lambda_0 \varphi_0 \geq L_{a_2, b_2, c_2, d_2} y_0 + \lambda_0 \varphi_0 \geq \lambda_0 \varphi_0. \quad (70)$$

Let $\lambda^* = \sup\{\lambda > 0: y_0 \geq \lambda \varphi_0\}$. Then, $\lambda_0 \in \{\lambda > 0: y_0 \geq \lambda \varphi_0\}$ and $y_0 \geq \lambda^* \varphi_0$. However, we note that $y_0 \geq L_{a_2, b_2, c_2, d_2} y_0 + \lambda_0 \varphi_0 \geq L_{a_2, b_2, c_2, d_2} \lambda^* \varphi_0 + \lambda_0 \varphi_0 = (\lambda^* r(L_{a_2, b_2, c_2, d_2}) + \lambda_0) \varphi_0$, and this contradicts the definition of λ^* for $r(L_{a_2, b_2, c_2, d_2}) \geq 1$. Therefore, (66) holds, as required. From Lemma 4, we have

$$i(A, B_{\rho_1} \cap P, P) = 0. \quad (71)$$

From (65) and (69), we have

$$i(A, (B_{R_1} \setminus \bar{B}_{\rho_1}) \cap P, P) = i(A, B_{R_1} \cap P, P) - i(A, B_{\rho_1} \cap P, P) = 1, \quad (72)$$

and hence A has at least one fixed point in $(B_{R_1}, \bar{B}_{\rho_1}) \cap P$, i.e., (1) has at least one positive solution. This completes the proof. $\square$

Theorem 2. Suppose that (H0–H2) and (H5–H7) hold. Then, (1) has at least one positive solution.

Proof. We show that

$$y \neq \lambda Ay, \quad \text{for } y \in \partial B_{\rho_2} \cap P, \lambda \in [0, 1]. \quad (73)$$

If the claim is false, then there exist $y_1 \in \partial B_{\rho_2} \cap P$ and $\lambda_1 \in [0, 1]$ such that $y_1(t) = \lambda_1 (Ay_1)(t)$, for $t \in [0, 1]$. For $t \in [0, 1]$, from (H5) we have

$$\begin{aligned} y_1(t) &\leq \int_0^1 G_1(t, s) f\left(s, \int_0^s \int_0^\tau y_1(r) dr d\tau, \int_0^s y_1(\tau) d\tau, y_1(s), y_1'(s)\right) ds \\ &\leq \int_0^1 G_1(t, s) \left(a_3 \int_0^s \int_0^\tau y_1(r) dr d\tau + b_3 \int_0^s y_1(\tau) d\tau + c_3 y_1(s) + d_3 y_1'(s)\right) ds \\ &= (L_{a_3, b_3, c_3, d_3} y_1)(t). \end{aligned} \quad (74)$$

Also, $y_1'(t) = \lambda_1 (Ay_1)'(t)$, for $t \in [0, 1]$ implies that

$$\begin{aligned} y_1'(t) &= \lambda \int_t^1 f\left(s, \int_0^s \int_0^\tau y_1(r) dr d\tau, \int_0^s y_1(\tau) d\tau, y_1(s), y_1'(s)\right) ds \\ &\leq \int_0^1 \frac{\partial G_1}{\partial t}(t, s) f\left(s, \int_0^s \int_0^\tau y_1(r) dr d\tau, \int_0^s y_1(\tau) d\tau, y_1(s), y_1'(s)\right) ds \\ &\leq \int_0^1 \frac{\partial G_1}{\partial t}(t, s) \left(a_3 \int_0^s \int_0^\tau y_1(r) dr d\tau + b_3 \int_0^s y_1(\tau) d\tau + c_3 y_1(s) + d_3 y_1'(s)\right) ds \\ &= (L_{a_3, b_3, c_3, d_3} y_1)'(t). \end{aligned} \quad (75)$$

Note that $L_{a_3, b_3, c_3, d_3}(P) \subset P$ and $r(L_{a_3, b_3, c_3, d_3}) < 1$, and we have

$$\begin{aligned} ((I - L_{a_3, b_3, c_3, d_3}) y_1)(t) &\leq 0, \\ ((I - L_{a_3, b_3, c_3, d_3}) y_1)'(t) &\leq 0, \end{aligned} \quad (76)$$

for $t \in [0, 1]$.

Therefore,

$$\begin{aligned} y_1(t) &= 0, \\ y_1'(t) &= 0, \end{aligned} \quad (77)$$

for $t \in [0, 1]$.

This contradicts the fact that $y_1 \in \partial B_{\rho_2} \cap P$. Hence, (71) is true, as required. From Lemma 5, we have

$$i(A, B_{\rho_2} \cap P, P) = 1. \quad (78)$$

On the contrary, from Lemma 2 there exists $\psi_1 \in P \setminus \{0\}$ such that $(L_{a_4, b_4, c_4}^* \psi_1)(s) = r(L_{a_4, b_4, c_4}^*) \psi_1(s)$, for $s \in [0, 1]$. Let $U = \{y \in P: y - Ay = \lambda \varphi_2, \text{ for } \lambda \geq 0\}$, where $\varphi_2(t) = t - (1/2)t^2 \in P$, for $t \in [0, 1]$. Note that $\lambda > 0$ (otherwise, the theorem is proved). We shall show that U is a bounded set in P . If $y \in P$, then from (H6), we have

$$\begin{aligned}
y(t) &= (Ay)(t) + \lambda \varphi_2(t) \geq (Ay)(t) \\
&\geq \int_0^1 G_1(t, s) \left(a_4 \int_0^s \int_0^\tau y(r) dr d\tau + b_4 \int_0^s y(\tau) d\tau + c_4 y(s) - e_2 \right) ds \\
&\geq \int_0^1 (a_4 G_2(t, s) + b_4 G_3(t, s) + c_4 G_1(t, s)) y(s) ds - e_2 \int_0^1 G_1(t, s) ds \\
&\geq \int_0^1 (a_4 G_2(t, s) + b_4 G_3(t, s) + c_4 G_1(t, s)) y(s) ds - \frac{1}{2} \kappa_2 e_2.
\end{aligned} \tag{79}$$

Multiplying both sides of the above inequality by $\psi_1(t)$ and integrating from 0 to 1 yields

$$\begin{aligned}
\int_0^1 y(t) \psi_1(t) dt &\geq \int_0^1 \psi_1(t) \int_0^1 (a_4 G_2(t, s) + b_4 G_3(t, s) \\
&\quad + c_4 G_1(t, s)) y(s) ds dt - \frac{1}{2} \kappa_2 e_2 \int_0^1 \psi_1(t) dt \\
&\geq r(L_{a_4, b_4, c_4}^*) \int_0^1 y(t) \psi_1(t) dt \\
&\quad - \frac{1}{2} \kappa_2 e_2 \int_0^1 \psi_1(t) dt.
\end{aligned} \tag{80}$$

This, together with $r(L_{a_4, b_4, c_4}^*) > 1$, implies that

$$\int_0^1 y(t) \psi_1(t) dt \leq \frac{\kappa_2 e_2 \int_0^1 \psi_1(t) dt}{2r(L_{a_4, b_4, c_4}^*) - 2}. \tag{81}$$

Note that $y \in U$, and we have

$$\begin{aligned}
y'(t) &= (Ay)'(t) + \lambda \varphi_2'(t) = \int_t^1 f\left(s, \int_0^s \int_0^\tau y(r) dr d\tau, \int_0^s y(\tau) d\tau, y(s), y'(s)\right) ds + \lambda(1-t) \geq 0, \\
y''(t) &= (Ay)''(t) + \lambda \varphi_2''(t) = -f\left(t, \int_0^t \int_0^\tau y(r) dr d\tau, \int_0^t y(\tau) d\tau, y(t), y'(t)\right) - \lambda \leq 0.
\end{aligned} \tag{82}$$

Then, y is a concave and increasing function on $[0, 1]$. Hence,

$$\begin{aligned}
\int_0^1 y(t) \psi_1(t) dt &= \int_0^1 y\left(\frac{t}{1} \cdot 1 + \frac{1-t}{1} \cdot 0\right) \psi_1(t) dt \\
&\geq \int_0^1 (ty(1) + (1-t)y(0)) \psi_1(t) dt \\
&\geq \int_0^1 ty(1) \psi_1(t) dt.
\end{aligned} \tag{83}$$

This enables us to obtain

$$\|y\|_C = y(1) \leq \frac{\kappa_2 e_2 \int_0^1 \psi_1(t) dt}{2r(L_{a_4, b_4, c_4}^*) - 2} \left[\int_0^1 t \psi_1(t) dt \right]^{-1}. \tag{84}$$

Now, note (82), and we see there is an $M > 0$ such that

$$\begin{aligned}
&\left| \int_0^s \int_0^\tau y(r) dr d\tau \right|, \\
&\left| \int_0^s y(\tau) d\tau \right|, |y(s)| \leq M, \quad \text{for } s \in [0, 1].
\end{aligned} \tag{85}$$

This, together with (H7), implies that

$$\begin{aligned}
f\left(s, \int_0^s \int_0^\tau y(r) dr d\tau, \int_0^s y(\tau) d\tau, y(s), y'(s)\right) &\leq H_M(y'(s)), \\
&\text{for } s \in [0, 1].
\end{aligned} \tag{86}$$

Note that $y \in U$, and we obtain

$$\begin{aligned}
-y''(t) &= -(Ay)''(t) - \lambda \varphi_2''(t) \\
&= f\left(t, \int_0^t \int_0^\tau y(r) dr d\tau, \int_0^t y(\tau) d\tau, y(t), y'(t)\right) \\
&\quad + \lambda \leq H_M(y'(t)) + \lambda.
\end{aligned} \tag{87}$$

This implies that

$$\frac{-y''(t)y'(t)}{H_M(y'(t)) + \lambda} \leq y'(t), \quad (88)$$

and then if we let $q = y'$, we have

$$\begin{aligned} \int_0^{\|y'\|_C} \frac{q dq}{H_M(q) + \lambda} &\leq \int_0^1 \int_0^1 y'(t) dt \\ &= y(1) - y(0) \leq \frac{\kappa_2 e_2 \int_0^1 \psi_1(t) dt}{2r(L_{a_4, b_4, c_4}^* - 2)} \\ &\quad \cdot \left[\int_0^1 t \psi_1(t) dt \right]^{-1}. \end{aligned} \quad (89)$$

Therefore, combining this and (H7), there exists $\mathcal{N}^* > 0$ such that

$$\|y'\|_C \leq \mathcal{N}^*. \quad (90)$$

Thus, U is bounded (see (82) and (88)). Taking $R_2 > \max\{\sup U, \rho_2\}$, we have

$$y - Ay \neq \lambda \varphi_2, \quad \text{for } y \in \partial B_{R_2} \cap P, \lambda \geq 0. \quad (91)$$

From Lemma 4, we have

$$i(A, B_{R_2} \cap P, P) = 0. \quad (92)$$

From (76) and (90), we have

$$\begin{aligned} i(A, (B_{R_2} \setminus \overline{B}_{\rho_2}) \cap P, P) &= i(A, B_{R_2} \cap P, P) \\ &\quad - i(A, B_{\rho_2} \cap P, P) = -1, \end{aligned} \quad (93)$$

and hence A has at least one fixed point in $(B_{R_2} \setminus \overline{B}_{\rho_2}) \cap P$, i.e., (1) has at least one positive solution. This completes the proof. $\square$

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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