Research Article

Competition among Predators and Allee Effect on Prey, Their Influence on a Gause-Type Predation Model

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Interference or competition among predators (CAP) has often been ruled out in depredation models, although there are varied mathematical forms to describe and incorporate it into this interaction. In this work, we present the most known of these descriptions and one of them will be used in a modified Volterra model. Moreover, of this ecological phenomenon, a simple and strong Allee effect affecting the prey population will be considered in the relationship. An important feature of the new model is to have until two positive equilibrium points, to the difference with the Volterra model (without Allee effect); hence different and interesting dynamic situations appear in the system. Conditions for the existence and local stability of equilibria are determined. The boundedness of solutions, the existence of a limit cycle and a separatrix curve are also proven. Besides, the main properties of the model are examined from an ecological point of view. To make a comparative discussion of our results, an Appendix is added with the main properties of models, in which neither the Allee effect nor the competition among predators is considered. Some simulations are shown to endorse our results.

1. Introduction

Usually, the analysis of predator-prey models considers different ecological phenomena affecting either one population or both populations. These phenomena can have strong consequences in the relationship between them and modifying the dynamical properties of a system describing it [1, 2].

Such is the case of an Allee effect affecting the prey population or else, the competition among predators (CAP) [3]; nonetheless, there are not enough studies to determine the real impact of these two phenomena in that dynamical relationship, when both act simultaneously in the interaction.

Moreover, different mathematical formulations have been given for each of these phenomena; as it has been shown in previous works [3], diverse mathematical forms for the same phenomenon can produce changes in the properties of the system describing the interaction [1]. Then, a comparative

study using another modeling for these phenomena must be also realized in the future. To determine the influence of these mathematical expressions in the dynamics of the systems is an interesting objective to the modelers.

So, in this work a modified Volterra model [4] is analyzed, in which the functional response is linear, assuming that (i) the prey growth rate is affected by a strong Allee effect [5] and (ii) there exists self-interference (interference) or competition among predators (CAP) [6].

1.1. Interference or Competition among Predators. The struggle among living things, for food, space, etc., is well known (competition in the survival of the fittest [7]); particularly, the competition among predators for prey or other resources [6] is one of them.

Obviously, intraspecific competitive interactions between individual predators can affect the birth and death rates

of their whole population [4]. Furthermore, antagonistic interactions may also affect predator efficiency in finding and killing prey [4], which can imply the modification of the predator functional response on the models.

Different behaviors of predator influencing the interrelation between prey and predator can be assumed, for example, behavior typical of territorial animals where individuals "waste time" in direct contests, thereby decreasing the time each could otherwise devote to foraging (searching for or handling prey); spatially aggregated predators and prey and so on [8].

Diverse mathematical forms to express the competition (or interference) among predators (CAP) have been proposed in the ecological literature and described in the following way:

(I) The early expression for CAP was formulated independently in 1975 by J. A. Beddington [9] and D. DeAngelis and coworkers [10], proposing a new functional response. They modify the hyperbolic Holling type II functional response, by adding a term in the denominator, obtaining a new functional response, dependent of two interacting populations and given by

$$h(x, y) = \frac{qx}{a + bx + cy}, \quad \text{with } q, a, b, c > 0, \qquad (1)$$

where x and y stand for the prey population size and the predator population size, respectively. The term cy in the denominator expresses the mutual interference among predators [11].

Although this function combines the hyperbolic response with self-interference among predators, [12] made a deduction involving prey refuges use instead of the CAP.

(II) A second form to express the competence among predators was formulated by Herbert I. Freedman in 1979 [13], modifying the assumption of the usual models that the total prey death rate is the predator functional response times the number of predators [13]. He proposed the function

$$h(x, y) = h(x) y^n, \qquad (2)$$

with n being the mutual interference constant such that $0 < n \leq 1$, and $h(x)$ is the prey-dependent functional response of predator.

Following C. W. Clark [14], the function $B(y) = y^n$ expresses the congestion among fishing vessels (the men as predators) harvesting a fish school, resulting in decreasing of catch-rates.

(III) The third form to describe the CAP is given by a negative quadratic term added into the predator growth equation [6]. Hence, the function takes the form

$$\phi(y) = -cy - ey^2, \quad \text{with } c, \text{ and } e > 0. \qquad (3)$$

In this case, it is assumed that the predator population be reduced by other causes as the size of the habitat suitable for the predator to live and reproduce there [6].

In our work, we model the CAP with this last form presented in the Bazykin's book [6]; moreover, we consider the linear functional response independent of predator density (i.e., only prey dependent), which means that any single

predator affects the prey population growth rate independently of its conspecifics [15].

We note that if in the function ϕ is added a quadratic positive term, i.e., $\phi(y) = -cy + ey^2$, with c and $e > 0$, it has the description of the *cooperation or collaboration* among predators [16]. This is also a frequent social behavior in nature, used by some predators to enhance their efficiency in the consumption of prey [16].

1.2. The Allee Effect. It is well known that any mechanism describing a positive correlation between a component of individual fitness and the population size of conspecifics can be named as an *Allee effect* [17–19].

This phenomenon, called after the American ecologist Warder Clayde Allee (1885-1955), is also known under different names; it has also been named as *density dependence* or *positive density dependence* and other names in Population Dynamics [17, 20] or *depensation* in Fisheries Sciences [14, 20].

Populations can exhibit Allee effect dynamics due to a wide range of biological phenomena, for example, reduced antipredator vigilance, social thermoregulation, genetic drift, mating difficulty, reduced antipredator defense, and deficient feeding at low densities; however, several other causes may lead to this phenomenon (see [21] or [5]).

This ecological phenomenon can be classified into three main types called *strong Allee effect* [22] or *critical depensation* [14, 20], *weak Allee effect* [19] or *noncritical depensation* [14, 20], and special weak Allee effect.

The strong Allee effect implies the existence of a threshold population level $m > 0$ [6, 23, 24], under which the population becomes extinct. This requires that the population growth rate be negative for population sizes minor than m.

Many continuous time equations have been used to model the Allee effect [23], although most of them are topologically equivalent [25]; i.e., solutions have the same qualitative behavior.

The most familiar equation is described by

$$\frac{dx}{dt} = r\left(1 - \frac{x}{K}\right)(x - m)x \qquad (4)$$

where $x = x(t)$ indicates the population size of a species in an environmental [24]. The parameters r and K are positives; meanwhile m can be positive, negative or zero. It has, respectively, a strong ($m > 0$), a weak ($m < 0$), or special ($m = 0$) weak Allee effect.

By ecological reason $m \ll K$ since is a population threshold under the growth population rate is negative [17, 18]. We note that if $m < -K$ there is not an Allee effect.

Diverse ecological research suggests that two or more Allee effects can lead to mechanisms acting simultaneously on a single population (see [21]); the combined influence of some of these phenomena is known as *multiple (double) Allee effect* [21]. In these cases, most complicated equations are proposed [3, 26], such as

$$\frac{dx}{dt} = r\left(1 - \frac{x}{K}\right)\left(1 - \frac{m + n}{x + n}\right)x,$$
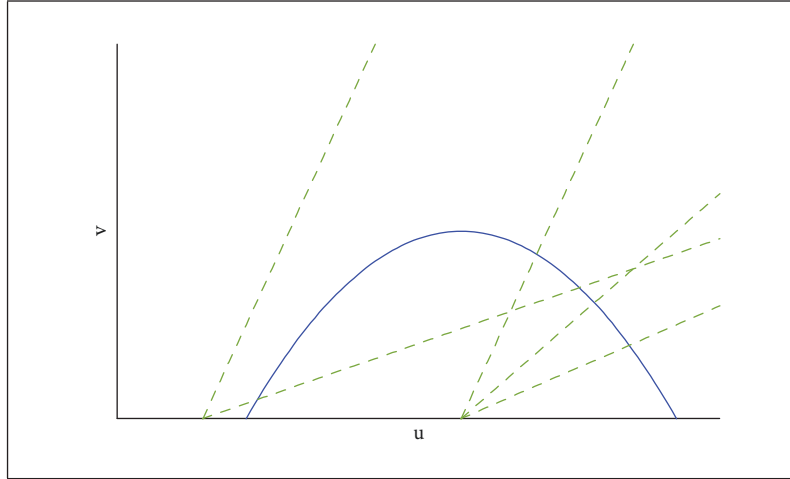
FIGURE 1: Intersection of isoclines of the prey $v = (1 - u)(u - M)$ (in blue) and of the predators $v = (u - C)/E$ (in green), according the different values of E, M and C. We note there exist two positive equilibrium points if $C < M/(1 + M)$.

$$\frac{dx}{dt} = r\left(1 - \frac{x}{K}\right)x - \frac{nx}{x + b},$$ (5)

which was presented in [23].

However, when they are considered in predator-prey models can produce changes in the dynamical of the system, particularly on the number of limit cycles of the system [3, 26], and the existence of separatrix curves for the trajectories.

This article is organized as follows: The model and general settings are presented in Section 2; in Section 3 we obtain the main results on the boundedness of trajectories, the local nature of equilibrium points and in the last Section the ecological implications of our findings are given.

The obtained results will be compared with the predator-prey model in which competition among predators is not considered, partially studied in the book by Kot [24]; also it will be compared with the model considering double Allee effect [5, 21], without self-interference among predators analyzed in [1, 2].

2. The Model

Using the simplest way to describe the Allee effect and expressing the action of many predators in the interaction, considering CAP (self-interference) as in the logistic growth rate. Thus, the model is described by the bidimensional system of Kolmogorov type [27]:

$$X_\mu : \begin{cases} \dfrac{dx}{dt} = r\left(1 - \dfrac{x}{K}\right)(x - m)\,x - qxy \\ \dfrac{dy}{dt} = (px - c - ey)\,y, \end{cases}$$ (6)

where $x = x(t)$ and $y = y(t)$ represent the prey and predator population size, respectively for $t \geq 0$ (measured in number of individuals, density by area or volume unit or biomass); the parameters are all positives, i.e., $\mu = (r, K, q, p, c, e, m) \in \mathbb{R}^7_+$, with $-K < m \ll K$.

In this work, we consider only $m > 0$, i.e., the prey population is affected by a strong Allee effect.

The parameters have the following ecological meanings:

r is the intrinsic prey growth rate or biotic potential

K is the prey environmental carrying capacity

$m > 0$ is the strong Allee effect threshold or the minimum of viable prey population

q is the quantity of prey that can be eaten by a predator in each time unit

p is the efficiency with which predators convert consumed prey into new predators

c is the natural death rate of predators

e is the level of the struggle among predators.

Particularly, the term $-ey^2$ represents interspecific density-restricted effects on the predators [6, 28].

System (6) describes a generalized Gause-type predator-prey model [27], which does not obey the mass action principle [29]. It is of Kolmogorov type [27], defined on

$$\Omega = \left\{ (x, y) \in \mathbb{R}^2 / 0 \leq x, 0 \leq y \right\}.$$ (7)

The equilibrium points are $(0, 0)$, $(K, 0)$ and the points (x_e, y_e) lie on the isoclines $r(1 - x/K)(x - m) - qy = 0$ and $px - c - ey = 0$.

It must be remembered that if $e = 0$, the obtained system is partially studied in the book by M. Kot [24]. Assuming that the prey is affected simoustanly by two Allee effects, the modified system is analyzed in [1], showing a richer dynamics than the model studied in Kot [24].

Setting $dy/dt = 0$, we find that equation (6) generates a slanted predator isocline that passes through the point $(0, -c/e)$, out of the first quadrant. With a predator isocline of this type, two positive equilibrium points can appear, one of multiplicity two or none (see Figure 1).

The positive slope of the predator isocline reflects direct density dependence and could have a stabilizing influence. Nonetheless, we prove the unique positive equilibrium point may be unstable, generating a stable limit cycle.

With the objective of to make a comparative study, we summarize in the Appendix, the properties of two related models with the model described by the system (6).

(i) Volterra model with competition among predators is studied in [6] and given by

$$X_\rho : \begin{cases} \dfrac{dx}{dt} = r\left(1 - \dfrac{x}{K}\right)x - qxy \\ \dfrac{dy}{dt} = (px - c - ey)\,y \end{cases} \tag{8}$$

with $\rho = (r, K, q, p, c, e) \in \mathbb{R}_+^6$.

(ii) The Volterra model with Allee effect on prey is studied in [24], described by

$$X_\sigma : \begin{cases} \dfrac{dx}{dt} = r\left(1 - \dfrac{x}{K}\right)(x - m)\,x - qxy \\ \dfrac{dy}{dt} = (px - c)\,y \end{cases} \tag{9}$$

with $\sigma = (r, K, q, p, c, m) \in \mathbb{R}_+^5 \times] - K, K[$.

3. Main Results

To simplify the calculations, following the methodology of [1, 2, 26], we make a reparameterization of the vector field X_μ considering the change of variables and a time rescaling given by a diffeomorphism [30, 31]

$\varphi : \overline{\Omega} \times \mathbb{R} \longrightarrow \Omega \times \mathbb{R}$, with $\overline{\Omega} = \{(u, v) \in \mathbb{R}^2 / u \geq 0, v \geq 0\}$, such that

$$\varphi(u, v, \tau) = \left(Ku, \dfrac{rK}{q}v, \dfrac{1}{rK}\tau\right) = (x, y, t). \tag{10}$$

Proposition 1. *Topologically equivalent systems*
System (6) is topologically equivalent to

$$Y_\eta : \begin{cases} \dfrac{du}{d\tau} = ((1 - u)(u - M) - v)\,u \\ \dfrac{dv}{d\tau} = B(u - C - Ev)\,v \end{cases} \tag{11}$$

defined in the set $\overline{\Omega} = \{(u, v) \in \mathbb{R}^2 / u \geq 0, \ v \geq 0\}$, with $\eta = (M, B, C, E) \in]-1, 1[\times\mathbb{R}_+^3$ where $M = m/K$, $B = p/r$, $C = c/pK$ and $E = er/pq$.

Proof. Using the change of variables given by $x = Ku$ and $y = (rK/q)v$, replacing in (6), we have $dx/dt = K(du/dt)$ and $dy/dt = (rK/q)(dv/dt)$ obtaining a new system given by

$$U_\mu : \begin{cases} \dfrac{du}{dt} = \left(rK(1 - u)\left(u - \dfrac{m}{K}\right) - rKv\right)u \\ \dfrac{dv}{dt} = pK\left(u - \dfrac{c}{pK} - \dfrac{er}{pq}v\right)v \end{cases} \tag{12}$$

or,

$$U_\mu : \begin{cases} \dfrac{du}{dt} = rK\left((1 - u)\left(u - \dfrac{m}{K}\right) - v\right)u \\ \dfrac{dv}{dt} = pK\left(u - \dfrac{c}{pK} - \dfrac{er}{pq}v\right)v \end{cases} \tag{13}$$

By means of the time rescaling given by $\tau = rKt$ and by using the chain rule, it follows that

$$V_\mu : \begin{cases} \dfrac{du}{d\tau} = \left((1 - u)\left(u - \dfrac{m}{K}\right) - v\right)u \\ \dfrac{dv}{d\tau} = \dfrac{p}{r}\left(u - \dfrac{c}{pK} - \dfrac{er}{pq}v\right)v \end{cases} \tag{14}$$

Renaming the parameters by $M = m/K$, $B = p/r$, $C = c/pK$ and $E = er/pq$ this becomes system (8). $\square$

Remark 2. The Jacobian matrix of φ is

$$D\varphi(u, v, \tau) = \begin{pmatrix} K & 0 & 0 \\ 0 & \dfrac{rK}{q} & 0 \\ 0 & 0 & \dfrac{1}{rK} \end{pmatrix} \tag{15}$$

and $\det D\varphi(u, v, \tau) = K/q > 0$.

Then, the diffeomorphism φ is a smooth change of variables with a rescaling of the time preserving the time orientation. Hence, from (6) we obtain a qualitatively (topologically) equivalent vector field $Y_\eta = \varphi \circ X_\mu$, which has the form $Y_\eta = P(u, v)(\partial/\partial u) + Q(u, v)(\partial/\partial v)$ [31], where $P(u, v)$ and $Q(u, v)$ are the right sides of system (11). Clearly, the associated second order differential equations system is the Kolmogorov type polynomial (11) [27].

Our study is divided into three cases: when $M \in]0, 1[$, the special case $M = 0$ and when $M < 0$.

3.1. Number of Positive Equilibria. The equilibrium points of system (11) or singularities of vector field Y_η are $(0, 0)$, $(1, 0)$, and $(M, 0)$ if $M > 0$; the positive equilibrium points lie in the intersection of the null clinic, which are the parabolic curve $v = (1 - u)(u - M)$ and the slanted straight line $v = (u - C)/E$ (see Figure 1).

It is clear that the straight line $v = (u - C)/E$ has slope $\mu = 1/E$ and it intercepts the x and $y - axis$ at the points $(0, -C/E)$ and $(C, 0)$, respectively. The number of the positive equilibrium points depends on the relation between M and C and the slope μ.

Furthermore, the abscise u of positive equilibrium points satisfies the equation of second degree:

$$p(u) = Eu^2 - ((M + 1)E - 1)u + (ME - C) = 0. \tag{16}$$

(a) If $M > 0$, and according to the Descartes's sign Rule, equation (16) can have two real positive roots, one of multiplicity 2, or none positive. So, (u_e, v_e) is a positive equilibrium point, if and only if $u_e - C > 0$ and $(1 - u_e)(u_e - M) > 0$, i.e., $0 < \max\{C, M\} < u_e < 1$.

A particular case is obtained when $E = 1$; equation (16) becomes

$$p(u) = u^2 - Mu + (M - C) = 0. \tag{17}$$

which can has two real positive roots, if $M > C$.

(b) If $M = 0$, equation (16) becomes

$$p(u) = Eu^2 - (E - 1)u - C = 0, \tag{*}$$

which has a unique real positive root, for any value of the parameter E.

(c) If $M < 0$, the factor $ME - C$ of equation (16) is negative. Then, any being the sign of the factor $(M + 1)E - 1$, equation (9) has a unique positive solution u_e. This implies a significant difference between the cases of strong and weak Allee effect.

Besides, we note that there is a difference in the dynamics of system (11) with the case in which $E = 0$, i.e., when the CAP does not exist [24].

The number of the positive equilibria of equation (16) can be resumed in the following proposition.

Lemma 3. *Number of positive equilibria considering strong Allee effect*

Assuming that $M > 0$, let us $\Delta = ((M+1)E-1)^2 - 4(ME - C)E$. For equation (16) we have the following classification:

(A) Supposing $ME + E - 1 > 0$ and $ME - C > 0$, then:
(A1) If $\Delta > 0$, there exist two real positive solutions

$$u_1 = \frac{1}{2E}\left(ME + E - 1 - \sqrt{\Delta}\right),$$
$$u_2 = \frac{1}{2E}\left(ME + E - 1 + \sqrt{\Delta}\right). \quad (18)$$

with, $0 < u_1 < u_2 < 1$. Notice that u_1 and u_2 do not depend on the parameter B.

(A2) If $\Delta = 0$, there exists a unique real positive solution

$$u^* = \frac{1}{2E}(ME + E - 1) \quad (19)$$

collapse of u_1 and u_2.
(A3) If $\Delta < 0$, there do not exist real solutions.
(B) $ME + E - 1 > 0$ and $ME - C = 0$, it has

$$u_3 = \frac{ME + E - 1}{E},$$
$$u_4 = 0. \quad (20)$$

(C) Assuming that $ME + E - 1 > 0$ and $ME - C < 0$, it has

$$u_1 < 0 < u_2 = \frac{1}{2E}\left(E(M+1) - 1 + \sqrt{\Delta}\right). \quad (21)$$

(D) $ME + E - 1 = 0$ and $ME - C > 0$, there do not exist real solutions.
(E) $ME + E - 1 = 0$ and $ME - C < 0$, there exists a unique real positive solution,

$$u_1 < 0 < u_2 = \sqrt{\frac{ME - C}{E}}. \quad (22)$$

(F) $ME + E - 1 < 0$ and $ME - C = 0$, there do not exist real solutions.
(G) $ME + E - 1 < 0$ and $ME - C < 0$, there exists a unique real positive solution,

$$u_2 = \frac{1}{2E}\left(E(M+1) - 1 + \sqrt{\Delta}\right). \quad (23)$$

Table 1: Number of positive real roots of equation (16).

Case	$ME + E - 1$	$ME - C$	Δ	Positive real roots
(A1)	+	+	+	2
(A2)	+	+	0	1
(A3)	+	+	−	0
(B)	+	0	+	1
(C)	+	−	+	1
(D)	0	+	−	0
(E)	0	−	+	1
(F)	−	0	+	0
(G)	−	−	+	1
(H)	0	0	0	0

(H) $ME + E - 1 = 0$ and $ME - C = 0$, there exists a unique real solution $u = 0$, of multiplicity 2.

Proof. It is immediate. □

Therefore, the number of positive equilibrium points follows from the lemma above, and the different cases obtained are displayed in Table 1.

So, in the cases $(A3)$, (D), (F) and (H), the points $(0, 0)$ and $(M, 0)$ are the equilibrium points of system.

The distinct positions in the plane (M, E) of the factors $ME + E - 1 = 0$ and $ME - C = 0$ of equation (16) are given in the Figure 2, for different values of C.

Remark 4. We note that
(i) if $M = 0$, then $\Delta = 4CE + (E - 1)^2$, confirming the existence of a unique solution.
(ii) if $M < 0$, then

$$\sqrt{\Delta} = \sqrt{(ME + E - 1)^2 - 4(ME - C)E}$$
$$> (ME + E - 1) \quad (24)$$

and it exists a unique solution of (9).

To determine the nature local of the equilibrium points we will use the Jacobian matrix given by

$$DY_\eta(u, v)$$
$$= \begin{pmatrix} -3u^2 + 2u(1 + M) - M - v & -u \\ Bv & B(u - C - 2Ev) \end{pmatrix} \quad (25)$$

For system (11) we have the following results:

Lemma 5. *Existence of a invariant region*
The set $\overline{\Gamma} = \{(u, v) \in \overline{\Omega}/0 \leq u \leq 1, 0 \leq v\}$ is a positively invariant region.

Proof. As system (11) is of the Kolmogorov type, the axes $u = 0$ and $v = 0$ are invariant set.

If $u = 1$, we have that $du/d\tau = -vu < 0$, and for any sign of $dv/d\tau = B(1 - C - Ev)v$, the trajectories enter the region $\overline{\Gamma}$. □
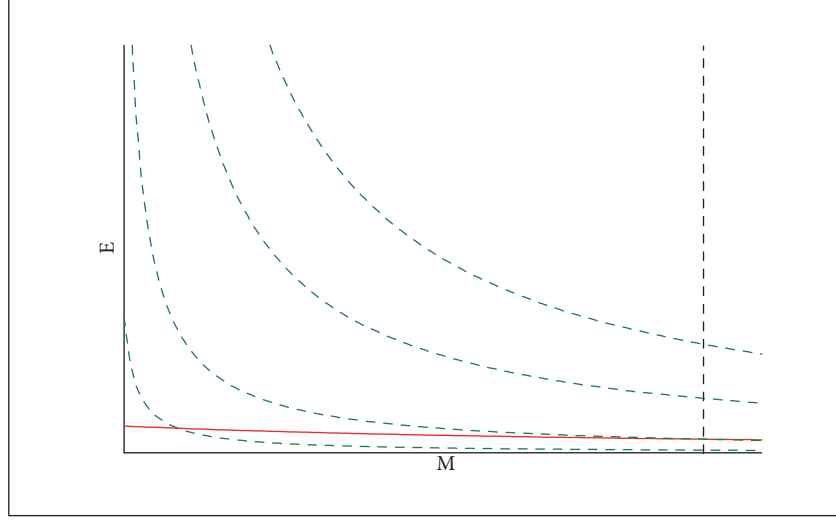
Figure 2: Distinct positions of the curves $E = 1/(M + 1)$ and $E = C/M$, for different values of C. It is clear that if $C < 1/2$, there exists an intersection point between both curves.

In system (6), the set $\Gamma = \{(x, y) \in \Omega/0 \le x \le K, 0 \le y\}$ is a positively invariant region.

Lemma 6. *Boundedness of trajectories*
The solutions are bounded.

Proof. The first equation of topological system (11) is

$$\frac{du}{d\tau} = ((1 - u)(u - M) - v)u \tag{26}$$

In absent of predators it becomes

$$\frac{du}{d\tau} = (1 - u)(u - M)u, \tag{27}$$

since

$$\frac{du}{d\tau} \le (1 - u)(u - M)u, \quad \forall v \in \mathbb{R}^2_+, \tag{*b}$$

having that

$$u(\tau) \longrightarrow 1, \quad \text{when } \tau \longrightarrow \infty, \ u > M.$$
$$u(\tau) \longrightarrow 0, \quad \text{when } \tau \longrightarrow \infty, \ u < M. \tag{28}$$

Furthermore,

$$u(\tau) \longrightarrow 1, \quad \text{when } \tau \longrightarrow \infty, \ u > 1. \tag{29}$$

Considering $L = \max\{u(0), 1\}$, from inequality (*b) we have

$$u(\tau) \le L, \quad \forall \tau \ge 0. \tag{30}$$

Let $W(\tau) = u + (1/B)v$. Clearly $0 < W(\tau), \ \forall \tau \ge 0$.
Deriving $W(\tau)$ respect to τ we obtain

$$\frac{dW(\tau)}{d\tau} = ((1 - u)(u - M) - v)u + (u - C - Ev)v$$
$$\frac{dW(\tau)}{d\tau} = -u^3 + (M + 1)u^2 + (-M)u - (C + vE)v \tag{31}$$

Therefore,

$$\frac{dW(\tau)}{d\tau} + W(\tau) = -u^3 + (M + 1)u^2 - Mu$$
$$- (C + vE)v + \left(u + \frac{1}{B}v\right). \tag{32}$$

After some algebraic manipulations it becomes

$$\frac{dW(\tau)}{d\tau} + W(\tau) = -u\left(u - \frac{(M + 1)}{2}\right)^2 + \frac{(M + 1)^2}{4}u$$
$$+ (1 - M)u$$
$$- E\left(v - \frac{1}{2E}\left(\frac{1}{B} - C\right)\right)^2$$
$$+ \frac{1}{4E}\left(\frac{1}{B} - C\right)^2 \tag{33}$$
$$\le \frac{(M + 1)^2}{4} + (1 - M)u$$
$$+ \frac{1}{4E}\left(\frac{1}{B} - C\right)^2.$$

Remembering that if $M < u$, then $u \longrightarrow 1$, when $\tau \longrightarrow \infty$, it becomes

$$\frac{dW(\tau)}{d\tau} + W(\tau) \le \frac{(M + 1)^2}{4} + (1 - M)$$
$$+ \frac{1}{4E}\left(\frac{1}{B} - C\right)^2. \tag{34}$$

Let us $Q = (M + 1)^2/4 + (1 - M) + (1/4E)(1/B - C)^2 > 0$, for any value of M, such that $-1 < M \ll 1$; then

$$\frac{dW(\tau)}{d\tau} + W(\tau) \le Q, \tag{35}$$

being a first order linear inequality. Applying the theorem of Comparison Theorem for differential inequality (Page 30 in [32]), we obtain

$$0 < W(u(\tau), v(\tau)) \le Q + Ce^{-\tau}. \tag{36}$$

Moreover,

$$0 < W(u(\tau), v(\tau)) \le Q + (W(u(0), v(0)) - Q)e^{-\tau}. \tag{37}$$

Clearly, when $\tau \longrightarrow \infty$, then $0 < W(u, v) \le Q$, for any sign of $(W(u(0), v(0)) - Q)$.

So, the solutions of system (11) are bounded.

Furthermore, there exists the set

$$S = \left\{ (u, v) \in \mathbb{R}_+^2 : 0 \le u + \frac{1}{B}v \le Q \right\} \tag{38}$$

which is an invariant region, where all the solutions of the system (11) starting in $\overline{\Omega}$ are confined. $\qquad \square$

Remark 7. (1) The result established in the above lemma implies the model is well posed [29], i.e., when the prey population size tends to zero, the predator population also tends to zero, since the prey is the unique source of food for predators.

(2) On the other hand,

If $u - C - Ev > 0$, i.e., $dv/d\tau > 0$; thus, $v < (u - C)/E$.
If $u - C - Ev < 0$, i.e., $dv/d\tau < 0$; thus, $v > (u - C)/E$.

Then, the subregion

$$\overline{\Gamma} = \left\{ (u, v) \in \overline{\Omega} / 0 \le u \le 1, \ 0 \le v \le \frac{u - C}{E} \right\} \tag{39}$$

is a bounded and closed region; i.e., it is a compact region and the Poincaré-Bendixon Theorem applies.

3.2. Strong Allee Effect.
In the follow, we assume $0 < M \ll 1$.

3.2.1. Nature of Equilibrium Point over the Axis

Lemma 8. *Nature of the point* $(0, 0)$.
The point $(0, 0)$ *is an attractor stable for all parameter values.*

Proof. Evaluating the Jacobian matrix we have that
At $(0, 0)$ it has

$$DY_\eta(0, 0) = \begin{pmatrix} -M & 0 \\ 0 & -BC \end{pmatrix}. \tag{40}$$

Then, $(0, 0)$ is an attractor equilibrium point. $\qquad \square$

Lemma 9. *Nature of the point* $(1, 0)$.
The point $(1, 0)$ *is*
(i) a hyperbolic saddle point, if and only if, $C < 1$,
(ii) a hyperbolic attractor point, if and only if, $C > 1$,
(iii) a saddle-node (non-hyperbolic), if and only if, $C = 1$.

Proof. At $(1, 0)$ we have

$$DY_\eta(1, 0) = \begin{pmatrix} -(1 - M) & -1 \\ 0 & B(1 - C) \end{pmatrix}. \tag{41}$$

Then, according the sign of factor $1 - C$, we have the thesis. $\qquad \square$

Lemma 10. *Nature of the point* $(M, 0)$.
The point $(M, 0)$ *is*
(i) a repeller, if and only if, $M > C$.
(ii) a saddle point, if and only if, $M < C$.
(iii) a saddle-node, if and only if, $M = C$.

Proof. The Jacobian matrix is

$$DY_\eta(M, 0) = \begin{pmatrix} M(1 - M) & -M \\ 0 & B(M - C) \end{pmatrix} \tag{42}$$

Therefore, according the sign of factor $M - C$, we have the lemma. $\qquad \square$

3.2.2. Nature of Positive Equilibrium Points.
As there are many cases to study, in the following we consider only a few cases.

Case A. First we consider
$ME + E - 1 > 0$ and $ME - C > 0$ and $\Delta = E^2 M^2 - 2E(E + 1)M + (4CE + (E - 1)^2) > 0$.

Then, in this case there exist two positive equilibrium points, $(u_1, (u_1 - C)/E)$ and $(u_2, (u_2 - C)/E)$, where $0 < u_1 < u_2$ are the solutions of equation (9). These points lie at the interior of the first quadrant, if and only if, $u_1 - C > 0$ and $u_2 - C > 0$.

The Jacobian matrix in the points $(u, (u - C)/E)$ is given by

$$DY_\eta\left(u, \frac{u - C}{E}\right)$$

$$= \begin{pmatrix} \dfrac{-3Eu^2 + (2E + 2ME - 1)u + (C - ME)}{E} & -u \\ \dfrac{B}{E}(u - C) & -B(u - C) \end{pmatrix}. \tag{43}$$

Then,

$$\det DY_\eta(u, v) = \frac{B}{E}(u - C)$$
$$\cdot \left(3Eu^2 - 2((M + 1)E - 1)u + (ME - C)\right), \tag{44}$$

which sign depends on the sign of

$$T = 3Eu^2 - 2((M + 1)E - 1)u + (ME - C). \tag{45}$$

Using equation (16) we have

$$T = (E(M + 1) - 1)u - 2(ME - C). \tag{46}$$

We note that T is independent of B.

Theorem 11. *Nature of the point* $(u_1, (u_1 - C)/E)$.

Assuming $u_1 - C > 0$, *the equilibrium point* $(u_1, (u_1 - C)/E)$ *is a saddle point.*

Proof. Substituting u_1 in T we have,

$$
\begin{aligned}
T &= (E(M+1) - 1)\left(\frac{1}{2E}\left(E(M+1) - 1 - \sqrt{\Delta}\right)\right) \\
&\quad - 2(ME - C) = \frac{1}{2E}\left((E(M+1) - 1)^2\right. \\
&\quad - (E(M+1) - 1)\sqrt{\Delta}) - 2(ME - C) \\
&= \frac{1}{2E}\left((E(M+1) - 1)^2 - 4E(ME - C)\right. \quad (47) \\
&\quad - (E(M+1) - 1)\sqrt{\Delta}\right). = \frac{1}{2E}\left(\Delta\right. \\
&\quad - (E(M+1) - 1)\sqrt{\Delta}) = \frac{1}{2E}\left(\sqrt{\Delta}\right. \\
&\quad - (E(M+1) - 1))\sqrt{\Delta} = -u_1\sqrt{\Delta} < 0.
\end{aligned}
$$

Then, $\det DY_\eta \det DY_\eta(u, v) < 0$. □

Then, the sign of tr $DY_\eta(u, v)$ depends on the sign of

$$
\begin{aligned}
V &= -E(u_2 - C)B - ((M+1)E - 2)u_2 \\
&\quad + 2(ME - C).
\end{aligned}
\quad (50)
$$

According the sign of V the results follow from Routh-Hurwitz criterion. □

Corollary 13. *Transversality condition*

The system undergoes a Hopf bifurcation with respect to bifurcation parameter B around the equilibrium point (u_2, v_2) if $B = ((2 - ME - E)u_2 + 2(ME - C))/E(u_2 - C)$.

Proof. If tr $DY_\eta(u_2, v_2) = 0$, then both eigenvalues will be purely imaginary provided $\det DY_\eta(u_2, v_2) > 0$. Therefore, the Implicit function Theorem assures that a Hopf bifurcation occurs where a periodic orbit is created as the stability of the equilibrium point (u_2, v_2) changes.

As $(\partial/\partial B)(\text{tr } DY_\eta(u_2, v_2)) = -E(u_2 - C) < 0$, this guarantees the existence of Hopf bifurcation around (u_2, v_2) and a stable limit cycle is generated from this point. □

Theorem 14. *Existence of a homoclinic curve and stability of the limit cycle.*

There are conditions on the parameter values for which

(a) A homoclinic curve exists, which is determined by the stable and unstable manifold of point (u_1, v_1).

Theorem 12. *Nature of the point* $(u_2, (u_2 - C)/E)$.

Assuming $C < u_2 < 1$, *the equilibrium point* $(u_2, (u_2 - C)/E)$ *is*

(i) an attractor, if and only if, $B < ((2 - ME - E)u_2 + 2(ME - C))/E(u_2 - C)$.

(ii) a repeller, if and only if, $B > ((2 - ME - E)u_2 + 2(ME - C))/E(u_2 - C)$.

(iii) a weak focus, if and only if, $B = ((2 - ME - E)u_2 + 2(ME - C))/E(u_2 - C)$.

Proof. Analogously the above proof we can demonstrate that $T > 0$; so, the nature of $(u_2, (u_2 - C)/E)$ depends on the sign of

$$
\begin{aligned}
&\text{tr } DY_\eta(u_2, v_2) \\
&= \frac{-3E(u_2)^2 + (2E + 2ME - 1)u_2 + (C - ME)}{E} \quad (48) \\
&\quad - B(u_2 - C).
\end{aligned}
$$

Using the expression for u_2 from equation (16) we have

$$
\begin{aligned}
\text{tr } DY_\eta(u_2, v_2) &= \frac{-3E((1/E)(((M+1)E - 1)u_2 - (ME - C))) + (2E + 2ME - 1)u_2 + (C - ME)}{E} - B(u_2 - C) \\
&= \frac{-E(u_2 - C)B + (2 - ME - E)u_2 + 2(ME - C)}{E}.
\end{aligned}
\quad (49)
$$

(b) There is a non-infinitesimal limit cycle bifurcating from the homoclinic [35] surrounding point (u_2, v_2), which is

(b1) unstable, if and only if, $B < ((-(E(M+1) - 1)u_1 + 2(ME - C)) + u_1)/E(u_1 - C)$.

(b2) stable, if and only if, $B > ((-(E(M+1) - 1)u_1 + 2(ME - C)) + u_1)/E(u_1 - C)$.

Proof. Let $W_+^u(u_1, v_1)$ and $W_+^s(u_1, v_1)$ be the right unstable manifold and the superior stable manifold of equilibrium point (u_1, v_1).

(a) As $\overline{\Gamma}$ is an invariant region, the orbits cannot cross the straight line $u = 1$ towards the right. By Existence and Uniqueness Theorem, the trajectory determined by the right unstable manifold $W_+^u(u_1, v_1)$ cannot meet or intersect the trajectory determined by the superior stable manifold $W_+^s(u_1, v_1)$.

Moreover, the $\alpha - limit$ of the $W_+^s(u_1, v_1)$ can lie at point $(M, 0)$ by the Boundedness Lemma or at infinity in the direction of $u - axis$.

On the other hand, the $\omega - limit$ of the right unstable manifold $W_+^u(u_1, v_1)$ can be either

(i) the point (u_2, v_2), when this is an attractor,

(ii) a stable limit cycle, if (u_2, v_2) is a repeller, or

(iii) the point $(0, 0)$.

Hence, there is a subset of the parameter space for which $W_+^u(u_1, v_1)$ intersects $W_+^s(u_1, v_1)$ and a homoclinic curve is

obtained. In this case, the same point (u_1, v_1) is the $\omega - limit$ of the right unstable manifold $W_+^u(u_1, v_1)$.

(b) We will analyze the stability of the homoclinic cycle obtained by the breaking of the homoclinic curve determined by the stable manifold $W_+^s(u_1, v_1)$ and the unstable manifold $W_+^u(u_1, v_1)$, of the point $W_+^s(u_1, v_1)$.

Denoting $\lambda_1^+(u_1, v_1)$ and $\lambda_1^-(u_1, v_1)$ the eigenvalues associated to point (u_1, v_1), where the upper indices correspond to the sign of the respective eigenvalue. We determine the neutrality of the homoclinic cycle considering $R = 1$ [3, 6]; then,

$$
\begin{aligned}
R &= \left| \frac{\lambda_1^-(u_1, v_1)}{\lambda_1^+(u_1, v_1)} \right| \\
&= \left| \frac{\operatorname{tr} DY_\eta(u_1, v_1) - \sqrt{\left(\operatorname{tr} DY_\eta(u_1, v_1) \right)^2 - 4 \det DY_\eta(u_1, v_1)}}{\operatorname{tr} DY_\eta(u_1, v_1) + \sqrt{\left(\operatorname{tr} DY_\eta(u_1, v_1) \right)^2 - 4 \det DY_\eta(u_1, v_1)}} \right|,
\end{aligned}
\tag{51}
$$

implies

$$
R = \operatorname{tr} DY_\eta(u_1, v_1) = 0,
\tag{52}
$$

i.e.,

$$
R = \frac{-3Eu_1^2 + (2E + 2ME - 1)u_1 + (C - ME)}{E}
- B(u_1 - C) = 0.
\tag{53}
$$

Replacing $u_1^2 = (((M + 1)E - 1)u_1 - (ME - C))/E$, we obtain

$$
R = -(u_1 - C)B - \frac{E(M + 1) - 2}{E}u_1 + \frac{2(ME - C)}{E}
\tag{54}
$$
$$
= 0.
$$

Then,

$$
B = \frac{-(E(M + 1) - 2)u_1 + 2(ME - C)}{E(u_1 - C)}.
\tag{55}
$$

So, the non-infinitesimal limit cycle generated by the breaking of the homoclinic curve and surrounding the equilibrium point (u_2, v_2) is

(b1) unstable, if and only if, $B < ((-(E(M + 1) - 1)u_1 + 2(ME - C)) + u_1)/E(u_1 - C)$.

(b2) stable, if and only if, $B > ((-(E(M+1)-1)u_1 + 2(ME-C)) + u_1)/E(u_1 - C)$. □

Theorem 15. *Collapse of the positive equilibrium points*

When coinciding the singularities (u_1, v_1) and (u_2, v_2), there exists a unique equilibrium point at the interior of the first quadrant, denoted by $(u^c, v^c) = (u^c, (u^c - C)/E)$ which is

(i) a non-hyperbolic attractor node, if and only if, $B > ((2 - ME - E)u^c + 2(ME - C))/E(u^c - C)$

(ii) a non-hyperbolic repeller node, if and only if, $B < ((2 - ME - E)u^c + 2(ME - C))/E(u^c - C)$ and the point $(0, 0)$ is a almost global attractor [33, 34].

(iii) a cusp point, if and only if, $B = ((2 - ME - E)u^c + 2(ME - C))/E(u^c - C)$.

In this case, there exists a unique trajectory which attains the point $(u^c, (u^c - C)/E)$.

Proof. The Jacobian matrix at (u^c, v^c) is

$$
DY_\eta\left(u^c, \frac{u^c - C}{E}\right) = \begin{pmatrix} \dfrac{-3E(u^c)^2 + (2E + 2ME - 1)u^c + (C - ME)}{E} & -u^c \\ \dfrac{B}{E}(u^c - C) & -B(u^c - C) \end{pmatrix}.
\tag{56}
$$

To determine the sign of $\det DY_\eta(u^c, (u^c - C)/E)$, we consider again the factor T from the above Theorem.

Replacing $u^c = (1/2E)(E(M + 1) - 1)$, we have

$$
\begin{aligned}
T &= (E(M + 1) - 1)\left(\frac{1}{2E}(E(M + 1) - 1) \right) \\
&\quad - 2(ME - C) = \frac{(E(M + 1) - 1)^2}{2E} - 2(ME - C) \\
&= \frac{(E(M + 1) - 1)^2 - 4(ME - C)E}{2E} = 0.
\end{aligned}
\tag{57}
$$

since $\Delta = (E(M + 1) - 1)^2 - 4(ME - C)E = 0$.

Then, the sign of $\det DY_\eta(u^c, (u^c - C)/E)$ depends on the sign of the factor

$$
T_1 = (1 - M)^2 E^2 - 2(1 + M - 2C)E + 1.
\tag{58}
$$

Thus, the $\operatorname{tr} Y_\eta(u^c, (u^c - C)/E)$ depends on the sign of

$$
V = -E(u^c - C)B + (2 - ME - E)u^c + 2(ME - C).
\tag{59}
$$

Considering different alternatives for V, we obtain the thesis. □

Remark 16. The point $(u^c, (u^c - C)/E)$ is the collapse of the equilibrium (u_1, v_1) and (u_2, v_2). Then,

$$
\det DY_\eta\left(u^c, \frac{u^c - C}{E}\right) = 0,
\tag{60}
$$

that is,

$$
C = \frac{1}{4E}\left(4ME - (E(M + 1) - 1)^2 \right).
\tag{61}
$$

When, $\operatorname{tr} DY_\eta(u^c, (u^c - C)/E) = 0$, it has the $B = ((2 - ME - E)u^c + 2(ME - C))/E(u^c - C)$, then, a Bogdanov-Takens bifurcation is obtained.

The point $(0, 0)$ is a almost global attractor [33, 34] since all the trajectories, except $(u^c, (u^c - C)/E)$ have that equilibrium as their $\omega - limit$.

Case B. Supposing that $ME + E - 1 > 0$ and $ME - C = 0$.
Thus, $C = ME$, $E > 1/(1 + M)$ and the system becomes

$$Y_\nu : \begin{cases} \dfrac{du}{d\tau} = ((1 - u)(u - M) - v)u \\ \dfrac{dv}{d\tau} = B(u - ME - Ev)v \end{cases} \tag{62}$$

where $\nu = (B, E, M) \in \mathbb{R}_+^2 \times]0, 1[$. The equilibrium points are $(0, 0)$, $(M, 0)$, $(1, 0)$, and the equilibrium $(u_3, (u_3 - ME)/E)$ with $u_3 = (ME + E - 1)/E$. The equilibrium over the $x - axis$ has the same nature expressed in the above lemmas.

Theorem 17. *Assuming that $u_3 - ME > 0$, the equilibrium point $(u_3, (u_3 - ME)/E)$ is at the interior of the first quadrant and it is*
(i) an attractor, if and only if, $B < -(ME + E - 1)(ME + E - 2)/E(ME - 1)(1 - E)$,
(ii) a repeller, if and only if, $B > -(ME + E - 1)(ME + E - 2)/E(ME - 1)(1 - E)$. Moreover, a stable limit cycle can exist.
(iii) a weak focus, if and only if, $B = -(ME + E - 1)(ME + E - 2)/E(ME - 1)(1 - E)$.

Proof. The Jacobian matrix in the unique positive equilibrium point $(u_3, (u_3 - ME)/E)$ is given by

$$DY_\nu \left(u_3, \frac{u_3 - ME}{E} \right)$$
$$= \begin{pmatrix} \dfrac{(-3Eu_3 + (2E + 2ME - 1))u_3}{E} & -u_3 \\ \dfrac{B}{E}(u_3 - ME) & -B(u_3 - ME) \end{pmatrix}. \tag{63}$$

We have that $\det DY_\nu(u_3, (u_3 - ME)/E) = Bu_3(u_3 - ME)((3u_3 - 2M - 2)E + 2)/E$,
As $u_3 - ME = (ME + E - 1)/E - ME = (ME - 1)(1 - E)/E > 0$,
then $1/(1 + M) < 1/M < E < 1$.
The sign of $\det DY_\eta(u_3, (u_3 - ME)/E)$ depends on the factor:
$T = (3u_3 - 2M - 2)E + 2 = E + ME - 1 > 0$, since $1/(1 + M) < E < 1$.
Thus, the nature of $(u_3, (u_3 - ME)/E)$ depends on the sign of the trace, i.e.,

$$\operatorname{tr} DY_\nu \left(u_3, \frac{u_3 - ME}{E} \right)$$
$$= \frac{(-3Eu_3 + (2E + 2ME - 1))u_3}{E} - B(u_3 - ME)$$

$$= -\frac{1}{E^2}(E + ME - 1)(E + ME - 2)$$
$$\quad - B\frac{(ME - 1)(1 - E)}{E}. \tag{64}$$

We note that $(E + ME - 2)/E = (E + ME - 1 - 1)/E < 0$.
$\operatorname{tr} DY_\nu(u_3, (u_3 - ME)/E) = 0$, if and only if, $B = -(E + ME - 1)(E + ME - 2)/E(ME - 1)(1 - E)$
and the other cases are obtained considering $\operatorname{tr} DY_\nu(u_3, (u_3 - ME)/E)$ positive or negative.
Moreover, the transversality condition is fulfilled since

$$\frac{\partial}{\partial B}\left(\operatorname{tr} DY_\nu\left(u_3, \frac{u_3 - ME}{E} \right) \right) = -\frac{(ME - 1)(1 - E)}{E} \tag{65}$$
$$< 0,$$

implying the existence of a Poincaré–Andronov–Hopf bifurcation. □

Case E. Supposing that $ME + E - 1 = 0$ and $ME - C < 0$.
Therefore, $E = 1/(1 + M)$ and the system becomes

$$Y_\lambda : \begin{cases} \dfrac{du}{d\tau} = ((1 - u)(u - M) - v)u \\ \dfrac{dv}{d\tau} = B\left(u - C - \dfrac{1}{1 + M}v \right)v \end{cases} \tag{66}$$

where $\lambda = (B, C, M) \in \mathbb{R}_+^2 \times]0, 1[$. The equilibrium points are $(0, 0)$, $(M, 0)$, $(1, 0)$, having the same nature expressed in the above lemmas and $(u_5, (u_5 - C)(1 + M))$ with $u_5 = \sqrt{(C - ME)/E} = \sqrt{C - M + CM}$.

Theorem 18. *Nature of the point $(u_5, (u_5 - C)(1 + M))$*
Assuming that $u_5 - C > 0$, the equilibrium point $(u_5, (u_5 - C)(1 + M))$ is at the interior of the first quadrant and it is
(i) an attractor, if and only if, $B > ((1 + M) - 2\sqrt{C - M + CM})\sqrt{C - M + CM}/(\sqrt{C - M + CM} - C)$,
(ii) a repeller, if and only if, $B < ((1 + M) - 2\sqrt{C - M + CM})\sqrt{C - M + CM}/(\sqrt{C - M + CM} - C)$. Moreover, a stable limit cycle can exist.
(iii) a weak focus, if and only if, $B = ((1 + M) - 2\sqrt{C - M + CM})\sqrt{C - M + CM}/(\sqrt{C - M + CM} - C)$

Proof. Now, the Jacobian matrix is given by

$$DY_\lambda (u_5, v_5)$$
$$= \begin{pmatrix} -3u_5^2 + u_5(1 + M) + (C(M + 1) - M) & -u_5 \\ B(u_5 - C)(1 + M) & -B(u_5 - C) \end{pmatrix}, \tag{67}$$

with $v_5 = (u_5 - C)(1 + M)$.
Then,

$$\det DY_\lambda ((u_5 - C)(1 + M))$$
$$= B(u_5 - C)(3u_5^2 - CM - C + M) \tag{68}$$
$$= 2B(u_5 - C)(C - M + CM) > 0.$$

So, the nature of $(u_5, (u_5 - C)(1 + M))$ depends only on the sign of the tr DY_λ, given by $C - M + CM$

$$
\begin{aligned}
&\operatorname{tr} DY_\lambda \left((u_5 - C)(1 + M) \right) \\
&\quad = -3u_5^2 + u_5(1 + M) + (C(M + 1) - M) \\
&\qquad - B(u_5 - C) \\
&\quad = -3(C - M + CM) + (C(M + 1) - M) \\
&\qquad - B(u_5 - C) + u_5(1 + M) \\
&\quad = -2(C - M + CM) - B(u_5 - C) + u_5(1 + M) \\
&\quad = -2(C - M + CM) + (1 + M)\sqrt{C - M + CM} \\
&\qquad - B\left(\sqrt{C - M + CM} - C\right)
\end{aligned} \tag{69}
$$

Then, tr $DY_\lambda((u_5 - C)(1 + M)) = 0$ implies

$$
B = \frac{\left((1 + M) - 2\sqrt{C - M + CM}\right)\sqrt{C - M + CM}}{\sqrt{C - M + CM} - C}. \tag{70}
$$

The denominator $\sqrt{C - M + CM} - C > 0$, if and only if, $M < C < 1$ (conditions for the existence of the a positive equilibrium); the numerator is positive, if and only if,

$$
(1 + M) - 2\sqrt{C - M + CM}, \tag{71}
$$

i.e.,

$$
(1 + M)^2 - 4(C - M + CM) > 0, \tag{72}
$$

or

$$
C < \frac{(1 + M)^2 + 4M}{4(M + 1)}. \tag{73}
$$

As $C < 1$, the constraint $((1 + M)^2 + 4M)/4(M + 1) < 1$, holds for all $M < 1$.

Analogously to the above case, the transversality condition is fulfilled since

$$
\frac{\partial}{\partial B}\left(\operatorname{tr} DY_\nu \left(u_5, \frac{u_5 - ME}{E} \right) \right) = -(u_5 - C) < 0. \tag{74}
$$
$\square$

Let us $P = (\operatorname{tr} DY_\lambda((u_5 - C)(1 + M)))^2 - 4 \det DY_\lambda((u_5 - C)(1 + M)) = (-2(C - M + CM) - B(u_5 - C) + u_5(1 + M))^2 - 8B(u_5 - C)(C - M + CM)$

We remember that if $P > 0$, it has a node, and when $P > 0$, it has a focus. Similarly in the other cases above.

Remark 19. If $C = ((1 + M)^2 + 4M)/4(M + 1)$, we have that tr $DY_\lambda((u_5 - C)(1 + M)) = -B(1/4)((1 - M)^2/(M + 1)) < 0$ and the point is a local attractor.

The same happens when $C > ((1 + M)^2 + 4M)/4(M + 1)$.

Cases C and G. In this case there exists a unique positive equilibrium point when and $ME + E - 1 > 0$ and $ME - C < 0$, or $ME + E - 1 < 0$ and $ME - C < 0$, respectively,

In both cases, the equilibrium points over the $x - axis$ are $(0, 0)$, $(M, 0)$, $(1, 0)$, having the same nature described in the above lemmas.

The Jacobian matrix in the points $(u_2, (u_2 - C)/E)$ is given by

$$
\begin{aligned}
&DY_\eta \left(u_2, \frac{u_2 - C}{E} \right) \\
&= \begin{pmatrix} \dfrac{-3Eu_2^2 + (2E + 2ME - 1)u_2 + (C - ME)}{E} & -u_2 \\ \dfrac{B}{E}(u_2 - C) & -B(u_2 - C) \end{pmatrix}.
\end{aligned} \tag{75}
$$

Then, according to above results (using Eu^2 from equation (16))

$$
\det DY_\eta \left(u_2, \frac{u_2 - C}{E} \right) = \frac{B}{E}(u_2 - C)T, \tag{76}
$$

with $T = (E(M + 1) - 1)u_2 - 2(ME - C)$.

Clearly $T > 0$, if and only if, $E(M + 1) - 1 \geq 0$.

Supposing $T < 0$, that is, $E(M + 1) - 1 < 0$ and $(E(M + 1) - 1)u_2 - 2(ME - C) < 0$, this leads to a contradiction (see proof below, for the case $M < 0$).

Then, the nature of $(u_2, (u_2 - C)/E)$ depends on the sign of

$$
\begin{aligned}
V &= -E(u_2 - C)B - ((M + 1)E - 2)u_2 \\
&\quad + 2(ME - C).
\end{aligned} \tag{77}
$$

So, we have

Theorem 20. *Nature of point $(u_2, (u_2 - C)/E)$*

The equilibrium point $(u_2, (u_2 - C)/E)$ is

(i) out of the first quadrant, if and only if, $u_2 < C$,

(ii) coincident with the point $(1, 0)$, if and ony if, $u_2 = C$; thus, $(u_2, (1 - u_2)u_2)$ is a saddle-node attractor.

(iii) in the interior the first quadrant, if and only if, $C < u_2 < 1$; furthermore, is

(iiia) a hyperbolic attractor, if and only if, $B < ((2 - ME - E)u_2 + 2(ME - C))/E(u_2 - C)$.

(iiib) a hyperbolic repeller, if and only if, $B > ((2 - ME - E)u_2 + 2(ME - C))/E(u_2 - C)$. Furthermore, a stable limit cycle can exist, surrounding the positive equilibrium point, by Hopf bifurcation.

(iiic) a weak focus, if and only if, $B = ((2 - ME - E)u_2 + 2(ME - C))/E(u_2 - C)$.

Proof. Immediate since $T > 0$ and the sign of tr $DY_\eta(u, v)$ depends on the sign of

$$
\begin{aligned}
V &= -E(u_2 - C)B - ((M + 1)E - 2)u_2 \\
&\quad + 2(ME - C).
\end{aligned} \tag{78}
$$

According the sign of V the results follow from Routh-Hurwitz criterion.

Moreover, $\partial V/\partial B = -E(u_2 - C) < 0$.

Then, the transversality condition is obtained. $\square$

In both cases the systems have at least a unique limit cycle surrounding the unique positive equilibrium point (see

Figure 6). Moreover, the points $(M, 0)$ and $(1, 0)$ are saddle points.

Theorem 21. *Existence of a heteroclinic curve*
There exists a heteroclinic curve joining the points $(1, 0)$ *and* $(M, 0)$ *(see Figure 7).*

Proof. Let us $W_+^s(M, 0)$ the upper stable manifold of the equilibrium $(M, 0)$ and the upper unstable manifold $W_+^u(1, 0)$, of the point $(1, 0)$.

The $\alpha - limit$ of the $W_+^s(M, 0)$ can be
(i) the point $(\infty, 0)$ (infinity in the direction of $u - axis$),
(ii) an unstable limit cycles surrounding the positive equilibrium (u_2, v_2), when this is an attractor.
(iii) the point (u_2, v_2), when this is a repeller.
At once, the $\omega - limit$ of $W_+^u(1, 0)$ cannot at infinity on the direction of $v - axis$ due to the boundedness of solutions. It can be either:
(i) the point (u_2, v_2), when this is an attractor,
(ii) a stable limit cycle, if (u_2, v_2) is a repeller, or
(iii) the equilibrium $(0, 0)$.
Then, there are points $(u^*, v^s) \in W_+^s(M, 0)$ and $(u^*, v^u) \in W_+^u(1, 0)$ where v^s and v^u are functions of the parameters M, B, C and E, that is, $v^s = f_1(M, B, C, E)$ and $v^u = f_2(M, B, C, E)$.

Clearly, if $0 < u^* \ll 1$, then, $v^s < v^u$; if $0 \ll u^* < 1$, then $v^s > v^u$. Since the vector field Y_η is continuous with respect to the parameter values, there is a subset of the parameter space for which $W_+^s(M, 0)$ intersects $W_+^u(1, 0)$ and a heteroclinic curve exists.

Moreover, there exists a point (u^*, v^*) in the invariant region, i.e., $(u^*, v^*) \in \overline{\Gamma}$, such as $v^* = v^s = v^u$.

This equation defines a surface in the parameter space for which the heteroclinic curve exists. □

Remark 22. (1) We note that a non-infinitesimal limit cycle can be generated by the breaking of the heteroclinic curve [35], which could coincide with the infinitesimal limit cycle generated by the Hopf bifurcation.

(2) The infinitesimal limit cycle generated by the Hopf bifurcation increases until attain the saddle point $(M, 0)$ and after it disappears, when the parameters change a little. Then, the equilibrium point $(0, 0)$ is a almost global attractor [33, 34] since all the trajectories, except (u_2, v_2) have that equilibrium as their $\omega - limit$.

3.3. Special Weak Allee Effect. Assuming $M = 0$ ($m = 0$) system (11) becomes

$$Y_\gamma : \begin{cases} \dfrac{du}{d\tau} = ((1 - u) u - v) u \\ \dfrac{dv}{d\tau} = B (u - C - Ev) v \end{cases} \tag{79}$$

with $\gamma = (B, C, E) \in \mathbb{R}_+^3$. The equilibrium points are $(0, 0)$, $(1, 0)$, and $(u^*, (1 - u^*)u^*) = (u^*, (u^* - C)/E)$, where u^* satisfy the equation

$$p(u) = Eu^2 - (E - 1) u - C = 0; \tag{80}$$

thus, any being the sign of $(E - 1)$ it has

$$u^* = \frac{1}{2E} \left(E - 1 + \sqrt{\Delta} \right) \tag{81}$$

with $\Delta = ((E - 1))^2 + 4EC > 0$.

Moreover, if $v^* = (u^* - C)/E = (1 - u^*)u^* > 0$, thus, $0 < C < u^* < 1$.

We note that when $E = 1$, a particular case is obtained; thus, it has $u^* = \sqrt{C}$ and $v^* = \sqrt{C} - C$ and $dv/d\tau = 0$, in the equilibrium $(\sqrt{C}, \sqrt{C} - C)$.

The Jacobian matrix of system (79) is

$$DY_\gamma (u, v) = \begin{pmatrix} -3u^2 + 2u - v & -u \\ Bv & -B (C - u + 2vE) \end{pmatrix} \tag{82}$$

3.3.1. Main Properties of System (79). In this case it has that the equilibriums
(a) $(1, 0)$ has the same properties of the case $M > 0$ (above Lemma).
(b) $(0, 0)$ is the collapse between $(M, 0)$ and $(0, 0)$ system (8). Then, the equilibrium $(0, 0)$ is a non-hyperbolic equilibrium.

Lemma 23. *Nature of the equilibrium* $(0, 0)$ *when* $M = 0$.
The equilibrium $(0, 0)$ *has hyperbolic and parabolic sectors, determined by the stable manifold* $W^s(0, 0)$.

Proof. It is immediate since when $M > 0$, the equilibrium $(0, 0)$ is an attractor and the equilibrium $(M, 0)$ is a saddle point, if and only if, $M < C$. □

The nature of the unique positive equilibrium point $(u^*, (1 - u^*)u^*)$ is given in the following theorem.

Theorem 24. *The positive equilibrium* $(u^*, (u^* - C)/E)$:
(i) is out of the first quadrant, if and only if, $u^* < C$,
(ii) coincides with the point $(1, 0)$, *if and only if,* $u^* = C$; *thus,* $(u^*, (1 - u^*)u^*)$ *is a saddle-node attractor.*
(iii) Assuming $0 < C < u^* < 1$, *it has*
(iiia) a hyperbolic attractor, if and only if, $B > ((2 - E)u^* - 2C)/(u^* - C)E$.
(iiib) a hyperbolic repeller, if and only if, $B < ((2 - E)u^* - 2C)/(u^* - C)E$.
(iiic) a weak focus, if and only if, $B = ((2 - E)u^* - 2C)/(u^* - C)E$.

Proof. (i) and (ii) are immediate
(iii) The Jacobian matrix evaluated in the positive equilibrium point is

$$DY_\gamma \left(u^*, \frac{u^* - C}{E} \right)$$

$$= \begin{pmatrix} -3 (u^*)^2 + 2u^* - \dfrac{u^* - C}{E} & -u^* \\ B\dfrac{u^* - C}{E} & -B (u^* - C) \end{pmatrix}. \tag{83}$$

Then,

$$\det DY_\gamma \left(u^*, \frac{u^* - C}{E} \right)$$

$$= B\left(u^* - C \right) \frac{2u^* - C + 3E\left(u^* \right)^2 - 2Eu^*}{E}, \tag{84}$$

which depends on the factor $2u^* - C + 3E(u^*)^2 - 2Eu^*$.

Considering equation (9) it has $E(u^*)^2 = C - (1 - E)u^*$; this becomes

$$\det DY_\gamma \left(u^*, \frac{u^* - C}{E} \right)$$

$$= \frac{B\left(u^* - C \right)\left((E - 1)u^* + 2C \right)}{E}. \tag{85}$$

As $u^* = (1/2E)(E - 1 + \sqrt{\Delta})$, it has
$(E - 1)u^* + 2C = ((E - 1)^2 + 4CE + \sqrt{\Delta}(E - 1))/2E = (\Delta + \sqrt{\Delta}(E - 1))/2E > 0$, any be the sign of $E - 1$, with $E \neq 1$.

Then, the nature of the equilibrium $(u^*, (u^* - C)/E)$ depends on the sign of the trace.

It has

$$\operatorname{tr} DY_\gamma \left(u^*, \frac{u^* - C}{E} \right) = -3\left(u^* \right)^2 + 2u^* - \frac{u^* - C}{E}$$

$$- B\left(u^* - C \right). \tag{86}$$

If $\operatorname{tr} DY_\gamma(u^*, (u^* - C)/E) = 0$, then $B = (-3E(u^*)^2 + 2Eu^* - u^* + C)/(u^* - C)E$.

Using the relation for $E(u^*)^2$, we have $B = ((2 - E)u^* - 2C)/(u^* - C)E$.

The other cases are obtained considering the sign of $\operatorname{tr} DY_\gamma(u^*, (u^* - C)/E)$. □

3.4. The Case $M < 0$ ($m < 0$). In this case system (11) has the equilibrium points $(0, 0)$, $(1, 0)$ and $(u_e, (1 - u_e)u_e)$, where u_e satisfy equation (9).

In that equation, $ME - C < 0$, with any being the sign of $ME + E - 1$, it has a unique solution given as

$$u_e = \frac{1}{2E}\left(ME + E - 1 + \sqrt{\Delta} \right). \tag{87}$$

with
$\Delta = (E(M + 1) - 1)^2 - 4(ME - C)E > 0$, which is fulfilled for all $C, E > 0$ and $M < 0$.

Then, there exists a unique positive equilibrium point $(u_e, (u_e - C)/E) = (u_e, (1 - u_e)u_e)$.

Lemma 25. *Nature of the equilibrium $(0, 0)$ when $M < 0$.*
The equilibrium $(0, 0)$ is a saddle point for all parameter values.

Proof. By the evaluating the Jacobian matrix in $(0, 0)$ it has

$$\det DY_\eta (0, 0) = MBC < 0. \tag{88}$$

□

Lemma 26. *Nature of the equilibrium $(1, 0)$ when $M < 0$.*
The point $(1, 0)$ is a
(i) hyperbolic saddle point, if and only if, $C < 1$,
(ii) hyperbolic attractor point, if and only if, $C > 1$, and
(iii) a saddle-node (non-hyperbolic), if and only if, $C = 1$.

Proof. Point $(1, 0)$ has the same dynamics of the case $M > 0$, according to the evaluation of the Jacobian matrix. □

Theorem 27. *Nature of the equilibrium $(u_e, (u_e - C)/E)$.*
The positive equilibrium $(u_e, (1 - u_e)u_e) = (u_e, (u_e - C)/E)$:
(i) is out of the first quadrant, if and only if, $u_e < C < 1$,
(ii) coincides with the point $(1, 0)$, if and only if, $u_e = C$; thus, $(u_e, (1 - u_e)u_e)$ is a saddle-node attractor.
(iii) The equilibrium $(u_e, (1 - u_e)u_e)$ is positive, if and only if, $0 < C < u_e < 1$. Moreover,
(iiia) a hyperbolic attractor, if and only if, $B < ((2 - (M + 1)E)u_e - 2(C - ME))/E(u_e - C)$,
(iiib) a hyperbolic repeller, if and only if, $B > ((2 - (M + 1)E)u_e - 2(C - ME))/E(u_e - C)$. Furthermore, there exist a limit cycle surrounded the positive equilibrium point.
(iiic) a weak focus, if and only if, $B = ((2 - (M + 1)E)u_e - 2(C - ME))/E(u_e - C)$.

Proof. (i) and (ii) are immediates.

(iii) The Jacobian matrix at the point $(u_e, (u_e - C)/E)$ after replacements is given by

$$DY_\eta \left(u_e, \frac{u_e - C}{E} \right)$$

$$= \begin{pmatrix} \dfrac{-((M + 1)E - 2)u_e + 2(ME - C)}{E} & -u \\ \dfrac{B}{E}(u - C) & -B(u - C) \end{pmatrix}. \tag{89}$$

Then, $\det DY_\eta(u_e, (u_e - C)/E) = (B/E)(u_e - C)T$, with $T = (E(M + 1) - 1)u_e - 2(ME - C)$, obtained in the theorem above.

Clearly $T > 0$, if and only if, $E(M + 1) - 1 \geq 0$.

Supposing $T < 0$, that is, $E(M + 1) - 1 < 0$ and

$$(E(M + 1) - 1)u_e - 2(ME - C) < 0. \tag{90}$$

It implies, $(E(M + 1) - 1)u_e < 2(ME - C) < 0$.
i.e., $(1 - E(M + 1))u_e > -2(ME - C) > 0$; thus

$$u_e > \frac{-2(ME - C)}{1 - E(M + 1)}. \tag{91}$$

After some algebraic manipulations, we obtain

$$\sqrt{\Delta} > \frac{(-2(ME - C))2E}{(1 - E(M + 1))} + (1 - E(M + 1)) > 0, \tag{92}$$

or

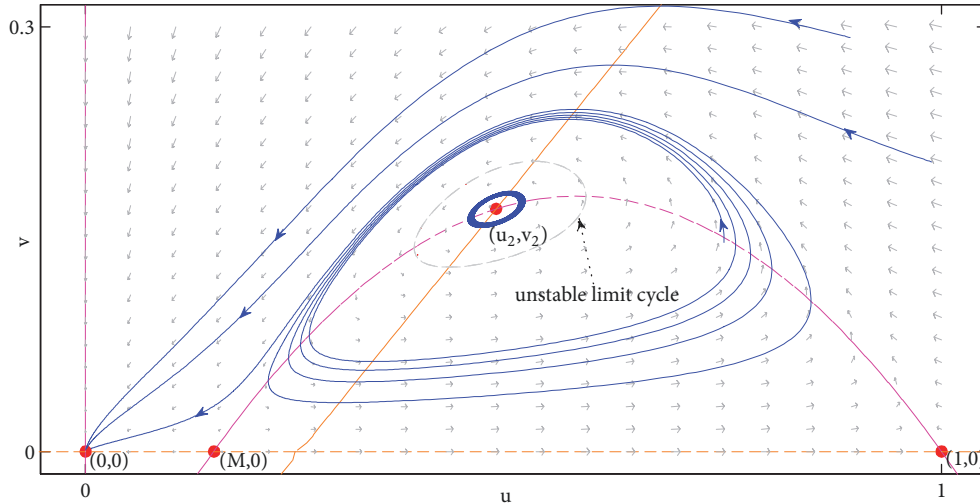$$\Delta_2 = \Delta - \left(\frac{(1 - E(M + 1))^2 + (-2(ME - C))2E}{(1 - E(M + 1))} \right)^2 \tag{93}$$

$$> 0.$$

FIGURE 3: For $B = 0.4$, $M = 0.15$, $C = 0.25$ and $E = 1.33825$ there exists a unique unstable limit cycle surrounding the unique equilibrium point (u_2, v_2) which is local attractor and the point $(0, 0)$ is also a local attractor. This limit cycle is the frontier of the basin of attraction of (u_2, v_2).

Replacing and simplifying this becomes

$$\Delta_2 = -4E\,(C - ME)$$

$$\cdot \frac{-E^2 M^2 - 2E\,(E + 1)\,M + \left(4CE + (E - 1)^2\right)}{(E + ME - 1)^2} \quad (94)$$

$$< 0,$$

and a contradiction is obtained.

Then, the nature of the equilibrium $(u_e, (u_e - C)/E)$ depends on the sign of

$$V = -E\,(u_e - C)\,B - ((M + 1)\,E - 2)\,u_e$$
$$- 2\,(C - ME), \quad (95)$$

corrected to the theorem above.

Again, according to the sign of V, the results follow from Routh-Hurwitz criterion.

When $(u_e, (u_e - C)/E)$ is a repeller it has

$$\frac{\partial V}{\partial B} = -E\,(u_e - C) < 0. \quad (96)$$

Then, the transversality condition is fulfilled. □

4. Some Numerical Simulations

In order to reinforce our analytical results, here we present some simulations, for the case $M > 0$. In Figures 3–9 some of different studied cases will be presented, mainly when there exists a unique positive equilibrium.

(A) Existence of a Unique Positive Equilibrium Point
(1) Existence of a unique unstable limit cycle and bi-stability phenomenon (Figure 3).
(2) The point $(0, 0)$ is an almost global attractor [33, 34] (Figure 4).

(3) Existence of two local attractors and the bi-stability phenomenon (Figure 5).
(4) Existence of a stable limit cycle (Figure 6).
(5) Existence of heteroclinic curve (Figure 7).
(6) Existence of a unstable imit cycle (Figure 8)

(B) Existence of Two Positive Equilibrium Points
(7) Existence of a unstable limit cycle and two equilibrium points (Figure 9).

5. Conclusions

In this work, we have analyzed a model considering competition among predators (CAP) and the prey population is affected by an Allee effect. According to the intensity of Allee effect, three cases were studied.

By means of a diffeomorphism [30], we analyzed the topologically equivalent system (11) to the original one, depending only on four parameters; conditions for the existence of positive equilibrium points and their nature were established in some cases.

When $M > 0$ $(m > 0)$, it has a strong Allee effect; one of the main mathematical consequences of the assumption of the existence of CAP is the apparition of a slanted isocline; this straight line can generate up to two positive equilibrium points when it intersects with the prey isocline.

Because it is assumed that the prey population be affected by the Allee effect, it can be proved that the equilibrium point $(0, 0)$ is always an attractor for all parameter values; meanwhile the nature of equilibria $(M, 0)$ and $(1, 0)$ depends on the relation between C with 1 and M (c, K and m in the original system).

The point $(M, 0)$ associated with the Allee effect determines a separatrix curve $\overline{\Sigma}$, determined by the stable manifold $W_+^s(M, 0)$, which divides the phase plane into two regions. The trajectories having initial conditions above this curve
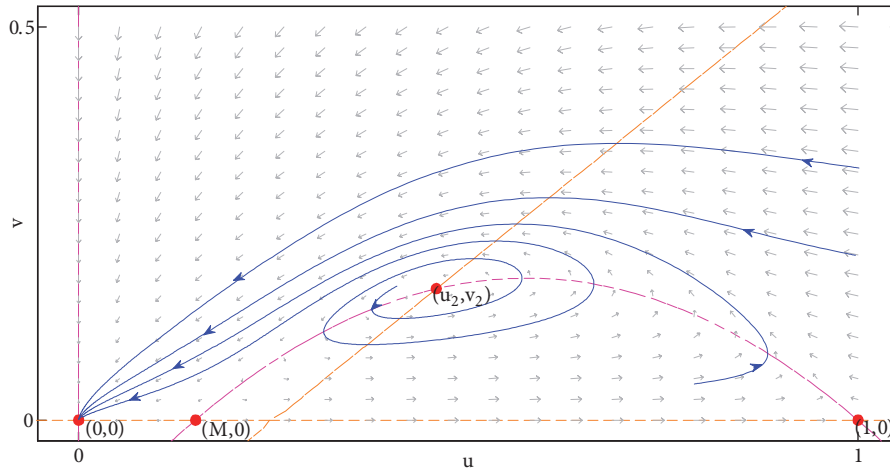
FIGURE 4: For $B = 0.4$, $M = 0.15$, $C = 0.25$ and $E = 1.25$, the point (u_2, v_2) is repeller focus and $(0, 0)$ is an almost global attractor [33, 34], since point (u_2, v_2) is the unique solution that does not attain $(0, 0)$.
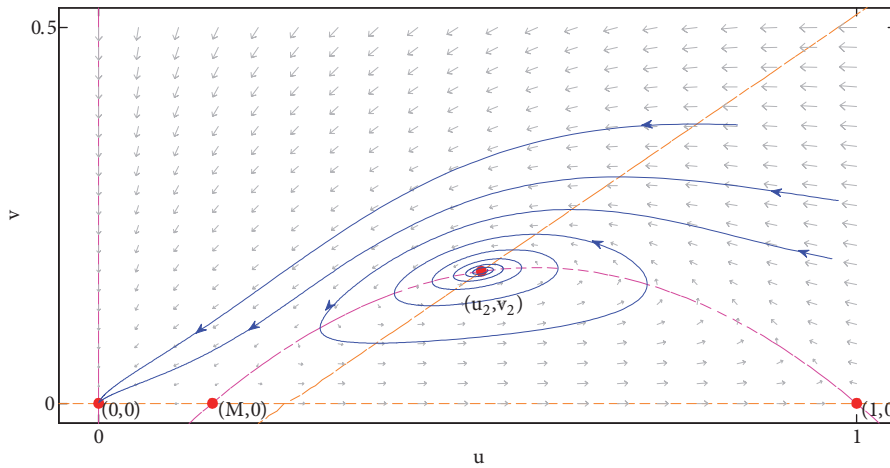

FIGURE 5: For $B = 0.4$, $M = 0.15$, $C = 0.25$, $E = 1.45$, point (u_2, v_2) is a local attractor focus and $(0, 0)$ is a local node attractor.
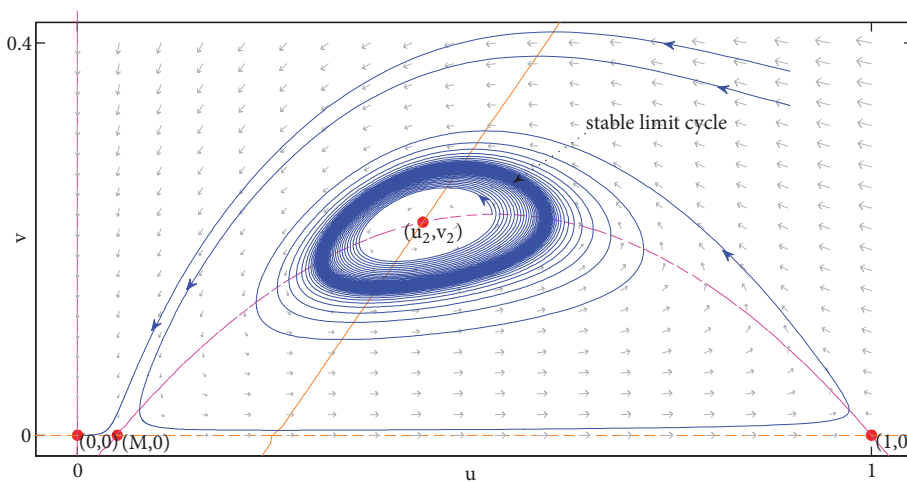

FIGURE 6: For $B = 0.4$, $M = 0.05$, $C = 0.25$, $E = 0.85$, point (u_2, v_2) is repeller focus, surrounding by a stable limit cycle and $(0, 0)$ is local attractor.
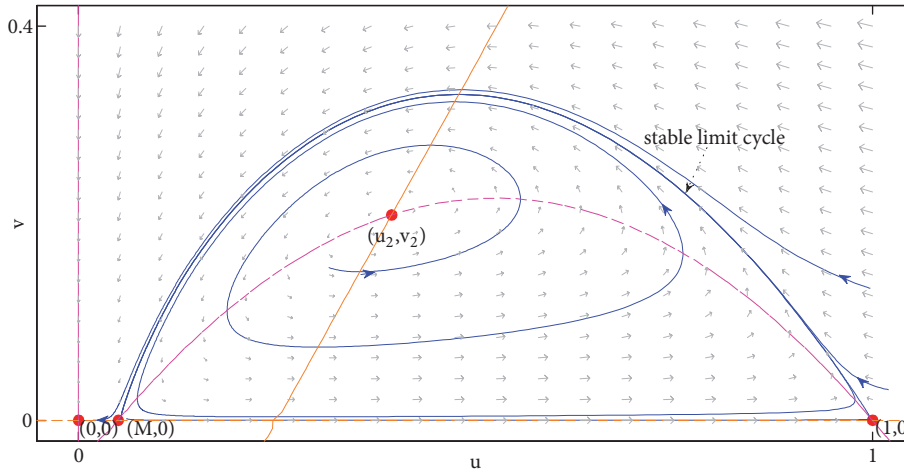
Figure 7: For $B = 0.4, M = 0.05, C = 0.25, E = 0.6915$, point (u_2, v_2) is repeller focus surrounding of a stable limit cycle (near of a heteroclinic) and $(0, 0)$ is a local attractor.
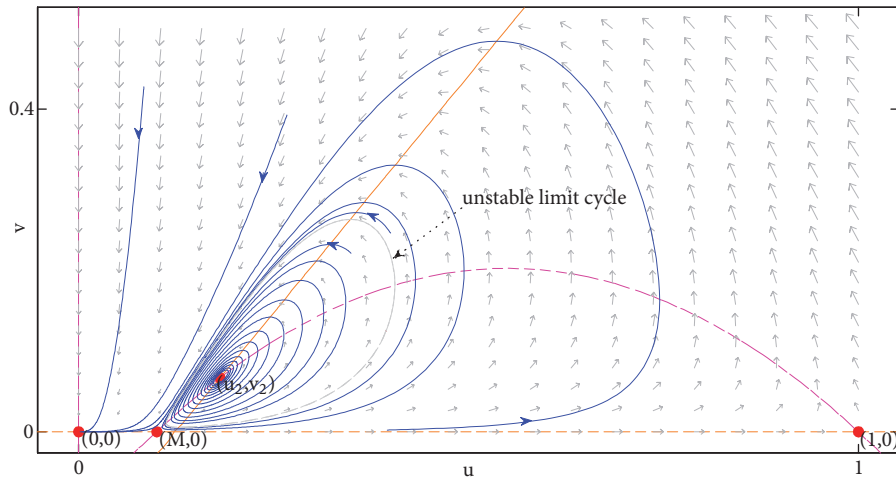

Figure 8: For $B = 2.5, M = 0.1, C = 0.12475$ and $E = 0.85$, there exist a unique equilibrium point (u_2, v_2), a unstable imit cycle surrounded this point; moreover point $(0, 0)$ is local attractor.
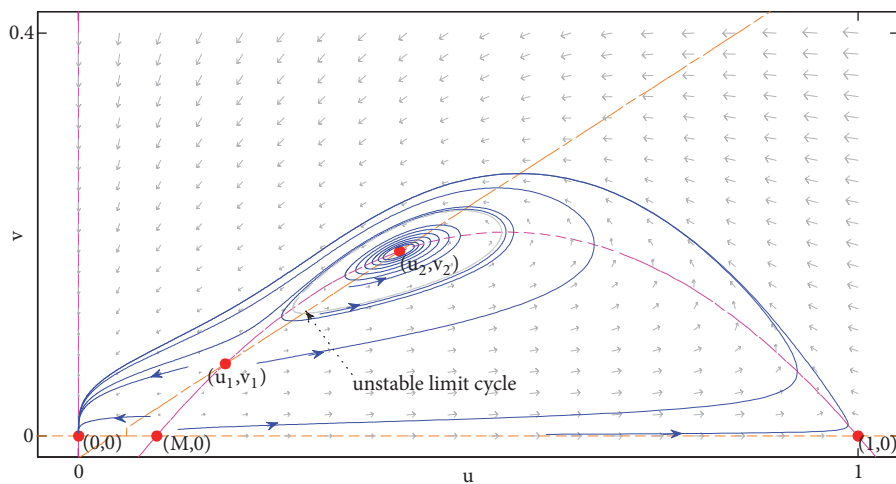

Figure 9: For $B = 0.334, M = 0.1, C = 0.045, E = 2.0$, point (u_2, v_2) is a attractor focus, surrounded of a unstable limit cycle; point (u_1, v_1) is saddle and $(0, 0)$ is a local attractor.

have the point $(0, 0)$ as their $\omega - limit$; meanwhile, those that lie below the separatrix can have a positive equilibrium point or a stable limit cycle as their $\omega - limit$.

This implies that there exists a great possibility of the prey population going to extinction, although the ratio prey-predator is high (many preys and few predators); thus, for the same set of parameter values, both populations can coexist, oscillating around specific population sizes or prey population can be depleted.

An important result was the determination of a subset of the parameter values, for which there exist two positive equilibrium points (u_1, v_1) and (u_2, v_2), the first of them being always a saddle point. The other equilibrium can be an attractor, a repeller or a weak focus, depending on the sign of the trace of the Jacobian matrix evaluated there, since its determinant is positive.

Then, the existence of a homoclinic curve determined by the stable and unstable manifolds of the positive saddle point (u_1, v_1), encircling the second positive equilibrium point (u_2, v_2), was proved. This means that the existence of the bi-stability phenomenon and population sizes can oscillate surround of the point (u_2, v_2) or they can go to extinction, since the equilibrium $(0, 0)$ is a local attractor.

Furthermore, both equilibrium points can collapse, obtaining a cusp point (Bogdanov-Takens bifurcation or codimension 2 bifurcation) [2, 36].

The studied system when $M > 0$ is undoubtedly rather sensitive to disturbances to others proposed in the ecological literature as the Rosenzweig-MacArthur model [4, 15], being an important question to ecologists and requiring a careful management in applied contexts of conservation and fisheries [14, 20]. The assumption that competition among predators originates a model with a more complex dynamics is fulfilled; as we have seen the interaction can have two or more attractors for the same set of parameter values, even existing the possibility of extinction of both populations.

When $M = 0$ $(m = 0)$, it has a special weak Allee effect; in system (11) the equilibriums $(0, 0)$ and $(M, 0)$ are coincident; the collapsed equilibrium has parabolic and hyperbolic sectors, generated by a separatrix curve $\overline{\Sigma}_0$, dividing the trajectories in the phase plane.

As in the case $M > 0$, there also exists a great possibility of the two populations going to extinction. This is because point $(0, 0)$ is an attractor for a wide set of trajectories, but for other it acts as a saddle point. For a same parameter constraint, depending on the predator population sizes, both populations go to extinction or coexist on a long time, as happens with the strong Allee effect.

When $-1 < M < 0$ $(-K < m < 0)$ the system has dynamics more simple that the case with strong Allee effect, existing a unique positive equilibrium point. There no exists a separatrix curve and the equilibrium $(0, 0)$ is saddle point, implying that the interacting species coexist either in a fix population sizes or in oscillating population sizes.

Unlike what happens with other models in which there is a marked difference between cases of strong and weak Allee effect [3], in the model studied here there exist minor differences among both cases.

Furthermore, all the obtained results imply a great difference with the well-known Volterra model, which has a unique global attractor equilibrium point at the first quadrant [3]. Also, system (6) has a different dynamic with the Volterra model incorporating only Allee effect on prey, analyzed partially in the Kot's book [24] (in which CAP is not considered); in that model the vertical predator isocline intersects the respective prey isocline in a unique positive equilibrium point.

An open problem after the obtained results in this work is to determine the uniqueness or non-uniqueness of the stable limit cycle shown in the different cases.

In short, the model considering self-interference among predators or CAP reveals a rich dynamic, which must be taken into account for the conservation of species and the modelers.

Appendix

With the objective of making a comparative analysis, we present the main properties of the system (8) and (9).

A. Common Properties for Both Systems (8) and (9)

(A.1) They are defined on

$$\Omega = \left\{(x, y) \in \mathbb{R}^2 / 0 \le x, 0 \le y\right\}. \tag{A.1}$$

(A.2) The set

$\Gamma = \{(x, y) \in \Omega / 0 \le x \le K, 0 \le y\}$ is an invariant region.

(A.3) The solutions are bounded.

(A.4) The equilibrium points over the axis are $(0, 0)$ and $(K, 0)$.

(A.5) There exists a unique positive equilibrium point (x_e, y_e).

B. Particular Properties

For system (8) it has:

(2.1) There exists a unique positive equilibrium point (x_e, y_e), if and only if, $Kp - c > 0$.

(2.2) The point $(0, 0)$ is a saddle point for all parameter values.

(2.3) The point $(K, 0)$ is:

(i) is a saddle point, if and only if, $Kp - c > 0$,

(ii) is an attractor node, if and only if, $Kp - c < 0$,

(iii) is a saddle-node, if and only if, $Kp - c = 0$.

(2.4) The unique positive equilibrium point $(x_e, y_e) = (K(re + cq)/(re + Kpq), r(Kp - c)/(re + Kpq))$ is

(i) an attractor, if and only if, $Kp - c > 0$,

(ii) a saddle-node, if and only if, $Kp - c = 0$.

(iii) out of the first quadrant, if and only if, $Kp - c < 0$.

(2.5) When $Kp - c > 0$, there no exist limit cycles. It can be proved applying the Dulac criterion, using the function $G(x, y) = 1/xy$.

When $0 < m$ for system (9) it get [24]:

(3.1) There exists a unique positive equilibrium point (x_e, y_e), if and only if, $0 < m < x_e < K$.

(3.2) There exists the equilibrium point $(m, 0)$, if and only if, $0 < m \ll K$.

(3.3) The point $(0, 0)$ is an attractor stable for all parameter values.

(3.4) The point $(1, 0)$ is:

(i) a saddle point, if and only if, $pK > c$,

(ii) an attractor point, if and only if, $pK < c$,

(iii) a saddle-node, if and only if, $pK = c$.

(3.5) When $m > 0$, the point $(m, 0)$ is:

(i) a repeller, if and only if, $m > c/p$.

(ii) a saddle point, if and only if, $m < c/p$.

(iii) a saddle-node, if and only if, $m = c/p$.

(3.6) The point (x_e, y_e) is:

(i) an attractor, if and only if, $-2c + Kp + mp > 0$,

(ii) a repeller, if and only if, $-2c + Kp + mp < 0$,

(iii) an order one weak focus, if and only if, $-2c + Kp + mp = 0$ [1].

(3.7) If $m = 0$ or $m < 0$, in the respective systems derived from (9), a unique limit cycle surrounding the unique positive equilibrium point, determined by Hopf bifurcation. The second Lyapunov quantity [30] is independent of m, when tr $DX_\sigma(x_e, y_e) = -2c + Kp + mp = 0$ [1].

(3.8) When $m < 0$, system (9) has a similar behavior to the well-known Rosenzweig-MacArthur model (without Allee effect) [2]. Both systems have a unique limit cycle surrounding the unique positive equilibrium point. Thus, the oscillatory behavior may be due to either the nonlinear functional response or to the prey growth equation.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] E. González-Olivares, B. González-Yañez, J. Mena Lorca, A. Rojas-Palma, and J. D. Flores, "Consequences of double Allee effect on the number of limit cycles in a predator-prey model," *Computers & Mathematics with Applications*, vol. 62, no. 9, pp. 3449–3463, 2011.

[2] E. González-Olivares, H. Meneses-Alcay, B. Gonzá lez-Yañez et al., "Multiple stability and uniqueness of the limit cycle in a Gause-type predator-prey model considering the Allee effect on prey," *Nonlinear Analysis: Real World Applications*, vol. 12, no. 6, pp. 2931–2942, 2011.

[3] E. González-Olivares and A. Rojas-Palma, "Allee effect in Gause type predator-prey models: existence of multiple attractors, limit cycles and separatrix curves. A brief review," *Mathematical Modelling of Natural Phenomena*, vol. 8, no. 6, pp. 143–164, 2013.

[4] P. Turchin, *Complex Population Dynamics: A Theoretical/Empirical Synthesis*, vol. 35, Princeton University Press, Princeton, NJ, USA, 2003.

[5] F. Courchamp, L. Berec, and J. Gascoigne, *Allee Effects in Ecology and Conservation*, Oxford University Press, 2008.

[6] A. D. Bazykin, *Nonlinear Dynamics of Interacting Populations*, vol. 11 of *Nonlinear Sciences Series A*, World Scientific Publishing Co. Pte. Ltd, NJ, USA, 1998.

[7] G. F. Gause, *The Struggle of Existence*, Dover Publications Inc, 1934.

[8] L. Berec, "Impacts of foraging facilitation among predators on predator-prey dynamics," *Bulletin of Mathematical Biology*, vol. 72, no. 1, pp. 94–121, 2010.

[9] J. R. Beddington, "Mutual interference between parasites or predators and its effect on searching efficiency," *Journal of Animal Ecology*, vol. 44, no. 1, pp. 331–340, 1975.

[10] D. L. DeAngelis, R. A. Goldstein, and R. V. O'Neill, "A model for trophic interaction," *Ecology*, vol. 56, pp. 881–892, 1975.

[11] M. Haque, "A detailed study of the Beddington-DeAngelis predator-prey model," *Mathematical Biosciences*, vol. 234, no. 1, pp. 1–16, 2011.

[12] S. Geritz and M. Gyllenberg, "A mechanistic derivation of the DeAngelis-Beddington functional response," *Journal of Theoretical Biology*, vol. 314, pp. 106–108, 2012.

[13] H. I. Freedman, "Stability analysis of a predator-prey system with mutual interference and density-dependent death rates," *Bulletin of Mathematical Biology*, vol. 41, no. 1, pp. 67–78, 1979.

[14] C. W. Clark, *Mathematical Bio-Economics: The Optimal Management of Renewable Resources*, John Wiley & Sons, New York, NY, USA, 2nd edition, 1990.

[15] R. M. May, *Stability and Complexity in Model Ecosystems*, Princeton University Press, 2nd edition, 2001.

[16] E. González-Olivares, S. Valenzuela-Figueroa, and A. Rojas-Palma, "A simple Gause type predator-prey model considering social predation," *Mathematical Methods in the Applied Sciences*, 2018.

[17] F. Courchamp, T. Clutton-Brock, and B. Grenfell, "Inverse density dependence and the Allee effect," *Trends in Ecology & Evolution*, vol. 14, no. 10, pp. 405–410, 1999.

[18] P. A. Stephens and W. J. Sutherland, "Consequences of the Allee effect for behaviour, ecology and conservation," *Trends in Ecology & Evolution*, vol. 14, no. 10, pp. 401–405, 1999.

[19] P. A. Stephens, W. J. Sutherland, and R. P. Freckleton, "What is the Allee effect?" *Oikos*, vol. 87, no. 1, pp. 185–190, 1999.

[20] M. Liermann and R. Hilborn, "Depensation: evidence, models and implications," *Fish and Fisheries*, vol. 2, no. 1, pp. 33–58, 2001.

[21] L. Berec, E. Angulo, and F. Courchamp, "Multiple allee effects and population management," *Trends in Ecology & Evolution*, vol. 22, no. 4, pp. 185–191, 2007.

[22] G. A. van Voorn, L. Hemerik, M. P. Boer, and B. W. Kooi, "Heteroclinic orbits indicate overexploitation in predator-prey systems with a strong Allee effect," *Mathematical Biosciences*, vol. 209, no. 2, pp. 451–469, 2007.

[23] D. S. Boukal and L. Berec, "Single-species models of the Allee effect: extinction boundaries, sex ratios and mate encounters," *Journal of Theoretical Biology*, vol. 218, no. 3, pp. 375–394, 2002.

[24] M. Kot, *Elements of Mathematical Biology*, Cambridge University Press, Cambridge, UK, 2001.

[25] E. González-Olivares, B. González-Yañez, J. Mena-Lorca, and R. Ramos-Jiliberto, "Modelling the Allee effect: are the different mathematical forms proposed equivalents?" in *Proceedings of International Symposium on Mathematical and Computational Biology*, R. Mondaini, Ed., pp. 53–71, E-papers Serviços Editoriais Ltda, 2007.

[26] E. Gonzalez-Olivares and J. D. Flores, "Consequences in an open access fishery model considering multiple Allee effects," *Journal of Biological Systems*, vol. 23, no. supp01, pp. S101–S121, 2015.

[27] H. I. Freedman, *Deterministic Mathematical Models in Population Ecology*, Marcel Dekker, New York, NY, USA, 1980.

[28] X. Qiu and H. Xiao, "Qualitative analysis of Holling type II predator-prey systems with prey refuges and predator restricts," *Nonlinear Analysis: Real World Applications*, vol. 14, no. 4, pp. 1896–1906, 2013.

[29] A. A. Berryman, A. P. Gutierrez, and R. Arditi, "Credible, parsimonious and useful predator-prey models - a reply to abrams, gleeson, and sarnelle," *Ecology*, vol. 76, no. 6, pp. 1980–1985, 1995.

[30] C. Chicone, *Ordinary Differential Equations with Applications*, vol. 34 of *Texts in Applied Mathematics*, Springer, New York, NY, USA, 2nd edition, 2006.

[31] F. Dumortier, J. Llibre, and J. C. Artés, *Qualitative Theory of Planar Differential Systems*, Springer, 2006.

[32] G. Birkhoff and G. Rota, *Ordinary Differential Equations*, John Wiley & Sons, New York, NY, USA, 4th edition, 1989.

[33] P. Monzón, "Almost global attraction in planar systems," *Systems & Control Letters*, vol. 54, no. 8, pp. 753–758, 2005.

[34] A. Rantzer, "A dual to Lyapunov's stability theorem," *Systems & Control Letters*, vol. 42, no. 3, pp. 161–168, 2001.

[35] V. A. Gaiko, *Global Bifurcation Theory and Hilbert's Sixteenth Problem*, vol. 559 of *Mathematics and its Applications*, Kluwer Academic Publishers, 2003.

[36] D. Xiao and S. Ruan, "Bogdanov-Takens bifurcations in predator-prey systems with constant rate harvesting," *Field Institute Communications*, vol. 21, pp. 493–506, 1999.