Research Article

Adaptively Random Weighted Cubature Kalman Filter for Nonlinear Systems

Zhaohui Gao [ID],[1] Dejun Mu [ID],[1] Yongmin Zhong,[2] Chengfan Gu,[3] and Chengcai Ren[4]

[1] School of Automatics, Northwestern Polytechnical University, Xi'an 710072, China
[2] School of Engineering, RMIT University, Bundoora, Melbourne 3083, Australia
[3] Department of Mechanical Engineering, University of Melbourne, Parkville, Melbourne 3010, Australia
[4] Baicheng Ordnance Test Centre of China, BaiCheng 137001, China

Correspondence should be addressed to Zhaohui Gao; alexandergao@mail.nwpu.edu.cn

This paper presents a new adaptive random weighting cubature Kalman filtering method for nonlinear state estimation. This method adopts the concept of random weighting to address the problem that the cubature Kalman filter (CKF) performance is sensitive to system noise. It establishes random weighting theories to estimate system noise statistics and predicted state and measurement together with their associated covariances. Subsequently, it adaptively adjusts the weights of cubature points based on the random weighting estimations to improve the prediction accuracy, thus restraining the disturbances of system noises on state estimation. Simulations and comparison analysis demonstrate the improved performance of the proposed method for nonlinear state estimation.

1. Introduction

Since navigation systems generally involve nonlinear characteristics, nonlinear filtering is a popular research topic in navigation and positioning. So far, various nonlinear filtering methods have been studied for spacecraft navigation. The extended Kalman filter (EKF) is a traditional nonlinear filtering method based on the first-order Taylor term of nonlinear functions [1, 2]. It maintains the computational efficiency of the linear Kalman filter. However, since it neglects higher-order terms of the nonlinear system model, EKF suffers from the degradation of estimation accuracy [3, 4]. The unscented Kalman filter (UKF) conducts nonlinear state estimation in the context of control theory, where a finite number of sigma points are used to propagate the probability of state distribution through nonlinear system dynamics [5–8]. In addition to the improved accuracy comparing to EKF, UKF has the merits such as simplicity in realization, high filtering precision, and good convergence [9–12]. However, UKF requires the a priori statistical characteristics of system noise to be precisely known. In practical applications, due to uncertainties in the dynamic environment, it is difficult to accurately describe noise statistics, leading to biased or even divergent filtering solutions [13–15].

In the recent ten years, the cubature Kalman filter (CKF) has received a great attention in nonlinear filtering [16, 17]. Based on the cubature rules, this method adopts a set of cubature points evaluated via nonlinear functions to approximate the Gaussian integration [18, 19]. Similar to UKF, CKF does not require the linearization of the nonlinear model and the calculation of Jacobian matrix, thus overcoming the problems caused by linear truncation such as poor positioning accuracy and error divergence. Different from UKF, CKF has a stronger nonlinearity due to the use of cubature points. However, CKF requires exact information on the statistical characteristics of system noise [20]. If the statistical characteristics of system noise are not known exactly, the CKF filtering solution will be deteriorated.

The random weighting is a statistical method with high estimation accuracy and low computational burden [21, 22]. This method can handle the calculation of large sample size without requiring the accurate distributions of model parameters. It has been used to solve many problems, for example,

multisensor data fusion [23–25], parameter estimation [26], M-test in linear models [27], analysis of asymptotic properties of function distribution [28], and dynamic navigation and positioning [29]. However, there has been very limited research on utilizing the random weighting method with CKF for spacecraft navigation.

This paper presents a new adaptive random weighting cubature Kalman filter (ARWCKF) by adopting the concept of random weighting to address the limitation that the CKF performance is sensitive to system noise. This method constructs random weighting estimations for system noise statistics, as well as predicted state and measurement vectors and their associated covariances. Based on the random weighting estimations, it adaptively adjusts the weights of cubature points to inhibit the disturbances of system noises on state estimation, leading to improved filtering robustness. Simulations and comparisons with CKF have been conducted to evaluate the performance of the proposed ARWCKF.

2. Concept of Random Weighting Methods

Denote $X_1, X_2, \ldots, X_n$ as a series of random variables in independent and identical distribution, and the common distribution function as $F(x)$. Then, the corresponding empirical distribution function can be defined as

$$F_n(x) = \frac{1}{n} \sum_{i=1}^{n} I_{(X_i \le x)} \tag{1}$$

where $I_{(X_i < x)}$ is the indicator function.

The random weighted estimation of $F_n(x)$ can be defined as

$$H_n(x) = \sum_{i=1}^{n} \lambda_i I_{(X_i < x)} \tag{2}$$

where $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ represents the random weighted vector subject to Dirichlet distribution D(1, 1, …, 1), i.e., $\sum_{i=1}^{n} \lambda_i = 1$, and the joint density function of $(\lambda_1, \lambda_2, \ldots, \lambda_{n-1})$ is $f(\lambda_1, \lambda_2, \ldots, \lambda_{n-1}) = \Gamma(n)$ with $(\lambda_1, \lambda_2, \ldots, \lambda_{n-1}) \in D_{n-1}$ and $D_{n-1} = \{(\lambda_1, \lambda_2, \ldots, \lambda_{n-1}) : \lambda_k \ge 0, k = 1, \ldots, n - 1, \sum_{k=1}^{n-1} \lambda_i \le 1\}$.

3. Nonlinear System Models

Consider the following nonlinear discrete-time system:

$$\begin{aligned} x_{k+1} &= f_k(x_k) + W_k \\ z_k &= h_k(x_k) + V_k \end{aligned} \tag{3}$$

where $x_k \in R^n$ is the state vector at epoch k, $z_k \in R^m$ is the measurement vector, $W_k \in R^n$ is the system process noise, $V_k \in R^m$ is the measurement noise, W_k and V_k are the Gaussian white noises, $f(\cdot)$ is the nonlinear system function, and $h(\cdot)$ is the nonlinear measurement function.

System process noise W_k and measurement noise V_k are assumed to be the uncorrelated Gaussian white noises of constant statistical properties; i.e.,

$$\begin{aligned} E[W_k] &= q \\ E[V_k] &= r \\ \text{cov}(W_k, W_j^T) &= E\left[(W_k - q)(W_j - q)^T\right] = Q\delta_{kj} \\ \text{cov}(V_k, V_j^T) &= E\left[(V_k - r)(V_j - r)^T\right] = R\delta_{kj} \\ \text{cov}(W_k, V_j^T) &= 0 \end{aligned} \tag{4}$$

where cov($\cdot$) is the covariance function, $Q \ge 0$ and $R > 0$ are the noise intensity matrices, and δ_{kj} is the *Kronecker-δ* function.

Let

$$\begin{aligned} W_k &= q + \mu_k \\ V_k &= r + \eta_k \\ &\Downarrow \\ \mu_k &= W_k - q \\ \eta_k &= V_k - r \end{aligned} \tag{5}$$

Substituting (5) into (3), the nonlinear system model can be expressed as

$$\begin{aligned} x_{k+1} &= f_k(x_k) + q + \mu_k \\ z_k &= h_k(x_k) + r + \eta_k \end{aligned} \tag{6}$$

where μ_k and η_k are the Gaussian white noises with zero means and variances Q and R, respectively.

4. Analysis of Cubature Kalman Filter

In order to improve the filtering performance of CKF, let us briefly review the filtering procedure of CKF, which is given as follows.

(i) Initialize State Estimate $\widehat{x}_0$ and Its Associated Error Covariance P_0

$$\begin{aligned} \widehat{x}_0 &= E[x_0] \\ P_0 &= \text{cov}(x_0, x_0^T) = E\left[(x_0 - \widehat{x}_0)(x_0 - \widehat{x}_0)^T\right] \end{aligned} \tag{7}$$

where x_0 obeys the Gaussian distribution.

(ii) Calculation of Cubature Points. Assume the posterior density function at epoch $k - 1$ is $p(x_{k-1}) = N(\widehat{x}_{k-1|k-1}, P_{k-1|k-1})$. By Cholesky decomposition, the error covariance $P_{k-1|k-1}$ at epoch $k-1$ can be written as $P_{k-1|k-1} = S_{k-1|k-1} S_{k-1|k-1}^T$, where $S_{k-1|k-1}$ is a diagonal matrix at epoch $k - 1$. The cubature points can be calculated by $x_{i,k-1|k-1} = S_{k-1|k-1} \xi_i + \widehat{x}_{k-1|k-1}$, where $x_{i,k-1|k-1}$ ($i = 1, 2, \ldots, m$) is the system state of the ith cubature point at epoch $k - 1$.

Cubature points set $[\xi_i, \omega_i]$ can be represented as

$$\xi_i = \sqrt{\frac{m}{2}} \, [\mathbf{1}]_i$$
$$\omega_i = \frac{1}{m}, \quad i = 1, \ldots, m. \ m = 2n \tag{8}$$

where m is the number of the cubature points and is set to twice of the state dimension, $[\mathbf{1}]$ is a set of points, and $[\mathbf{1}]_i$ is the i-th point of $[\mathbf{1}]$.

(iii) State Prediction. The estimated state of the ith cubature point from epoch $k - 1$ to k is described as

$$x^*_{i,k|k-1} = f\left(x_{i,k-1|k-1}\right) \tag{9}$$

Based on (9), the predicted state of the ith cubature point from epoch $k - 1$ to k is

$$\widehat{x}_{k|k-1} = \frac{1}{m} \sum_{i=1}^{m} x^*_{i,k|k-1} \tag{10}$$

and its covariance is $\boldsymbol{P}_{k|k-1} = (1/m) \sum_{i=1}^{m} x^*_{i,k|k-1} x^{*\mathrm{T}}_{i,k|k-1} - \widehat{x}_{k|k-1} \widehat{x}^{\mathrm{T}}_{k|k-1} + \boldsymbol{Q}_{k-1}$.

Let

$$\widehat{x}_{k+1|k} = f_k\left(x_k\right)\big|_{x_k \leftarrow \widehat{x}_k} + q = \widehat{E}\left[f_k\left(x_k\right) \mid z_k\right] + q \tag{11}$$

and

$$\boldsymbol{\rho}_k = f_k\left(x_k\right) - f_k\left(x_k\right)\big|_{x_k \leftarrow \widehat{x}_k}$$
$$= f_k\left(x_k\right) - \widehat{E}\left[f_k\left(x_k\right) \mid z_k\right] \tag{12}$$

where $f_k(x_k)\big|_{x_k \leftarrow \widehat{x}_k}$ is the posterior mean of the state estimation $\widehat{x}_k$, which is propagated by the nonlinear system function $f_k(\cdot)$.

Then

$$\boldsymbol{P}_{k+1|k} = E\left(\boldsymbol{\rho}_k \boldsymbol{\rho}_k^{\mathrm{T}}\right) + \boldsymbol{Q} \tag{13}$$

(iv) Observation Update. The estimated observation of the ith cubature point from epoch $k - 1$ to k is represented by

$$z^*_{i,k|k-1} = h\left(x_{i,k|k-1}\right) \tag{14}$$

Based on (14), the predicted observation of the ith cubature point from epoch $k - 1$ to epoch k can be represented by

$$\widehat{z}_{k|k-1} = \frac{1}{m} \sum_{i=1}^{m} z^*_{i,k|k-1} \tag{15}$$

and its autocovariance and cross-covariance matrices are $\boldsymbol{P}_{zz,k|k-1} = (1/m) \sum_{i=1}^{m} z^*_{i,k|k-1} z^{*\mathrm{T}}_{i,k|k-1} - \widehat{z}_{k|k-1} \widehat{z}^{\mathrm{T}}_{k|k-1} + \boldsymbol{R}_k$ and $\boldsymbol{P}_{xz,k|k-1} = (1/m) \sum_{i=1}^{m} x_{i,k|k-1} z^{*\mathrm{T}}_{i,k|k-1} - \widehat{x}_{k|k-1} \widehat{z}^{\mathrm{T}}_{k|k-1}$.

Let

$$\widehat{z}_{k|k-1} = h_{k+1}\left(x_{k+1}\right)\big|_{x_{k+1} \leftarrow x_{k+1|k}} + r$$
$$= \widehat{E}\left[h_{k+1}\left(x_{k+1}\right) \mid z_k\right] + r \tag{16}$$

and

$$\boldsymbol{\zeta}_k = h_{k+1}\left(x_{k+1}\right) - h_{k+1}\left(x_{k+1}\right)\big|_{x_{k+1} \leftarrow x_{k+1|k}}$$
$$= h_{k+1}\left(x_{k+1}\right) - \widehat{E}\left[h_{k+1}\left(x_{k+1}\right) \mid z_k\right] \tag{17}$$

where $h_{k+1}(x_{k+1})\big|_{x_{k+1} \leftarrow x_{k+1|k}}$ is the posterior mean of the state prediction $\widehat{x}_{k+1|k}$, which is propagated by the nonlinear measurement function $h_{k+1}(\cdot)$, and $\widetilde{x}_{k+1|k} = x_{k+1} - \widehat{x}_{k+1|k}$.

Then

$$\boldsymbol{P}_{\widetilde{z}_{k+1}} = E\left(\boldsymbol{\zeta}_{k+1} \boldsymbol{\zeta}_{k+1}^{\mathrm{T}}\right) + \boldsymbol{R}$$
$$\boldsymbol{P}_{\widetilde{x}_{k+1} \widetilde{z}_{k+1}} = E\left(\widetilde{x}_{k+1|k} \boldsymbol{\zeta}_{k+1}^{\mathrm{T}}\right) \tag{18}$$

(v) State Update

$$\widehat{x}_{k+1} = \widehat{x}_{k+1|k} + \boldsymbol{K}_{k+1}\left(z_{k+1} - \widehat{z}_{k+1|k}\right)$$
$$\boldsymbol{K}_{k+1} = \boldsymbol{P}_{\widetilde{x}_{k+1} \widetilde{z}_{k+1}} \boldsymbol{P}_{\widetilde{z}_{k+1}}^{-1} \tag{19}$$
$$\boldsymbol{P}_{k+1} = \boldsymbol{P}_{k+1|k} - \boldsymbol{K}_{k+1} \boldsymbol{P}_{\widetilde{z}_{k+1}} \boldsymbol{K}_{k+1}^{\mathrm{T}}$$

It can be seen from (11), (13), (16), and (18) that CKF requires the statistical characteristics of the system and measured noises to be known. However, in engineering practice, it is difficult to accurately know the statistical characteristics of the system and measurement noises, thus degrading the filtering accuracy. In the following, we shall discuss how to adaptively estimate the system noise and measurement noise via the concept of random weighting.

5. Adaptive Random Weighting Estimation of Noise Statistics

5.1. Estimation of Noise Statistics. Denote the maximum posterior estimates of $\boldsymbol{q}, \boldsymbol{Q}, \boldsymbol{r}$, and $\boldsymbol{R}$ by $\widehat{\boldsymbol{q}}, \widehat{\boldsymbol{Q}}, \widehat{\boldsymbol{r}}$, and $\widehat{\boldsymbol{R}}$, respectively, and the posterior estimate of state x_j ($j = 0, 1, \ldots, k + 1$) by $\widehat{x}_{j|k+1}$. Assuming $J = p[x_{k+1}, \boldsymbol{q}, \boldsymbol{Q}, \boldsymbol{r}, \boldsymbol{R}, z_{k+1}]$ is a joint probability density function, we have [30]

$$J = p\left[x_{k+1}, \boldsymbol{q}, \boldsymbol{Q}, \boldsymbol{r}, \boldsymbol{R}, z_{k+1}\right]$$
$$= p\left[z_{k+1} \mid x_{k+1}, \boldsymbol{q}, \boldsymbol{Q}, \boldsymbol{r}, \boldsymbol{R}\right] p\left[x_{k+1} \mid \boldsymbol{q}, \boldsymbol{Q}, \boldsymbol{r}, \boldsymbol{R}\right] \tag{20}$$
$$\cdot p\left[\boldsymbol{q}, \boldsymbol{Q}, \boldsymbol{r}, \boldsymbol{R}\right]$$

where $p[\boldsymbol{q}, \boldsymbol{Q}, \boldsymbol{r}, \boldsymbol{R}]$ can be obtained from prior information, and thus it is considered as a constant.

Consider the conditional density function

$$J^* = p\left[x_{k+1}, \boldsymbol{q}, \boldsymbol{Q}, \boldsymbol{r}, \boldsymbol{R} \mid z_{k+1}\right]$$
$$= \frac{p\left[x_{k+1}, \boldsymbol{q}, \boldsymbol{Q}, \boldsymbol{r}, \boldsymbol{R}, z_{k+1}\right]}{p\left[z_{k+1}\right]} \tag{21}$$

where $x_{k+1} = \{x_0, x_1, \ldots, x_{k+1}\}$ and $z_{k+1} = \{z_0, z_1, \ldots, z_{k+1}\}$.

Consequently, the maximum posterior estimates $\widehat{\boldsymbol{q}}, \widehat{\boldsymbol{Q}}, \widehat{\boldsymbol{r}}$, and $\widehat{\boldsymbol{R}}$ can be obtained by solving for the conditional density function J^*. It is evident that the maximum posterior

estimates $\widehat{q}$, $\widehat{Q}$, $\widehat{r}$, and $\widehat{R}$ are not related to the denominator $p[z_{k+1}]$ in (21). Therefore, the problem of calculating $\widehat{q}$, $\widehat{Q}$, $\widehat{r}$, and $\widehat{R}$ can be equivalently obtained by maximizing the following density function $p[x_{k+1}, q, Q, r, R, z_{k+1}]$ in (21).

According to the multiplication theorem of conditional probability, we can obtain [30]

$$p\left[x_{k+1} \mid q, Q, r, R\right] = p\left[x_0\right] \prod_{j=0}^{k} p\left[x_{j+1} \mid x_j, q, Q\right]$$

$$= \frac{1}{(2\pi)^{n/2} |P_0|^{1/2}} \exp\left\{-\frac{1}{2} \|x_0 - \widehat{x}_0\|_{P_0^{-1}}^2\right\}$$

$$\times \prod_{j=0}^{k} \frac{1}{(2\pi)^{n/2} |Q|^{1/2}} \exp\left\{-\frac{1}{2} \|x_{j+1} - f_j(x_j)\right.$$

$$\left. - q\|_{Q^{-1}}^2\right\} = \frac{1}{(2\pi)^{n/2+n(k+1)/2}} |P_0|^{-1/2} |Q|^{-(k+1)/2}$$

$$\cdot \exp\left\{-\frac{1}{2}\left(\|x_0 - \widehat{x}_0\|_{P_0^{-1}}^2\right.\right. \tag{22}$$

$$\left.\left. + \sum_{j=0}^{k} \|x_{j+1} - f_j(x_j) - q\|_{Q^{-1}}^2\right)\right\} = C_1 |P_0|^{-1/2}$$

$$\cdot |Q|^{-(k+1)/2} \exp\left\{-\frac{1}{2}\left(\|x_0 - \widehat{x}_0\|_{P_0^{-1}}^2\right.\right.$$

$$\left.\left. + \sum_{j=0}^{k} \|x_{j+1} - f_j(x_j) - q\|_{Q^{-1}}^2\right)\right\}$$

where n is the state dimension, $C_1 = 1/(2\pi)^{n(k+2)/2}$ is a constant, and $|A|$ is the determinant of A.

Assume that the measurements $z_1, z_2, \ldots, z_{k+1}$ are known and are independent of each other. Then,

$$p\left[z_{k+1} \mid x_{k+1}, q, Q, r, R\right] = \prod_{j=0}^{k} p\left[z_{j+1} \mid x_{j+1}, r, R\right]$$

$$= \prod_{j=0}^{k} \frac{1}{(2\pi)^{m/2} |R|^{1/2}}$$

$$\cdot \exp\left\{-\frac{1}{2} \|z_{j+1} - h_{j+1}(x_{j+1}) - r\|_{R^{-1}}^2\right\} \tag{23}$$

$$= C_2 |R|^{(k+1)/2}$$

$$\cdot \exp\left\{-\frac{1}{2}\sum_{j=0}^{k} \|z_{j+1} - h_{j+1}(x_{j+1}) - r\|_{R^{-1}}^2\right\}$$

where m is the measurement dimension and $C_2 = 1/(2\pi)^{m(k+1)/2}$ is a constant.

Substituting (22) and (23) into (20) yields

$$J = C_1 C_2 |P_0|^{-1/2} \cdot |Q|^{-(k+1)/2} \cdot |R|^{-(k+1)/2} \cdot p\left[q, Q, r,\right.$$

$$\left. R\right] \cdot \exp\left\{-\frac{1}{2}\left[\|x_0 - \widehat{x}_0\|_{P_0^{-1}}^2\right.\right.$$

$$+ \sum_{j=0}^{k} \|x_{j+1} - f_j(x_j) - q\|_{Q^{-1}}^2$$

$$\left.\left. + \sum_{j=0}^{k} \|z_{j+1} - h_{j+1}(x_{j+1}) - r\|_{R^{-1}}^2\right]\right\} \tag{24}$$

$$= C |Q|^{-(k+1)/2} \cdot |R|^{-(k+1)/2}$$

$$\times \exp\left\{-\frac{1}{2}\left[\sum_{j=0}^{k} \|x_{j+1} - f_j(x_j) - q\|_{Q^{-1}}^2\right.\right.$$

$$\left.\left. + \sum_{j=0}^{k} \|z_{j+1} - h_{j+1}(x_{j+1}) - r\|_{R^{-1}}^2\right]\right\}$$

where $C = C_1 C_2 |P_0|^{-1/2} p[q, Q, r, R] \exp\{-(1/2)\|x_0 - \widehat{x}_0\|_{P_0^{-1}}^2\}$.

Taking the logarithm on both sides of (24) yields

$$\ln J = -\frac{k+1}{2} \ln |Q| - \frac{k+1}{2} \ln |R|$$

$$- \frac{1}{2}\sum_{j=0}^{k} \|x_{j+1} - f_j(x_j) - q\|_{Q^{-1}}^2 \tag{25}$$

$$- \sum_{j=0}^{k} \|z_{j+1} - h_{j+1}(x_{j+1}) - r\|_{R^{-1}}^2 + \ln C$$

Let

$$\left.\frac{\partial \ln J}{\partial q}\right|_{q=\widehat{q}_{k+1}} = 0$$

$$\left.\frac{\partial \ln J}{\partial Q}\right|_{Q=\widehat{Q}_{k+1}} = 0 \tag{26}$$

The maximum posterior estimates $\widehat{q}$ and $\widehat{Q}$ can be obtained as (see Appendices A and B for details)

$$\widehat{q}_{k+1} = \frac{1}{k+1}\sum_{j=0}^{k} \left[\left.\widehat{x}_{j+1|k+1} - f_j(x_j)\right|_{x_j \leftarrow \widehat{x}_{j|k+1}}\right]$$

$$\widehat{Q}_{k+1} = \frac{1}{k+1}\sum_{j=0}^{k} \left\{\left[\left.\widehat{x}_{j+1|k+1} - f_j(x_j)\right|_{x_j \leftarrow \widehat{x}_{j|k+1}} - q\right]\right. \tag{27}$$

$$\left. \cdot \left[\left.\widehat{x}_{j+1|k+1} - f_j(x_j)\right|_{x_j \leftarrow \widehat{x}_{j|k+1}} - q\right]^{\mathrm{T}}\right\}$$

where $f_j(x_j)|_{x_j \leftarrow \widehat{x}_{j|k+1}}$ is the posterior mean of state estimate $\widehat{x}_j$, which is propagated through nonlinear system function $f_j(\cdot)$.

Similarly, let

$$\left.\frac{\partial \ln J}{\partial r}\right|_{r=\widehat{r}_{k+1}} = 0$$

$$\left.\frac{\partial \ln J}{\partial R}\right|_{R=\widehat{R}_{k+1}} = 0 \tag{28}$$

The maximum posterior estimates $\widehat{r}$ and $\widehat{R}$ can be obtained as (see Appendices C and D for details)

$$\widehat{r}_{k+1} = \frac{1}{k+1}\sum_{j=0}^{k}\left[z_{j+1} - h_{j+1}\left(x_{j+1}\right)\big|_{x_{j+1}\leftarrow\widehat{x}_{j+1|k+1}}\right]$$

$$\widehat{R}_{k+1} = \frac{1}{k+1}$$

$$\cdot \sum_{j=0}^{k}\left\{\left[z_{j+1} - h_{j+1}\left(x_{j+1}\right)\big|_{x_{j+1}\leftarrow\widehat{x}_{j+1|k+1}} - r\right]\right.$$

$$\left.\cdot\left[z_{j+1} - h_{j+1}\left(x_{j+1}\right)\big|_{x_{j+1}\leftarrow\widehat{x}_{j+1|k+1}} - r\right]^{\mathrm{T}}\right\} \tag{29}$$

where $h_{j+1}(x_{j+1})|_{x_{j+1}\leftarrow\widehat{x}_{j+1|j}}$ is the posterior mean of one-step state prediction $\widehat{x}_{j+1|j}$ which is propagated through nonlinear measurement function $h_{j+1}(\cdot)$.

In (27) and (29), replacing the filtering estimations $\widehat{x}_{j|k+1}$ and $\widehat{x}_{j+1|k+1}$ with the estimations $\widehat{x}_{j}$ and $\widehat{x}_{j+1}$ and replacing $f_{j}(x_{j})|_{x_{j}\leftarrow\widehat{x}_{j|k+1}} + q$ with state prediction $\widehat{x}_{j+1|j}$, we have

$$\widehat{q}_{k+1} = \frac{1}{k+1}\sum_{j=0}^{k}\left\{\widehat{x}_{j+1} - \widehat{E}\left[f_{j}\left(x_{j}\right) \mid z_{k}\right]\right\}$$

$$\widehat{Q}_{k+1} = \frac{1}{k+1}\sum_{j=0}^{k}\left\{\left[\widehat{x}_{j+1} - \widehat{E}\left[f_{j}\left(x_{j}\right) \mid z_{k}\right] - q\right]\right.$$

$$\left.\cdot\left[\widehat{x}_{j+1} - \widehat{E}\left[f_{j}\left(x_{j}\right) \mid z_{k}\right] - q\right]^{\mathrm{T}}\right\} = \frac{1}{k+1}$$

$$\cdot \sum_{j=0}^{k}\left\{\left[\widehat{x}_{j+1} - \widehat{x}_{j+1|j}\right]\left[\widehat{x}_{j+1} - \widehat{x}_{j+1|j}\right]^{\mathrm{T}}\right\}$$

$$\tag{30}$$

$$\widehat{r}_{k+1} = \frac{1}{k+1}\sum_{j=0}^{k}\left\{ z_{j+1} - \widehat{E}\left[h_{j+1}\left(x_{j+1}\right) \mid z_{k}\right]\right\}$$

$$\widehat{R}_{k+1} = \frac{1}{k+1}\sum_{j=0}^{k}\left\{\left[z_{j+1} - \widehat{E}\left[h_{j+1}\left(x_{j+1}\right) \mid z_{k}\right] - r\right]\right.$$

$$\left.\cdot\left[z_{j+1} - \widehat{E}\left[h_{j+1}\left(x_{j+1}\right) \mid z_{k}\right] - r\right]^{\mathrm{T}}\right\} = \frac{1}{k+1}$$

$$\cdot \sum_{j}^{k}\left\{\left[z_{j+1} - \widehat{z}_{j+1|j}\right]\left[z_{j+1} - \widehat{z}_{j+1|j}\right]^{\mathrm{T}}\right\}$$

where $f_{k}(x_{k})|_{x_{k}\leftarrow\widehat{x}_{k}} = \widehat{E}[f_{k}(x_{k}) \mid z_{k}]$ and $h_{k+1}(x_{k+1})|_{x_{k+1}\leftarrow\widehat{x}_{k+1|k}} = \widehat{E}[h_{k+1}(x_{k+1}) \mid z_{k}]$.

Accordingly, the random weighted estimations of q_{k+1}, Q_{k+1}, r_{k+1}, and R_{k+1} can be represented as

$$\widehat{q}_{k+1}^{*} = \sum_{j=0}^{k}\lambda_{j}\left\{\widehat{x}_{j+1} - \widehat{E}\left[f_{j}\left(x_{j}\right) \mid z_{k}\right]\right\}$$

$$\widehat{Q}_{k+1}^{*} = \sum_{j=0}^{k}\lambda_{j}\left\{\left[\widehat{x}_{j+1} - \widehat{E}\left[f_{j}\left(x_{j}\right) \mid z_{k}\right] - q\right]\right.$$

$$\left.\cdot\left[\widehat{x}_{j+1} - \widehat{E}\left[f_{j}\left(x_{j}\right) \mid z_{k}\right] - q\right]^{\mathrm{T}}\right\}$$

$$= \sum_{j=0}^{k}\lambda_{j}\left\{\left[\widehat{x}_{j+1} - \widehat{x}_{j+1|j}\right]\left[\widehat{x}_{j+1} - \widehat{x}_{j+1|j}\right]^{\mathrm{T}}\right\}$$

$$\tag{31}$$

$$\widehat{r}_{k+1}^{*} = \sum_{j=0}^{k}\lambda_{j}\left\{ z_{j+1} - \widehat{E}\left[h_{j+1}\left(x_{j+1}\right) \mid z_{k}\right]\right\}$$

$$\widehat{R}_{k+1}^{*} = \sum_{j=0}^{k}\lambda_{j}\left\{\left[z_{j+1} - \widehat{E}\left[h_{j+1}\left(x_{j+1}\right) \mid z_{k}\right] - r\right]\right.$$

$$\left.\cdot\left[z_{j+1} - \widehat{E}\left[h_{j+1}\left(x_{j+1}\right) \mid z_{k}\right] - r\right]^{\mathrm{T}}\right\}$$

$$= \sum_{j=0}^{k}\lambda_{j}\left\{\left[z_{j+1} - \widehat{z}_{j+1|j}\right]\left[z_{j+1} - \widehat{z}_{j+1|j}\right]^{\mathrm{T}}\right\}$$

where λ_{j} ($j = 1, 2, \ldots, k$) are the random weighting factors.

5.2. Determination of Random Weighting Factors. Suppose the estimation and prediction of the state at epoch $k - j$ ($j = 1, 2, \ldots, n$) are $\widehat{x}_{k-j}$ and $\widehat{x}_{k-j/k-1-j}$, respectively. The residual vector of the prediction is assumed to be Δx_{k-j}, which is expressed as

$$\Delta x_{k-j} = \widehat{x}_{k-j} - \widehat{x}_{k-j/k-1-j} \quad (j = 1, 2, \ldots, n) \tag{32}$$

The residual vector of the measurement is expressed as

$$\Delta z_{k-j} = \widehat{z}_{k-j} - z_{k-j} \quad (j = 1, 2, \ldots, n) \tag{33}$$

where $\widehat{z}_{k-j} = h_{k}(\widehat{x}_{k-j})$.

When the statistics of the system process noise are changed, the contribution of the state prediction $\widehat{x}_{k-j/k-1-j}$ to the state estimate will decrease, leading the prediction to be biased. As a result, the magnitude of the residual vector of the state prediction Δx_{k-j} will increase. Similarly, when the statistics of the measurement noise are changed, the measurement residual $\Delta\widehat{z}_{k-j}$ will be biased and its magnitude will increase.

In order to capture the changes of the system noises, the random weighting factors are required to satisfy

$$\lambda_{j} \propto \|\Delta x_{k-j}\| \cdot \|\Delta z_{k-j}\| \quad (j = 1, 2, \ldots, n) \tag{34}$$

where $\|\Delta x_{k}\| = \sqrt{\Delta x_{k}^{\mathrm{T}} \cdot \Delta x_{k}}$, $\|\Delta z_{k}\| = \sqrt{\Delta z_{k}^{\mathrm{T}} \cdot \Delta z_{k}}$, and the symbol "$\propto$" indicates the proportional operation.

Equation (34) implies that the larger the value of $\|\Delta \boldsymbol{x}_k\| \cdot \|\Delta \boldsymbol{z}_k\|$ is, the larger the weighting factor is.

Therefore, the random weighting factors λ_j can be determined as follows.

Let

$$w_j = \left\| \Delta \boldsymbol{x}_{k-j} \right\| \cdot \left\| \Delta \boldsymbol{z}_{k-j} \right\| \quad (j = 1, 2, \ldots, n) \qquad (35)$$

Normalizing w_j ($j = 1, 2, \ldots, n$), the random weighting factors are obtained as

$$\lambda_j = \frac{w_j}{\sum_{j=1}^n w_j} \quad (j = 1, 2, \ldots, n) \qquad (36)$$

The random weighted estimation of noise statistics enables adaptively adjusting the weights of the cubature points to improve the accuracy of the state prediction and measurement prediction and further inhibit the interference of system process noise and measurement noise on state estimation, leading to improved filtering accuracy and reliability.

6. Adaptive Random Weighting Cubature Kalman Filter

Assume the posterior density function at epoch $k - 1$ is $p(\boldsymbol{x}_{k-1}) = N(\widehat{\boldsymbol{x}}_{k-1|k-1}, \boldsymbol{P}_{k-1|k-1})$. The ARWCKF algorithm for the nonlinear system defined by (6) can be described as follows.

(i) Initialize State Estimate $\widehat{\boldsymbol{x}}_0$ and Error Covariance $\boldsymbol{P}_0$

$$\widehat{\boldsymbol{x}}_0 = E\left[\boldsymbol{x}_0\right]$$
$$\boldsymbol{P}_0 = \text{cov}\left(\boldsymbol{x}_0, \boldsymbol{x}_0^{\mathrm{T}}\right) = E\left[\left(\boldsymbol{x}_0 - \widehat{\boldsymbol{x}}_0\right)\left(\boldsymbol{x}_0 - \widehat{\boldsymbol{x}}_0\right)^{\mathrm{T}}\right] \qquad (37)$$

(ii) Time Update and Calculation of Cubature Points. Denote the covariance of state prediction at time $k - 1$ by $\boldsymbol{P}_{k-1|k-1}$. By Cholesky decomposition [16, 17], $\boldsymbol{P}_{k-1|k-1}$ can be written as

$$\boldsymbol{P}_{k-1|k-1} = \boldsymbol{S}_{k-1|k-1} \boldsymbol{S}_{k-1|k-1}^{\mathrm{T}} \qquad (38)$$

and the cubature points can be calculated by

$$\boldsymbol{x}_{i,k-1|k-1} = \boldsymbol{S}_{k-1|k-1} \xi_i + \widehat{\boldsymbol{x}}_{k-1|k-1} \qquad (39)$$

where $\boldsymbol{x}_{i,k-1|k-1}$ ($i = 1, 2, \ldots, m$) is the system state of the ith cubature point at epoch $k-1$, and $\boldsymbol{S}_{k-1|k-1}$ is a diagonal matrix at time $k - 1$.

Then, the estimated state of the ith cubature point from epoch k to $k - 1$ can be written as

$$\boldsymbol{x}_{i,k|k-1}^* = f\left(\boldsymbol{x}_{i,k|k-1}\right) \qquad (40)$$

(iii) State Prediction. Based on (40), the predicted state of the ith cubature point from epoch $k - 1$ to k is

$$\widehat{\boldsymbol{x}}_{k|k-1} = \frac{1}{m}\sum_{i=1}^m \boldsymbol{x}_{i,k|k-1}^* + \widehat{\boldsymbol{q}}_{k-1} \qquad (41)$$

Accordingly, the random weighted estimation of $\widehat{\boldsymbol{x}}_{k|k-1}$ can be written as

$$\widehat{\boldsymbol{x}}_{k|k-1}^* = \sum_{i=1}^m \lambda_i \boldsymbol{x}_{i,k|k-1}^* + \widehat{\boldsymbol{q}}_{k-1}^* \qquad (42)$$

The covariance of the state prediction $\boldsymbol{P}_{k|k-1}$ is described by

$$\boldsymbol{P}_{k|k-1} = \frac{1}{m}\sum_{i=1}^m \boldsymbol{x}_{i,k|k-1}^* \boldsymbol{x}_{i,k|k-1}^{*T} - \widehat{\boldsymbol{x}}_{k|k-1}\widehat{\boldsymbol{x}}_{k|k-1}^T + \widehat{\boldsymbol{Q}}_{k-1} \qquad (43)$$

Accordingly, the random weighted estimation of $\boldsymbol{P}_{k|k-1}$ can be written as

$$\boldsymbol{P}_{k|k-1}^* = \sum_{i=1}^m \lambda_i \boldsymbol{x}_{i,k|k-1}^* \boldsymbol{x}_{i,k|k-1}^{*\mathrm{T}} - \widehat{\boldsymbol{x}}_{k|k-1}\widehat{\boldsymbol{x}}_{k|k-1}^{\mathrm{T}} + \widehat{\boldsymbol{Q}}_{k-1}^* \qquad (44)$$

(iv) Measurement Prediction. The estimated measurement of the ith cubature point from epoch $k - 1$ to k can be written as

$$\boldsymbol{z}_{i,k|k-1}^* = h\left(\boldsymbol{x}_{i,k|k-1}\right) \qquad (45)$$

The predicted measurement of the ith cubature point from epoch $k - 1$ to k can be expressed as

$$\widehat{\boldsymbol{z}}_{k|k-1} = \frac{1}{m}\sum_{i=1}^m \boldsymbol{z}_{i,k|k-1}^* + \widehat{\boldsymbol{r}}_k \qquad (46)$$

Accordingly, the random weighted estimation of $\widehat{\boldsymbol{z}}_{k|k-1}$ can be written as

$$\widehat{\boldsymbol{z}}_{k|k-1}^* = \sum_{i=1}^m \lambda_i \boldsymbol{z}_{i,k|k-1}^* + \widehat{\boldsymbol{r}}_k^* \qquad (47)$$

The autocovariance of the measurement prediction $\boldsymbol{P}_{zz,k|k-1}$ is described by

$$\boldsymbol{P}_{zz,k|k-1} = \frac{1}{m}\sum_{i=1}^m \boldsymbol{z}_{i,k|k-1}^* \boldsymbol{z}_{i,k|k-1}^{*\mathrm{T}} - \widehat{\boldsymbol{z}}_{k|k-1}\widehat{\boldsymbol{z}}_{k|k-1}^{\mathrm{T}} + \widehat{\boldsymbol{R}}_k \qquad (48)$$

Accordingly, the random weighting estimation for the autocovariance of the measurement prediction is

$$\boldsymbol{P}_{zz,k|k-1}^* = \sum_{i=1}^m \lambda_i \left(\boldsymbol{z}_{i,k|k-1}^* \boldsymbol{z}_{i,k|k-1}^{*\mathrm{T}}\right) - \widehat{\boldsymbol{z}}_{k|k-1}\widehat{\boldsymbol{z}}_{k|k-1}^{\mathrm{T}} + \widehat{\boldsymbol{R}}_k^* \qquad (49)$$

The cross-covariance of the measurement prediction $\boldsymbol{P}_{xz,k|k-1}$ is

$$\boldsymbol{P}_{xz,k|k-1} = \frac{1}{m}\sum_{i=1}^m \boldsymbol{x}_{i,k|k-1}^* \boldsymbol{z}_{i,k|k-1}^{*\mathrm{T}} - \widehat{\boldsymbol{x}}_{k|k-1}\widehat{\boldsymbol{z}}_{k|k-1}^{\mathrm{T}} \qquad (50)$$

Accordingly, the random weighting estimation for the cross-covariance of the measurement prediction is

$$\boldsymbol{P}_{xz,k|k-1}^* = \sum_{i=1}^m \lambda_i \left(\boldsymbol{x}_{i,k|k-1}^* \boldsymbol{z}_{i,k|k-1}^{*\mathrm{T}}\right) - \widehat{\boldsymbol{x}}_{k|k-1}\widehat{\boldsymbol{z}}_{k|k-1}^{\mathrm{T}} \qquad (51)$$

(v) State Update. The filter gain at epoch k is

$$K_k = P^*_{xz,k|k-1} \left(P^*_{zz,k|k-1} \right)^{-1} \tag{52}$$

The state estimation at epoch k is

$$\widehat{x}_{k|k} = \widehat{x}^*_{k|k-1} + K_k \left(z_k - \widehat{z}^*_{k|k-1} \right) \tag{53}$$

The covariance estimation of the state error at epoch k is

$$P^*_{k|k} = P^*_{k|k-1} - K_k P^*_{zz,k|k} K_k^{\mathrm{T}} \tag{54}$$

From the above, it can be seen that the proposed ARWCKF incorporates random weighting factors in CKF, enabling us to adaptively refrain from the disturbances of system noises by online adjusting random weights to improve the accuracy of state estimation. The ARWCKF algorithm is shown in Figure 1.

$$X(t) = \begin{bmatrix} \delta v_x & \delta v_y & \delta v_z & \delta P_x & \delta P_y & \delta P_z & \psi_x & \psi_y & \psi_z & \theta_x & \theta_y & \theta_z & \zeta_x & \zeta_y & \zeta_z \end{bmatrix}^{\mathrm{T}} \tag{55}$$

where $(\delta v_x, \delta v_y, \delta v_z)$, $(\delta P_x, \delta P_y, \delta P_z)$, and (ψ_x, ψ_y, ψ_z) are the errors in velocity, position, and attitude from SINS; $(\theta_x, \theta_y, \theta_z)$ is the constant bias of the gyros; and $(\zeta_x, \zeta_y, \zeta_z)$ is the zero bias of the accelerometers.

The system state equation of the SINS/SRS integration is described by

$$\dot{X}(t) = f(X(t)) + G(t)W(t) \tag{56}$$

where $f(\cdot)$ is the nonlinear function given by (57), $X(t)$ is the system state vector, and $W(t)$ is the system noise.

$$f(X(t))$$

$$= \begin{bmatrix} C_\omega^{-1} \left[(I - C_n^c) \widehat{\omega}_{in}^n + C_n^c \delta\omega_{in}^n - C_b^c \delta\omega_{ib}^b \right] \\ \left[I - (C_n^c)^{\mathrm{T}} \right] C_b^c \widehat{f}_{sf}^b + (C_n^c)^{\mathrm{T}} C_b^c \delta f_{sf}^b \\ - \left(2\delta\omega_{ie}^n + \delta\omega_{en}^n \right) \times V - \left(2\widehat{\omega}_{ie}^n + \widehat{\omega}_{en}^n \right) \times \delta V \\ + \left(2\omega_{ie}^n + \omega_{en}^n \right) \times \delta V + \delta g \\ \dfrac{v_y}{R_M + h} - \dfrac{\left(v_y - \delta v_y \right)}{\left(R_M - \delta R_M \right) + \left(h - \delta h \right)} \\ \dfrac{v_x \sec\varphi}{R_N + h} - \dfrac{\left(v_x - \delta v_x \right) \sec\left(\varphi - \delta\varphi \right)}{\left(R_N - \delta R_N \right) + \left(h - \delta h \right)} \\ \delta v_z \\ \mathbf{0}_{1\times7} \end{bmatrix} \tag{57}$$

where C_ω is the Euler platform error angle matrix; C_n^c is the transformation matrix from the navigation (n) to computer (c) frames in terms of attitude; C_b^c is the transformation matrix from the body (b) to computer (c) frames in terms of attitude; $\widehat{\omega}_{ie}^n$ and ω_{ie}^n are the projections of the actual and ideal angular velocities from the Earth coordinate system (e) to the inertial coordinate system (i) into the navigation frame (n); $\delta\omega_{in}^n$ and $\delta\omega_{ib}^b$ are the calculation errors of ω_{in}^n and ω_{ib}^b; $\widehat{f}_{sf}^b$ and δf_{sf}^b are the real force and its associated error of the accelerometer; V and δV are the real velocity and its associated error; δg is the error of gravity acceleration; φ and h are the latitude and altitude values; and R_M and R_N are the radii of curvatures of local meridian and prime vertical.

The noise coefficient matrix is defined as

$$G(t) = \begin{bmatrix} C_{b(3\times3)}^c & \mathbf{0}_{(3\times3)} \\ \mathbf{0}_{(3\times3)} & C_{b(3\times3)}^c \\ \mathbf{0}_{(9\times3)} & \mathbf{0}_{(9\times3)} \end{bmatrix}_{(15\times6)} \tag{58}$$

The system noise is described by $W(t) = [w_{\varepsilon_x} \ w_{\varepsilon_y} \ w_{\varepsilon_z} \ w_{\nabla_x} \ w_{\nabla_y} \ w_{\nabla_z}]^{\mathrm{T}}_{(6\times1)}$, where $(w_{\varepsilon_x}, w_{\varepsilon_y}, w_{\varepsilon_z})$ and $(w_{\nabla_x}, w_{\nabla_y}, w_{\nabla_z})$ are the gyro and accelerometer random noise, respectively.

7. Performance Evaluation and Discussion

Simulation trials were conducted to evaluate the effectiveness of the proposed ARWCKF for a SINS/SRS (strap-down inertial navigation system/spectrum red-shift) integrated navigation system. The comparison analysis with CKF was also conducted to demonstrate the improved performance of the proposed ARWCKF for the SINS/SRS integrated navigation system.

7.1. System State Model. Due to the high accuracy in the velocity of SRS, the velocity error of SRS is much smaller than that of SINS. Therefore, the velocity error of SRS is assumed as a Gaussian white noise to reduce the dimension of the SINS/SRS integrated navigation system. Accordingly, the state vector of the SINS/SRS integrated navigation system consists of SINS errors only.

The SINS/SRS integrated navigation system adopts the inertial coordinate frame as the navigation frame. Its system state vector is defined as

7.2. Measurement Model. In order to overcome the drawback that the velocity error of SINS is accumulated in time series, the velocity information from SRS is used to correct the velocity error of SINS. Further, a radar altimeter is used to provide the altitude information to correct the altitude channel of SINS.

The measurement of the SINS/SRS integrated navigation system can be chosen as the difference in velocity between SINS and SRS as well as the difference in altitude between radar altimeter and SINS.
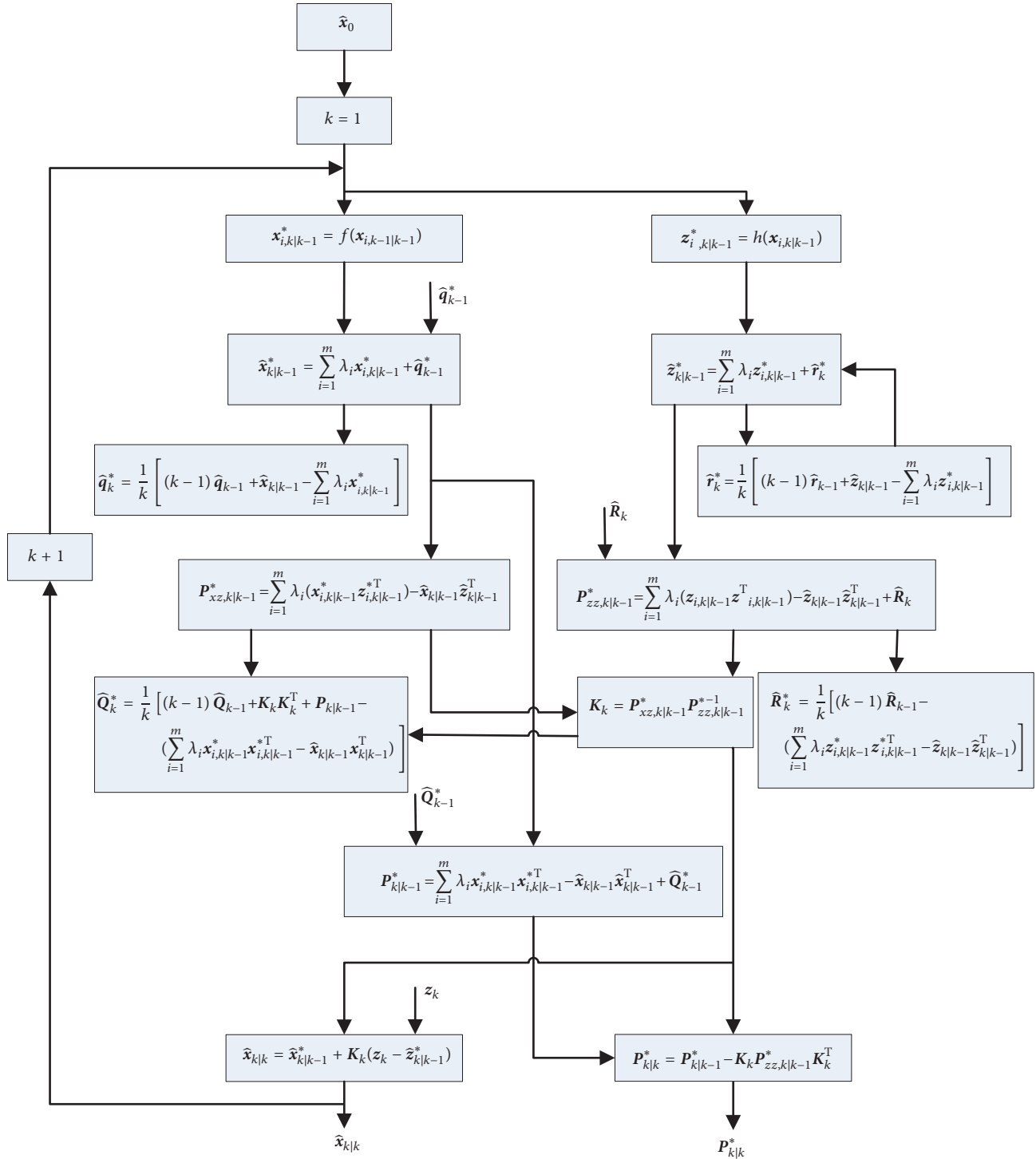
$$\widehat{\boldsymbol{x}}_0$$

$$k = 1$$

$$\boldsymbol{x}^*_{i,k|k-1} = f(\boldsymbol{x}_{i,k-1|k-1})$$

$$\boldsymbol{z}^*_{i,k|k-1} = h(\boldsymbol{x}_{i,k|k-1})$$

$$\widehat{\boldsymbol{q}}^*_{k-1}$$

$$\widehat{\boldsymbol{x}}^*_{k|k-1} = \sum_{i=1}^{m} \lambda_i \boldsymbol{x}^*_{i,k|k-1} + \widehat{\boldsymbol{q}}^*_{k-1}$$

$$\widehat{\boldsymbol{z}}^*_{k|k-1} = \sum_{i=1}^{m} \lambda_i \boldsymbol{z}^*_{i,k|k-1} + \widehat{\boldsymbol{r}}^*_k$$

$$\widehat{\boldsymbol{q}}^*_k = \frac{1}{k}\left[(k-1)\widehat{\boldsymbol{q}}_{k-1} + \widehat{\boldsymbol{x}}_{k|k-1} - \sum_{i=1}^{m} \lambda_i \boldsymbol{x}^*_{i,k|k-1}\right]$$

$$\widehat{\boldsymbol{r}}^*_k = \frac{1}{k}\left[(k-1)\widehat{\boldsymbol{r}}_{k-1} + \widehat{\boldsymbol{z}}_{k|k-1} - \sum_{i=1}^{m} \lambda_i \boldsymbol{z}^*_{i,k|k-1}\right]$$

$$\widehat{\boldsymbol{R}}_k$$

$$k + 1$$

$$\boldsymbol{P}^*_{xz,k|k-1} = \sum_{i=1}^{m} \lambda_i (\boldsymbol{x}^*_{i,k|k-1} \boldsymbol{z}^{*\mathrm{T}}_{i,k|k-1}) - \widehat{\boldsymbol{x}}_{k|k-1}\widehat{\boldsymbol{z}}^{\mathrm{T}}_{k|k-1}$$

$$\boldsymbol{P}^*_{zz,k|k-1} = \sum_{i=1}^{m} \lambda_i (\boldsymbol{z}_{i,k|k-1} \boldsymbol{z}^{\mathrm{T}}_{i,k|k-1}) - \widehat{\boldsymbol{z}}_{k|k-1}\widehat{\boldsymbol{z}}^{\mathrm{T}}_{k|k-1} + \widehat{\boldsymbol{R}}_k$$

$$\widehat{\boldsymbol{Q}}^*_k = \frac{1}{k}\left[(k-1)\widehat{\boldsymbol{Q}}_{k-1} + \boldsymbol{K}_k \boldsymbol{K}^{\mathrm{T}}_k + \boldsymbol{P}_{k|k-1} - \left(\sum_{i=1}^{m}\lambda_i \boldsymbol{x}^*_{i,k|k-1}\boldsymbol{x}^{*\mathrm{T}}_{i,k|k-1} - \widehat{\boldsymbol{x}}_{k|k-1}\boldsymbol{x}^{\mathrm{T}}_{k|k-1}\right)\right]$$

$$\boldsymbol{K}_k = \boldsymbol{P}^*_{xz,k|k-1} \boldsymbol{P}^{*-1}_{zz,k|k-1}$$

$$\widehat{\boldsymbol{R}}^*_k = \frac{1}{k}\left[(k-1)\widehat{\boldsymbol{R}}_{k-1} - \left(\sum_{i=1}^{m}\lambda_i \boldsymbol{z}^*_{i,k|k-1}\boldsymbol{z}^{*\mathrm{T}}_{i,k|k-1} - \widehat{\boldsymbol{z}}_{k|k-1}\widehat{\boldsymbol{z}}^{\mathrm{T}}_{k|k-1}\right)\right]$$

$$\widehat{\boldsymbol{Q}}^*_{k-1}$$

$$\boldsymbol{P}^*_{k|k-1} = \sum_{i=1}^{m} \lambda_i \boldsymbol{x}^*_{i,k|k-1}\boldsymbol{x}^{*\mathrm{T}}_{i,k|k-1} - \widehat{\boldsymbol{x}}_{k|k-1}\widehat{\boldsymbol{x}}^{\mathrm{T}}_{k|k-1} + \widehat{\boldsymbol{Q}}^*_{k-1}$$

$$\boldsymbol{z}_k$$

$$\widehat{\boldsymbol{x}}_{k|k} = \widehat{\boldsymbol{x}}^*_{k|k-1} + \boldsymbol{K}_k(\boldsymbol{z}_k - \widehat{\boldsymbol{z}}^*_{k|k-1})$$

$$\boldsymbol{P}^*_{k|k} = \boldsymbol{P}^*_{k|k-1} - \boldsymbol{K}_k \boldsymbol{P}^*_{zz,k|k-1} \boldsymbol{K}^{\mathrm{T}}_k$$

$$\widehat{\boldsymbol{x}}_{k|k}$$

$$\boldsymbol{P}^*_{k|k}$$

FIGURE 1: ARWCKF algorithm.

Suppose the velocities obtained by SRS and SINS are $\boldsymbol{V}_{SRS} = (v_{Sx}, v_{Sy}, v_{Sz})$ and $\boldsymbol{V}_{SINS} = (v_x, v_y, v_z)$. The difference in velocity between SRS and SINS is defined as

$$\Delta \boldsymbol{Z}_v = \begin{bmatrix} \Delta Z_{S1} \\ \Delta Z_{S2} \\ \Delta Z_{S3} \end{bmatrix} = \begin{bmatrix} v_x - v_{Sx} \\ v_y - v_{Sy} \\ v_z - v_{Sz} \end{bmatrix} = \boldsymbol{H}_v \boldsymbol{X}(t) + \boldsymbol{V}_v(t) \quad (59)$$

where $\boldsymbol{V}_v$ is the velocity measurement noise matrix, and $\boldsymbol{H}_v = [\boldsymbol{I}_{3\times3} \quad \boldsymbol{0}_{3\times12}]^{\mathrm{T}}$ is the measurement matrix.

Further, the difference in altitude between the radar altimeter and SINS can be described as

$$\Delta \boldsymbol{Z}_h = [h_{SINS} - h_H] = \boldsymbol{H}_h \boldsymbol{X}(t) + \boldsymbol{V}_h(t) \quad (60)$$

TABLE 1: The orbit parameters of the spacecraft.

Orbit Parameters	Value
Semi-major axis	6865.112418km
Eccentricity	0.001248
Orbit inclination	29.059°
Right ascension of ascending node	338.82°
Argument of perigee	351.167°
True anomaly	229.46°
Orbit cycle	50076s
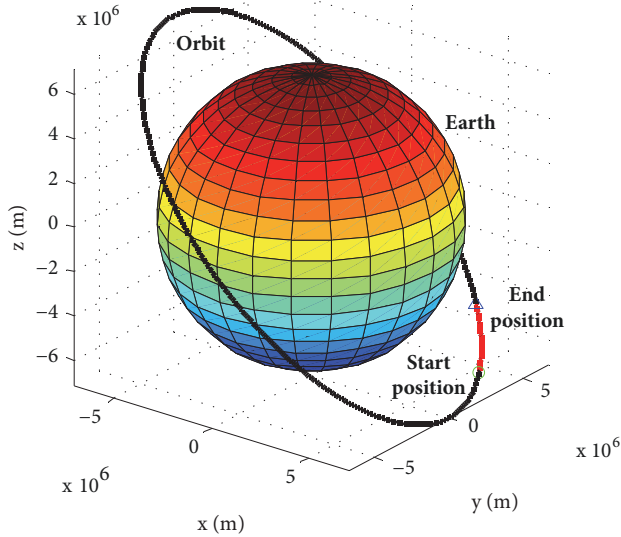
FIGURE 2: Flight trajectory of the spacecraft.

FIGURE 3: Position and velocity errors by CKF and ARWCKF.

where h_{SINS} and h_H are the altitudes obtained by SINS and the radar altimeter; $V_h(t)$ is the measurement noise matrix in altitude; and $H_h = [\mathbf{0}_{1\times5} \quad 1 \quad \mathbf{0}_{1\times9}]^T$ is the measurement matrix.

Combining (59) with (60), the measurement equation of the SINS/SRS integration subsystem is established as

$$\Delta Z_1(t) = \begin{bmatrix} H_v \\ H_h \end{bmatrix} X(t) + \begin{bmatrix} V_v(t) \\ V_h(t) \end{bmatrix}$$
$$= H_1(t) X(t) + V_1(t) \tag{61}$$

7.3. Simulation and Analysis. Assume the spacecraft orbits the earth, where the orbit parameters are described in Table 1. A flight period of 4000s was selected for the simulation test, where the initial position was "6359800m, 3076100m, -6268900m," and the end position was "4550800m, 5222600m, -4356700." The flight trajectory is shown in Figure 2.

In the simulation, the SINS gyro drift is 0.05(°)/h, the random walk of the gyro is 0.005(°)/√h, and the zero bias and random drift of the accelerator are $1\times10^{-3}g$ and $1\times10^{-4}g/\sqrt{s}$. The SINS initial alignment error is $0'$. The initial position and velocity errors of the spacecraft are 10m, 10m, and 10m and
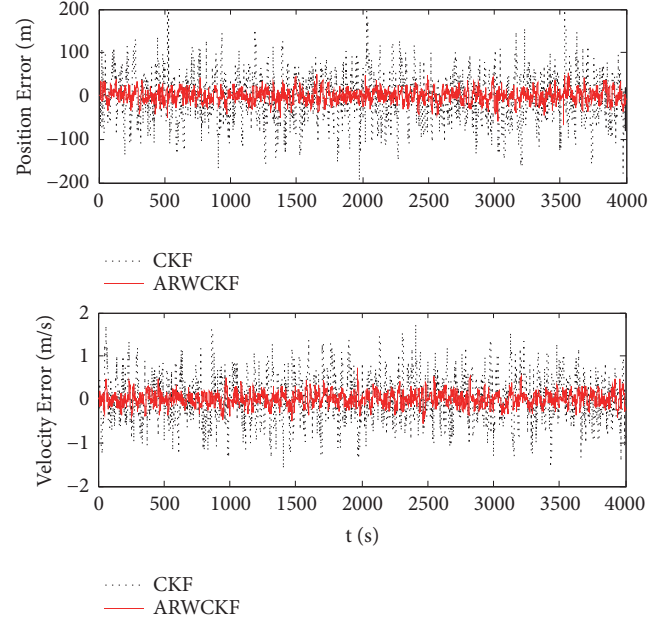
0.2m/s, 0.2m/s, and 0.2m/s. The sampling period of SINS is 0.01s, the data measurement frequency of SRS is 1Hz, and the velocity error of SRS is 0.05m/s. The noises of the system state model and measurement model are set as 0.01 and 0.02, respectively. The unscented transformation parameters are $\alpha = 0.5$ and $\beta = 2$.

The position error is defined as

$$\delta P = \sqrt{(\delta P_x)^2 + (\delta P_y)^2 + (\delta P_z)^2} \tag{62}$$

The velocity error is defined as

$$\delta V = \sqrt{(\delta v_x)^2 + (\delta v_y)^2 + (\delta v_z)^2} \tag{63}$$

For comparison analysis, 100 Monte Carlo simulation trials were conducted under the same conditions by CKF and ARWCKF, respectively.

Figure 3 shows the estimation errors in position and velocity by both CKF and ARWCKF. It can be seen that the position and velocity errors are within "-165m, 145m" and "-1.72m/s, 1.68m/s." This poor estimation accuracy is because CKF does not have the capability to inhibit the disturbance of system noises. In contrast, as shown in Figure 4, ARWCKF can effectively estimate the noise statistics of the system state and measurement models to inhibit the disturbances of system noises, leading to the significantly improved accuracy. As shown in Figure 3, the resultant position and velocity errors by ARWCKF are within "-30m, 32m" and "-0.31m/s, 0.35m/s." Table 2 summaries the means of the root mean square errors (RMSEs) in terms of position and velocity by both methods as well as the RMSEs of the estimated noises by the proposed ARWCKF for the SINS/SRS integrated navigation system.

TABLE 2: RMSEs of CKF and ARWCKF.

Filtering algorithms	Position RMSEs (m)	Velocity RMSEs (m/s)	Estimation RMSEs	
			q	r
CKF	39.629	0.357	-	-
ARWCKF	11.289	0.068	3.811×10^{-4}	3.782×10^{-4}
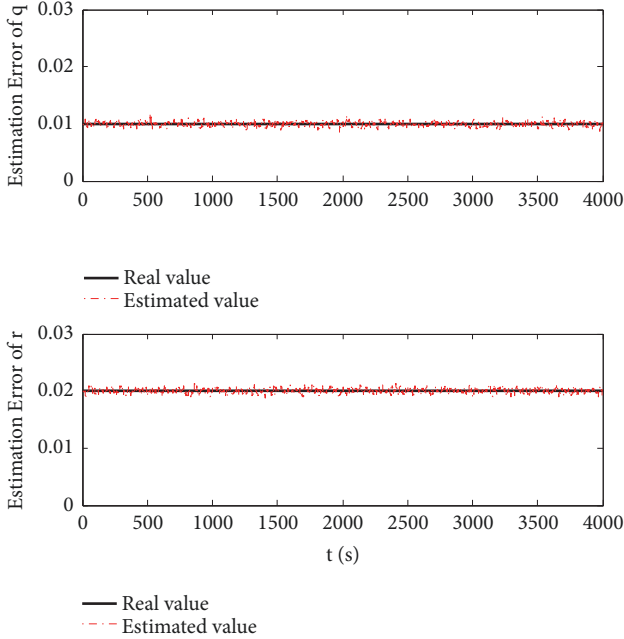
FIGURE 4: The estimations of system noise statistics by ARWCKF.

TABLE 3: ANEESs of the CKF and ARWCKF algorithms.

Filtering algorithm	Position ANEESs	Velocity ANEESs
CKF	0.5~22.3	0.4~20.6
ARWCKF	0.08~1.12	0.07~1.08

FIGURE 5: ANEESs of CKF and ARWCKF.

The performance evaluation was further analyzed in terms of average normalized estimation error squared (ANEES). Figure 5 shows the ANEES values in terms of position and velocity errors for both CKF and ARWCKF. It can be seen that there are significant oscillations involved in the ANEES curves of CKF. This means that a considerable portion of the error covariance matrices obtained by CKF are unreliable due to the existence of system noises. However, the ANEES values of ARWCKF are small during the entire simulation test, showing the estimation consistency of ARWCKF is not affected by system noises. Table 3 summaries the ANEES values of both CKF and ARWCKF. The results in terms of ANEES demonstrate the covariance matrices obtained by ARWCKF are much more reliable than those of CKF.

The above results demonstrate that the proposed ARWCKF can effectively inhibit the disturbances of system noises for spacecraft navigation, leading to the improved accuracy and robustness in comparison with CKF.

8. Conclusions

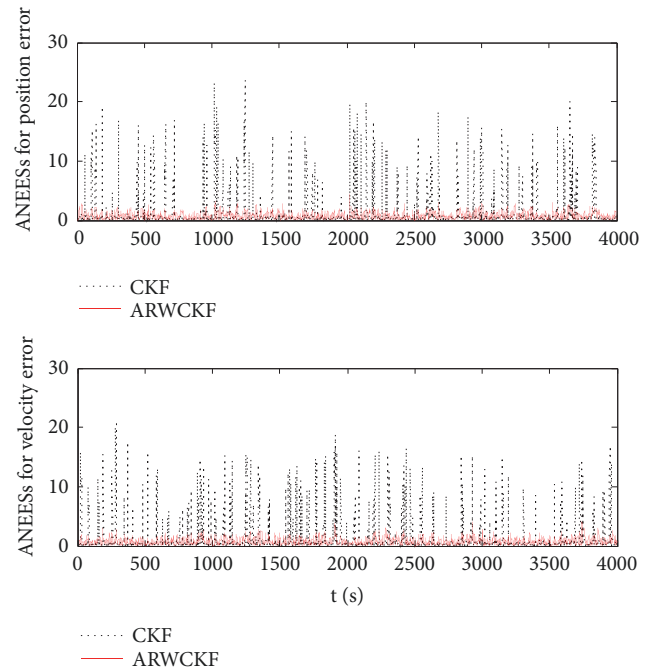This paper presents a new ARWCKF for state estimation in nonlinear systems. It adopts the concept of random weighting for the first time to address the problem that the CKF performance is sensitive to system noise. The proposed method incorporates random weights in CKF to estimate system noise statistics as well as predicted state and measurement together with their associated covariances. Based on this, it adaptively adjusts random weights of cubature points to inhibit the disturbances of system noises on state estimation, leading to improved filtering accuracy and robustness. Simulations and comparison analysis demonstrate that the proposed ARWCKF is superior to CKF for spacecraft navigation.

Future work will focus on improvement of the proposed ARWCKF. By combining the proposed ARWCKF with advanced artificial intelligence techniques such as deep learning neural networks, swarming algorithms, and fuzzy logic, it is expected to develop an intelligent filtering algorithm for automatic estimation of system noise statistics from various error sources in practical engineering applications.

Appendix

A. Proof of $\widehat{q}_{k+1}$ in (27)

From (24), we readily have

$$
\begin{aligned}
\ln J = &-\frac{k+1}{2}\ln|\mathbf{Q}| - \frac{k+1}{2}\ln|\mathbf{R}| \\
&- \frac{1}{2}\sum_{j=0}^{k}\left\|\mathbf{x}_{j+1} - f_j(\mathbf{x}_j) - \mathbf{q}\right\|_{\mathbf{Q}^{-1}}^{2} \\
&- \sum_{j=0}^{k}\left\|\mathbf{z}_{j+1} - h_{j+1}(\mathbf{x}_{j+1}) - \mathbf{r}\right\|_{\mathbf{R}^{-1}}^{2} + \ln C
\end{aligned}
\tag{A.1}
$$

Taking the partial derivatives of $\mathbf{q}$ on both sides of (A.1) yields

$$
\begin{aligned}
\frac{\partial \ln J}{\partial \mathbf{q}} &= -0-0 \\
&\quad + \frac{1}{2}\sum_{j=0}^{k}\left[\mathbf{Q}^{-1} + \left(\mathbf{Q}^{-1}\right)^{\mathrm{T}}\right]\left(\mathbf{x}_{j+1} - f_j(\mathbf{x}_j) - \mathbf{q}\right) - 0 \\
&\quad + 0
\end{aligned}
\tag{A.2}
$$

Let

$$
\begin{aligned}
&\frac{\partial \ln J}{\partial \mathbf{q}} \\
&= \frac{1}{2}\left[\mathbf{Q}^{-1} + \left(\mathbf{Q}^{-1}\right)^{\mathrm{T}}\right]\sum_{j=0}^{k}\left(\mathbf{x}_{j+1} - f_j(\mathbf{x}_j) - \mathbf{q}\right)\Big|_{\mathbf{q}=\widehat{\mathbf{q}}_{k+1}} \\
&= 0
\end{aligned}
\tag{A.3}
$$

Since $(1/2)[\mathbf{Q}^{-1} + (\mathbf{Q}^{-1})^{\mathrm{T}}] \neq 0$, we have

$$
\sum_{j=0}^{k}\left(\mathbf{x}_{j+1} - f_j(\mathbf{x}_j) - \mathbf{q}\right)\Big|_{\mathbf{q}=\widehat{\mathbf{q}}_{k+1}} = 0
\tag{A.4}
$$

i.e.,

$$
\sum_{j=0}^{k}\left[\mathbf{x}_{j+1} - f_j(\mathbf{x}_j)\right] - (k+1)\mathbf{q}\Big|_{\mathbf{q}=\widehat{\mathbf{q}}_{k+1}} = 0
\tag{A.5}
$$

Thus,

$$
\widehat{\mathbf{q}}_{k+1} = \frac{1}{k+1}\sum_{j=0}^{k}\left[\widehat{\mathbf{x}}_{j+1|k+1} - f_j(\mathbf{x}_j)\big|_{\mathbf{x}_j \leftarrow \widehat{\mathbf{x}}_{j|k+1}}\right]
\tag{A.6}
$$

The proof of $\widehat{\mathbf{q}}_{k+1}$ is completed.

B. Proof of $\widehat{\mathbf{Q}}_{k+1}$ in (27)

Suppose we have the following conditions:

① $\mathbf{Q}$ and $\mathbf{R}$ are symmetric matrices; i.e., $\mathbf{Q}^{\mathrm{T}} = \mathbf{Q}$ and $\mathbf{R}^{\mathrm{T}} = \mathbf{R}$

② $\partial \ln|\mathbf{A}|/\partial \mathbf{A} = 1/|\mathbf{A}| \cdot \mathbf{A}^{*} = \mathbf{A}^{-1}(\partial|\mathbf{A}|/\partial \mathbf{A} = \mathbf{A}^{*})$

③ $\partial \mathbf{A}^{-1}/\partial \mathbf{A} = -\mathbf{A}^{-T} \times \mathbf{A}^{-1}$

where $|\mathbf{A}|$ is the determinant of $\mathbf{A}$.

Taking the partial derivatives of $\mathbf{Q}$ on both sides of (A.1) yields

$$
\begin{aligned}
&\frac{\partial \ln J}{\partial \mathbf{Q}}\bigg|_{\mathbf{Q}=\widehat{\mathbf{Q}}_{k+1}}^{\mathbf{x}_j=\widehat{\mathbf{x}}_{j|j+1},\,\mathbf{x}_{j+1}=\widehat{\mathbf{x}}_{j+1|j+1}} = \left[-\frac{k+1}{2}\cdot\frac{\partial \ln|\mathbf{Q}|}{\partial \mathbf{Q}} - \frac{1}{2}\right. \\
&\left.\cdot\sum_{j=0}^{k}\left\|\mathbf{x}_{i+1} - f_j(\mathbf{x}_j) - \mathbf{q}\right\|^2 \frac{\mathbf{Q}^{-1}}{\partial \mathbf{Q}}\right]\bigg|_{\mathbf{Q}=\widehat{\mathbf{Q}}_{k+1}}^{\mathbf{x}_j=\widehat{\mathbf{x}}_{j|j+1},\,\mathbf{x}_{j+1}=\widehat{\mathbf{x}}_{j+1|j+1}} \\
&= \left\{-\frac{k+1}{2}\mathbf{A}^{-1} - \frac{1}{2}\sum_{j=0}^{k}\left[\mathbf{x}_{i+1} - f_j(\mathbf{x}_j) - \mathbf{q}\right]\right. \\
&\quad\times\left[\mathbf{x}_{i+1} - f_j(\mathbf{x}_j) - \mathbf{q}\right]^{\mathrm{T}} \times \left(-\mathbf{A}^{-T}\right. \\
&\left.\left.\times \mathbf{A}^{-1}\right)\right\}_{\mathbf{Q}=\widehat{\mathbf{Q}}_{k+1}}^{\mathbf{x}_j=\widehat{\mathbf{x}}_{j|j+1},\,\mathbf{x}_{j+1}=\widehat{\mathbf{x}}_{j+1|j+1}} = 0
\end{aligned}
\tag{B.1}
$$

Equation (B.1) can be rewritten as

$$
\begin{aligned}
\frac{k+1}{2}\widehat{\mathbf{Q}}_{k+1}^{-1} = &\frac{1}{2}\sum_{j=0}^{k}\left[\mathbf{x}_{i+1} - f_j(\mathbf{x}_j)\big|_{\mathbf{x}_j \leftarrow \widehat{\mathbf{x}}_{j|k+1}} - \mathbf{q}\right] \\
&\cdot\left[\mathbf{x}_{i+1} - f_j(\mathbf{x}_j)\big|_{\mathbf{x}_j \leftarrow \widehat{\mathbf{x}}_{j|k+1}} - \mathbf{q}\right]^{\mathrm{T}} \\
&\cdot\left(\widehat{\mathbf{Q}}_{k+1}^{-T}\cdot\widehat{\mathbf{Q}}_{k+1}^{-1}\right)
\end{aligned}
\tag{B.2}
$$

Multiplying both sides of (B.2) by $\widehat{\mathbf{Q}}_{k+1}$ yields

$$
\begin{aligned}
\frac{k+1}{2}\mathbf{I} = &\frac{1}{2}\sum_{j=0}^{k}\left[\mathbf{x}_{i+1} - f_j(\mathbf{x}_j)\big|_{\mathbf{x}_j \leftarrow \widehat{\mathbf{x}}_{j|k+1}} - \mathbf{q}\right] \\
&\cdot\left[\mathbf{x}_{i+1} - f_j(\mathbf{x}_j)\big|_{\mathbf{x}_j \leftarrow \widehat{\mathbf{x}}_{j|k+1}} - \mathbf{q}\right]^{\mathrm{T}}\cdot\widehat{\mathbf{Q}}_{k+1}^{-T}
\end{aligned}
\tag{B.3}
$$

Substituting $\widehat{\mathbf{Q}}_{k+1}^{-T} = \widehat{\mathbf{Q}}_{k+1}^{-1}$ into (B.3), we have

$$
\begin{aligned}
\widehat{\mathbf{Q}}_{k+1} = \frac{1}{k+1}\sum_{j=0}^{k}&\left\{\left[\widehat{\mathbf{x}}_{j+1|k+1} - f_j(\mathbf{x}_j)\big|_{\mathbf{x}_j \leftarrow \widehat{\mathbf{x}}_{j|k+1}} - \mathbf{q}\right]\right. \\
&\left.\cdot\left[\widehat{\mathbf{x}}_{j+1|k+1} - f_j(\mathbf{x}_j)\big|_{\mathbf{x}_j \leftarrow \widehat{\mathbf{x}}_{j|k+1}} - \mathbf{q}\right]^{\mathrm{T}}\right\}
\end{aligned}
\tag{B.4}
$$

The proof of $\widehat{\mathbf{Q}}_{k+1}$ is completed.

C. Proof of $\widehat{r}_{k+1}$ in (29)

Taking the partial derivatives of r on both sides of (A.1) yields

$$\frac{\partial \ln g}{\partial r} = \text{-0-0} + \sum_{j=0}^{k} \left[R^{-1} + \left(R^{-1} \right)^{\mathrm{T}} \right]$$
$$\cdot \left(z_{j+1} - h_{j+1} \left(x_{j+1} \right) - r \right)\big|_{r=\widehat{r}_{k+1}} \text{-0} + 0 \tag{C.1}$$

Let

$$\frac{\partial \ln g}{\partial r} = \left[R^{-1} + \left(R^{-1} \right)^{\mathrm{T}} \right]$$
$$\cdot \sum_{j=0}^{k} \left(z_{j+1} - h_{j+1} \left(x_{j+1} \right) - r \right)\big|_{r=\widehat{r}_{k+1}} = 0 \tag{C.2}$$

Since $R^{-1} + \left(R^{-1} \right)^{\mathrm{T}} \neq 0$, we have

$$\sum_{j=0}^{k} \left(z_{j+1} - h_{j+1} \left(x_{j+1} \right) - r \right)\Bigg|_{\substack{x_{j+1} \leftarrow \widehat{x}_{j+1|k+1} \\ r=\widehat{r}_{k+1}}} = 0 \tag{C.3}$$

i.e.,

$$\sum_{j=0}^{k} \left[z_{j+1} - h_{j+1} \left(x_{j+1} \right)\big|_{x_{j+1} \leftarrow \widehat{x}_{j+1|k+1}} \right] - (k+1)\, r|_{r=\widehat{r}_{k+1}} \tag{C.4}$$
$$= 0$$

Thus,

$$\widehat{r}_{k+1} = \frac{1}{k+1} \sum_{j=0}^{k} \left[\widehat{z}_{j+1} - h_{j+1} \left(x_{j+1} \right)\big|_{x_{j+1} \leftarrow \widehat{x}_{j+1|k+1}} \right] \tag{C.5}$$

The proof of $\widehat{r}_{k+1}$ is completed.

D. Proof of $\widehat{R}_{k+1}$ in (29)

Assume we have the same conditions as in Appendix B. Taking the partial derivatives of R on both sides of (A.1) yields

$$\frac{\partial \ln g}{\partial R}\Bigg|_{\substack{z_{j+1}=\widehat{z}_{j+1|j+1},\, x_{j+1}=\widehat{x}_{j+1|j+1} \\ R=\widehat{R}_{k+1}}} = \left[-\frac{k+1}{2} \cdot \frac{\partial \ln |R|}{\partial R} \right.$$
$$- \sum_{j=0}^{k} \left\| z_{j+1} - h_{j+1} \left(x_{j+1} \right) - r \right\|^{2}$$
$$\left. \cdot \frac{R^{-1}}{\partial R} \right]\Bigg|_{\substack{z_{j+1}=\widehat{z}_{j+1|j+1},\, x_{j+1}=\widehat{x}_{j+1|j+1} \\ R=\widehat{R}_{k+1}}} = \left\{ -\frac{k+1}{2} A^{-1} - \frac{1}{2} \right. \tag{D.1}$$
$$\cdot \sum_{j=0}^{k} \left[z_{j+1} - h_{j+1} \left(x_{j+1} \right) - r \right] \times \left[z_{j+1} - h_{j+1} \left(x_{j+1} \right) \right.$$
$$\left. - r \right]^{\mathrm{T}} \times \left(-A^{-T} \times A^{-1} \right) \Bigg\}\Bigg|_{\substack{z_{j+1}=\widehat{z}_{j+1|j+1},\, x_{j+1}=\widehat{x}_{j+1|j+1} \\ R=\widehat{R}_{k+1}}} = 0$$

According to the second equal sign in the above formula, we can get

$$\frac{k+1}{2}\widehat{R}_{k+1}^{-1} = \frac{1}{2}$$
$$\cdot \sum_{j=0}^{k} \left[z_{j+1} - h_{j+1} \left(x_{j+1} \right)\big|_{x_{j+1} \leftarrow \widehat{x}_{j+1|k+1}} - r \right]$$
$$\cdot \left[z_{j+1} - h_{j+1} \left(x_{j+1} \right)\big|_{x_{j+1} \leftarrow \widehat{x}_{j+1|k+1}} - r \right]^{\mathrm{T}}$$
$$\cdot \left(\widehat{R}_{k+1}^{-T} \cdot \widehat{R}_{k+1}^{-1} \right) \tag{D.2}$$

Multiplying both sides of the above formula by $\widehat{R}_{k+1}$ yields

$$\frac{k+1}{2}\mathrm{I} = \frac{1}{2}$$
$$\cdot \sum_{j=0}^{k} \left[\left[z_{j+1} - h_{j+1} \left(x_{j+1} \right)\big|_{x_{j+1} \leftarrow \widehat{x}_{j+1|k+1}} - r \right] \right] \tag{D.3}$$
$$\cdot \left[\left[z_{j+1} - h_{j+1} \left(x_{j+1} \right)\big|_{x_{j+1} \leftarrow \widehat{x}_{j+1|k+1}} - r \right] \right]^{\mathrm{T}} \cdot \widehat{R}_{k+1}^{-T}$$

Substituting $\widehat{R}_{k+1}^{-T} = \widehat{R}_{k+1}^{-1}$ into the above formula yields

$$\widehat{R}_{k+1} = \frac{1}{k+1}$$
$$\cdot \sum_{j=0}^{k} \left\{ \left[z_{j+1} - h_{j+1} \left(x_{j+1} \right)\big|_{x_{j+1} \leftarrow \widehat{x}_{j|k+1}} - r \right] \right. \tag{D.4}$$
$$\left. \cdot \left[z_{j+1} - h_{j+1} \left(x_{j+1} \right)\big|_{x_{j+1} \leftarrow \widehat{x}_{j|k+1}} - r \right]^{\mathrm{T}} \right\}$$

The proof of $\widehat{R}_{k+1}$ is completed.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

Acknowledgments

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