

## Research Article

# Threshold Dynamics of the Switched Multicity Epidemic Models with Pulse Control

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Received 27 November 2018; Revised 19 March 2019; Accepted 27 March 2019; Published 9 April 2019

Academic Editor: Andrzej Swierniak

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This paper mainly studies the threshold dynamics of new multicity HIV (Human Immunodeficiency Virus) epidemic models with switching parameters and pulse control. The model's parameters are assumed to be time-varying functions and switching functional forms in time due to seasonal changes. And the susceptible population is assumed to become infected via shared injections or sexual contacts with infected individuals and pre-AIDS patients (following infection with HIV but before the full development of acquired immune deficiency syndrome). New threshold conditions are established to ensure the extinction of the disease by using Razumikhin-type approach. Pulse control strategies are then applied to the multicity epidemic model and analyzed to guarantee their success in eradicating the disease. Numerical examples are performed to support the analytical results.

## 1. Introduction

Since the first patients with HIV infection were found in 1981, HIV/AIDS has become a global transmission disease. In 2013, it was reported that 35 million individuals (including 3.2 million children) lived with the disease worldwide, and 2.1 million people became newly infected and 1.5 million died of AIDS-related illnesses in the same year [1]. Viral transmission may have many routes such as sexual transmission among homosexual men, heterosexual transmission, parenteral transmission among drug injectors, through needle sharing, mother-to-newborn transmission, and so on [2]. Particularly, multiperson use (sharing) of needles and syringes for injecting drugs has led to extremely rapid transmission of HIV among injecting drug users in many places, with incidence rates of 20% per year or higher [3, 4]. In many developing countries, travelling conditions in mass transit, such as sanitation, can be relatively poor, leading to an increase in the spread of the disease due to infected individuals who travel [5]. Thus, it has become a major issue that how to effectively prevent or control epidemics as they spread through population.

Mathematical models have been developed to understand and analyze the dynamics of systems and much work has

been done by researchers [5–9]. In particular, Naresh et al. [10] proposed a nonlinear mathematical model of the spread of HIV/AIDS in a population of varying size with immigration of infections and analyzed that the disease was always persistent if the direct immigration of infected individuals was allowed in the community, but the spread of infections could be slowed down if the direct immigration of infected individuals was restricted. Cai et al. [11] established the HIV/AIDS treatment model with two infective stages and studied the global dynamics by the basic reproduction number. Dividing individual hosts into several classes, according to transmission, contact patterns, and different geography [12], multigroup epidemic models have been developed to describe the transmission dynamics, and specific issues have been addressed by researchers [13–16]. However, to the best of our knowledge, there are no studies on multicity HIV model with switching parameters. Based on the models in the literatures [10–12], assume that the population size is divided into four subclasses of the susceptible population, infections population, pre-AIDS, and AIDS population, and assume that susceptible population can become infected via sharing needles with infective and pre-AIDS patients. Multicity epidemic models with HIV infections are established by further partitioning the total population into some

distinct cities according to different geographic spread of the disease.

In the real world, due to the seasonal variety, which makes the population behave periodically, the parameters involved with the modeling approach of epidemic systems always fluctuate in time. For instance, Samanta assumed the coefficients of the HIV/AIDS epidemic model are periodic functions and studied its global asymptotic stability [17, 18]. Researchers [19–23] added seasonal variety factor into their models. Some researchers [24, 25] assumed that the coefficients of the models were smooth-varying and periodic and analyzed the influence of season on the infectious disease. However, there are few discussions assuming that the parameters of HIV infections models are smooth-varying. This is because these parameters are also abruptly changed due to changes in weather and temperature [6, 10]. Particularly in the winter, the HIV infection leads to lowering the human immune, so that HIV-infected patients are at high risk of disease in susceptible populations [26]. From a biological point of view, it is reasonable to assume that the model's parameters are time-varying and time-dependent that are piecewise constants and assume that these parameters are governed by a switching rule. Thus, it is important to formulate a switched model by incorporating switching parameters into the multicity HIV infections model.

There have been numerous attempts to control and eradicate the infectious disease, such as use of condoms, sex education, and treatments with cocktails of drugs [27–29]. Pulse vaccination strategy, due to its highly successful application in the control of the transmission of diseases, such as measles, hepatitis, parotitis, smallpox, and phthisis, has been further considered in the literatures [30–35]. Pulse vaccination strategy distinguishes from the traditional constant vaccination. The vaccine doses are applied in a time which is very short with respect to the dynamics of the target disease [36]. In this paper, assume that a constant fraction of infections population and pre-AIDS patients is vaccinated, new switched multicity HIV infections models with pulse control are proposed, and some new threshold conditions are established to ensure the eradication of the disease.

The organization of this paper is as follows. In Section 2, a switched multicity HIV epidemic model is constructed and some new threshold conditions are established to ensure eradication of the disease. Pulse vaccination is applied to the above model in Section 3, and some conditions of the eradication of the disease are derived. In Section 4, we provide numerical examples to illustrate the validity of our results.

## 2. The Switched Multicity HIV Infection Model

Based on the models in the literatures [10–12] and the fact that it takes a long time ( $\geq 10$  years) for infectious population to become full-blown AIDS population, assume that there are  $n$  cities or groups and the population  $N_j(t)$  for city  $j$  ( $j = 1, 2, \dots, n$ ) at time  $t$  are partitioned into susceptible

population  $S_j(t)$ , infectious population before the onset of AIDS  $I_j(t)$ , pre-AIDS population  $R_j(t)$ , and the full-blown AIDS population  $A_j(t)$ . Here, the total population  $N(t)$  of  $n$  cities at time  $t$  is  $N(t) = \sum_{j=1}^n N_j$ , in which  $N_j = S_j + I_j + R_j + A_j$ .

Due to the seasonal variety, biological and environmental parameters are naturally subjected to fluctuation in time. To investigate this kind of problems, assume that the coefficients of the model are time-varying. Assume that these parameters are modeled as switching parameters and are governed by a switching rule  $\sigma(t) : (t_{k-1}, t_k] \rightarrow \{1, 2, \dots, m\}, k = 1, 2, \dots$ , which is a piecewise continuous (from the left) function of time, i.e.,  $\sigma(t_k) = \lim_{t \rightarrow t_k^-} \sigma(t)$  for all  $k$ . Assume that  $t_{i_k}$  denotes that the  $i$ th subsystem is switched in the  $k$ th time, i.e.,  $\sigma(t_{i_k}^-) = i$ , for  $i \in \{1, 2, \dots, m\}, k = 1, 2, \dots$ . Assume that the switching time  $\{t_k\}$  satisfies  $t_k > t_{k-1}$  and  $t_k \rightarrow \infty$  as  $k \rightarrow \infty$ . Denote the set of all switching rules by  $\mathcal{S}$ . Motivated by the above discussion, a new switched multicity HIV epidemic model is formulated, for  $t \in (t_{k-1}, t_k]$ ,

$$\begin{aligned} \dot{S}_j &= u_{j_{i_k}}(t) N_j \\ &\quad - \left( \frac{m_{jj_{i_k}}(t) a_{jj_{i_k}}(t) S_j I_j}{N_j} + \frac{n_{jj_{i_k}}(t) a_{jj_{i_k}}(t) S_j R_j}{N_j} \right) \\ &\quad + \sum_{l=1}^n b_{lj_{i_k}}(t) S_j - u_{j_{i_k}}(t) S_j \\ &\quad - \sum_{\substack{l=1 \\ l \neq j}}^n b_{lj_{i_k}}(t) \frac{m_{lj_{i_k}}(t) a_{lj_{i_k}}(t) S_l I_l}{N_l} \\ &\quad - \sum_{\substack{l=1 \\ l \neq j}}^n b_{lj_{i_k}}(t) \frac{n_{lj_{i_k}}(t) a_{lj_{i_k}}(t) S_l R_l}{N_l}, \\ \dot{I}_j &= \left( \frac{m_{jj_{i_k}}(t) a_{jj_{i_k}}(t) S_j I_j}{N_j} + \frac{n_{jj_{i_k}}(t) a_{jj_{i_k}}(t) S_j R_j}{N_j} \right) \\ &\quad + \sum_{l=1}^n b_{lj_{i_k}}(t) I_j - (\delta + u_{j_{i_k}}(t)) I_j \\ &\quad + \sum_{\substack{l=1 \\ l \neq j}}^n b_{lj_{i_k}}(t) \frac{m_{lj_{i_k}}(t) a_{lj_{i_k}}(t) S_l I_l}{N_l} \\ &\quad + \sum_{\substack{l=1 \\ l \neq j}}^n b_{lj_{i_k}}(t) \frac{n_{lj_{i_k}}(t) a_{lj_{i_k}}(t) S_l R_l}{N_l}, \\ \dot{R}_j &= d\delta I_j + \sum_{l=1}^n b_{lj_{i_k}}(t) R_j - (\delta + u_{j_{i_k}}(t)) R_j, \end{aligned}$$

$$\begin{aligned} \dot{A}_j &= (1-d)\delta I_j + \sum_{l=1}^n b_{lj_{ik}}(t) A_j + \delta R_j \\ &\quad - (\beta + u_{j_{ik}}(t)) A_j. \end{aligned} \tag{1}$$

Here, the initial conditions satisfy  $S_j(0) > 0$ ,  $I_j(0) \geq 0$ ,  $R_j(0) \geq 0$ , and  $A_j(0) \geq 0$  for  $j = 1, 2, \dots, n$ . It is natural biologically to make some assumptions as follows:

$u_{j_{ik}}$  is the recruitment rate function of susceptible population from the larger embedding population, which is equal to the death rate of susceptible population, infections population, pre-AIDS population, and AIDS population in city  $j$ .

$a_{jj_{ik}}$  is the average number of contacts of an individual per unit of time in city  $j$ .

$m_{jj_{ik}}$  is the contact rate between susceptible and pre-AIDS individuals in city  $j$ .

$n_{jj_{ik}}$  is the contact rate between susceptible and AIDS individuals in city  $j$ .

$b_{lj_{ik}}$  is the dispersal rate of individuals from city  $l$  to city  $j$ .

$-b_{jj_{ik}} \geq 0$  is the immigration rate of individuals from city  $j$  to other cities.

$\delta$  is the movement rate from infections, which is equal to the movement rate from pre-AIDS individuals to AIDS individuals in every city.

$d$  is a fraction of infectious individuals into pre-AIDS individuals in city  $j$ .

$(1-d)$  is a fraction of infectious individuals into AIDS individuals in city  $j$ .

$\beta$  is the contact disease-induced death rate of AIDS individuals in every city. The values of all the parameters in the model are assumed to be nonnegative.

Assume that the total population size is constant by 1. The meaningful physical domain for system (1) is  $\Gamma = \{(S_1, \dots, S_n, I_1, \dots, I_n, R_1, \dots, R_n, A_1, \dots, A_n) \in \mathbb{R}_+^{4n} : N(t) = 1\}$ . Note that  $\dot{N}(t) = 0$ , and hence this domain is invariant to system (1). It is apparent that  $P_0 = (S_1^0, \dots, S_n^0, 0, \dots, 0, 0, \dots, 0, 0, \dots, 0)$  is a disease-free equilibrium of system (1), where  $S_j^0 = N_j$ ,  $j = 1, 2, \dots, n$ .

Motivated by the fact that  $I_j = 0$  is an equilibrium solution for the variable  $I_j(t)$ , for  $j = 1, 2, \dots, n$ , it follows from the third equation of system (1) that each  $R_j$  approaches zero with  $\delta + u_{j_{ik}} > 0$ . Then, for  $t \in (t_{k-1}, t_k]$ , we have the following limit system:

$$\begin{aligned} \dot{S}_j &= u_{j_{ik}}(t) A_j + \sum_{l=1}^n b_{lj_{ik}}(t) S_j \\ \dot{A}_j &= \sum_{l=1}^n b_{lj_{ik}}(t) A_j - (\beta + u_{j_{ik}}(t)) A_j. \end{aligned} \tag{2}$$

It is clear that  $R_j$  approaches zero as  $t \rightarrow \infty$  for all  $j$ . Assume that  $b_{kj}(t) = b_{kj}(t + \omega)$ , for some  $\omega > 0$ , then system (2)

has a periodic solution  $Q_0 = (\tilde{S}_1(t), \tilde{S}_2(t), \dots, \tilde{S}_n(t), 0, \dots, 0)$  (see [37]). Furthermore, system (1) has a periodic disease-free solution:

$$E_0 = (\tilde{S}_1(t), \dots, \tilde{S}_n(t), 0, \dots, 0, 0, \dots, 0, 0, \dots, 0). \tag{3}$$

The basic reproduction number  $R_0$  can be defined as the average number of secondary cases produced by a single infective population introduced into entirely susceptible population. For the switched HIV epidemic model (1), the basic reproduction number  $R_0$  can be the spectral radius of a next generation integral operator,  $R_0 = \rho(L)$ , where the next infection operator  $L$  is given by [38, 39]

$$(L\varphi)(t) = \int_0^\infty K(t, x) \varphi(t-x) dx, \tag{4}$$

where  $\varphi$  is a continuous functions and  $K(t, x)$  is the rate of new infections produced by the infected individuals who were introduced at time  $t-x$ .

The basic reproduction number has become an important tool to study the stability of epidemic models. In the following, we will first present an approximate basic reproduction number which has a similar threshold proposed to that of  $R_0$  and then prove the stability of the proposed model (i.e., system (1)).

**Theorem 1.** Assume that there exist continuous functions  $m_i(t) > 0$ ,  $a_i(t) > 0$ ,  $u_i(t) > 0$ , and  $b(t)$  such that  $m_{jj_i}(t) \leq m_i(t)$ ,  $n_{jj_i}(t) \leq n_i(t)$ ,  $a_{jj_i}(t) \leq a_i(t)$ ,  $u_j(t) \geq u_i(t)$ , and  $b_j(t) \leq b(t)$  for  $i = 1, 2, \dots, m$ ,  $l, j = 1, 2, \dots, n$  ( $l \neq j$ ). If  $\sigma \in \mathcal{S}$  and  $\bar{R}_0 < 1$ , where

$$\begin{aligned} \bar{R}_0 &= \sup_{t \geq h} \frac{\int_{t_0}^t (m_\sigma(s) + n_\sigma(s)) (1 + (n-1)b(s)) a_\sigma(s) ds}{\int_{t_0}^t ((1-d)\delta + u_\sigma(s)) ds}, \end{aligned} \tag{5}$$

then the solutions of system (1) converge to the disease-free solution  $E_0$ .

*Proof.* For any piecewise constant switching rule  $\sigma(t) \in \mathcal{S}$ , and given that  $t > t_0$ , let  $t_1 < t_2 < \dots < t$  be the switching points of  $\sigma(t)$  over the interval  $(t_0, t]$ . Assume that  $i = i_k$ , for  $t \in (t_{k-1}, t_k]$ , where  $i_k$  follows the switching rule  $\sigma(t)$ , which means that the  $i_k$ th subsystem is activated with  $t \in (t_{k-1}, t_k]$ .

Let  $X_j = I_j + R_j$ ,  $j = 1, 2, \dots, n$ . Considering the trajectory of the  $i_k$ th subsystem, it follows that

$$\begin{aligned} \dot{X}_j &\leq (m_{jj_{ik}}(t) + n_{jj_{ik}}(t)) a_{jj_{ik}}(t) X_j + \sum_{l=1}^n b_{lj_{ik}}(t) X_j \\ &\quad + \sum_{\substack{l=1 \\ l \neq j}}^n b_{lj_{ik}}(t) a_{lj_{ik}}(t) m_{lj_{ik}}(t) X_l \end{aligned}$$

$$\begin{aligned}
 & + \sum_{\substack{l=1 \\ l \neq j}}^n b_{l j_{i_k}}(t) a_{l j_{i_k}}(t) n_{l j_{i_k}}(t) X_l \\
 & - \left( (1-d)\delta + u_{j_{i_k}}(t) \right) X_j.
 \end{aligned} \tag{6}$$

By assumptions of Theorem 1, we obtain that

$$\sum_{j=1}^n \dot{X}_j \leq \lambda_{i_k} \sum_{j=1}^n X_j, \tag{7}$$

where  $\lambda_{i_k} = (m_{i_k}(t) + n_{i_k}(t))a_{i_k}(t) + (n-1)b(t)a_{i_k}(t)(m_{i_k}(t) + n_{i_k}(t)) - ((1-d)\delta + u_{i_k}(t))$ .

For  $t \in (t_{k-1}, t_k]$ , taking integration over both sides of the above inequality, it results in

$$\sum_{j=1}^n X_j \leq \sum_{j=1}^n X_j(t_{k-1}) \exp \left( \int_{t_{k-1}}^t \lambda_{i_k}(s) ds \right). \tag{8}$$

Apply (7) successively on each subinterval. For  $t \in (t_{k-1}, t_k]$ ,

$$\begin{aligned}
 \sum_{j=1}^n X_j & \leq \sum_{j=1}^n X_j(t_0) \exp \left[ \int_{t_0}^{t_1} \lambda_{i_1}(s) ds + \int_{t_1}^{t_2} \lambda_{i_2}(s) ds \right. \\
 & \left. + \dots + \int_{t_{k-1}}^t \lambda_{i_k}(s) ds \right].
 \end{aligned} \tag{9}$$

Since  $\bar{R}_0 < 1$ , and it is obvious that there is  $\epsilon > 0$  such that

$$\begin{aligned}
 & \frac{\int_{t_0}^t (m_\sigma(s) + n_\sigma(s))(1 + (n-1)b(s))a_\sigma(s) ds}{\int_{t_0}^t ((1-d)\delta + u_\sigma(s)) ds} \\
 & \leq 1 - \epsilon, \quad \text{for all } t \geq h,
 \end{aligned} \tag{10}$$

i.e.,  $\int_{t_0}^t [(m_\sigma(s) + n_\sigma(s))(1 + (n-1)b(s))a_\sigma(s) - ((1-d)\delta + u_\sigma(s))] ds \leq -\epsilon \int_{t_0}^t ((1-d)\delta + u_\sigma(s)) ds$ , and hence,

$$\int_{t_0}^t \lambda_\sigma(s) ds \leq -\epsilon \pi (t - t_0), \tag{11}$$

where  $\pi = \inf_{t \geq t_0} ((1-d)\delta + u_\sigma(t)) > 0$ . Consequently, from (9), we have

$$\begin{aligned}
 \sum_{j=1}^n X_j & \leq \sum_{j=1}^n X_j(t_0) \exp \left[ \int_{t_0}^t \lambda_\sigma(s) ds \right] \\
 & \leq \sum_{j=1}^n X_j(t_0) \exp [-\epsilon \pi (t - t_0)], \quad \text{for } t \geq h,
 \end{aligned} \tag{12}$$

which implies that  $\sum_{j=1}^n X_j$  converges to zero exponentially as  $t \rightarrow \infty$ . Note that  $X_1, X_2, \dots, X_n \geq 0$ , and hence,  $X_j$  converges to zero exponentially as  $t \rightarrow \infty$ , for  $j = 1, 2, \dots, n$ . Furthermore, we can get that  $I_j$  and  $R_j$  converge to zero exponentially as  $t \rightarrow \infty$  for  $j = 1, 2, \dots, n$ , respectively. Therefore, system (1) reduces to system (2), and the solutions of system (1) converge to the disease-free equilibrium  $E_0$ .  $\square$

*Remark 2.* Theorem 1 implies that if the basic reproduction number  $\bar{R}_0 < 1$ , then the disease always dies out from all cities.

Since biological and environmental parameters are naturally varying due to the seasonal variety, which makes the population behave periodically, it is very necessary to consider the effects of a periodically varying environment. Assume that a periodic switching rule  $\sigma$  satisfies  $t_k - t_{k-1} = \tau_k$  with  $\tau_{k+m} = \tau_k$ , for  $t \in (t_{k-1}, t_k]$ . Suppose that  $m_{i_k}(t) = m_{i_k}(t)$ ,  $a_{i_k}(t) = a_{i_k}(t)$ , and  $u_{i_k}(t) = u_{i_k}(t)$  wherever  $t \in (t_{k-1}, t_k]$ . Suppose that  $m_{jj(k+m)}(t) = m_{jjk}(t)$ ,  $n_{jj(k+m)}(t) = n_{jjk}(t)$ ,  $a_{jj(k+m)}(t) = a_{jjk}(t)$ ,  $u_{j(k+m)}(t) = u_{jk}(t)$ , and  $m_{jjk}(t) = m_{jjk}(t + \omega)$ ,  $n_{jjk}(t) = n_{jjk}(t + \omega)$ ,  $a_{jjk}(t) = a_{jjk}(t + \omega)$ ,  $u_{jk}(t) = u_{jk}(t + \omega)$ , and  $b_{jk}(t) = b_{jk}(t + \omega)$ , where  $\omega = \tau_1 + \tau_2 + \dots + \tau_m$  is one period of the switching rule. Let  $\mathcal{F}_{Periodic}$  be the set of periodic switching rule and  $\mathcal{F}_{Periodic} \subset \mathcal{F}$ .

**Theorem 3.** Assume that there exist continuous functions  $m_i(t) = m_i(t + \omega) > 0$ ,  $n_i(t) = n_i(t + \omega) > 0$ ,  $a_i(t) = a_i(t + \omega) > 0$ ,  $u_i(t) = u_i(t + \omega) > 0$ , and  $b(t)$  such that  $m_{jj_i}(t) \leq m_i(t)$ ,  $n_{jj_i}(t) \leq n_i(t)$ ,  $a_{jj_i}(t) \leq a_i(t)$ ,  $u_{j_i}(t) \geq u_i(t)$ , and  $b_{l_j}(t) \leq b(t)$  for  $i = 1, 2, \dots, m$ ,  $l, j = 1, 2, \dots, n$  ( $l \neq j$ ). If  $\sigma \in \mathcal{F}_{Periodic}$  and  $\hat{R}_0 < 1$ , where

$$\begin{aligned}
 \hat{R}_0 & = \frac{\int_{t_0}^{t_0+\omega} (m_\sigma(s) + n_\sigma(s))(1 + (n-1)b(s))a_\sigma(s) ds}{\int_{t_0}^{t_0+\omega} ((1-d)\delta + u_\sigma(s)) ds},
 \end{aligned} \tag{13}$$

then the solutions of system (1) converge to the disease-free solution  $E_0$ .

*Proof.* Assume that  $i = i_k$ , for  $t \in (t_{k-1}, t_k]$ , where  $i_k$  follows the switching rule  $\sigma(t) \in \mathcal{F}_{Periodic}$ . Let  $X_j = I_j + R_j$ , it follows from (9) that

$$\begin{aligned}
 \sum_{j=1}^n X_j(t_0 + \omega) & \leq \sum_{j=1}^n X_j(t_0) \exp \left[ \int_{t_0}^{t_1} \lambda_{i_1}(s) ds \right. \\
 & \left. + \int_{t_1}^{t_2} \lambda_{i_2}(s) ds + \dots + \int_{t_{m-1}}^{t_m} \lambda_{i_m}(s) ds \right],
 \end{aligned} \tag{14}$$

which implies that  $\sum_{j=1}^n X_j(t_0 + \omega) \leq \xi \sum_{j=1}^n X_j(t_0)$ , where  $\xi = \exp[\int_{t_0}^{t_1} \lambda_{i_1}(s) ds + \int_{t_1}^{t_2} \lambda_{i_2}(s) ds + \dots + \int_{t_{m-1}}^{t_m} \lambda_{i_m}(s) ds]$ .

Note that  $\hat{R}_0 < 1$ , it follows that  $\xi < 1$ . For any integer  $h = 1, 2, \dots$ , it follows that

$$\begin{aligned}
 \sum_{j=1}^n X_j(t_0 + h\omega) & \leq \xi \sum_{j=1}^n X_j(t_0 + (h-1)\omega) \leq \dots \\
 & \leq \xi^h \sum_{j=1}^n X_j(t_0).
 \end{aligned} \tag{15}$$

It follows that

$$\lim_{h \rightarrow \infty} \sum_{j=1}^n X_j(t_0 + h\omega) = \lim_{h \rightarrow \infty} \xi^h \sum_{j=1}^n X_j(t_0) = 0. \tag{16}$$

In general, for  $t \in (t_{k-1}, t_k]$  and  $t_0 + h\omega < t_k \leq t_0(h+1)\omega$ , we have

$$\sum_{j=1}^n X_j(t) \leq M \sum_{j=1}^n X_j(t_0 + h\omega), \quad (17)$$

where  $M = \max_{t_{k-1} < t \leq t_k} \exp[\int_{t_0}^{t_1} \lambda_{i_1}(s)ds + \int_{t_1}^{t_2} \lambda_{i_2}(s)ds + \dots + \int_{t_{k-1}}^t \lambda_{i_k}(s)ds]$ . Note that the sequence  $\{\sum_{j=1}^n X_j(t_0 + h\omega)\}_{h=0}^\infty$  converges to zero exponentially as  $h \rightarrow \infty$ , which implies that  $\sum_{j=1}^n X_j(t)$  approaches zero as  $t \rightarrow \infty$ . Since  $X_1, X_2, \dots, X_j \geq 0$ , it follows that  $X_j$  converges to zero exponentially as  $t \rightarrow \infty$ , for  $j = 1, 2, \dots, n$ . Furthermore, we can get that  $I_j$  and  $R_j$  converge to zero exponentially, respectively, as  $t \rightarrow \infty$ , for  $j = 1, 2, \dots, n$ . Then, system (1) reduces to system (2), and the solutions of system (1) converge to the disease-free equilibrium  $E_0$ .  $\square$

*Remark 4.* For periodic case, we have established threshold conditions which ensure eradication of the disease in all cities.

### 3. Pulse Vaccination Applied to a Switched Multicity HIV Infection Model

Distinguishing from the traditional constant vaccination, pulse vaccination strategy, the repeated application of vaccine across an age range, is gaining prominence as a control strategy for the elimination of the transmission of diseases and has been considered in the literatures [36, 40, 41]. Motivated by the work in [33], assume that the vaccination does not have perfect efficacy (since no vaccine can guarantee complete immunization). More specifically, assume that a constant fraction  $0 \leq p_j \leq 1$  of infections population in city  $j$  is vaccinated. Assume that a constant fraction  $0 \leq p_j \leq 1$  of pre-AIDS population in city  $j$  is given a pulse vaccination. Immediately following each vaccination pulse, both  $p_j^1 I_j$  infections population and  $p_j^2 R_j$  pre-AIDS population are transferred to the AIDS class of the population. And assume that all the vaccine doses are applied in a very short periodic time  $\omega$  with respect to the dynamics of the target disease. When pulse vaccination is incorporated into the multicity HIV infection model (1), the system becomes impulsive and may be rewritten as follows:

$$\begin{aligned} \dot{S}_j &= u_{j_{i_k}}(t) N_j \\ &- \left( \frac{m_{jj_{i_k}}(t) a_{jj_{i_k}}(t) S_j I_j}{N_j} + \frac{n_{jj_{i_k}}(t) a_{jj_{i_k}}(t) S_j R_j}{N_j} \right) \\ &+ \sum_{l=1}^n b_{lj_{i_k}}(t) S_j - u_{j_{i_k}}(t) S_j \\ &- \sum_{\substack{l=1 \\ l \neq j}}^n b_{lj_{i_k}}(t) \frac{m_{lj_{i_k}}(t) a_{lj_{i_k}}(t) S_l I_l}{N_l} \end{aligned}$$

$$- \sum_{\substack{l=1 \\ l \neq j}}^n b_{lj_{i_k}}(t) \frac{n_{lj_{i_k}}(t) a_{lj_{i_k}}(t) S_l R_l}{N_l}, \quad t \neq t_0 + k\omega$$

$$\begin{aligned} \dot{I}_j &= \left( \frac{m_{jj_{i_k}}(t) a_{jj_{i_k}}(t) S_j I_j}{N_j} + \frac{n_{jj_{i_k}}(t) a_{jj_{i_k}}(t) S_j R_j}{N_j} \right) \\ &+ \sum_{l=1}^n b_{lj_{i_k}}(t) I_j - (\delta + u_{j_{i_k}}(t)) I_j \\ &+ \sum_{\substack{l=1 \\ l \neq j}}^n b_{lj_{i_k}}(t) \frac{m_{lj_{i_k}}(t) a_{lj_{i_k}}(t) S_l I_l}{N_l} \\ &+ \sum_{\substack{l=1 \\ l \neq j}}^n b_{lj_{i_k}}(t) \frac{n_{lj_{i_k}}(t) a_{lj_{i_k}}(t) S_l R_l}{N_l}, \end{aligned}$$

$$\dot{R}_j = d\delta I_j + \sum_{l=1}^n b_{lj_{i_k}}(t) R_j - (\delta + u_{j_{i_k}}(t)) R_j,$$

$$\begin{aligned} \dot{A}_j &= (1-d)\delta I_j + \sum_{l=1}^n b_{lj_{i_k}}(t) A_j + \delta R_j \\ &- (\beta + u_{j_{i_k}}(t)) A_j, \end{aligned}$$

$$S_j(t^+) = S_j(t), \quad t = t_0 + k\omega$$

$$I_j(t^+) = (1-p_j) I_j(t),$$

$$R_j(t^+) = (1-p_j) R_j(t),$$

$$A_j(t^+) = p_j^1 I_j(t) + p_j^2 R_j(t) + A_j(t),$$

(18)

where  $k = 1, 2, \dots, j = 1, 2, \dots, n$ , and  $i_k \in \{1, 2, \dots, m\}$  following a switching rule  $\sigma \in \mathcal{F}^{Periodic-Pulse}$ . Denote  $\mathcal{F}^{Periodic-Pulse}$  as the set of switching pulse-periodic rules. The initial conditions are  $S_j(t_0^+) = S_{j,0} > 0$ ,  $I_j(t_0^+) = I_{j,0}$ ,  $R_j(t_0^+) = R_{j,0}$ , and  $A_j(t_0^+) = A_{j,0}$ .

If  $I_j = 0$  and  $R_j = 0$ , the dynamics is governed by the reduced system

$$\dot{S}_j = u_{j_{i_k}}(t) A_j + \sum_{l=1}^n b_{lj_{i_k}}(t) S_j, \quad t \neq t_0 + k\omega$$

$$\dot{A}_j = \sum_{l=1}^n b_{lj_{i_k}}(t) A_j - (\beta + u_{j_{i_k}}(t)) A_j, \quad (19)$$

$$S_j(t^+) = S_j(t), \quad t = t_0 + k\omega$$

$$A_j(t^+) = A_j(t).$$

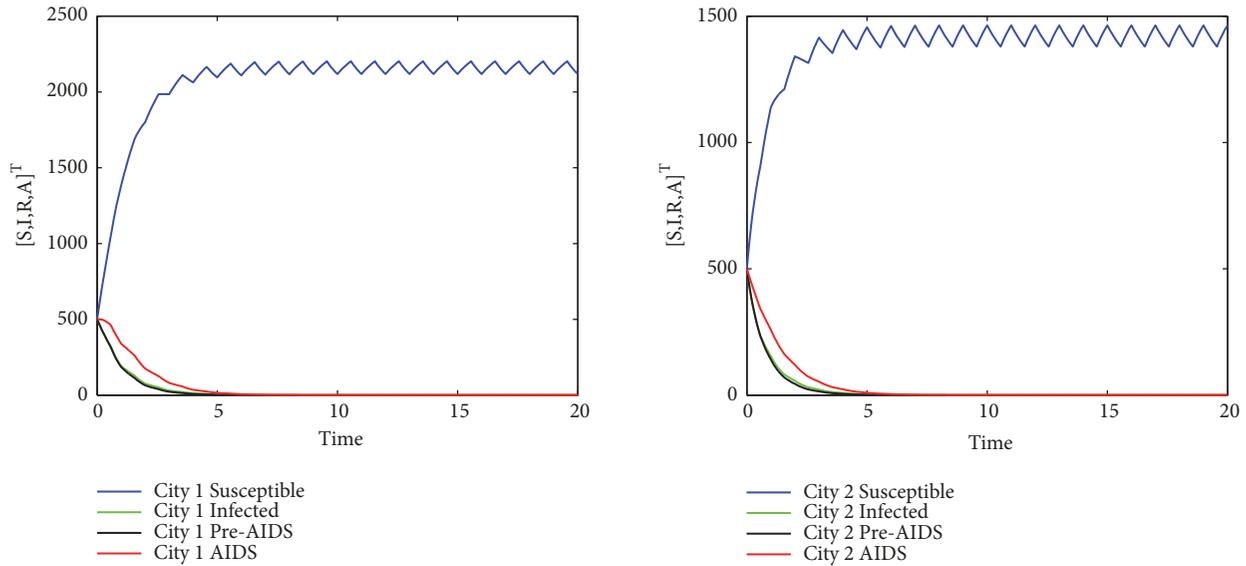


FIGURE 1: Solutions of the two-city HIV epidemic model with  $\bar{R}_0 = 0.5859 < 1$ .

Assume that  $b_{kj}(t) = b_{kj}(t + \omega)$ , then system (18) has the same disease-free solution  $E_0$  as that of system (1).

Now, we are ready to show the stability conditions of disease-free solution based on an approximate basic reproduction number.

**Theorem 5.** Assume that there exist continuous functions  $m_i(t) = m_i(t + \omega) > 0$ ,  $n_i(t) = n_i(t + \omega) > 0$ ,  $a_i(t) = a_i(t + \omega) > 0$ ,  $u_i(t) = u_i(t + \omega) > 0$ , and  $b(t)$  such that  $m_{jj}(t) \leq m_i(t)$ ,  $n_{jj}(t) \leq n_i(t)$ ,  $a_{jj}(t) \leq a_i(t)$ ,  $u_j(t) \geq u_i(t)$ , and  $b_j(t) \leq b(t)$  for  $i = 1, 2, \dots, m$ ,  $l, j = 1, 2, \dots, n$  ( $l \neq j$ ). Assume that there exists constant  $p$ , such that  $p \leq p_j$ , for  $j = 1, 2, \dots, n$ , and  $\int_{t_0}^{t_0+\omega} (u_\sigma(s) + (1-d)\delta) ds > \ln(1-p)$ . If  $\sigma \in \mathcal{F}_{\text{Periodic-Pulse}}$  and  $\bar{R}_0 < 1$ , where

$$\bar{R}_0 = \frac{\int_{t_0}^{t_0+\omega} (m_\sigma(s) + n_\sigma(s)) (1 + (n-1)b(s)) a_\sigma(s) ds}{\int_{t_0}^{t_0+\omega} (u_\sigma(s) + (1-d)\delta) ds - \ln(1-p)}, \quad (20)$$

then the solutions of system (18) converge to the disease-free equilibrium  $E_0$ .

**Remark 6.** In the special case that  $p_j = p_j^1 = p_j^2 = 0$  for  $j = 1, 2, \dots, n$ , system (18) will be reduced to system (1). In comparison with the approximation reproduction number,  $\bar{R}_0$  of system (18) is less than one by adding pulse control, which implies that the disease-free equilibrium  $E_0$  is stable.

#### 4. Numerical Simulations

In this section, some numerical examples of system (1) and system (18) are given to confirm the above analysis. For these simulations, most of the parametric values are chosen

the same as those in [2, 5, 6, 10–12, 14, 19]. To make the simulations easy, assume that  $n = 2$  and  $t_0 = 0$  in the whole paper and that the switching for all the simulations is periodic:

$$\sigma(t) = \begin{cases} 1, & \text{if } t \in (k, k + 0.75], \\ 2, & \text{if } t \in (k + 0.75, k + 1]. \end{cases} \quad (21)$$

First, consider system (1) with initial conditions  $(S_{i,0}, I_{i,0}, R_{i,0}, A_{i,0}) = (500, 500, 500, 500)$  for  $i = 1, 2$ . Let the constant parameters be  $\delta = 0.4$ ,  $d = 0.25$ ,  $\beta = 0.25$ , and  $N_1 = N_2 = 2000$ . Take the switching parameters as follows:  $m_{111} = 0.5$ ,  $m_{121} = 0.28$ ,  $m_{211} = 0.45$ ,  $m_{221} = 0.3$ ,  $m_{112} = 0.18$ ,  $m_{122} = 0.13$ ,  $m_{212} = 0.16$ ,  $m_{222} = 0.2$ ,  $n_{111} = 0.1$ ,  $n_{121} = 0.15$ ,  $n_{211} = 0.1$ ,  $n_{221} = 0.3$ ,  $n_{112} = 0.15$ ,  $n_{122} = 0.3$ ,  $n_{212} = 0.15$ ,  $n_{222} = 0.35$ ,  $a_{111} = 0.5$ ,  $a_{121} = 0.45$ ,  $a_{211} = 0.35$ ,  $a_{221} = 0.45$ ,  $a_{112} = 0.35$ ,  $a_{122} = 0.67$ ,  $a_{212} = 0.65$ ,  $a_{222} = 0.55$ ,  $b_{111} = -0.26$ ,  $b_{121} = 0.26$ ,  $b_{211} = 0.5$ ,  $b_{221} = -0.5$ ,  $b_{112} = -0.45$ ,  $b_{122} = 0.45$ ,  $b_{212} = 0.55$ ,  $b_{222} = -0.55$ ,  $u_{11} = 0.65$ ,  $u_{12} = 0.85$ ,  $u_{21} = 0.8$ , and  $u_{22} = 0.87$ . For  $l, j, i = 1, 2$ , choose  $m_1 = \max\{m_{lj1}\} = 0.5$ ,  $m_2 = \max\{m_{lj2}\} = 0.2$ ,  $n_1 = \max\{n_{lj1}\} = 0.3$ ,  $n_2 = \max\{n_{lj2}\} = 0.35$ ,  $a_1 = \max\{a_{lj1}\} = 0.5$ ,  $a_2 = \max\{a_{lj2}\} = 0.67$ ,  $u_1 = \min\{u_{l1}\} = 0.65$ ,  $u_2 = \min\{u_{l2}\} = 0.85$ , and  $b = \max\{b_{lj1}\} = 0.6$ ; then,  $\bar{R}_0 = 0.5859 < 1$ . Thus, the conditions of Theorem 3 are satisfied, which implies that the disease is eradicated in both cities. It is clearly seen from Figure 1 that the numerical simulations agree well with the theoretical results.

On the other hand, for system (1), we take the switching parameter as follows:  $m_{111} = 0.8$ ,  $m_{121} = 0.78$ ,  $m_{211} = 0.85$ ,  $m_{221} = 0.75$ ,  $m_{112} = 0.68$ ,  $m_{122} = 0.73$ ,  $m_{212} = 0.76$ ,  $m_{222} = 0.76$ ,  $n_{111} = 0.81$ ,  $n_{121} = 0.85$ ,  $n_{211} = 0.71$ ,  $n_{221} = 0.73$ ,  $n_{112} = 0.65$ ,  $n_{122} = 0.73$ ,  $n_{212} = 0.75$ ,  $n_{222} = 0.65$ ,  $a_{111} = 0.65$ ,  $a_{121} = 0.6$ ,  $a_{211} = 0.65$ ,  $a_{221} = 0.7$ ,  $a_{112} = 0.75$ ,  $a_{122} = 0.67$ ,  $a_{212} = 0.65$ ,  $a_{222} = 0.58$ ,  $b_{111} = -0.6$ ,  $b_{121} = 0.55$ ,  $b_{211} = 0.6$ ,  $b_{221} = -0.55$ ,  $b_{112} = -0.8$ ,  $b_{122} = 0.8$ ,  $b_{212} = 0.55$ ,  $b_{222} = -0.55$ ,

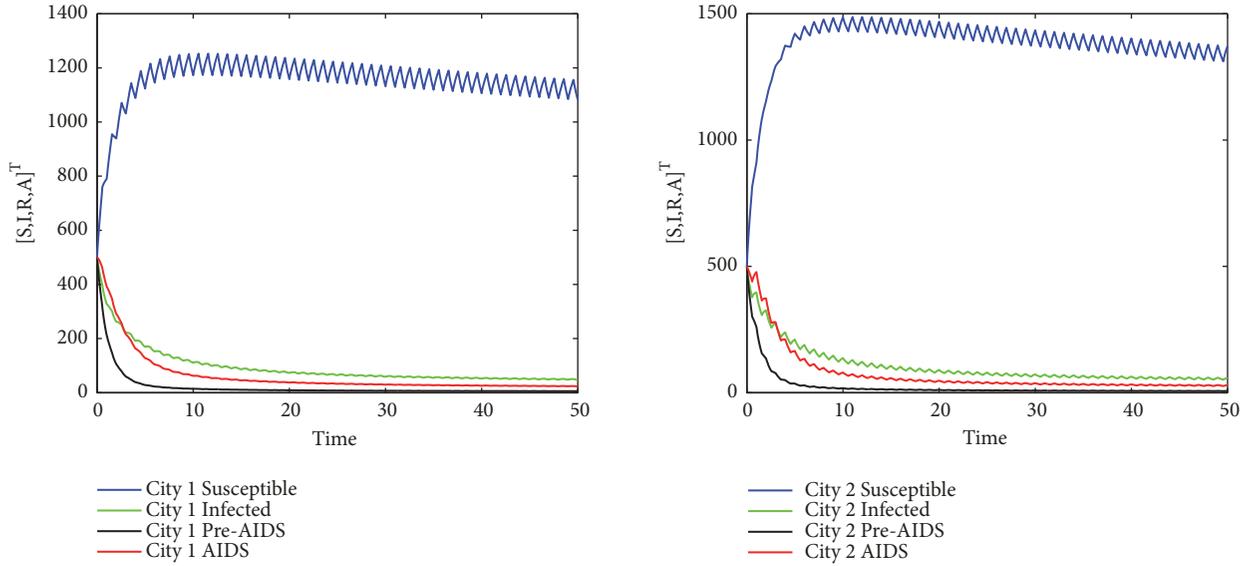


FIGURE 2: Solutions of the two-city HIV epidemic model with  $\bar{R}_0 = 3.3051 > 1$ .

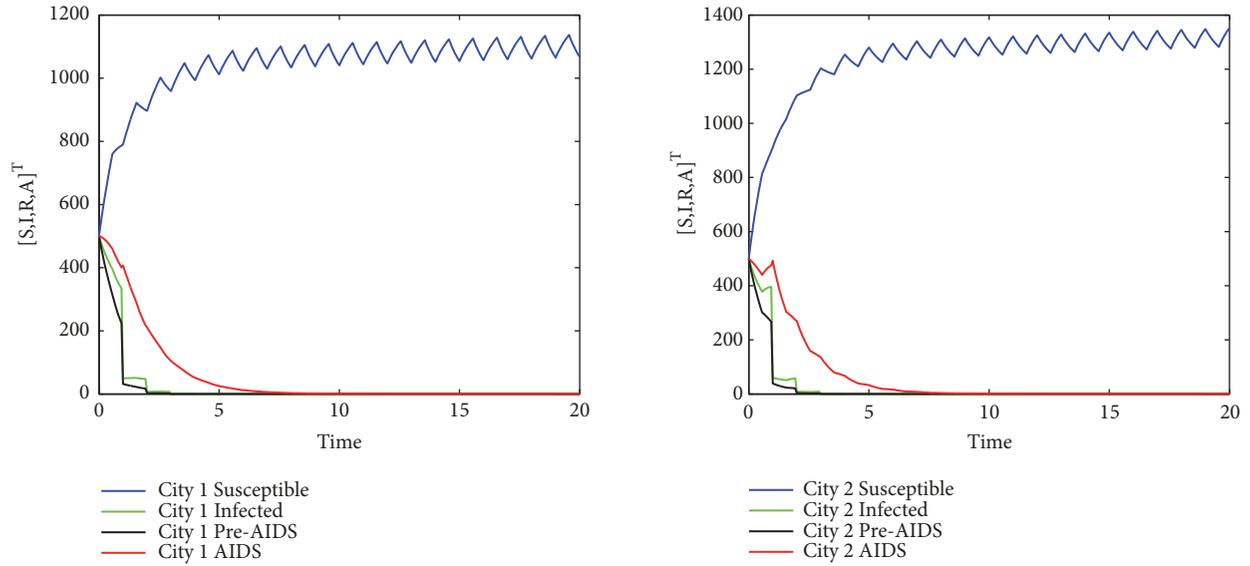


FIGURE 3: Solutions of the two-city HIV epidemic model with  $\bar{R}_0 = 0.815 < 1$ .

$u_{11} = 0.5, u_{12} = 0.4, u_{21} = 0.6,$  and  $u_{22} = 0.26$ . The other parameters have the same values as in Figure 1. We choose  $m_1 = \max\{m_{ij1}\} = 0.8, m_2 = \max\{m_{ij2}\} = 0.76,$   $n_1 = \max\{n_{ij1}\} = 0.85, n_2 = \max\{n_{ij2}\} = 0.75, a_1 = \max\{a_{ij1}\} = 0.7, a_2 = \max\{a_{ij2}\} = 0.75, u_1 = \min\{u_{i1}\} = 0.4,$   $u_2 = \min\{u_{i2}\} = 0.26,$  and  $b = \max\{b_{j1}\} = 0.8;$  then,  $\bar{R}_0 = 3.3051$ . Figure 2 suggests that the disease persists in both cities when  $\bar{R}_0 > 1$ .

Consider system (18) with pulse control  $p_1 = p_1^1 = p_1^2 = 0.85,$  and  $p_2 = p_2^1 = p_2^2 = 0.85$ . We take  $\omega_1 = 0.5, \omega_2 = 0.5,$  and  $\omega = 1$ . The other parameters have the same value as in Figure 2. According to Theorem 5, we can obtain  $\bar{R}_0 = 0.815 < 1,$  which implies that the disease will be clear. Comparing Figure 3 with Figure 2, the numerical solutions

(see Figure 3) converge to the disease-free equilibrium due to pulse control effect.

### 5. Conclusions

In this paper, we have presented new switched HIV models with transported-related infections and studied the transmission dynamics of the disease under the influence of switching factors and pulse control. The obtained approximation basic reproduction numbers, which change with respect to switch time, differ from the previous results in [10, 11] for single-group models. The cause of this difference is thought to be that the obtained approximation basic reproduction numbers are not calculated straightforwardly as the spectral radius of a next generation matrix, due to multiple

infected compartments. Moreover, by using Razumikhin-type approach, it is shown that the disease will die out if the obtained approximation basic reproduction number is less than one. The theoretical results are able to provide some useful guidance for making effective preventable diseases. Finally, numerical simulations are given to illustrate theoretical analysis results.

## Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

## Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

## Acknowledgments

This work was supported by the National Natural Science Foundation of China (Grant No. 11847081), the Key Scientific Research Projects of Henan Province (Grant No. 17A110039), and the Fundamental Research Funds for the Henan Provincial Colleges and Universities in Henan University of Technology (Grant No. 2017QNJH18), and High-Level Personal Foundation of Henan University of Technology (No. 2017BS009).

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