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*Research Article*

# A Novel Wirtinger-Based Inequality to $H_\infty$ Filtering for Discrete-Time Systems with Time-Varying Delay

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This article is committed to $H_\infty$ filtering for linear discrete-time systems with time-varying delay. The novelty of the paper comes from the consideration of the new Wirtinger-based inequality with double accumulation terms and the idea of delay-partitioning, which guarantees a better asymptotic stability and is less conservative than the celebrated free-weighting matrix or Jensen's inequality methods. In combination with the improved Wirtinger-based inequality to handle the modified Lyapunov-Krasovskii (L-K) functionals, a new delay-dependent bound real lemma (BRL) is gained. In the light of the derived $H_\infty$ performance analysis results, the $H_\infty$ filter will be designed in response to linear matrix inequality (LMI). The validness of the proposed methods will be expressed via some numerical examples by the comparison of existing results.

## 1. Introduction

Concerning the design of the $H_\infty$ filter for control systems, a large number of results were investigated in the literature (see, for example, [1–13] and reference therein). As we all know, one of its goals is to design a more suitable and stable filter to make the maximum ratio of noise signal to estimated error less than a certain positive value, thus ensuring that the $H_\infty$ norm from the interference input to the estimated error output is minimized. Because the classical Kalman filter provides the optimal estimation with the minimum root mean square error as the objective function and makes idealized assumptions in the filtering process, for example, the system model or system disturbance is available. However, in practical engineering applications, when problems such as model uncertainty and noise assumptions are not clear, the Kalman filter becomes incapable. Therefore, in order to cope with model uncertainty and parameter uncertainty, robust Kalman filtering and adaptive Kalman filtering have emerged, respectively. But the results are not satisfactory. Afterwards, researchers began to explore more realistic filters from the perspective of robustness, including the $H_\infty$ filter we have mentioned earlier. Compared with $H_2$ and Kalman filter, the $H_\infty$ filter merely requires the noise energy to be limited without statistical information or the prior knowledge of noise. Obviously, $H_\infty$ filter has better robustness and it is more suitable for practical engineering applications.

In addition, time delay is common and inevitable phenomenon existing in control systems, which often makes our systems property decline or even course instability. However, the results show that the stability condition of delay-dependent filter is less conservative than delay-independent filter [14–18]. In order to reduce the conservativeness of the results obtained, many pieces of literature have studied the bounding techniques for cross-terms and model transformation and selected the appropriate Lyapunov-Krasovskii (L-K) functionals deeply in [19–28]. As mentioned in [29, 30], for delay-dependent $H_\infty$ filter design for discrete-time systems, the authors had proposed using delay-partitioning idea [30, 31] to divide the constant lower bound $d_m$ into m partial intervals. As a result, this delay-partitioning method was included in the newly defined L-K functionals, which means the results acquired also rely on the number of divisions m for the lower bound $d_m$. Therefore, the idea of delay-partitioning plays an important role in reducing the conservativeness of time-delay systems. Furthermore, in combination with some conventional methods, such as free-weighting matrix [32], Jensen's inequality [33], and Wirtinger-based integral inequality with single summation [34], to deal with the chosen L-K functionals, we gained efficient stability analysis criteria for discrete-time systems with time-delay. In the latest article, researchers have proposed Wirtinger-based integral inequality with double summation [35] and reciprocally convex combination inequality [36], which have been applied in practice. And we note that the novel Wirtinger-based integral inequality also has a significant effect in dealing with some nonlinear systems. For example, in [37], Wei and Qiu skillfully combine the new Wirtinger-based integral inequality with the improved reciprocally convex approach and S-processes to obtain a novel $H_\infty$ performance analysis criterion for the underlying closed-loop system. Thus, it effectively reduces the conservativeness of the original results. As the same, for the nonlinear systems with time-varying delays, choosing the appropriate L-K functional and combining an improved Wirtinger-based inequality to handle with the quadratic accumulation term encountered in the derivation of L-K functional are equally effective, as in [38]. Accordingly, this novel summation inequality is more effective than the routine methods—free-weighting matrix, Jensen's inequality, and Wirtinger-based inequality with single summation, which has introduced more system elements while adding a quadratic accumulation term. Thus, we can grasp more systematic information to achieve the purpose of reducing the conservativeness of the original results.

Inspired by the abovementioned, the main contribution of this paper is to further improve the stability criteria for discrete-time systems with time-varying delay and we work on reducing the conservativeness of the stability results. By introducing the new quadratic summation term, more system information is added to our consideration as much as possible in order to make full use of the novel Wirtinger-based inequality to deal with the integral term in L-K functional. For the effective means delay-partitioning and a novel summation inequality, an improved delay-dependent $H_\infty$ filter design for discrete-time systems with a state delay will be presented, importantly, utilizing the new Wirtinger-based inequality with double summation to further process the modified L-K functional, which can acquire a lower conservativeness contrast to what we have mentioned above. Finally, some elaborate numerical examples will be given to prove the effectiveness and the advantage of our ways.

Throughout our paper, the superscript "$T$" represents the matrix transposition, $\mathbb{N}$ denotes the set of nonnegative, $\mathbb{R}^n$ expresses the n-dimensional Euclidean space, $\mathbb{R}^{m\times n}$ means the set of all $m \times n$ real matrices, $0_{m\times n}$ stands for $m \times n$ zero matrix, $\mathbf{I}$ and $\mathbf{0}$ indicate an identity matrix and a zero matrix with appropriate dimension, respectively, and $P > 0$ $(\geq 0)$ is used to denote the matrix $P$ is positively defined and symmetric. Moreover, the $sym\{X\}$ shows $X+X^T$, and diag$\{\ldots\}$ means a block-diagonal matrix. And in the symmetric matrix, the symmetric term is defined as, e.g.,

$$\begin{bmatrix} A & B \\ * & C \end{bmatrix} = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}. \tag{1}$$

## 2. System Description and Problem Analysis

Define the following linear time-invariant discrete-time system with a state delay:

$$\begin{aligned} x(t+1) &= Ax(t) + A_d x(t-\tau(t)) + B\omega(t), \\ y(t) &= Cx(t) + C_d x(t-\tau(t)) + D\omega(t), \\ z(t) &= Hx(t) + H_d x(t-\tau(t)) + L\omega(t), \\ x(t) &= \phi(t), \quad t = -\overline{\tau}, -\overline{\tau}+1, \ldots, 0, \end{aligned} \tag{2}$$

where $x(t) \in \mathbb{R}^n$ stands for the state vector; $y(t) \in \mathbb{R}^q$ represents the measured output; $z(t) \in \mathbb{R}^p$ means the signal to be measured; $\omega(t) \in \mathbb{R}^l$ is the noise input content $\omega \triangleq \{\omega(t)\} \in l_2[0,\infty)$; $x(t) = \phi(t)$, $t = -\overline{\tau}, -\overline{\tau}+1, \ldots, 0$, is a given initial condition sequence; $\tau(t)$ expresses the time delay and we assume that it satisfies

$$1 \leq \underline{\tau} \leq \tau(t) \leq \overline{\tau}, \quad t = 1, 2, \ldots, \tag{3}$$

and besides, $\underline{\tau}$ and $\overline{\tau}$ indicate the known lower bounds and upper bounds of time delay, respectively. At the same time, $A, A_d, B, C, C_d, D, H, H_d$, and $L$ are known constant systems matrices.

Given system (2), we are committed to design a linear time-invariant filter to guarantee the $z_F(t)$ (output of the filter) tracks the original $z(t)$ to be estimated. Consider a linear full-order filter described by

$$\begin{aligned} x_F(t+1) &= A_F x_F(t) + B_F y(t), \quad x_F(0) = 0, \\ z_F(t) &= C_F x_F(t) + D_F y(t), \end{aligned} \tag{4}$$

where $x_F(t) \in \mathbb{R}^n$ is the filter state, and $A_F, B_F, C_F$, and $D_F$ are filer matrices to be determined.

Defining the augmented state vector $\zeta(t) = col[x(t), x_F(t)]$, and the estimation error $e_t = z(t) - z_F(t)$, we receive the following filtering error system:

$$\begin{aligned} \zeta(t+1) &= \overline{A}\zeta(t) + \overline{A_d}K\zeta(t-\tau(t)) + \overline{B}\omega(t), \\ e(t) &= \overline{C}\zeta(t) + \overline{C_d}K\zeta(t-\tau(t)) + \overline{D}\omega(t), \\ \zeta(t) &= \left[\phi^T(t), 0\right]^T, \quad t = -\overline{\tau}, -\overline{\tau}+1, \ldots, 0, \end{aligned} \tag{5}$$

where

$$\overline{A} = \begin{bmatrix} A & 0 \\ B_F C & A_F \end{bmatrix},$$

$$\overline{A_d} = \begin{bmatrix} A_d \\ B_F C_d \end{bmatrix},$$

$$
\begin{aligned}
\overline{B} &= \begin{bmatrix} B \\ B_F D \end{bmatrix}, \\
\overline{C} &= \begin{bmatrix} H - D_F C & -C_F \end{bmatrix}, \\
\overline{C_d} &= H_d - D_F C_d, \\
\overline{D} &= L - D_F D, \\
K &= \begin{bmatrix} I & 0 \end{bmatrix}.
\end{aligned} \tag{6}
$$

Owing to the fact that the original system model has no control input, to prove the asymptotic stability of the filtering error system, we need to assume that system (2) is asymptotically stable.

The propose of our paper is to design a stable full-order filter of the form (4) and meet the following two points: Firstly, the filter error system (5) is asymptomatically stable and secondly, the filtering error system in (5) has a prescribed level $\gamma$ of $H_\infty$ noise attenuation, and under zero-initial condition, $\|e\|_2 \leq \gamma\|\omega\|_2$ is satisfied for all nonzero $\omega \in l_2[0, \infty)$.

Before entering the next section, a vital lemma will be introduced to be used in the derivation of our conclusion. Then, for any sequence of discrete-time variable $x$ in $[\alpha, \beta] \longmapsto \mathbb{R}^n$, we define $l = \beta - \alpha$, $\kappa_1 = (1/(l+1)) \sum_{i=\alpha}^{\beta} x(i)$, $\kappa_2 = (2/(l+1)(l+2)) \sum_{i=\alpha}^{\beta} \sum_{j=i}^{\beta} x(j)$.

**Lemma 1** (see [35, 39]). *For any sequence of discrete-time variable $x$ in $[\alpha, \beta] \longmapsto \mathbb{R}^n$, matrices $R > 0$, $L_i \in \mathbb{R}^{4n\times n}$, $i \in \{0, 1, 2\}$, and an arbitrary vector $\delta$, the following summation inequality holds:*

$$
-\sum_{i=\alpha}^{\beta-1} y^T(i) R y(i) \leq \delta^T \psi_0 \delta + sym\left\{\delta^T \mathrm{F}\right\}, \tag{7}
$$

*where*

$$
\begin{aligned}
\delta &\triangleq col\{x(\beta), x(\alpha), \kappa_1, \kappa_2\}, \\
\psi_0 &\triangleq \sigma_0 L_0 R^{-1} L_0^T + \sigma_1 L_1 R^{-1} L_1^T + \sigma_2 L_2 R^{-1} L_2^T, \\
\mathrm{F} &\triangleq L_0 \mathrm{F}_0 + L_1 \mathrm{F}_1 + L_2 \mathrm{F}_2, \\
\sigma_0 &= l, \\
\sigma_1 &= \frac{l(l-1)}{3(l+1)}, \\
\sigma_2 &= \frac{l(l-1)(l-2)}{5(l+1)(l+2)}, \\
\mathrm{F}_0 &= x(\beta) - x(\alpha), \\
\mathrm{F}_1 &= x(\beta) + x(\alpha) - 2\kappa_1, \\
\mathrm{F}_2 &= x(\beta) - x(\alpha) + 6\kappa_1 - 6\kappa_2.
\end{aligned} \tag{8}
$$

*Remark 2.* As Lemma 1 described, if $(l-1)/(l+1) < 1$ and $(l-1)(l-2)/(l+1)(l+2) < 1$, inequality (7) could be represented into the following inequality:

$$
-\sum_{i=\alpha}^{\beta-1} y^T(i) R y(i) \leq \delta^T \psi_1 \delta + sym\left\{\delta^T \mathrm{F}\right\}, \tag{9}
$$

where

$$
\psi_1 \triangleq \sigma_0 \left(L_0 R^{-1} L_0^T + \frac{1}{3} L_1 R^{-1} L_1^T + \frac{1}{5} L_2 R^{-1} L_2^T\right). \tag{10}
$$

## 3. $H\infty$ Performance Analysis

In this part, we will conduct $H_\infty$ filtering stability analysis by using aforementioned Lemma 1. Before we begin, considering the idea of the delay-partitioning, we note that the lower bound $\underline{\tau}$ of time-varying delay will be described as $\underline{\tau} = \mu\lambda$, where $\mu$ and $\lambda$ are positive integers. Meanwhile, several definitions are as follows:

$$
\begin{aligned}
&\tau_t \triangleq \tau(t), \quad \tau = \overline{\tau} - \underline{\tau}, \\
&\tau_1 \triangleq \tau_t - \underline{\tau}, \\
&\tau_2 \triangleq \overline{\tau} - \tau_t, \\
&\Gamma(t) \triangleq col\{x(t), x(t-\lambda), \ldots, x(t-(\mu-1)\lambda)\}, \\
&\eta_1(t) \triangleq col\{\zeta(t), \Gamma(t-\lambda), x(t-\tau_t), x(t-\overline{\tau})\}, \\
&\eta_2(t) \triangleq col\left\{\frac{1}{\lambda+1}\sum_{i=t-\lambda}^{t} x(i), \frac{1}{\tau_1+1}\sum_{i=t-\tau_t}^{t-\underline{\tau}} x(i), \frac{1}{\tau_2+1}\right. \\
&\qquad \left. \cdot \sum_{i=t-\overline{\tau}}^{t-\tau_t} x(i)\right\}, \\
&\eta_3(t) \triangleq col\left\{\frac{2}{(\lambda+1)(\lambda+2)}\right. \\
&\qquad \cdot \sum_{i=t-\lambda}^{t}\sum_{j=i}^{t} x(j), \frac{2}{(\tau_1+1)(\tau_1+2)} \\
&\qquad \left. \cdot \sum_{i=t-\tau_t}^{t-\underline{\tau}}\sum_{j=i}^{t-\underline{\tau}} x(j), \frac{2}{(\tau_2+1)(\tau_2+2)} \sum_{i=t-\overline{\tau}}^{t-\tau_t}\sum_{j=i}^{t-\tau_t} x(j)\right\}, \\
&\eta(t) \triangleq col\{\eta_1(t), \eta_2(t), \eta_3(t)\}, \\
&\eta_*(t) \triangleq col\{\eta(t), \omega(t)\}, \\
&e_i \triangleq \{0_{n\times(i-1)n} \;\; I_n \;\; 0_{n\times(10+\mu-i)n}\}, \quad i \in \{1, \ldots, 10+\mu\}, \\
&\Pi_1 \triangleq col\{e_1, e_2\}, \\
&\Pi_2 \triangleq col\{e_1, e_3\}, \\
&\Pi_3 \triangleq col\left\{e_{2+j}\right\}, \quad j \in \{1, \ldots, \mu\}, \\
&\Pi_4 \triangleq col\left\{e_1, e_3, e_{(2+\mu)+3}, e_{(2+\mu)+6}\right\},
\end{aligned}
$$

$$\begin{aligned}
&\Pi_5 \triangleq e_1 - e_3,\\
&\Pi_6 \triangleq e_1 + e_3 - 2e_{(2+\mu)+3},\\
&\Pi_7 \triangleq e_1 - e_3 + 6e_{(2+\mu)+3} - 6e_{(2+\mu)+6},\\
&\Pi_8 \triangleq col\left\{e_{2+\mu}, e_{(2+\mu)+1}, e_{(2+\mu)+4}, e_{(2+\mu)+7}\right\},\\
&\Pi_9 \triangleq e_{2+\mu} - e_{(2+\mu)+1},\\
&\Pi_{10} \triangleq e_{2+\mu} + e_{(2+\mu)+1} - 2e_{(2+\mu)+4},\\
&\Pi_{11} \triangleq e_{2+\mu} - e_{(2+\mu)+1} + 6e_{(2+\mu)+4} - 6e_{(2+\mu)+7},\\
&\Pi_{12} \triangleq col\left\{e_{(2+\mu)+1}, e_{(2+\mu)+2}, e_{(2+\mu)+5}, e_{(2+\mu)+8}\right\},\\
&\Pi_{13} \triangleq e_{(2+\mu)+1} - e_{(2+\mu)+2},\\
&\Pi_{14} \triangleq e_{(2+\mu)+1} + e_{(2+\mu)+2} - 2e_{(2+\mu)+5},\\
&\Pi_{15} \triangleq e_{(2+\mu)+1} - e_{(2+\mu)+2} + 6e_{(2+\mu)+5} - 6e_{(2+\mu)+8}.
\end{aligned} \tag{11}$$

**Theorem 3.** *In view of system (2), we assume integers $\mu > 0$ and $\lambda > 0$ with satisfying $\underline{\tau} = \mu\lambda$. For an admissible $H_\infty$ filter and a prescribed scalar $\gamma > 0$, the filtering error system (5) is asymptotically stable if there exist real matrices*

$$\begin{aligned}
&P = P^T > 0,\\
&Q = \begin{bmatrix} Q_1 & Q_2 \\ * & Q_3 \end{bmatrix} > 0,\\
&R_j = R_j^T > 0,\\
&S_j = S_j^T > 0,\\
&\qquad j = 1, 2,\\
&G, F, L_q, M_q, N_q, \quad q = 0, 1, 2,
\end{aligned} \tag{12}$$

*such that following LMI*

$$\begin{bmatrix}
\phi_4 & \phi_1^T G & \phi_2^T F & \phi_3^T & \phi_L & \phi_M(\tau_t) & \phi_N(\tau_t) \\
* & \phi_5 & 0 & 0 & 0 & 0 & 0 \\
* & * & \phi_6 & 0 & 0 & 0 & 0 \\
* & * & * & -I & 0 & 0 & 0 \\
* & * & * & * & -S_{1t} & 0 & 0 \\
* & * & * & * & * & -S_{2t} & 0 \\
* & * & * & * & * & * & -S_{3t}
\end{bmatrix} < 0 \tag{13}$$

*holds for $\tau_t \in \{\underline{\tau}, \overline{\tau}\}$, where*

$$\begin{aligned}
&\phi_1 = \left[\overline{A}\Pi_1 \;\; \overline{A_d}e_{(2+\mu)+1} \;\; \overline{B}\right],\\
&\phi_2 = \left[(A - I)e_1 \;\; A_d e_{(2+\mu)+1} \;\; B\right],\\
&\phi_3 = \left[\overline{C}\Pi_1 \;\; \overline{C_d}e_{(2+\mu)+1} \;\; \overline{D}\right],\\
&\phi_4 = \begin{bmatrix} \Xi_0 & 0 \\ * & -\gamma^2 I \end{bmatrix},\\
&\phi_5 = P - X - X^T,\\
&\phi_6 = \Omega - F - F^T, \quad \Omega = \lambda S_1 + \tau S_2,\\
&\phi_L = \lambda\Pi_4^T\left[L_0 \;\; L_1 \;\; L_2\right],\\
&\phi_M(\tau_t) = \tau_1\Pi_8^T\left[M_0 \;\; M_1 \;\; M_2\right],\\
&\phi_N(\tau_t) = \tau_2\Pi_{12}^T\left[N_0 \;\; N_1 \;\; N_2\right],\\
&\Xi_0\\
&\quad = -\Pi_1^T P\Pi_1 + sym\left\{e_1^T Q_2 e_3\right\} + e_1^T Q e_1 + e_3^T Q_3 e_3\\
&\quad\quad - \Pi_3^T Q\Pi_3 + e_1^T\left(R_1 + (\tau + 1)R_2\right)e_1\\
&\quad\quad - e_{(2+\mu)+2}^T R_1 e_{(2+\mu)+2} - e_{(2+\mu)+1}^T R_2 e_{(2+\mu)+1}\\
&\quad\quad + sym\left\{\Pi_4^T L_0\Pi_5 + \Pi_4^T L_1\Pi_6 + \Pi_4^T L_2\Pi_7\right\}\\
&\quad\quad + sym\left\{\Pi_8^T M_0\Pi_9 + \Pi_8^T M_1\Pi_{10} + \Pi_8^T M_2\Pi_{11}\right\}\\
&\quad\quad + sym\left\{\Pi_{12}^T N_0\Pi_{13} + \Pi_{12}^T N_1\Pi_{14} + \Pi_{12}^T N_2\Pi_{15}\right\},\\
&\widetilde{S}_1 = \mathrm{diag}\left\{S_1, 3S_1, 5S_1\right\},\\
&\widetilde{S}_2 = \mathrm{diag}\left\{S_2, 3S_2, 5S_2\right\},\\
&S_{1t} = \mathrm{diag}\left\{\lambda S_1, 3\lambda S_1, 5\lambda S_1\right\},\\
&S_{2t}(\tau_t) = \mathrm{diag}\left\{\tau_1 S_2, 3\tau_1 S_2, 5\tau_1 S_2\right\},\\
&S_{3t}(\tau_t) = \mathrm{diag}\left\{\tau_2 S_2, 3\tau_2 S_2, 5\tau_2 S_2\right\}.
\end{aligned} \tag{14}$$

*Proof.* Now, considering the idea of a slack-variable [40] and using the similar lines as in [29], the following LMI (15) is equivalent to (13):

$$\begin{bmatrix}
\phi_4 & \phi_1^T P & \phi_2^T \Omega & \phi_3^T & \phi_L & \phi_M(\tau_t) & \phi_N(\tau_t) \\
* & -P & 0 & 0 & 0 & 0 & 0 \\
* & * & -\Omega & 0 & 0 & 0 & 0 \\
* & * & * & -I & 0 & 0 & 0 \\
* & * & * & * & -S_{1t} & 0 & 0 \\
* & * & * & * & * & -S_{2t} & 0 \\
* & * & * & * & * & * & -S_{3t}
\end{bmatrix} < 0. \tag{15}$$

Then, we consider the modified L-K functional candidate which contains the delay partitioning items:

$$V(t) \triangleq V_0(t) + V_1(t) + V_2(t) + V_3(t), \tag{16}$$

with

$$\begin{aligned} V_0(t) &\triangleq \zeta^T(t) P\zeta(t), \\ V_1(t) &\triangleq \sum_{i=t-\lambda}^{t-1} \Gamma^T(i) Q\Gamma(i), \\ V_2(t) &\triangleq \sum_{i=t-\overline{\tau}}^{t-1} x^T(i) R_1 x(i) \\ &\quad + \sum_{j=-\overline{\tau}+1}^{-\underline{\tau}+1} \sum_{i=t-1+j}^{t-1} x^T(i) R_2 x(i), \\ V_3(t) &\triangleq \sum_{i=-\lambda}^{-1} \sum_{j=t+i}^{t-1} \theta^T(j) S_1 \theta(j) \\ &\quad + \sum_{i=-\overline{\tau}}^{-\underline{\tau}-1} \sum_{j=t+i}^{t-1} \theta^T(j) S_2 \theta(j). \end{aligned} \tag{17}$$

And we note that $\theta(t) \triangleq x(t+1) - x(t) = \phi_2 \eta_*(t)$, $P > 0$, $Q > 0$, $R_j > 0$, $S_j > 0$, $j = 1, 2$. Then utilizing the previously described $V(t)$, we will obtain the equation as follows:

$$\begin{aligned} \Delta V_0(t) &= \zeta^T(t+1) P\zeta(t+1) - \zeta^T(t) P\zeta(t) = \eta^T(t) \\ &\quad \cdot \Big[\Pi_1^T \left(\overline{A}^T P\overline{A} - P\right) \Pi_1 \\ &\quad + sym\left(\Pi_1^T \overline{A}^T P\overline{A_d} e_{(2+\mu)+1}\right) \\ &\quad + e_{(2+\mu)+1}^T \overline{A_d}^T P\overline{A_d} e_{(2+\mu)+1}\Big] \eta(t), \end{aligned} \tag{18}$$

$$\begin{aligned} \Delta V_1(t) &= \Gamma^T(t) Q\Gamma(t) - \Gamma^T(t-\lambda) Q\Gamma(t-\lambda) \\ &= \eta^T(t) \left[\Pi_2^T Q\Pi_2 - \Pi_3^T Q\Pi_3\right] \eta(t), \end{aligned} \tag{19}$$

$$\begin{aligned} \Delta V_2(t) &= x^T(t) R_1 x(t) - x^T(t-\overline{\tau}) R_1 x(t-\overline{\tau}) \\ &\quad + \sum_{j=-\overline{\tau}+1}^{-\underline{\tau}+1} x^T(t) R_2 x(t) - \sum_{j=t-\overline{\tau}}^{t-\overline{\tau}} x^T(j) R_2 x(j) \\ &\le \eta^T(t) \Big[e_1^T (R_1 + (\tau+1) R_2) e_1 \\ &\quad - e_{(2+\mu)+2}^T R_1 e_{(2+\mu)+2} - e_{(2+\mu)+1}^T R_2 e_{(2+\mu)+1}\Big] \eta(t), \end{aligned} \tag{20}$$

$$\begin{aligned} \Delta V_3(t) &= \theta^T(t) (\lambda S_1 + \tau S_2) \theta(t) - \sum_{i=t-\lambda}^{t-1} \theta^T(i) S_1 \theta(i) \\ &\quad - \sum_{i=t-\overline{\tau}}^{t-\underline{\tau}-1} \theta^T(i) S_2 \theta(i) = \eta^T(t) \\ &\quad \cdot \Big[e_1^T (A-I)^T \Omega (A-I) e_1 \\ &\quad + sym\left\{e_1^T (A-I)^T \Omega A_d e_{(2+\mu)+1}\right\} \\ &\quad + e_{(2+\mu)+1}^T A_d^T \Omega A_d e_{(2+\mu)+1}\Big] \eta(t) - \sum_{i=t-\lambda}^{t-1} \theta^T(i) S_1 \theta(i) \\ &\quad - \sum_{i=t-\tau_t}^{t-\underline{\tau}-1} \theta^T(i) S_2 \theta(i) - \sum_{i=t-\overline{\tau}}^{t-\tau_t-1} \theta^T(i) S_2 \theta(i). \end{aligned} \tag{21}$$

Applying inequality (9), to the summation terms in (21), and we can set $\delta = col\{x(t), x(t-\lambda), (1/(\lambda+1))\sum_{i=t-\lambda}^{t} x(i), (2/(\lambda+1)(\lambda+2)) \sum_{i=t-\lambda}^{t} \sum_{j=i}^{t} x(j)\}$, the following inequalities hold:

$$\begin{aligned} -\sum_{i=t-\lambda}^{t-1} \theta^T(i) S_1 \theta(i) &\le \eta^T(t) \Big\{\lambda \Pi_4^T L_j \widetilde{S}_1^{-1} L_j^T \Pi_4 \\ &\quad + sym\left\{\Pi_4^T L_0 \Pi_5 + \Pi_4^T L_1 \Pi_6 + \Pi_4^T L_2 \Pi_7\right\}\Big\} \eta(t). \end{aligned} \tag{22}$$

Also, apply inequality (9), and set $\delta = col\{x(t-\underline{\tau}), x(t-\tau_t), (1/(\tau_1+1))\sum_{i=t-\tau_t}^{t-\underline{\tau}} x(i), (2/(\tau_1+1)(\tau_1+2)) \sum_{i=t-\tau_t}^{t-\underline{\tau}} \sum_{j=i}^{t-\underline{\tau}} x(j)\}$; we have

$$\begin{aligned} -\sum_{i=t-\tau_t}^{t-\underline{\tau}-1} \theta^T(i) S_2 \theta(i) &\le \eta^T(t) \Big\{\tau_1 \Pi_8^T M_j \widetilde{S}_2^{-1} M_j^T \Pi_8 \\ &\quad + sym\left\{\Pi_8^T M_0 \Pi_9 + \Pi_8^T M_1 \Pi_{10} + \Pi_8^T M_2 \Pi_{11}\right\}\Big\} \eta(t). \end{aligned} \tag{23}$$

Similarly, apply inequality (9), and set $\delta = col\{x(t-\tau_t), x(t-\overline{\tau}), (1/(\tau_2+1)) \sum_{i=t-\overline{\tau}}^{t-\tau_t} x(i), (2/(\tau_2+1)(\tau_2+2)) \sum_{i=t-\overline{\tau}}^{t-\tau_t} \sum_{j=i}^{t-\tau_t} x(j)\}$; then we have

$$\begin{aligned} -\sum_{i=t-\overline{\tau}}^{t-\tau_t-1} \theta^T(i) S_2 \theta(i) &\le \eta^T(t) \Big\{\tau_2 \Pi_{12}^T N_j \widetilde{S}_2^{-1} N_j^T \Pi_{12} \\ &\quad + sym\left\{\Pi_{12}^T N_0 \Pi_{13} + \Pi_{12}^T N_1 \Pi_{14} + \Pi_{12}^T N_2 \Pi_{15}\right\}\Big\} \\ &\quad \cdot \eta(t). \end{aligned} \tag{24}$$

Then, by combining (18)–(24), and we suppose that the input has zero noise, i.e., $\omega(t) = 0$, and we have

$$\begin{aligned} \Delta V(t) &= \Delta V_0(t) + \Delta V_1(t) + \Delta V_2(t) + \Delta V_3(t) \\ &\le \eta^T(t) \left[\Xi_0 + \Xi_1 + \Xi_2(\tau_t)\right] \eta(t), \end{aligned} \tag{25}$$

where

$$\begin{aligned} \Xi_1 &= \Pi_1^T \overline{A}^T P\overline{A} \Pi_1 + sym\left(\Pi_1^T \overline{A}^T P\overline{A_d} e_{(2+\mu)+1}\right) \\ &\quad + e_{(2+\mu)+1}^T \overline{A_d}^T P\overline{A_d} e_{(2+\mu)+1} \\ &\quad + e_1^T (A-I)^T \Omega (A-I) e_1 \\ &\quad + sym\left\{e_1^T (A-I)^T \Omega A_d e_{(2+\mu)+1}\right\} \\ &\quad + e_{(2+\mu)+1}^T A_d^T \Omega A_d e_{(2+\mu)+1}, \\ \Xi_2(\tau_t) &= \lambda \Pi_4^T L\widetilde{S}_1^{-1} L^T \Pi_4 + \tau_1 \Pi_8^T M\widetilde{S}_2^{-1} M^T \Pi_8 \\ &\quad + \tau_2 \Pi_{12}^T N\widetilde{S}_2^{-1} N^T \Pi_{12}. \end{aligned} \tag{26}$$

According to the Schur complement, for all nonzero $\zeta(t)$, $\Xi_0 + \Xi_1 + \Xi_2(\tau_t) < 0$ is equivalent to the LMI in (13). Thus, the negative definition of the $\Delta V(t)$ will guarantee the filtering error system (5) is asymptotically stable.

Furthermore, in order to construct the $H_\infty$ performance index, we define $J \triangleq \sum_{t=0}^{\infty}[e^T(t)e(t) - \gamma^2\omega^T(t)\omega(t)]$. And under zero initial condition $V(0) = 0$, $V(\infty) > 0$, and $\omega(t) \neq 0$, one obtains

$$\begin{aligned} J &\leq \sum_{t=0}^{\infty}\left[e^T(t)e(t) - \gamma^2\omega^T(t)\omega(t)\right] + V(0) - V(\infty) \\ &= \sum_{t=0}^{\infty}\left[e^T(t)e(t) - \gamma^2\omega^T(t)\omega(t) + \Delta V(t)\right] \\ &\leq \sum_{t=0}^{\infty}\eta_*^T(t)\left[\phi_4 + \phi_1^T P\phi_1 + \phi_2^T\Omega\phi_2 + \phi_3^T\phi_3 + \phi_\star\right] \\ &\cdot \eta_*(t), \end{aligned} \quad (27)$$

where $\phi_\star = \phi_L\widetilde{S}_1\phi_L^T + \phi_M\widetilde{S}_2\phi_M^T + \phi_N\widetilde{S}_2\phi_N^T$. By using the Schur complement, inequality (27) indicates that $J < 0$, which is equivalent to the LMI (15). Besides, for all $\omega(t) \in l_2[0, +\infty)$, $\|e\|_2 < \gamma\|\omega\|_2$. Then, this completes the proof. □

*Remark 4.* Different from the previous methods where free-weighting matrix and Jensen's inequality are used to handle the double summation function $\Delta V_3(t)$ in the proof of Theorem 3, the novel Wirtinger-based inequality with double summation could carry out an accurate summation inequality to bound the signal summation term and give rise to the derivative of the L-K functional to get vital in reducing the conservatism.

## 4. $H_\infty$ Filter Design

In this section, we will focus on the design of full-order $H_\infty$ filter for state delay in system (2).

**Theorem 5.** *In view of system (2), we assume integers $\mu > 0$ and $\lambda > 0$ with satisfying $\underline{\tau} = \mu\lambda$. For an admissible filter of form (4) ensure a prescribed $H_\infty$ norm bound $\gamma$ and the asymmetric stability of the filtering error system (5) exist if there exist matrices $V_1, V_2, V_3, F, \overline{A}_F, \overline{B}_F, \overline{C}_F, \overline{D}_F$ of appropriate dimensions and*

$$\begin{aligned} \overline{P} &= \begin{bmatrix} \overline{P}_1 & \overline{P}_2 \\ * & \overline{P}_3 \end{bmatrix} > 0, \\ Q &= \begin{bmatrix} Q_1 & Q_2 \\ * & Q_3 \end{bmatrix} > 0, \\ R_j &= R_j^T, \\ S_j &= S_j^T, \\ & \qquad j = 1, 2, \\ G, F, L_q&, M_q, N_q, \quad q = 0, 1, 2, \end{aligned} \quad (28)$$

*such that following LMIs are satisfied:*

$$\Xi = \begin{bmatrix} \phi & \phi_L & \phi_M(\tau_t) & \phi_N(\tau_t) \\ * & -S_{1t} & 0 & 0 \\ * & * & -S_{2t} & 0 \\ * & * & * & -S_{3t} \end{bmatrix} < 0, \quad (29)$$

*where*

$$\begin{aligned} \phi &= \begin{bmatrix} \Xi_0 & 0 & \Xi_1^1 \\ * & -\gamma^2 I & \Xi_1^2 \\ * & * & \Xi_1^3 \end{bmatrix}, \\ \Xi_1^1 &= col\left[\Theta_0 \;\; \Theta_1 \;\; 0_{2\times 4} \;\; \Theta_2 \;\; 0_{7\times 4}\right], \\ \Theta_0 & \\ &= \left[A^T V_1 + C^T\overline{B}_F^T \;\; A^T V_3 + C^T\overline{B}_F^T \;\; (A - I)^T F \;\; H^T - C^T\overline{D}_F^T\right], \\ \Theta_1 &= \left[\overline{A}_F^T \;\; \overline{A}_F^T \;\; 0 \;\; -\overline{C}_F^T\right], \\ \Theta_2 &= \left[A_d^T V_1 + C_d^T\overline{B}_F^T \;\; A_d^T V_3 + C_d^T\overline{B}_F^T \;\; A_d^T F \;\; H_d^T - C_d^T\overline{D}_F^T\right], \\ \Xi_1^2 &= \left[B^T V_1 + D^T\overline{B}_F^T \;\; B^T V_3 + D^T\overline{B}_F^T \;\; B^T F \;\; L^T - D^T\overline{D}_F^T\right], \\ \Xi_1^3 &= \begin{bmatrix} \overline{P}_1 - V_1 - V_1^T & \overline{P}_2 - V_2 - V_3 & 0 & 0 \\ * & \overline{P}_3 - V_2 - V_2^T & 0 & 0 \\ * & * & \phi_6 & 0 \\ * & * & * & -I \end{bmatrix}. \end{aligned} \quad (30)$$

*Then, the filtering problem can be solved. In addition, a suitable filter state space realization is presented by*

$$\begin{aligned} A_F &= V_2^{-1}\overline{A}_F, \\ B_F &= V_2^{-1}\overline{B}_F, \\ C_F &= \overline{C}_F, \\ D_F &= \overline{D}_F. \end{aligned} \quad (31)$$

*Proof.* Due to $\overline{P} > 0$ and LMI in (29), it follows that $\overline{P}_3 > 0$ and $-\overline{P}_3 + V_2 + V_2^T > 0$, respectively, which means $V_2 + V_2^T > 0$. Thus, we can conclude that $V_2$ is nonsingular and for arbitrary square and nonsingular matrices $X_3$ and $X_4$, meeting $V_2 = X_3^T X_4^{-1} X_3$. Then, we define the matrices:

$$\begin{aligned} J_1 &= \begin{bmatrix} I & 0 \\ 0 & X_4^{-1}X_3 \end{bmatrix} > 0, \\ X_2 &= V_3 X_3^{-1} X_4, \\ X &= \begin{bmatrix} V_1 & X_2 \\ X_3 & X_4 \end{bmatrix}, \\ P &= J_1^{-T}\begin{bmatrix} \overline{P}_1 & \overline{P}_2 \\ * & \overline{P}_3 \end{bmatrix} J_1^{-1}, \\ J_2 &= \operatorname{diag}\left\{J_1, I\cdots I, J_1, I\cdots I\right\}, \end{aligned} \quad (32)$$

$$\begin{aligned} A_F &= X_3^{-T}\overline{A}_F X_3^{-1} X_4, \\ B_F &= X_3^{-T}\overline{B}_F, \\ C_F &= \overline{C}_F X_3^{-1} X_4, \\ D_F &= \overline{D}_F. \end{aligned} \tag{33}$$

Then, matric $\overline{P}$ and LMI (29) can be expressed as

$$J_1^T P J_1 > 0, \tag{34}$$

$$J_2^T \Xi J_2 < 0, \tag{35}$$

So, by way of congruent transformation to LMI (34) and (35), respectively, with $J_1^{-1}$ and $J_2^{-1}$, it is clear that LMI in (29) is equivalent to (13). Consequently, from Theorem 5, we could draw a conclusion that the filtering error system in (5) is asymptotically stable with a $H_\infty$ norm bound $\gamma$.

The next job is to get the filer state-space realization as (31). Through some conventional matrix processing in [10], substituting $V_2 = X_3^T X_4^{-1} X_3$ into (33), then

$$\begin{bmatrix} A_F & B_F \\ C_F & D_F \end{bmatrix} \tag{36}$$

could be directly rewritten as

$$\begin{bmatrix} X_3^{-1}X_4 & 0 \\ 0 & I \end{bmatrix}^{-1} \begin{bmatrix} V_2^{-1}\overline{A}_F & V_2^{-1}\overline{B}_F \\ \overline{C}_F & \overline{D}_F \end{bmatrix} \begin{bmatrix} X_3^{-1}X_4 & 0 \\ 0 & I \end{bmatrix}. \tag{37}$$

Due to the fact that matrices $X_3$ and $X_4$ are nonsingular, we can infer that

$$\begin{bmatrix} X_3^{-1}X_4 & 0 \\ 0 & I \end{bmatrix} \tag{38}$$

is also nonsingular. And this makes the point that the following systems are algebraically equivalent:

$$\begin{bmatrix} A_F & B_F \\ C_F & D_F \end{bmatrix} \Longleftrightarrow \begin{bmatrix} V_2^{-1}\overline{A}_F & V_2^{-1}\overline{B}_F \\ \overline{C}_F & \overline{D}_F \end{bmatrix}, \tag{39}$$

and thus, a state-space realization $(A_F, B_F, C_F, D_F)$, which is defined in (4), of the desired filter could be acquired from (39). This completes the proof. □

## 5. Numerical Examples

*Example 1* (see [29]). Firstly, we consider system (2) with varying-delay:

$$\begin{aligned} A &= \begin{bmatrix} 0.8 & 0 \\ 0.1 & 0.9 \end{bmatrix}, \\ A_d &= \begin{bmatrix} -0.1 & 0.15 \\ -0.1 & -0.15 \end{bmatrix}, \\ B &= \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \\ C &= \begin{bmatrix} 1 & 1 \end{bmatrix}, \\ C_d &= \begin{bmatrix} 0.4 & 0.6 \end{bmatrix}, \\ D &= 1, \\ H &= \begin{bmatrix} 1 & 2 \end{bmatrix}, \\ H_d &= \begin{bmatrix} 0.5 & 0.6 \end{bmatrix}, \\ L &= -0.5, \end{aligned} \tag{40}$$

and $\underline{\tau} = 6$, $\overline{\tau} = 8$, $\mu = 2$, by processing of Theorem 5, the optimal $\gamma^* = 9.9691$, which is much less than $\gamma^* = 13.8580$[1], 8.7038[29], 10.3752[29], respectively. Our achieved results have lower conservativeness than presently existing methods. And when $\overline{\tau} = 9$ the $H_\infty$ filters parameters

$$\left[\begin{array}{c|c} A_F & B_F \\ \hline C_F & D_F \end{array}\right] \tag{41}$$

are given as follows:

$$\left[\begin{array}{cc|c} 0.1250 & 0.1086 & -0.2255 \\ 0.1380 & 0.3049 & -0.3008 \\ \hline 1.4573 \times 10^{-5} & -1.8674 \times 10^{-5} & 1.4699 \end{array}\right]. \tag{42}$$

More detailed data will be listed in Table 1, which content the minimum $\gamma^*$ with different upper bounds $\overline{\tau}$, and set $\underline{\tau} = 6$.

*Example 2* (see [10]). Then, we consider system (2) with

$$\begin{aligned} A &= \begin{bmatrix} 0.85 & 0.1 \\ -0.1 & 0.7 \end{bmatrix}, \\ A_d &= \begin{bmatrix} 0.2 & 0 \\ -0.2 & 0.1 \end{bmatrix}, \\ B &= \begin{bmatrix} 0.1 \\ 0.4 \end{bmatrix}, \\ C &= \begin{bmatrix} 0.2 & 2.5 \end{bmatrix}, \\ C_d &= \begin{bmatrix} -0.5 & 0.5 \end{bmatrix}, \\ D &= -1, \\ H &= \begin{bmatrix} 0 & 2.2 \end{bmatrix}, \\ H_d &= \begin{bmatrix} 1.5 & -0.4 \end{bmatrix}, \\ L &= -0.1, \end{aligned} \tag{43}$$

and let $\mu = 1, \mu = 2$, respectively, with different lower and upper bounds of time-delay. Thus, we could have the clear discovery that when $\mu = 2$, $\underline{\tau} = 2$, and $\overline{\tau} = 6$, by processing of Theorem 5, we could acquire $\gamma^* = 3.9910$, which is much less

Table 1: The minimum $\gamma^*$ with different $\overline{\tau}$ when $\underline{\tau} = 6$.

| $\overline{\tau}$ | 7 | 8 | 9 |
|---|---|---|---|
| [1] | 7.5050 | 13.8580 | $\infty$ |
| [29]Theorem 4 | 6.0430 | 8.7038 | 13.9799 |
| [29]Theorem 5 | 6.2928 | 10.3752 | 29.9862 |
| Theorem 5 | 5.7470 | 7.5334 | 9.9691 |

Table 2: The minimum $\gamma^*$ with different $\mu, \overline{\tau}$ and $\underline{\tau}$.

| $\mu$ | 1 | 1 | 2 | 2 |
|---|---|---|---|---|
| $\underline{\tau}$ | 1 | 1 | 2 | 2 |
| $\overline{\tau}$ | 4 | 5 | 5 | 6 |
| [10] | 4.9431 | 6.1608 | 5.3551 | 6.7581 |
| [29]Theorem 4 | 3.6545 | 4.6494 | 5.3458 | 6.7185 |
| Theorem 5 | 1.7342 | 2.2883 | 3.9551 | 3.9910 |

than $\gamma^* = 6.7185, 6.7581$, respectively, in [10, 29]. The $H_\infty$ filters parameters are given in (44)

$$\left[\begin{array}{cc|c} 1.0252 & 1.4341 & 0.3874 \\ -0.8278 & -1.1579 & -0.0675 \\ \hline -0.3177 & -0.4441 & 0.8472 \end{array}\right]. \tag{44}$$

Similarly, more specific data will be listed in Table 2, which content the minimum $\gamma^*$ with different delay partitioning number $\mu$ and delay boundary.

## 6. Conclusions

By this we have completed the study for the delay-dependent $H_\infty$ filter design for discrete-time system with varying-delay. In the research process, by introducing the novel Wirtinger-based inequality to further handle the quadratic accumulative items coming from the modified Lyapunov-Krasovskii functionals and taking advantage of the delay-partitioning idea, a new BRL for the filtering system has been gained. Furthermore, the achieved result has lower conservativeness than presently existing methods, for instance, Jensen's inequality and free-weighting matrix methods, etc. Eventually, a numerical example has been provided to verify the validness of our method.

## Data Availability

The numerical simulation data used to support the findings of this study are available from the corresponding author upon request.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

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