Research Article

Stabilization of Discrete-Time Delayed Systems in Presence of Actuator Saturation Based on Wirtinger Inequality

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This paper examines the stability analysis of discrete-time control systems particularly during the event of actuator saturation and time-varying state delay. With the help of Wirtinger inequality along with Lyapunov-Krasovskii functional gain of state feedback controller is determined for stabilization of above system. The saturation nonlinearity is represented in the terms of convex hull. A new linear matrix inequality (LMI) criterion is settled with reciprocally convex combination based inequality which is dependent on delay. The proposed criterion is less conservative in concern to increase the delay bound and a controller is also simulated for real time problem of missile control system in this paper. It is also attained that projected stability criterion is less conservative compared to other outcomes. Furthermore, an optimization procedure together with LMI constraints has been proposed to maximize the attraction of domain.

1. Introduction

Practically, every system is distressed due to common nonlinearities like saturation, deadzone, backlash, etc. Among them actuator saturation is more general in industrial control systems since there is always limitation in inputs of control in the form of amplitude as well as rate saturation. Saturation nonlinearity leads to degradation of system stability and is responsible for poor performance of closed loop system. In most of the research works, sector based conditions [1] and convex hull approach [2] are selected to work on the actuator or input saturation nonlinearity. The problem of control system stability with actuator saturation has received extensive attention [3–10].

Time delay is another phenomenon which exists in dynamical systems due to communication between data processing elements in computerised control systems, e.g., processes controlled via SCADA in chemical industries [11], optimal communication network systems [12], metrology systems [13], power systems, and neural network control system [14, 15]. Performance deterioration and instability are other

issues created by this phenomenon. An adequate amount of work has been carried out on delayed systems [[16–21] and the references therein]. There are two common and popular ways to deal with time delayed systems, i.e., dependent on delay and not dependent on delay. The information about delay is utilized in delay-dependent method which yields less conservative results for limited delay range. Presently, many researchers are devoting their attention in stability analysis of time delayed processes using various approaches (such as model transformation, free weighting matrix, free matrix based integral inequality, Park's inequality, Jensen's inequality, Convex combination method and Wirtinger-based integral inequality, etc.) for achieving less conservative results.

Numerous versions of Wirtinger-based integral inequality have been applied for stability of open loop system in presence of delay, which provides less conservative result as compared to well-known Jensen's inequality [22–28]. In the same direction, the above cited new advancements are tested on delayed discrete systems [29–31]. In [28], an accurate integral inequality [32] is considered to reduce the conservatism and it is dependent on $x(t)$, $x(t-d)$ as well

as on $\int_{t-d}^t x(s)ds$. A new summation inequality is utilized for discrete-time delay systems in [30], which is extension of integral inequality given in [27]. The discrete version of Wirtinger's inequality is derived in [30] and also applied for stabilization of delayed discrete systems along with interval like time-varying delay. A novel summation inequality has been projected for delayed discrete systems by considering Lyapunov-Krasovskii functional with two additional terms in difference of forward calculation in [31]. An advanced sequence of summation inequality has been obtained in [22], which includes Jensen, Wirtinger and Free matrix summation based inequality ones as specific case. Moreover, advancement in previous results has been obtained by Zhang et al. in 2018 [33] by extending the Wirtinger's inequality. In [34], discrete inequalities for single summation and double summation have been proposed which are based on multiple auxiliary functions and include the Jensen discrete inequality and the discrete Wirtinger-based inequality as special cases. In [35], a new augmented Lyapunov-Krasovskii functional with single and double summation terms is used to derive the stability conditions for discrete-time neural networks by using Wirtinger-based inequality and decomposing the time-delay interval into two non-equidistant subintervals. Delay-dependent stability criteria are established by using Wirtinger inequality approach, and based upon the memoryless H_∞ state feedback controller is designed for uncertain linear system with time-varying delay and disturbance [36]. They have utilized all possible configurations of the delay such as its, upper bounds, lower bounds and derivative of upper bounds. A new method of delay partitioning is introduced by [37], in which tractable conditions for stability are deduced for discrete time- delayed systems and the obtained conditions are also dependent on the partitioning part.

Further, the problems reported in the above literature are related to stability analysis of considered system in the absence of controller design part, which is essentially required to stabilize the practical systems. For stabilization of closed loop systems, controller design is a very challenging task in industrial processes inflicted to different nonlinearities in which what is more challenging is the input saturation along with time-varying state delay. Very few results are obtained for controller design for time delayed systems [36]. Stability analysis and control synthesis problems incorporated with both nonlinearities (i.e., saturation and delay) have appeared in [8, 21, 38, 40] using LMI technique.

Inspired by the above literature, especially [30], in this manuscript a controller based on feedback of state is invented to stabilize discrete systems along with input saturation and time-varying state delays through novel summation inequality in LKF. A new and less conservative stability criterion is established by using Lyapunov-Krasovskii functional based on reciprocal convex approach along with Wirtinger inequality. In this work, novelty is to synthesize a feedback controller for discrete-time systems in presence of nonlinearities like actuator saturation and time-varying state delays, and maximized domain of attraction is estimated. The anticipated criterion is simulated for stabilization of missile control system.

The manuscript is arranged as follows. In Section 2, problem is formalized and necessary lemmas are recalled. A

delay-dependent approach is utilized for deriving asymptotic stability conditions in Section 3. To show the adequacy of the projected work numerical examples are demonstrated in Section 4. Finally, the paper ends with general remarks.

Notations. Throughout this paper, the following representations are used:

$\mathfrak{R}^{m \times n}$ - $m \times n$ real matrices;

$\mathfrak{R}^m$ - $m \times 1$ real matrices;

I - identity matrix of suitable dimension;

$\mathbf{0}$ - null matrix or null vector;

Ω^T - transpose matrix of Ω ;

$\lambda_{\max}(\Omega)$ - maximum eigenvalue of any given matrix Ω ;

$\text{diag}\{a_1, a_2, \dots, a_n\}$ - diagonal matrix with diagonal elements $a_1, a_2, \dots, a_n$;

$*$ - symmetricity in matrices;

$\|\cdot\|$ - Euclidean norm

$\text{He}(M) = M + M^T$.

2. Problem Formulation and Prefaces

A Discrete-time system is considered in which the effect of actuator saturation and delay is present,

$$x(q+1) = Ax(q) + A_d x(q-d(q)) + B \text{sat}(u(q)) \quad (1a)$$

$$y(q) = C_p x(q) \quad (1b)$$

$$x(q) = \phi(q) \quad q \in [-d_2, 0] \quad (1c)$$

where

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad (1d)$$

$$A_d = \begin{bmatrix} A_{d11} & A_{d12} \\ A_{d21} & A_{d22} \end{bmatrix},$$

$q \in \bar{z}_+$ and $\bar{z}_+$ specify non-negative integers. State vector is represented by $x(q) \in \mathfrak{R}^n$, control vector as $u(q) \in \mathfrak{R}^p$ whereas output vector is $y(q) \in \mathfrak{R}^t$. Matrices $A \in \mathfrak{R}^{n \times n}$, $A_d \in \mathfrak{R}^{n \times n}$, $B \in \mathfrak{R}^{n \times p}$, $C_p \in \mathfrak{R}^{t \times n}$ are known value matrices, $\phi(q)$ is the initial condition and $d(q)$ is time-varying delay.

The saturation function is illustrated as below for $n = 1, 2, \dots, 2^p$ as

$$\text{sat}(\bar{u}_{(n)}) = \begin{cases} 1 & \text{if } \bar{u}_{(n)} > 1 \\ \bar{u}_{(n)} & \text{if } -1 \leq \bar{u}_{(n)} \leq 1 \\ -1 & \text{if } \bar{u}_{(n)} < -1 \end{cases} \quad (2)$$

A $d(q)$ is time-varying delay satisfying the following equation (3) as

$$d_1 \leq d(q) \leq d_2 \quad (3)$$

for positive lower delay bounds d_1 and upper delay bounds d_2 , respectively.

A state feedback controller is considered as $u(q) = Kx(q)$ for stabilization of closed loop system (1a), (1b), and (1d)

$$x(q+1) = Ax(q) + A_d x(q-d(q)) + B \text{sat}(Kx(q)) \quad (4a)$$

$$y(q) = C_p x(q) \quad (4b)$$

In this work the saturation nonlinearity is tackled by convex hull approach which is stated in Lemma 1.

Lemma 1 (see [2]). *The set of convex hull is described as*

$$\text{sat}(Kx(q)) \in \text{co}(D_m K + D_m^- \bar{H}) x(q), \quad m \in [1, \bar{M}] \quad (5)$$

for matrices K and $\bar{H} \in \mathfrak{R}^{p \times n}$ so that all $x(q) \in \mathfrak{R}^n$ satisfying $|\bar{H}_s x(q)| \leq 1, s = 1, 2, \dots, p$

where $\bar{H}_t$ is t^{th} row of the matrix $\bar{H}$ and $\text{co}\{\cdot\}$ represents convex hull. In this case there exist $\delta_1 \geq 0, \dots, \delta_m \geq 0$, with $\sum_{m=1}^{\bar{M}} \delta_m = 1$ where $\bar{M} = 2^p$.

Taking a set $\Psi \in \mathfrak{R}^{p \times p}$ make up of elements either 1 or 0 which are diagonal, such as for $p = 2$, $\Psi = \{\bar{D}_1, \bar{D}_2, \bar{D}_3, \bar{D}_4\}$

$$= \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\} \quad (6a)$$

For $0 < P \in \mathfrak{R}^{n \times n}$, ellipsoid $\varepsilon(P, 1)$ is described as

$$\varepsilon(P, 1) = \{x(q) \in \mathfrak{R}^n : x^T(q) P x(q) \leq 1\}. \quad (6b)$$

A set $\ell(\bar{H})$ is deduced as follows:

$$\ell(\bar{H}) = \{x(q) \in \mathfrak{R}^n : |\bar{H}_s x(q)| \leq \rho, s = 1, 2, \dots, p\} \quad (6c)$$

When $x(q) \in \ell(H)$, it follows from Lemma 1 that

$$\text{sat}(Kx(q)) = \sum_{m=1}^{\bar{M}} \delta_m ((D_m K + D_m^- \bar{H}) x(q)) \quad (7)$$

Next, using (5)–(7), equation (4a) and (4b) becomes as

$$x(q+1) = \sum_{m=1}^{\bar{M}} \delta_m \bar{A}_m x(q) + A_d x(q-d(q)) \quad (8a)$$

$$= \bar{A} x(q) + A_d x(q-d(q))$$

$$y(q) = C_p x(q) \quad (8b)$$

with

$$\bar{A} = \sum_{m=1}^{\bar{M}} \delta_m \bar{A}_m; \quad \bar{A}_m = \begin{bmatrix} \bar{A}_{11}^m & \bar{A}_{12}^m \\ \bar{A}_{21}^m & \bar{A}_{22}^m \end{bmatrix}; \quad (9)$$

$$\bar{A}_{ij}^m = A_{ij} + B_i D_m K_j + B_i D_m^- H_j.$$

The initial condition for closed loop system (8a) and (8b) is given as

$$x_0 = \phi_x(q), q = -d_2, -d_2 + 1, \dots, 0 \text{ be } \phi(q, x_0)$$

and domain of attraction of the origin of system (8a) and (8b) is

$$\Gamma \triangleq \left\{ \phi_x(q), q = -d_2, -d_2 + 1, \dots, 0 : \lim_{q \rightarrow \infty} \phi_x(q, x_0) = 0 \right\} \quad (10a)$$

Domain of attraction is estimated as $E_\delta \subset \Gamma$ where

$$E_\delta \triangleq \{\phi_x(q), -d_2 \leq q \leq 0 : \max \|\phi_x(q)\| \leq \delta\}. \quad (10b)$$

Lemma 2 (see [29, 30]). *For a given symmetric positive definite matrix $W \in \mathfrak{R}^{m \times m}$, any sequence of discrete-time variable $x(q)$ in $[-d \ 0] \cap \mathbb{Z} \rightarrow \mathfrak{R}^m$, the following inequality holds:*

$$\sum_{i=-d+1}^0 \omega^T(i) W \omega(i) \geq \frac{1}{d} \begin{bmatrix} \theta_0 \\ \theta_1 \end{bmatrix}^T \begin{bmatrix} W & 0 \\ 0 & 3\left(\frac{d+1}{d-1}\right)W \end{bmatrix} \begin{bmatrix} \theta_0 \\ \theta_1 \end{bmatrix} \quad (11)$$

where $d > 1$, $\omega(i) = x(i) - x(i-1)$ and

$$\theta_0 = x(0) - x(-h),$$

$$\theta_1 = x(0) + x(-d) - \frac{2}{d+1} \sum_{i=-d}^0 x(i). \quad (12)$$

Since the factor $((d+1)/(d-1))$ is difficult to handle in some practical systems with time-varying delay, the following Lemma 3 is given to make this term disappear from the inequality.

Lemma 3 (see [30]). *For a given symmetric positive definite matrix $W \in \mathfrak{R}^{m \times m}$, any sequence of discrete-time variable x in $[-d \ 0] \cap \mathbb{Z} \rightarrow \mathfrak{R}^m$, where $d \geq 1$, then the following inequality holds:*

$$\sum_{i=-d+1}^0 \omega^T(i) W \omega(i) \geq \frac{1}{d} \begin{bmatrix} \theta_0 \\ \theta_1 \end{bmatrix}^T \begin{bmatrix} W & 0 \\ 0 & 3W \end{bmatrix} \begin{bmatrix} \theta_0 \\ \theta_1 \end{bmatrix} \quad (13)$$

where $\omega(i) = x(i) - x(i-1)$ and

$$\theta_0 = x(0) - x(-h),$$

$$\theta_1 = x(0) + x(-d) - \frac{2}{d+1} \sum_{i=-d}^0 x(i). \quad (14)$$

Lemma 4 (see [30, 41]). *For any vectors σ_1, σ_2 matrices T, S and real numbers $\alpha_1 \geq 0, \alpha_2 \geq 0$ satisfying*

$$\begin{bmatrix} T & S \\ * & T \end{bmatrix} \geq 0, \quad (15a)$$

$$\alpha_1 + \alpha_2 = 1,$$

$$\sigma_k = 0 \quad \text{if } \alpha_k = 0 \quad (k = 1, 2) \quad (15b)$$

then

$$-\frac{1}{\alpha_1} \sigma_1^T T \sigma_1 - \frac{1}{\alpha_2} \sigma_2^T T \sigma_2 \leq - \begin{bmatrix} \sigma_1 \\ \sigma_2 \end{bmatrix}^T \begin{bmatrix} T & S \\ * & T \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \end{bmatrix} \quad (15c)$$

3. Main Results

The main outcomes of the research are as follows:

Theorem 5. For the scalars d_1, d_2 fulfilling the condition $0 < d_1 < d_2$ and any time-varying delay $d(q) \in [d_1, d_2]$, if there exist symmetric matrices $0 < P = \text{diag}(P_1, P_2, P_3) \in \mathfrak{R}^{3n \times 3n}$, $0 < Q_q (q = 1, 2) \in \mathfrak{R}^{n \times n}$, $0 < Z_q (q = 1, 2) \in \mathfrak{R}^{n \times n}$, $0 < X_q (q = 1, \dots, 3) \in \mathfrak{R}^{n \times n}$, controller gain matrix $K \in \mathfrak{R}^{p \times n}$, $\bar{H} \in \mathfrak{R}^{p \times n}$, matrix $S_q (q = 1, \dots, 4)$ with pertinent dimension fulfilling the following LMIs (16)–(18)

$$\phi = \begin{bmatrix} \bar{Z}_1 & 0 & 0 \\ * & \bar{Z}_2 & S \\ * & * & \bar{Z}_2 \end{bmatrix} > 0 \quad (16)$$

$$\begin{bmatrix} P_1 & H_s^T \\ * & \rho^2 \end{bmatrix} > 0, \quad s = 1, 2, \dots, p \quad (17)$$

$$\begin{bmatrix} \psi_{11} & -2Z_1 & 0 & 0 & 6Z_1 & 0 & 0 & \bar{A}^T & d_1(\bar{A} - I)^T & d_{12}(\bar{A} - I)^T \\ * & \psi_{22} & \psi_{23} & \psi_{24} & 6Z_1 & 6Z_2 & 2S_2 + 2S_4 & 0 & 0 & 0 \\ * & * & \psi_{33} & \psi_{34} & 0 & \psi_{36} & \psi_{37} & A_d^T & d_1 A_d^T & d_{12} A_d^T \\ * & * & * & \psi_{44} & 0 & -2S_3^T + 2S_4^T & 6Z_2 & 0 & 0 & 0 \\ * & * & * & * & -12Z_1 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & -12Z_2 & -4S_4 & 0 & 0 & 0 \\ * & * & * & * & * & * & -12Z_2 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & -P_1^{-1} & 0 & 0 \\ * & * & * & * & * & * & * & * & -Z_1^{-1} & 0 \\ * & * & * & * & * & * & * & * & * & -Z_2^{-1} \end{bmatrix} < 0 \quad (18)$$

where

$$\begin{aligned} \bar{Z}_1 &= \text{diag}(Z_1, 3Z_1), \\ \bar{Z}_2 &= \text{diag}(Z_2, 3Z_2), \\ d_{12} &= d_2 - d_1, \end{aligned} \quad (19)$$

$$S = \begin{bmatrix} S_1 & S_2 \\ S_3 & S_4 \end{bmatrix}$$

then, the closed loop system defined by (8a) and (8b) is asymptotically stable using feedback controller of gain K . The predicted domain-attraction for (8a) and (8b) is represented by $\Gamma_\delta \leq 1$

where

$$\begin{aligned} \Gamma_\delta &= \delta^2 [\lambda_{\max}(P_1) + d_1 \lambda_{\max}(P_2) \\ &+ (d_2 - d_1) \lambda_{\max}(P_3) + d_1 \lambda_{\max}(Q_1) \\ &+ (d_2 - d_1) \lambda_{\max}(Q_2) + 2d_1(d_1 + 1) \lambda_{\max}(Z_1) \\ &+ 2(d_2 - d_1)(d_2 + d_1 + 1) \lambda_{\max}(Z_2)]. \end{aligned} \quad (20)$$

Proof. Following quadratic Lyapunov-Krasovskii functional is considered

$$\widehat{V}(x(q)) = \sum_{a=1}^3 \widehat{V}_a(x(q)) \quad (21)$$

where

$$\begin{aligned} V_1(x(q)) &= \begin{bmatrix} x^T(q) & \sum_{i=q-d_1}^{q-1} x^T(i) & \sum_{i=k-d_2}^{q-d_1-1} x^T(i) \end{bmatrix} P \begin{bmatrix} x(q) \\ \sum_{i=q-d_1}^{q-1} x(i) \\ \sum_{i=q-d_2}^{q-d_1-1} x(i) \end{bmatrix} \end{aligned} \quad (22)$$

$$V_2(x(q)) = \sum_{i=q-d_1}^{q-1} x^T(i) Q_1 x(i) + \sum_{i=k-d_2}^{q-d_1-1} x^T(i) Q_2 x(i) \quad (23)$$

$$\begin{aligned} V_3(x(k)) &= d_1 \sum_{i=-d_1+1}^0 \sum_{j=q+i}^q \eta^T(j) Z_1 \eta(j) \\ &+ d_{12} \sum_{i=-d_2+1}^{-d_1} \sum_{j=q+i}^q \eta^T(j) Z_2 \eta(j) \end{aligned} \quad (24)$$

where $\eta(i) = x(i) - x(i-1)$.

An augmented vector $\xi(q)$ is defined as

$$\xi(q) = \begin{bmatrix} x^T(q) & x^T(q-d_1) & x^T(q-d(k)) & x^T(q-d_2) & v_1^T(q) & v_2^T(q) & v_3^T(q) \end{bmatrix}^T \quad (25)$$

where

$$v_1(q) = \frac{1}{d_1+1} \sum_{i=q-d_1}^q x(i) \quad (26)$$

$$v_2(q) = \frac{1}{d(q)-d_1+1} \sum_{i=q-d(q)}^{q-d_1} x(i) \quad (27)$$

$$v_3(q) = \frac{1}{d_2-d(q)+1} \sum_{i=q-d_2}^{q-d(q)} x(i) \quad (28)$$

For system (8a) and (8b) the difference of above LKF will be

$$\begin{aligned} \Delta V(x(q)) &= \sum_{a=1}^3 (V_a(x(q+1)) - V_a(x(q))) \\ &= \sum_{a=1}^3 (\Delta V_a(x(q))) \end{aligned} \quad (29)$$

where

$$\begin{aligned} \Delta V_1(x(q)) &= \xi^T(q) \left[(\omega_1 + \beta(d(q)))^T P (\omega_1 \right. \\ &\quad \left. + \beta(d(q)) - (\omega_2 + \beta(d(q)))^T P (\omega_2 + \beta(d(q))) \right] \\ &\quad \cdot \xi(q) \\ &= \xi^T(q) \left[\omega_1^T P \omega_1 - \omega_2^T P \omega_2 \right] \\ &\quad + \text{He}(\beta^T(d(q)) P (\omega_1 - \omega_2)) \xi(q) \end{aligned} \quad (30)$$

$\Delta V_1(x(q))$ is obtained by following relation

$$\begin{aligned} &\begin{bmatrix} x(q+1) \\ \sum_{i=q-d_1+1}^q x(i) \\ \sum_{i=q-d_2+1}^{q-d_1} x(i) \end{bmatrix} \\ &= \begin{bmatrix} (\widehat{A}-I)x(q) + A_d x(q-d(q)) \\ -x(q-d_1) + v_1(q) \\ v_2(q) + v_3(q) - x(q-d(q)) - x(q-d_2) \end{bmatrix} \end{aligned}$$

$$\begin{aligned} &+ \begin{bmatrix} x(q) \\ d_1 v_1(q) \\ (d(q)-d_1)v_2(q) + (d_2-d(q))v_3(q) \end{bmatrix} \\ &= (\omega_1 + \beta(d(q))) \xi(q) \end{aligned} \quad (31)$$

$$\begin{aligned} &\begin{bmatrix} x(q) \\ \sum_{i=q-d_1}^{q-1} x(i) \\ \sum_{i=q-d_2}^{q-d_1-1} x(i) \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ -x(q) + v_1(q) \\ v_2(q) + v_3(q) - x(q-d_1) - x(q-d(q)) \end{bmatrix} \\ &\quad + \begin{bmatrix} x(q) \\ d_1 v_1(q) \\ (d(q)-d_1)v_2(q) + (d_2-d(q))v_3(q) \end{bmatrix} \\ &= (\omega_2 + \beta(d(q))) \xi(q) \end{aligned} \quad (32)$$

where

$$\begin{aligned} \omega_1 &= \begin{bmatrix} \widehat{A}-I & 0 & A_d & 0 & 0 & 0 & 0 \\ 0 & -I & 0 & 0 & I & 0 & 0 \\ 0 & 0 & -I & -I & 0 & I & I \end{bmatrix}, \\ \omega_2 &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -I & 0 & 0 & 0 & I & 0 & 0 \\ 0 & -I & -I & 0 & 0 & I & I \end{bmatrix}, \end{aligned} \quad (33)$$

$\beta(d(q))$

$$= \begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & d_1 I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & (d(q)-d_1)I & (d_2-d(q))I \end{bmatrix}$$

$\Delta V_2(x(q))$

$$\begin{aligned} &= \sum_{i=q-d_1+1}^q x^T(i) Q_1 x(i) + \sum_{i=q-d_2+1}^{q-d_1} x^T(i) Q_2 x(i) \\ &\quad - \sum_{i=q-d_1}^{q-1} x^T(i) Q_1 x(i) + \sum_{i=q-d_2}^{q-d_1-1} x^T(i) Q_2 x(i) \end{aligned}$$

$$\begin{aligned}
&= x^T(q) Q_1 x(q) - x^T(q-d_1) (Q_1 - Q_2) x(q-d_1) \\
&\quad - x^T(q-d_2) Q_2 x(q-d_2) \\
&= \xi^T(q) \widehat{Q} \xi(q)
\end{aligned} \tag{34}$$

where

$$\begin{aligned}
\widehat{Q} &= \text{diag}(Q_1, -Q_1 + Q_2, 0, -Q_2, 0, 0, 0) \\
\Delta V_3(x(q)) &= d_1 \sum_{i=-d_1+1}^0 \sum_{j=q+i+1}^{q+1} \eta^T(j) Z_1 \eta(j) \\
&\quad + d_{12} \sum_{i=-d_2+1}^{-d_1} \sum_{j=q+i+1}^{q+1} \eta^T(j) Z_2 \eta(j) \\
&\quad - d_1 \sum_{i=-d_1+1}^0 \sum_{j=q+i}^q \eta^T(j) Z_1 \eta(j) \\
&\quad - d_{12} \sum_{i=-d_2+1}^{-d_1} \sum_{j=q+i}^q \eta^T(j) Z_2 \eta(j) \\
&= \eta^T(q+1) d_1^2 Z_1 \eta(q+1) \\
&\quad + \eta^T(q+1) d_{12}^2 Z_2 \eta(q+1)
\end{aligned}$$

$$\begin{aligned}
&- d_1 \sum_{i=q-d_1+1}^q \eta^T(i) Z_1 \eta(i) \\
&- d_{12} \sum_{i=q-d_2+1}^{q-d_1} \eta^T(i) Z_1 \eta(i)
\end{aligned} \tag{35}$$

The summation limit in the last term of (35) can be split into two parts: first one containing the terms $q-d_1$ to $q-d(q)+1$ and next one from $q-d(q)$ to $q-d_2+1$

$$\begin{aligned}
&= [(x^T(q+1) - x^T(q)) (d_1^2 Z_1 + d_{12}^2 Z_2) \\
&\quad \cdot (x(q+1) - x(q))] - d_1 \sum_{i=q-d_1+1}^q \eta^T(i) Z_1 \eta(i) \\
&\quad - \frac{d_{12}}{(d(q)-d_1)} (d(q)-d_1) \sum_{i=k-d(q)+1}^{q-d_1} \eta^T(i) Z_2 \eta(i) \\
&\quad - \frac{d_{12}}{(d_2-d(q))} (d_2-d(q)) \sum_{i=q-d_2+1}^{q-d(q)} \eta^T(i) Z_2 \eta(i)
\end{aligned} \tag{36}$$

On applying Lemma 2 on first term and Lemma 3 to the last three terms of (36), we get following relation:

$$\begin{aligned}
\Delta V_3(x(q)) &\leq \xi^T(q) \omega_1^T \bar{Z}_{12} \omega_1 \xi(q) - \left[\begin{array}{c} x(q) - x(q-d_1) \\ x(q) + x(q-d_1) - \frac{2}{d_1+1} \sum_{i=q-d_1}^q x(i) \end{array} \right]^T \left[\begin{array}{cc} Z_1 & 0 \\ 0 & 3Z_1 \end{array} \right] \\
&\quad \times \left[\begin{array}{c} x(q) - x(q-d_1) \\ x(q) + x(q-d_1) - \frac{2}{d_1+1} \sum_{i=q-d_1}^q x(i) \end{array} \right] - \left[\begin{array}{c} x(q-d_1) - x(q-d(q)) \\ x(q-d_1) + x(q-d(q)) - \frac{2}{d(q)-d_1+1} \sum_{i=q-d(q)}^{q-d_1} x(i) \end{array} \right]^T \\
&\quad - \left[\begin{array}{cc} \frac{d_{12}}{d(q)-d_1} Z_2 & 0 \\ 0 & 3 \frac{d_{12}}{d(q)-d_1} Z_2 \end{array} \right] \left[\begin{array}{c} x(q-d_1) - x(q-d(q)) \\ x(q-d_1) + x(q-d(q)) - \frac{2}{d(q)-d_1+1} \sum_{i=q-d(q)}^{q-d_1} x(i) \end{array} \right] \\
&\quad - \left[\begin{array}{c} x(q-d(q)) - x(q-d_2) \\ x(q-d(q)) + x(q-d_2) - \frac{2}{d_2-d(q)+1} \sum_{i=q-d_2}^{q-d(q)} x(i) \end{array} \right]^T \left[\begin{array}{cc} \frac{d_{12}}{d_2-d(q)} Z_2 & 0 \\ 0 & 3 \frac{d_{12}}{d_2-d(q)} Z_2 \end{array} \right] \\
&\quad \times \left[\begin{array}{c} x(k-d(q)) - x(q-d_2) \\ x(q-d(q)) + x(q-d_2) - \frac{2}{d_2-d(q)+1} \sum_{i=q-d_2}^{q-d(q)} x(i) \end{array} \right]
\end{aligned} \tag{37}$$

where

$$\begin{aligned}\tilde{Z}_{12} &= \begin{bmatrix} d_1^2 Z_1 + d_{12}^2 Z_2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\ Z_3 &= \begin{bmatrix} \tilde{Z}_1 & 0 & 0 \\ 0 & \frac{d_{12}}{d(q) - d_1} \tilde{Z}_2 & 0 \\ 0 & 0 & \frac{d_{12}}{d_2 - d(q)} \tilde{Z}_2 \end{bmatrix}, \\ \tilde{Z}_1 &= \text{diag}(Z_1, 3Z_1), \\ \tilde{Z}_2 &= \text{diag}(Z_2, 3Z_2),\end{aligned}\quad (38)$$

$$\theta(q) = \begin{bmatrix} I & -I & 0 & 0 & 0 & 0 & 0 \\ I & I & 0 & 0 & -2I & 0 & 0 \\ 0 & I & -I & 0 & 0 & 0 & 0 \\ 0 & I & I & 0 & 0 & -2I & 0 \\ 0 & 0 & I & -I & 0 & 0 & 0 \\ 0 & 0 & I & I & 0 & 0 & -2I \end{bmatrix}$$

Further, if there exists matrix $S = \begin{bmatrix} S_1 & S_2 \\ S_3 & S_4 \end{bmatrix} \in \mathfrak{R}^{2n \times 2n}$ such that (16) holds then based on Lemma 4, the above term can be modified as

$$\Delta V_3(q) \leq \xi^T(q) \omega_1^T \tilde{Z}_{12} \omega_1 \xi(q) - \theta^T(q) \phi \theta(q) \quad (39)$$

where

$$\begin{aligned}\phi &= \begin{bmatrix} \tilde{Z}_1 & 0 & 0 \\ 0 & \tilde{Z}_2 & S \\ 0 & S^T & \tilde{Z}_2 \end{bmatrix}, \\ \theta(q) &= \begin{bmatrix} I & -I & 0 & 0 & 0 & 0 & 0 \\ I & I & 0 & 0 & -2I & 0 & 0 \\ 0 & I & -I & 0 & 0 & 0 & 0 \\ 0 & I & I & 0 & 0 & -2I & 0 \\ 0 & 0 & I & -I & 0 & 0 & 0 \\ 0 & 0 & I & I & 0 & 0 & -2I \end{bmatrix}\end{aligned}\quad (40)$$

Employing (30)–(39), $\Delta V(x(q))$ can be deduced as

$$\Delta V(x(q)) \leq \xi^T(q) \tilde{Q} \xi(q) + \xi^T(q) \left[\omega_1^T P \omega_1 - \omega_2^T P \omega_2 \right] + \text{He} \left(\beta^T(d(q)) P (\omega_1 - \omega_2) \right) \xi(q) \quad (41)$$

$$+ \xi^T(q) \omega_1^T \tilde{Z}_{12} \omega_1 \xi(q) - \theta^T(q) \phi \theta(q)$$

$$\Delta V(x(q)) \leq \xi^T(q) \psi \xi(q) \quad (42)$$

where

$$\psi = \begin{bmatrix} \psi_{11} + \mu_{11} & -2Z_1 & \psi_{13} & 0 & 6Z_1 & 0 & 0 \\ * & \psi_{22} & \psi_{23} & \psi_{24} & 6Z_1 & 6Z_2 & 2S_2 + 2S_4 \\ * & * & \mu_{33} + \psi_{33} & \psi_{34} & 0 & \psi_{36} & \psi_{37} \\ * & * & * & \psi_{44} & 0 & -2S_3^T + 2S_4^T & 6Z_2 \\ * & * & * & * & -12Z_1 & 0 & 0 \\ * & * & * & * & * & -12Z_2 & -4S_4 \\ * & * & * & * & * & * & -12Z_2 \end{bmatrix} \quad (43)$$

$$\mu_{11} = (\bar{A} - I)^T d_1^2 Z_1 (\bar{A} - I) + (\bar{A} - I)^T d_{12}^2 Z_2 (\bar{A} - I) + \bar{A}^T P_1 \bar{A} \quad (44a)$$

$$\psi_{11} = -P_1 + P_2 + Q_1 - 4Z_1 \quad (44b)$$

$$\psi_{13} = \bar{A}^T P_1 A_d + (\bar{A} - I)^T (d_1^2 Z_1 + d_{12}^2 Z_2) A_d \quad (44c)$$

$$\psi_{22} = -P_2 + P_3 - Q_1 + Q_2 - 4Z_1 - 4Z_2 \quad (45a)$$

$$\psi_{23} = -2Z_2 - S_1 - S_2 - S_3 - S_4 \quad (45b)$$

$$\psi_{24} = S_1 - S_2 + S_3 - S_4 \quad (45c)$$

$$\mu_{33} = A_d^T P_1 A^T + A_d^T d_1^2 Z_1 A_d + A_d^T d_{12}^2 Z_2 A_d \quad (46a)$$

$$\psi_{33} = -8Z_2 + S_1^T + S_1 + S_2^T + S_2 - S_3^T - S_3 - S_4^T - S_4 \quad (46b)$$

$$\psi_{34} = -2Z_2 - S_1 + S_2 + S_3 - S_4 \quad (46c)$$

$$\psi_{36} = 6Z_2 + 2S_3^T + 2S_4^T \quad (47a)$$

$$\psi_{37} = 6Z_2 - 2S_2 + 2S_4 \quad (47b)$$

$$\psi_{44} = -4Z_2 - P_3 - Q_2 \quad (47c)$$

On applying Schur's complement on (43), we get

$$\psi_1 = \begin{bmatrix} \psi_{11} & -2Z_1 & 0 & 0 & 6Z_1 & 0 & 0 & \bar{A}^T & d_1(\bar{A} - I)^T & d_{12}(\bar{A} - I)^T \\ * & \psi_{22} & \psi_{23} & \psi_{24} & 6Z_1 & 6Z_2 & 2S_2 + 2S_4 & 0 & 0 & 0 \\ * & * & \psi_{33} & \psi_{34} & 0 & \psi_{36} & \psi_{37} & A_d^T & d_1 A_d^T & d_{12} A_d^T \\ * & * & * & \psi_{44} & 0 & -2S_3^T + 2S_4^T & 6Z_2 & 0 & 0 & 0 \\ * & * & * & * & -12Z_1 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & -12Z_2 & -4S_4 & 0 & 0 & 0 \\ * & * & * & * & * & * & -12Z_2 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & -P_1^{-1} & 0 & 0 \\ * & * & * & * & * & * & * & * & -Z_1^{-1} & 0 \\ * & * & * & * & * & * & * & * & * & -Z_2^{-1} \end{bmatrix} \quad (48)$$

For symmetric matrices $0 < X_k$ ($k = 1, \dots, 3$), we have

$$\begin{aligned} -P^{-1} &\leq -2X_1 + X_1 P X_1, \\ -Z_1^{-1} &\leq -2X_2 + X_2 Z_1 X_2, \\ -Z_2^{-1} &\leq -2X_3 + X_3 Z_2 X_3 \end{aligned} \quad (49)$$

□

Remark 6. For finding the values X_k ($k = 1, 2, 3$) the method discussed in [38, 42] is adopted. These values can be determined iteratively.

It may be perceived that $\Delta V(x(q)) < 0$ for $\xi(q) \neq 0$ along with (16)–(17) are LMI conditions for asymptotic stability of closed system given by (8a) and (8b). The condition $\Delta V(x(q)) < 0$ can be written as $\psi_1 < 0$.

Equation (17) indicates that the set $\varepsilon(P_1) = \{x \in \mathfrak{R}^n; x^T P_1 x \leq 1\}$ is contained in polyhedral set $\ell(\bar{H})$. It can be seen that $\varepsilon(P_1) = \{x \in \mathfrak{R}^n; x^T P_1 x \leq 1\}$ is equivalent to [43]

$$P_1 - H_{(s)}^T H_{(s)} \rho^{-2} > 0, \quad s = 1, 2, \dots, p \quad (50)$$

The relation of $x \in \ell(\bar{H})$ can be found by pre and post multiply of (50) with x^T and x for all $x \in \varepsilon(P_1)$. Finally, (17) may be obtained by using Schur's complement of (50).

3.1. Maximization the Domain of Attraction. The domain of attraction for $\phi(q; x)$, set of initial state $x(q)$, is defined as $\lim_{q \rightarrow 0} \phi(q; x) = 0$ [44].

Next, theorem is illustrating the procedure for finding the estimate of domain of attraction.

Theorem 7. The maximized estimate of domain of attraction may be found by the following convex optimization problem.

minimize r ,
with

$$\begin{aligned} r = r_1 + d_1 r_2 + (d_2 - d_1) r_3 + d_1 r_4 + (d_2 - d_1) r_5 \\ + 2d_1(d_1 + 1) r_6 + 2(d_2 - d_1)(d_2 + d_1 + 1) r_7 \end{aligned} \quad (51)$$

subject to (16)–(18) and

$$\begin{aligned} r_1 I - P_1 &\geq 0, \\ r_2 I - P_2 &\geq 0, \\ r_3 I - P_3 &\geq 0, \\ r_4 I - Q_1 &\geq 0, \\ r_5 I - Q_2 &\geq 0, \\ r_6 I - Z_1 &\geq 0, \\ r_7 I - Z_2 &\geq 0 \end{aligned} \quad (52)$$

has a feasible solution for the weighting parameters $r_i > 0, i = 1, 2, \dots, 7$, positive definite symmetric matrices $P_1 \in \mathfrak{R}^{n \times n}$, $P_2 \in \mathfrak{R}^{n \times n}$, $P_3 \in \mathfrak{R}^{n \times n}$, $Q_1 \in \mathfrak{R}^{n \times n}$, $Q_2 \in \mathfrak{R}^{n \times n}$, $Z_1 \in \mathfrak{R}^{n \times n}$, $Z_2 \in \mathfrak{R}^{n \times n}$, $X_k \in \mathfrak{R}^{n \times n}, k = 1, 2, 3$, matrix $K \in \mathfrak{R}^{p \times n}$, matrix $H \in \mathfrak{R}^{p \times n}$ and matrix $S \in \mathfrak{R}^{2n \times 2n}$.

In this situation, state feedback controller gain matrix K provides a maximized estimation of domain of attraction given by $\delta_{\max} = 1/\sqrt{\tau}$, where

$$\begin{aligned} \tau = & \lambda_{\max}(P_1) + d_1 \lambda_{\max}(P_2) + (d_2 - d_1) \lambda_{\max}(P_3) \\ & + d_1 \lambda_{\max}(Q_1) + (d_2 - d_1) \lambda_{\max}(Q_2) \\ & + 2d_1(d_1 + 1) \lambda_{\max}(Z_1) \\ & + 2(d_2 - d_1)(d_2 + d_1 + 1) \lambda_{\max}(Z_2). \end{aligned} \quad (53)$$

Proof. If the conditions given by (52) hold true, then $r_1 I \geq \lambda_{\max}(P_1)$, $r_2 I \geq \lambda_{\max}(P_2)$, $r_3 I \geq \lambda_{\max}(P_3)$, $r_4 I \geq \lambda_{\max}(Q_1)$, $r_5 I \geq \lambda_{\max}(Q_2)$, $r_6 I \geq \lambda_{\max}(Z_1)$, $r_7 I \geq \lambda_{\max}(Z_2)$.

From (20), we get $\delta = \Gamma_\delta/\sqrt{\tau}$. Thus, if we minimize the value of r which is given by (51), estimate of domain of attraction δ is being maximized. In other words, the optimization procedure in Theorem 7 orients the solutions of (16)–(18) in order to get the domain of attraction as large as possible. $\square$

Remark 8. By using LKF to deal with delay, conservative results are obtained because in this approach complication arises in model transformation and bound of some cross terms for finding the difference of above function.

3.2. Robust Stabilization. Consider the discrete time delayed system (1a), (1b), and (1d) in presence of actuator saturation and uncertainty as follows:

$$\begin{aligned} x(q+1) &= \sum_{i=0}^N \mu_i \{A_i x(q) + A_{d_i} x(q-d(q)) + B \text{sat}(u(q))\} \quad (54a) \\ &\forall q > 0 \end{aligned}$$

$$y(q) = C_p x(q) \quad (54b)$$

$$x(q) = \phi(q) \quad q \in [-d_2, 0] \quad (55)$$

where $\phi(q)$ is the initial condition and time varying $\mu_i(q)$ belongs to unit simplex with $\mu_i(q) > 0, \forall i = 0, \dots, N$ and $\sum_{i=0}^N \mu_i(q) = 1$.

By using Lemma 1 and $u(q) = Kx(q)$ the closed loop system (8a) and (8b) becomes

$$x(q+1) = \sum_{i=0}^N \mu_i \{\tilde{A}_i x(q) + A_{d_i} x(q-d(q))\} \quad (56a)$$

$$y(q) = C_p x(q) \quad (56b)$$

The delay $d(q)$ is pretended to gratify the restraint given in (3). The extension of Theorem 5 is given as follows:

Theorem 9. For given scalars d_1, d_2 fulfilling the condition $0 < d_1 < d_2$ and any time-varying delay $d(q) \in [d_1, d_2]$, if there exist symmetric matrices $0 < P = \text{diag}(P_1, P_2, P_3) \in \mathfrak{R}^{3n \times 3n}$, $0 < Q_k (k = 1, 2) \in \mathfrak{R}^{n \times n}$, $0 < Z_k (k = 1, 2) \in \mathfrak{R}^{n \times n}$, $0 < X_k (k = 1, \dots, 3) \in \mathfrak{R}^{n \times n}$, matrix $K \in \mathfrak{R}^{p \times n}$, $\bar{H} \in \mathfrak{R}^{p \times n}$, matrix $S_k (k = 1, \dots, 4)$ with pertinent dimension fulfilling the conditions (16), (17) and the following set of LMI

$$\begin{bmatrix} \psi_{11} & -2Z_1 & 0 & 0 & 6Z_1 & 0 & 0 & \bar{A}_i^T & d_1(\bar{A}_i - I)^T & d_{12}(\bar{A}_i - I)^T \\ * & \psi_{22} & \psi_{23} & \psi_{24} & 6Z_1 & 6Z_2 & 2S_2 + 2S_4 & 0 & 0 & 0 \\ * & * & \psi_{33} & \psi_{34} & 0 & \psi_{36} & \psi_{37} & A_{d_i}^T & d_1 A_{d_i}^T & d_{12} A_{d_i}^T \\ * & * & * & \psi_{44} & 0 & -2S_3^T + 2S_4^T & 6Z_2 & 0 & 0 & 0 \\ * & * & * & * & -12Z_1 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & -12Z_2 & -4S_4 & 0 & 0 & 0 \\ * & * & * & * & * & * & -12Z_2 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & -2X_1 + X_1 P_1 X_1 & 0 & 0 \\ * & * & * & * & * & * & * & * & -2X_2 + X_2 Z_1 X_2 & 0 \\ * & * & * & * & * & * & * & * & * & -2X_3 + X_3 Z_2 X_3 \end{bmatrix} < 0 \quad (57)$$

where

$$\begin{aligned}\bar{Z}_1 &= \text{diag}(Z_1, 3Z_1), \\ \bar{Z}_2 &= \text{diag}(Z_2, 3Z_2), \\ d_{12} &= d_2 - d_1, \\ S &= \begin{bmatrix} S_1 & S_2 \\ S_3 & S_4 \end{bmatrix}\end{aligned}\quad (58)$$

then, the closed loop system given by (8a) and (8b) is asymptotically stable by feedback controller gain K .

Proof. The proof of the above theorem is a direct use of Schur's complement with ensuring the convexity of LMI (18) with respect to the system matrices $\bar{A}$ and A_d . $\square$

4. Numerical Examples

The numerical example are given to demonstrate the main results.

Example 1. Consider a discrete-time delayed systems

$$\begin{aligned}A &= \begin{bmatrix} 0.8 & 0 \\ 0 & 0.97 \end{bmatrix}, \\ A_d &= \begin{bmatrix} -0.1 & -0.1 \\ 0 & -0.1 \end{bmatrix}, \\ B &= \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \\ C_p &= [1 \ 0.5].\end{aligned}\quad (59)$$

The LMI conditions (16)–(18) and (52) in Theorem 7 are found to be feasible for delay range $1 \leq d(k) \leq 11$ by using LMI Toolbox [45], and maximized estimated domain of attraction is 0.0647.

The other unknown parameters and controller K which stabilizes the above system are obtained as-

$$\begin{aligned}K &= [0.0185 \ -0.0679], \\ P_1 &= \begin{bmatrix} 9.3159 & 1.0839 \\ 1.0839 & 15.4223 \end{bmatrix}, \\ P_2 &= \begin{bmatrix} 0.5946 & 0.2122 \\ 0.2122 & 0.0826 \end{bmatrix}, \\ P_3 &= \begin{bmatrix} 0.3442 & 0.1344 \\ 0.1344 & 0.5680 \end{bmatrix}\end{aligned}\quad (60)$$

In Table 1, the obtained result is compared with the existing one.

It is seen that the delay bounds in [38] are obtained as $1 \leq d(q) \leq 5$ but in the proposed method the delay bounds are obtained as $1 \leq d(q) \leq 11$.

Example 2. Consider discrete-time delayed systems represented by (1a), (1b), (1d), (2), (3), (4a), (4b), (5), (6a),

TABLE 1: Maximized Delay bounds.

Method	$d_1 \leq d(q) \leq d_2$
Theorem 2 [38]	$1 \leq d(q) \leq 5$
Theorem 2 [proposed work]	$1 \leq d(q) \leq 11$

FIGURE 1: Characteristic of saturation.

(6b), (6c), (7), (8a), and (8b) with the parameters as follows:

$$\begin{aligned}A &= \begin{bmatrix} 1.2 & 0.5 \\ 0.8 & 0.7 \end{bmatrix}, \\ A_d &= \begin{bmatrix} -0.015 & -0.01 \\ 0 & -0.013 \end{bmatrix}, \\ B &= \begin{bmatrix} 1 \\ 0.45 \end{bmatrix}, \\ C_p &= [-0.0019 \ -0.0017]\end{aligned}\quad (61)$$

Since 1.6301 and 0.2699 are calculated as the eigen values of A , so it is unstable model. The considered system is stabilized through proposed criterion given in Theorem 7. On using LMI toolbox, conditions (16)–(18) and (52) stated in Theorem 7 are found to be feasible for delay range $1 \leq d(q) \leq 19$ and maximized estimated domain of attraction is 0.0737.

The controller gain K along with other unknown parameters are found as

$$\begin{aligned}K &= [-1.1593 \ -0.5452], \\ P_1 &= \begin{bmatrix} 130.4658 & 24.7357 \\ 24.7357 & 91.3822 \end{bmatrix}, \\ P_2 &= \begin{bmatrix} 0.4365 & -0.1219 \\ -0.1219 & 0.3462 \end{bmatrix}, \\ P_3 &= \begin{bmatrix} 0.2800 & 0.0077 \\ 0.0077 & 0.3048 \end{bmatrix}\end{aligned}\quad (62)$$

In Figure 2(a), state trajectories are sketched for preliminary state of above system as $x_0 = [1.2 \ -2.5]^T$.

The control effort is shown in Figure 2(b). From Figure 2, it can be concluded that the above unstable system is asymptotically stable under the effect of controller $u(q) = Kx(q)$ with saturation limit of ± 1 (Figure 1).

FIGURE 2: State trajectories of system and control effort.

FIGURE 3: Control system of missile.

Example 3. A missile control system given in [46] is considered in this example.

Overall closed loop transfer function $T(z)$ can be obtained as

$$T(z) = \frac{1.31z + 0.5566}{z^2 + 0.2428z + 0.6238} \quad (63)$$

with the sampling interval $T=0.1$ sec, amplifier gain $K_a = 20$ and airframe dynamics is given by

$$D_m = \frac{27}{s(s+27)}. \quad (64)$$

In state space the above system is represented by

$$\tilde{x}(n+1) = \begin{bmatrix} -0.2428 & -0.6238 \\ 1 & 0 \end{bmatrix} \tilde{x}(n) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tilde{u}(n); \quad (65)$$

$$\tilde{y}(n) = [1.31 \quad 0.5566] \tilde{x}(n).$$

The model of missile control system must be robust and have an ability to adapt significant changes in atmosphere/environment. In this framework, model of block diagram is given by the Figure 3, digital computer operated as a controller by building up the proper controlling input. The impact of delay likewise is considered for is missile control system because of consuming time in calculation/computation and transmission of flag/data.

In this way, the above system may be expressed in terms of (1a), (1b), (1d), (2), (3), (4a), (4b), (5), (6a), (6b), (6c), (7), (8a), and (8b) with the accompanying parameters

$$A = \begin{bmatrix} -0.2428 & -0.6238 \\ 1 & 0 \end{bmatrix},$$

$$A_d = \begin{bmatrix} -0.016 & -0.012 \\ 0 & -0.013 \end{bmatrix}, \quad (66)$$

$$B = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$C_p = [1.31 \quad 0.5566].$$

FIGURE 4: State trajectories of system and control effort.

TABLE 2: Maximized Delay bounds.

Method	$d_1 \leq d(q) \leq d_2$
Theorem 1 [39]	$1 \leq d(q) \leq 10$
Theorem 1 [proposed work]	$1 \leq d(q) \leq 15$

By using LMI Toolbox (Gahinet et al., 1995), the LMI conditions (16)–(18) stated in Theorem 5 are found feasible for delay range $1 \leq d(q) \leq 15$.

The controller gain K and other unknown parameters are obtained as

$$\begin{aligned}
 K &= [0.5555 \quad 0.6290], \\
 P_1 &= 10^3 \times \begin{bmatrix} 1.5853 & -0.0930 \\ -0.0930 & 0.0912 \end{bmatrix}, \\
 P_2 &= \begin{bmatrix} 398.2891 & -3.9665 \\ -3.9665 & 6.4803 \end{bmatrix}, \\
 P_3 &= \begin{bmatrix} 270.6726 & -2.8491 \\ -2.8491 & 4.8553 \end{bmatrix}
 \end{aligned} \tag{67}$$

In Table 2, the obtained result is compared with the existing one.

For $x_0 = [0.4 \quad -0.2]^T$ (initial condition), the system trajectories are presented in the Figure 4(a) whereas control effort is predicted by Figure 4(b) for saturation limit of ± 1 . So, from the application of above approach, the overall system is asymptotically stable.

Remark 4. From Table 2, it is clear that when stability criteria given in Theorem 1 of [39] are applied, it is found that the system is asymptotic stable for delay range $1 \leq d(k) \leq 10$ in presence of actuator saturation and time-varying delay,

whereas in the present approach using Wirtinger based inequality the system is found to be asymptotically stable for delay range $1 \leq d(k) \leq 15$.

5. Conclusion

The stability investigation of time delay control system using Wirtinger's inequality has been carried out in this paper for optimized performance and improvement of discrete systems used in various industries and practical systems. A qualitative analysis of the delay dependent LKF is considered and LMI based stability criterion is derived. The state feedback technique is utilized for designing of controller which stabilizes the closed loop delayed practical systems. The set of convex hull is considered to deal with input saturation. The potency of the projected work has been exposed in numerical examples section.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

The authors have made the same contribution. All authors read and approved the final manuscript.

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