Research Article

Fixed Points of Subadditive Maps with an Application to a System of Volterra–Fredholm Type Integro-differential Equations

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In this paper, some fixed-point theorems are established for strongly subadditive maps on $C(\Omega, Y)$ (where $C(\Omega, Y)$ denotes the space of Y -valued continuous functions on a compact Hausdorff space Ω and Y is a unital Banach algebra). Finally, the result is applied to prove the existence and uniqueness of a solution for a system of nonlinear integro-differential equations.

1. Introduction

Volterra–Fredholm integro differential equations [1–10] appear in a number of physical models, and an important question is whether these equations can support periodic solutions. This question has been studied extensively by a number of authors; in fact, they investigated the existence of solutions for a class of these kinds of equations using Schauder's fixed-point theorem and Green's function. The distance between two zeros of nontrivial solutions to integro-differential equations was estimated by Domoshnitsky [11]. Ezzinbi and Ndambomve [12] considered the control system governed by some partial functional integro-differential equations with finite delay in Banach spaces. On the other hand, the fixed-point theory is one of the most rapidly growing topics of nonlinear functional analysis. It is a vast and interdisciplinary subject whose study belongs to several mathematical domains such as classical analysis, functional analysis, operator theory, topology, and algebraic topology. This topic has grown very rapidly perhaps due to its interesting applications in various fields within and outside mathematics such as integral equations, initial and boundary

value problems for ordinary and partial differential equations, and others. In our work, using a tool of fixed-point theory, we will ensure the existence and uniqueness of a solution for a system of nonlinear integro-differential equations.

In [13], some properties of subadditive separating maps have been investigated. A fixed-point theorem was also given, and it was used to study the existence and uniqueness of a solution of nonlinear integral type equations. Consider Ω and Λ as compact Hausdorff spaces. Take Y as a unital Banach algebra. Denote by $C(\Omega, Y)$ (resp. $C(\Lambda, Y)$) the set of Y -valued continuous functions on Ω (resp. Λ). The operator $\Delta : C(\Omega, Y) \longrightarrow C(\Lambda, Y)$ is said to be separating if $\|\zeta(\xi)\| \cdot \|\varphi(\xi)\| = 0$ implies that $\|\Delta\zeta(\varrho)\| \cdot \|\Delta\varphi(\varrho)\| = 0$ for all $\zeta, \varphi \in C(\Omega, Y)$ and $\xi \in \Omega, \varrho \in \Lambda$. Recently, an attractive work on separating maps between different spaces of functions (as well as, operator algebras) is considered (see [14–17] and references therein). The existence, uniqueness, continuation, and other properties of solutions for nonlinear integral and integro-differential equations are studied in [18–25].

The purpose of this work is to prove some new fixed-point theorems for strongly subadditive maps and to ensure

the existence and uniqueness of solutions for the following system of Volterra–Fredholm type integrodifferential equations:

$$\begin{cases} x'(t) = \zeta(t) + \int_0^t k(t, s, x(s))ds + \int_0^b h(t, s, x(s))ds, & t \in J = [0, b], \\ x(0) = x_0, \end{cases} \quad (1)$$

where $\zeta : J \rightarrow \Omega$, $k, h : J \times J \times \Omega \rightarrow \Omega$ are continuous and $x_0 \in \Delta$ (Δ is a Banach space with norm $\|\cdot\|$). Note that a solution of problem (1) is as follows:

$$\begin{aligned} x(t) = x_0 + \int_0^t \zeta(s)ds + \int_0^t \left[\int_0^s k(s, \varrho, x(\varrho))d\varrho \right. \\ \left. + \int_0^b h(s, \varrho, x(\varrho))d\varrho \right] ds, \quad t \in J = [0, b]. \end{aligned} \quad (2)$$

2. Preliminaries

In the sequel, denote by $B = C([0, b], \Omega)$ the Banach space of all continuous functions from J into Ω endowed with the norm

$$\|x\| = \sup_{s \in J} |x(s)|. \quad (3)$$

Let Ω and Λ be two compact Hausdorff spaces. Let Y be a unital Banach algebra with a unit element e_0 . For $\zeta \in C(\Omega, Y)$, denote by $\text{coz}(\zeta)$ the cozero set of ζ , i.e., $\text{coz}(\zeta) = \{\xi \in X : \zeta(\xi) \neq 0\}$. $\text{Cl}U$ (resp. CU) denotes the topological closure (resp. the compliment) of the set U . For a separating map $\Delta : C(\Omega, Y) \rightarrow C(\Lambda, Y)$ and $\varrho \in \Lambda$, $\widehat{\varrho} \circ \Delta : C(\Omega, Y) \rightarrow Y$ and $\widehat{\varrho} \circ \Delta(f) = \Delta\zeta(\varrho)$, for all $\zeta \in C(\Omega, Y)$.

Definition 1 (see [26]).

- (a) The operator $\Delta : C(\Omega, Y) \rightarrow C(\Lambda, Y)$ is said to be subadditive, if

$$\|\Delta(\zeta + \varphi)\| \leq \|\Delta\zeta\| + \|\Delta\varphi\|, \quad (4)$$

for all $\zeta, \varphi \in C(\Omega, Y)$.

- (b) Given $M > 0$ and a subadditive map Δ , if for any $\zeta \in C(\Omega, Y)$ there is $\varepsilon > 0$ so that $\|\Delta\zeta - \delta\varphi\| \leq M\|\Delta(\zeta - \varphi)\|$, for all $\varphi \in C(\Omega, Y)$ verifying $\|\zeta - \varphi\| < \varepsilon$, then Δ is said to be strongly subadditive.

Example 1. Given a continuous map $\varphi : \Lambda \rightarrow \Lambda$, take $\sigma \in C(\Lambda, \mathbb{C})$. The operator $\Delta : C(\Omega, \mathbb{C}) \rightarrow C(\Lambda, \mathbb{C})$, defined by $\zeta \rightarrow |\sigma \cdot (\zeta \circ \varphi)|$, is separating and strongly subadditive. Note that Δ is not additive.

Definition 2 (see [26]). The operator $\Delta : C(\Omega, Y) \rightarrow C(\Lambda, Y)$ is said to be pointwise subadditive if

$$\|\Delta(\zeta + \varphi)(\tau)\| \leq \|\Delta\zeta(\tau)\| + \|\Delta\varphi(\tau)\|, \quad (5)$$

for all $\zeta, \varphi \in C(\Omega, Y)$ and $\tau \in \Lambda$.

Definition 3 (see [26]). Let $\Delta : C(\Omega, Y) \rightarrow C(\Lambda, Y)$ be a separating map and U be an open subset of Ω . Such U is said to be a vanishing set for $\widehat{\tau} \circ \Delta$ if for each $\zeta \in C(\Omega, Y)$, $\text{coz}(\zeta) \subseteq U$ implies that $\|\widehat{\tau} \circ \Delta(\zeta)\| = 0$. The support of $\widehat{\tau} \circ \Delta$ is given as follows:

$$\text{supp } \widehat{\tau} \circ \Delta = \overline{\Omega \setminus \bigcup_{V \subseteq \Omega : V \text{ is a vanishing set for } \widehat{\tau} \circ \Delta} V}. \quad (6)$$

For $e \in Y$, define $1_e : \Omega \rightarrow Y$ by $1_e(\xi) = e$ for each $\xi \in \Omega$. Here, $1_e \in C(\Omega, Y)$. Suppose that $\|\Delta 1_{e_0}(\tau)\| \neq 0$ for every $\tau \in \Lambda$. Due to the fact that $\|1_{e_0}\| \cdot 0 = 0$ and Δ is separating, we have $\Delta 0 = 0$. The use of 1_e is to extend the scalar-valued case to vector-valued case.

Remark 1 (decomposition of the identity). If $K = \mathbb{R}$ or $\mathbb{C}$, then for each finite cover $\{U_i\}_{i=1}^n$ of open subsets of Ω , there exists a continuous decomposition of identity, $\{e_i\}_{i=1}^n \subseteq C(\Omega, Y)$, subordinate to the U_i . That is, $\sum_{i=1}^n e_i = 1$ and $\text{coz}(e_i) \subseteq U_i$, for any $i = 1, 2, \dots, n$ [27].

Note that $\Delta 1_{e_0}$ is not equal to zero everywhere, so $\Delta(C(\Omega, Y))$ separates Λ . Also, each compact Hausdorff space is a Tychonoff space. The following remark is based on Theorem 1.1 of [28].

Remark 2. Let $\Delta : C(\Omega, Y) \rightarrow C(\Lambda, Y)$ be a pointwise subadditive separating map. Then, for each $\tau \in \Lambda$, $\text{supp } (\widehat{\tau} \circ \Delta)$ is a singleton.

Theorem 1 (see [13]). Let $\Delta : C(\Omega, Y) \rightarrow C(\Lambda, Y)$ be a strongly subadditive map. Suppose that the following assumptions hold:

- (A1) For any $\zeta \in C(\Omega, Y)$, there is $\varepsilon > 0$ such that $\|\Delta\zeta - \zeta\| < \varepsilon$
 (A2) There is $c > 0$ such that $0 < cM < 1$ and $\|\Delta\zeta\| \leq c\|\zeta\|$, for any $\zeta \in C(\Omega, Y)$, where M follows using the strong subadditivity of Δ

Then, Δ possesses a unique fixed point ζ_0 and $\lim_{n \rightarrow \infty} \Delta^n \zeta = \zeta_0$, for any $\zeta \in C(\Omega, Y)$.

Examples A.1. and A.2. given in Appendix support Theorem 1.

3. Main Results

Consider the following statement:

- (H) Let $y_0 = \zeta$. Define in $C(\Omega, Y)$ a sequence $\{y_n\}$ approximately to $\Delta(y_{n-1})$, i.e., $\{y_n\}$ satisfies

$$\|y_n - \Delta(y_{n-1})\| \leq \sigma_n, \quad (7)$$

where $\{\sigma_n\}$ is a sequence of positive numbers, for which $\lim_{n \rightarrow \infty} \sigma_n = 0$.

Theorem 2. Let $\Delta : C(\Omega, Y) \longrightarrow C(\Omega, Y)$ be a strongly subadditive map. Suppose that the following assumptions hold:

- (B1) For each $\zeta \in C(\Omega, Y)$, there is $\varepsilon > 0$ such that $\|\Delta^n(\zeta) - y_{n-1}\| < \varepsilon$, for all $n \geq 1$, where $\{y_n\}$ is defined in (H)
- (B2) There is $c > 0$ such that $0 < cM < 1$ and $\|\Delta\zeta\| \leq c\|\zeta\|$, for each $\zeta \in C(\Omega, Y)$, where M comes from strong subadditivity of Δ

Then, Δ admits a unique fixed point ζ_0 in $C(\Omega, Y)$. Moreover, for each $\zeta \in C(\Omega, Y)$, $\lim_{n \rightarrow \infty} \Delta^n \zeta = \zeta_0$.

Proof. It is sufficient to take $n = 1$; then, by Theorem 1, Δ has a unique fixed point ζ_0 in $C(\Omega, Y)$. Also, for each $\zeta \in C(\Omega, Y)$, $\lim_{n \rightarrow \infty} \Delta^n \zeta = \zeta_0$.

Now, a natural question arises. If in Theorem 2, one replaces the sequence $\{\Delta^n \zeta\}$ by $\{y_n\}$, then under what conditions we have the case that

$$\lim_{n \rightarrow \infty} y_n = \zeta_0? \quad (8)$$

In the following, we propose a positive answer to this question. $\square$

Theorem 3. Consider the assumptions of Theorem 2 and let ζ_0 be the unique fixed point of Δ . Then, $\lim_{n \rightarrow \infty} y_n = \zeta_0$, where $\{y_n\}$ is defined by (H).

Proof. Let $y_0 = \zeta$. By strong subadditivity of Δ and conditions (B1) and (B2), we have

$$\begin{aligned} \|\Delta^{m+1}(y_0) - y_m\| &= \|\Delta^{m+1}(\zeta) - y_m\| \\ &\leq \|\Delta(\Delta^m(\zeta)) - H(y_{m-1})\| \\ &\quad + \|\Delta(y_{m-1}) - y_m\| \\ &\leq M\|\Delta(\Delta^m(\zeta) - y_{m-1})\| + \sigma_m \\ &\leq cM\|\Delta^m(\zeta) - y_{m-1}\| + \sigma_m \\ &\leq cM(cM\|\Delta^{m-1}(\zeta) - y_{m-2}\| + \sigma_{m-1}) + \sigma_m \\ &\vdots \\ &\leq cM(cM(\dots(cM\|\Delta(\zeta) - y_0\| + \sigma_1)\dots) \\ &\quad + \sigma_{m-1}) + \sigma_m, \end{aligned} \quad (9)$$

where $M, c > 0$, M comes from strong subadditivity of Δ , and $0 < cM < 1$. We get that

$$\|\Delta^{m+1}(\zeta) - y_m\| \leq \sum_{i=1}^m (cM)^{m-i} \sigma_i. \quad (10)$$

Thus,

$$\begin{aligned} \|y_m - \zeta_0\| &\leq \|y_m - \Delta^{m+1}(\zeta)\| + \|\Delta^{m+1}(\zeta) - \zeta_0\| \\ &\leq \sum_{i=1}^m (cM)^{m-i} \sigma_i + \|\Delta^{m+1}(\zeta) - \zeta_0\|. \end{aligned} \quad (11)$$

Now, let $\sigma > 0$. Since $\lim_{n \rightarrow \infty} \sigma_n = 0$, there is $N \in \mathbb{N}$ so that for $m \geq N$, $\sigma_m \leq \sigma$. Thus,

$$\begin{aligned} \sum_{i=1}^m (cM)^{m-i} \sigma_i &= \sum_{i=1}^N (cM)^{m-i} \sigma_i + \sum_{i=N+1}^m (cM)^{m-i} \sigma_i \\ &\leq (cM)^{m-N} \sum_{i=1}^N (cM)^{N-i} \sigma_i + \sigma \sum_{i=N+1}^m (cM)^{m-i}. \end{aligned} \quad (12)$$

Therefore,

$$\lim_{m \rightarrow \infty} \sum_{i=1}^m (cM)^{m-i} \sigma_i \leq \sigma \left(\frac{(cM)^{N+1}}{1 - cM} \right). \quad (13)$$

Since $\sigma > 0$ is arbitrary, $\lim_{m \rightarrow \infty} \sum_{i=1}^m (cM)^{m-i} \sigma_i = 0$. Also, by Theorem 2, $\lim_{m \rightarrow \infty} \|\Delta^{m+1}(\zeta) - \zeta_0\| = 0$. Thus, $\lim_{m \rightarrow \infty} y_m = \zeta_0$. $\square$

Theorem 4. Suppose $\Delta : C(\Omega, Y) \longrightarrow C(\Omega, Y)$ is a map such that all the conditions of Theorem 1 are satisfied for Δ^N (Δ^N is N th iteration of Δ for some positive integer N). Then, Δ admits a unique fixed point.

Proof. By Theorem 1, $\Delta^N : C(\Omega, Y) \longrightarrow C(\Omega, Y)$ has a unique fixed point $\zeta \in C(\Omega, Y)$. However,

$$\Delta^N(\Delta\zeta) = \Delta^{N+1}\zeta = \Delta(\Delta^N\zeta) = \Delta\zeta, \quad (14)$$

so $\Delta\zeta$ is also fixed point of Δ^N . Since the fixed point of Δ^N is unique, one writes that $\Delta\zeta = \zeta$. Also, if $\Delta\varphi = \varphi$, then $\Delta^N\varphi = \varphi$. By uniqueness, we obtain that $\zeta = \varphi$.

Now, we present an extension of Theorem 1.

Let $\mathcal{S}$ denote the class of functions $\alpha : \mathbb{R}^+ \longrightarrow [0, 1]$ such that $k\alpha(t_n) \longrightarrow 1 \implies t_n \longrightarrow 0$ and $k\alpha(t) \leq 1$ for every $t \in \mathbb{R}^+$ (where $k > 0$). $\square$

Theorem 5. Let $\Delta : C(\Omega, Y) \longrightarrow C(\Omega, Y)$ be a strongly subadditive map. Suppose that the following assumptions hold:

- (C1) For all $\zeta \in C(\Omega, Y)$, there is $\varepsilon > 0$ so that $\|\Delta^{n+1}(\zeta) - \Delta^n(\zeta)\| < \varepsilon$, for each $n \in \mathbb{N}$
- (C2) There is $\alpha \in \mathcal{S}$ so that $0 < M\alpha(\|\zeta\|) < 1$ and for each $\zeta \in C(\Omega, Y)$, $\|\Delta\zeta\| \leq \alpha(\|\zeta\|)\|\zeta\|$, where M comes from strong subadditivity of Δ

Then, Δ admits a unique fixed point ζ_0 and $\lim_{n \rightarrow \infty} \Delta^n \zeta = \zeta_0$, for each $\zeta \in C(\Omega, Y)$.

Proof. Fix $\zeta \in C(\Omega, Y)$ and let $\zeta_n = \Delta^n \zeta$ for $n = 1, 2, 3, \dots$. The proof is divided in two steps.

Step 1. We claim that $\lim_{n \rightarrow \infty} \|\zeta_{n+1} - \zeta_n\| = 0$.

By conditions (C1) and (C2) and strong subadditivity of Δ , the sequence $\{\|\zeta_{n+1} - \zeta_n\|\}$ is monotonously decreasing and is bounded below. Hence, $\lim_{n \rightarrow \infty} \|\zeta_{n+1} - \zeta_n\| = r \geq 0$. Suppose that $r \neq 0$. Using the strong subadditivity of H and condition (C2), we have

$$\begin{aligned} \frac{\|\zeta_{n+2} - \zeta_{n+1}\|}{\|\zeta_{n+1} - \zeta_n\|} &= \frac{\|\Delta^{n+2}(\zeta) - \Delta^{n+1}(\zeta)\|}{\|\Delta^{n+1}(\zeta) - \Delta^n(\zeta)\|} \\ &\leq \frac{M\|\Delta(\Delta^{n+1}(\zeta) - \Delta^n(\zeta))\|}{\|\Delta^{n+1}(\zeta) - \Delta^n(\zeta)\|} \\ &\leq \frac{M\alpha(\|\Delta^{n+1}(\zeta) - \Delta^n(\zeta)\|)\|\Delta^{n+1}(\zeta) - \Delta^n(\zeta)\|}{\|\Delta^{n+1}(\zeta) - \Delta^n(\zeta)\|} \\ &= M\alpha(\|\Delta^{n+1}(\zeta) - \Delta^n(\zeta)\|) \\ &= M\alpha(\|\zeta_{n+1} - \zeta_n\|), \quad n = 1, 2, 3, \dots \end{aligned} \quad (15)$$

Taking $n \rightarrow \infty$, we get $1 \leq \lim_{n \rightarrow \infty} M\alpha(\|\zeta_{n+1} - \zeta_n\|) < 1$ (by (C2)). Since $\alpha \in \mathcal{S}$, this implies that $r = 0$, that is, a contradiction, and so step 1 is completed.

Step 2. We shall prove that $\{\zeta_n\}$ is a Cauchy sequence.

Suppose that $\limsup_{m,n \rightarrow \infty} \|\zeta_m - \zeta_n\| > 0$. By triangle inequality, strong subadditivity of Δ , and condition (C2), we have

$$\begin{aligned} \|\zeta_m - \zeta_n\| &\leq \|\zeta_m - \zeta_{m+1}\| + \|\zeta_{m+1} - \zeta_{n+1}\| + \|\zeta_{n+1} - \zeta_n\| \\ &\leq \|\zeta_m - \zeta_{m+1}\| + M\alpha(\|\zeta_m - \zeta_n\|)\|\zeta_m - \zeta_n\| + \|\zeta_{n+1} - \zeta_n\| \\ &\leq (1 - M\alpha(\|\zeta_m - \zeta_n\|))^{-1} [\|\zeta_m - \zeta_{m+1}\| + \|\zeta_{n+1} - \zeta_n\|]. \end{aligned} \quad (16)$$

Under the assumption that $\limsup_{m,n \rightarrow \infty} \|\zeta_m - \zeta_n\| > 0$, Step 1 now implies that

$$\limsup_{m,n \rightarrow \infty} (1 - M\alpha(\|\zeta_m - \zeta_n\|))^{-1} = +\infty. \quad (17)$$

Therefore,

$$\limsup_{m,n \rightarrow \infty} M\alpha(\|\zeta_m - \zeta_n\|) = 1. \quad (18)$$

However, since $\alpha \in \mathcal{S}$, this implies that $\limsup_{m,n \rightarrow \infty} \|\zeta_m - \zeta_n\| = 0$, which is again a contradiction. Therefore, Step 2 is established.

Now, let $\zeta \in C(\Omega, Y)$. Since $\{\zeta_n\} = \{\Delta^n \zeta\}$ is a Cauchy sequence in $C(\Omega, Y)$, which is complete, we have $\lim_{n \rightarrow \infty} \Delta^n \zeta = \zeta_0 \in C(\Omega, Y)$. The continuity of Δ yields that $\Delta \zeta_0 = \zeta_0$. Now, we shall show that ζ_0 is the unique fixed point. Let $\varphi \in C(\Omega, Y)$ be another fixed point of Δ . So, $\zeta_0 = \lim_{n \rightarrow \infty} \Delta^n \varphi = \varphi$, and this completes the proof. $\square$

4. An Application to Nonlinear Integrodifferential Equations

Consider the following assumptions:

(i) For all $x, y \in C([0, b], \Omega)$,

$$\begin{aligned} &\left| x_0 + \int_0^t \zeta(s) ds + \int_0^t \left[\int_0^s k(s, \varsigma, (x + y)(\varsigma)) d\varsigma \right. \right. \\ &\quad \left. \left. + \int_0^b h(s, \varsigma, (x + y)(\varsigma)) d\varsigma \right] ds \right| \\ &\leq \left| x_0 + \int_0^t \zeta(s) ds + \int_0^t \left[\int_0^s k(s, \varsigma, x(\varsigma)) d\varsigma \right. \right. \\ &\quad \left. \left. + \int_0^b h(s, \varsigma, x(\varsigma)) d\varsigma \right] ds \right| \\ &\quad + \left| x_0 + \int_0^t \zeta(s) ds + \int_0^t \left[\int_0^s k(s, \varsigma, y(\varsigma)) d\varsigma \right. \right. \\ &\quad \left. \left. + \int_0^b h(s, \varsigma, y(\varsigma)) d\varsigma \right] ds \right| \end{aligned}$$

for all $t \in [0, b]$.

(19)

(ii) Let $M > 0$. For each $x \in C([0, b], \Omega)$, there is $\varepsilon > 0$ such that

$$\begin{aligned} &\left| \int_0^t \left[\int_0^s (k(s, \varsigma, x(\varsigma)) - k(s, \varsigma, y(\varsigma))) d\varsigma \right. \right. \\ &\quad \left. \left. + \int_0^b (h(s, \varsigma, x(\varsigma)) - h(s, \varsigma, y(\varsigma))) d\varsigma \right] ds \right| \\ &\leq M \left| x_0 + \int_0^t \zeta(s) ds + \int_0^t \left[\int_0^s k(s, \varsigma, (x - y)(\varsigma)) d\varsigma \right. \right. \\ &\quad \left. \left. + \int_0^b h(s, \varsigma, (x - y)(\varsigma)) d\varsigma \right] ds \right|. \end{aligned} \quad (20)$$

for each $y \in C([0, b], \Omega)$ satisfying $\|x - y\| < \varepsilon$ and for each $t \in [0, b]$.

(iii) For each $x \in C([0, b], \Omega)$, there is $\varepsilon > 0$ such that

$$\begin{aligned} &\left\| x_0 + \int_0^t \zeta(s) ds + \int_0^t \left[\int_0^s k(s, \varsigma, x(\varsigma)) d\varsigma \right. \right. \\ &\quad \left. \left. + \int_0^b h(s, \varsigma, x(\varsigma)) d\varsigma \right] ds - x(t) \right\| < \varepsilon. \end{aligned} \quad (21)$$

(iv) For each $x \in C([0, b], \Omega)$, there is $L > 0$ such that

$$\left\| x_0 + \int_0^t \zeta(s)ds + \int_0^t \left[\int_0^s k(s, \varsigma, x(\varsigma))d\varsigma + \int_0^b h(s, \varsigma, x(\varsigma))d\varsigma \right] ds \right\| \leq L\|x\|. \quad (22)$$

Theorem 6. Let Ω be a Banach space. Suppose that the assumptions (i) – (iv) hold. Then, the initial value problem (1) has a unique solution in $C([0, b], \Omega)$, for which $0 < LM < 1$.

Proof. We shall prove that the operator $G : C([0, b], \Omega) \longrightarrow C([0, b], \Omega)$, given as

$$Gx(t) = x_0 + \int_0^t \zeta(s)ds + \int_0^t \left[\int_0^s k(s, \varsigma, x(\varsigma))d\varsigma + \int_0^b h(s, \varsigma, x(\varsigma))d\varsigma \right] ds, \quad t \in [0, b], \quad (23)$$

admits a unique fixed point, which is a solution of (1). For this, by condition (i), we have

$$\begin{aligned} \|G(x+y)\| &= \left\| x_0 + \int_0^t \zeta(s)ds + \int_0^t \left[\int_0^s k(s, \varsigma, (x+y)(\varsigma))d\varsigma + \int_0^b h(s, \varsigma, (x+y)(\varsigma))d\varsigma \right] ds \right\| \\ &= \sup_{0 \leq t \leq b} \left| x_0 + \int_0^t \zeta(s)ds + \int_0^t \left[\int_0^s k(s, \varsigma, (x+y)(\varsigma))d\varsigma + \int_0^b h(s, \varsigma, (x+y)(\varsigma))d\varsigma \right] ds \right| \\ &\leq \sup_{0 \leq t \leq b} \left| x_0 + \int_0^t \zeta(s)ds + \int_0^t \left[\int_0^s k(s, \varsigma, x(\varsigma))d\varsigma + \int_0^b h(s, \varsigma, x(\varsigma))d\varsigma \right] ds \right| + \sup_{0 \leq t \leq b} \left| x_0 + \int_0^t \zeta(s)ds + \int_0^t \left[\int_0^s k(s, \varsigma, y(\varsigma))d\varsigma + \int_0^b h(s, \varsigma, y(\varsigma))d\varsigma \right] ds \right| \\ &= \|Gx\| + \|Gy\|, \end{aligned} \quad (24)$$

for all $x, y \in C([0, b], \Omega)$. Hence, G is subadditive. Let $M > 0$. By condition (ii), for each $x \in C([0, b], \Omega)$, there is $\epsilon > 0$ such that

$$\begin{aligned} \|Gx - Gy\| &= \left\| \int_0^t \left[\int_0^s k(s, \varsigma, x(\varsigma))d\varsigma + \int_0^b h(s, \varsigma, x(\varsigma))d\varsigma \right] ds - \int_0^t \left[\int_0^s k(s, \varsigma, y(\varsigma))d\varsigma + \int_0^b h(s, \varsigma, y(\varsigma))d\varsigma \right] ds \right\| \\ &= \sup_{0 \leq t \leq b} \left| \int_0^t \left[\int_0^s k(s, \varsigma, x(\varsigma))d\varsigma + \int_0^b h(s, \varsigma, x(\varsigma))d\varsigma \right] ds - \int_0^t \left[\int_0^s k(s, \varsigma, y(\varsigma))d\varsigma + \int_0^b h(s, \varsigma, y(\varsigma))d\varsigma \right] ds \right| \\ &= \sup_{0 \leq t \leq b} \left| \int_0^t \left[\int_0^s (k(s, \varsigma, x(\varsigma)) - k(s, \varsigma, y(\varsigma)))d\varsigma + \int_0^b (h(s, \varsigma, x(\varsigma)) - h(s, \varsigma, y(\varsigma)))d\varsigma \right] ds \right| \\ &\leq M \sup_{0 \leq t \leq b} \left| x_0 + \int_0^t \zeta(s)ds + \int_0^t \left[\int_0^s k(s, \varsigma, (x-y)(\varsigma))d\varsigma + \int_0^b h(s, \varsigma, (x-y)(\varsigma))d\varsigma \right] ds \right| \\ &= M\|G(x-y)\|, \end{aligned} \quad (25)$$

for each $y \in C([0, b], \Omega)$ satisfying $\|x - y\| < \epsilon$, so G is strongly subadditive. By condition (iii), for each $x \in C([0, b], \Omega)$, there is $\epsilon > 0$ so that $\|Gx - x\| < \epsilon$. Also, by condition (iv), we conclude that for each $x \in C([0, b], \Omega)$, there is $L > 0$ such that $\|Gx\| \leq L\|x\|$, for which $0 < LM < 1$. Therefore, all conditions of Theorem 1 hold, and so (1) has a unique solution.

Now, an example is provided in order to make use of our result. $\square$

Example 2. Let l_a ($a \geq 2$) be the map as in Example A.1. (given later in Appendix section). Take $\Omega = \mathbb{R}^+ \cup \{0\}$ and $x \in C([0, 1], \Omega)$. We consider the following initial value problem:

$$\begin{cases} x'(t) = \frac{1}{\sqrt{1 + (1200(5)^2/1199)}} \left(\int_0^t t l_{s+2}(x(s))ds + \int_0^1 s |x(s)|ds \right), & t \in [0, 1], \\ x(0) = 0. \end{cases} \quad (26)$$

For all $s, t \in [0, 1]$ and $x \in C([0, 1], \Omega)$, we define in (1):

$$\zeta(t) = 0, k(t, s, x(s)) = \frac{1}{\sqrt{1 + (1200(5)^2/1199)}} t l_{s+2}(x(s))$$

and the norm $\|x - y\| = \sup_{0 \leq t \leq 1} |x(t) - y(t)|$ on $C([0, 1], \Omega)$. Then, $C([0, 1], \Omega)$ is a Banach space.

(i) For all $x, y \in C([0, 1], \Delta)$ and $t \in [0, 1]$, we have

$$\text{and } h(t, s, x(s)) = \frac{1}{\sqrt{1 + (1200(5)^2/1199)}} s |x(s)|, \quad (27)$$

$$\begin{aligned} & \left| x_0 + \int_0^t \zeta(s) ds + \int_0^t \left[\int_0^s k(s, \varsigma, (x+y)(\varsigma)) d\varsigma + \int_0^1 h(s, \varsigma, (x+y)(\varsigma)) d\varsigma \right] ds \right| \\ &= \frac{1}{\sqrt{1 + (1200(5)^2/1199)}} \left| \int_0^t \left[\int_0^s s l_{\varsigma+2}((x+y)(\varsigma)) d\varsigma + \int_0^1 \varsigma |(x+y)(\varsigma)| d\varsigma \right] ds \right| \\ &\leq \frac{1}{\sqrt{1 + (1200(5)^2/1199)}} \left| \int_0^t \left[\int_0^s s l_{\varsigma+2}(x(\varsigma)) d\varsigma + \int_0^1 \varsigma |x(\varsigma)| d\varsigma \right] ds \right| \\ &\quad + \frac{1}{\sqrt{1 + (1200(5)^2/1199)}} \left| \int_0^t \left[\int_0^s s l_{\varsigma+2}(y(\varsigma)) d\varsigma + \int_0^1 \varsigma |y(\varsigma)| d\varsigma \right] ds \right| \\ &= \left| x_0 + \int_0^t \zeta(s) ds + \int_0^t \left[\int_0^s k(s, \varsigma, x(\varsigma)) d\varsigma + \int_0^1 h(s, \varsigma, x(\varsigma)) d\varsigma \right] ds \right| \\ &\quad + \left| x_0 + \int_0^t \zeta(s) ds + \int_0^t \left[\int_0^s k(s, \varsigma, y(\varsigma)) d\varsigma + \int_0^1 h(s, \varsigma, y(\varsigma)) d\varsigma \right] ds \right|. \end{aligned} \quad (28)$$

(ii) For every $t \in [0, 1]$, we have

$$\begin{aligned} & \left| \int_0^t \left[\int_0^s (k(s, \varsigma, x(\varsigma)) - k(s, \varsigma, y(\varsigma))) d\varsigma + \int_0^1 (h(s, \varsigma, x(\varsigma)) - h(s, \varsigma, y(\varsigma))) d\varsigma \right] ds \right| \\ &= \frac{1}{\sqrt{1 + (1200(5)^2/1199)}} \left| \int_0^t \left[\int_0^s (s l_{\varsigma+2}(x(\varsigma)) - s l_{\varsigma+2}(y(\varsigma))) d\varsigma + \int_0^1 (\varsigma |x(\varsigma)| - \varsigma |y(\varsigma)|) d\varsigma \right] ds \right| \\ &\leq \frac{1}{\sqrt{1 + (1200(5)^2/1199)}} \left| \int_0^t \left[\int_0^s \sqrt{1 + \frac{1200(1+\varsigma)^2}{1199}} s l_{\varsigma+2}((x-y)(\varsigma)) d\varsigma + \int_0^1 \varsigma |x(\varsigma) - y(\varsigma)| d\varsigma \right] ds \right| \\ &\leq \frac{\sqrt{1 + (1200(3)^2/1199)}}{\sqrt{1 + (1200(5)^2/1199)}} \left| \int_0^t \left[\int_0^s s l_{\varsigma+2}((x-y)(\varsigma)) d\varsigma + \int_0^1 \varsigma |x(\varsigma) - y(\varsigma)| d\varsigma \right] ds \right| \\ &= M \left| x_0 + \int_0^t \zeta(s) ds + \int_0^t \left[\int_0^s k(s, \varsigma, (x-y)(\varsigma)) d\varsigma + \int_0^1 h(s, \varsigma, (x-y)(\varsigma)) d\varsigma \right] ds \right|, \end{aligned} \quad (29)$$

with $M = \sqrt{1 + (1200(3)^2/1199)}$ and for each $y \in C([0, 1], \Omega)$ satisfying $\|x - y\| < \varepsilon$.

(iii) For every $x \in C([0, 1], \Omega)$, we have

$$\begin{aligned}
& \left\| x_0 + \int_0^t \zeta(s) ds + \int_0^t \left[\int_0^s k(s, \varsigma, x(\varsigma)) d\varsigma + \int_0^1 h(s, \varsigma, x(\varsigma)) d\varsigma \right] ds - x(t) \right\| \\
&= \left\| \frac{1}{\sqrt{1 + (1200(5)^2/1199)}} \int_0^t \left[\int_0^s s l_{\varsigma+2}(x(\varsigma)) d\varsigma + \int_0^1 \varsigma |x(\varsigma)| d\varsigma \right] ds - x(t) \right\| \\
&\leq \left\| \frac{1}{\sqrt{1 + (1200(5)^2/1199)}} \int_0^t \left[\int_0^s s l_{\varsigma+2}(x(\varsigma)) d\varsigma + \int_0^1 \varsigma |x(\varsigma)| d\varsigma \right] ds \right\| + \|x(t)\| \\
&\leq \sup_{0 \leq t \leq 1} \left| \int_0^t \left[\int_0^s s |x(\varsigma)| d\varsigma + \int_0^1 \varsigma |x(\varsigma)| d\varsigma \right] ds \right| + M_1 \\
&\leq \sup_{0 \leq t \leq 1} \left| \int_0^t \left[s^2 M_1 + \frac{1}{2} M_1 \right] ds \right| + M_1 \\
&< 2M_1 = \epsilon.
\end{aligned} \tag{30}$$

Again by (iii), we have

$$\begin{aligned}
& \left\| x_0 + \int_0^t \zeta(s) ds + \int_0^t \left[\int_0^s k(s, \varsigma, x(\varsigma)) d\varsigma \right. \right. \\
& \quad \left. \left. + \int_0^1 h(s, \varsigma, x(\varsigma)) d\varsigma \right] ds \right\| \\
&\leq \frac{1}{\sqrt{1 + (1200(5)^2/1199)}} \left(\frac{5}{6} \|x\| \right) \\
&< \frac{1}{\sqrt{1 + (1200(5)^2/1199)}} \|x\|.
\end{aligned} \tag{31}$$

Therefore,

$$\begin{aligned}
& \left\| x_0 + \int_0^t \zeta(s) ds + \int_0^t \left[\int_0^s k(s, \varsigma, x(\varsigma)) d\varsigma \right. \right. \\
& \quad \left. \left. + \int_0^1 h(s, \varsigma, x(\varsigma)) d\varsigma \right] ds \right\| \leq L \|x\|,
\end{aligned} \tag{32}$$

with $L = 1/(\sqrt{1 + (1200(5)^2/1199)})$. Also, $0 < LM = (\sqrt{1 + (1200(3)^2/1199)})/(\sqrt{1 + (1200(5)^2/1199)}) < 1$. Thus, all conditions of Theorem 6 hold, and so the initial value problem (26) possesses a unique solution in $C([0, 1], \Omega)$.

5. Conclusion

We established some fixed-point theorems for strongly subadditive maps. We applied the obtained results to ensure the existence and uniqueness of a solution for a system of nonlinear integrodifferential equations. Moreover, some

related examples have been presented in support of the given results.

Appendix

The following examples are in support of Theorem 1.

Example A.1. Given $a \geq 2$ and $z = xe^{i\theta} \in \mathbb{C}$ with $-\pi < \theta < \pi$, take $l_a : \mathbb{C} \rightarrow \mathbb{C}$ given as

$$l_a : xe^{i\theta} \mapsto \frac{1}{2 + (2\sqrt{2}a)^2} xe^{ia\theta}. \tag{A.1}$$

Then, l_a is subadditive and strongly subadditive. Here, 0 is the unique fixed point of l_a .

For this, since $|l_a(z)| = (1/(2 + (2\sqrt{2}a)^2))|z|$, it follows that

$$\begin{aligned}
|l_a(z + z')| &= \frac{1}{2 + (2\sqrt{2}a)^2} |z + z'| \\
&\leq \frac{1}{2 + (2\sqrt{2}a)^2} |z| + \frac{1}{2 + (2\sqrt{2}a)^2} |z'| \\
&= |l_a(z)| + |l_a(z')|.
\end{aligned} \tag{A.2}$$

Therefore, l_a is subadditive. The continuity of l_a is clear. To show that l_a is strongly subadditive: given any $z_0 \in \mathbb{C}$, we should find $\epsilon_{z_0} > 0$ and $M > 0$ so that $|l_a(z) - l_a(z_0)| \leq M|l_a(z - z_0)|$ when $|z - z_0| < \epsilon_{z_0}$. It suffices to ensure that

$$f(z, z_0) = \frac{|l_a(z) - l_a(z_0)|^2}{(2 + (2\sqrt{2}a)^2)^2 |z - z_0|^2} \tag{A.3}$$

is bounded in some neighborhood of z_0 . Let $z = xe^{i\theta}$ and $z_0 = x_0e^{i\theta_0}$. Note that

$$\begin{aligned} f(z, z_0) &= \frac{\left| \frac{1}{(2 + (2\sqrt{2}a)^2)}xe^{ia\theta} - \frac{1}{(2 + (2\sqrt{2}a)^2)}x_0e^{ia\theta_0} \right|^2}{(2 + (2\sqrt{2}a)^2)^2 |xe^{i\theta} - x_0e^{i\theta_0}|^2} \\ &= \frac{\left| (xe^{ia\theta}/x_0e^{ia\theta_0}) - 1 \right|^2}{\left| (xe^{i\theta}/x_0e^{i\theta_0}) - 1 \right|^2} = f\left(\frac{z}{z_0}, 1\right). \end{aligned} \quad (\text{A.4})$$

First, we consider the case $z_0 = 1$. Then,

$$f(z, 1) = \frac{|1 - xe^{ia\theta}|^2}{|1 - xe^{i\theta}|^2} = \frac{1 + x^2 - 2x \cos(a\theta)}{1 + x^2 - 2x \cos(\theta)}. \quad (\text{A.5})$$

Now, we show that $f(z, 1)$ is bounded from above for z in a neighborhood of 1. Using power series expansion of $\cos(\theta)$, we get

$$\begin{aligned} 1 - \frac{\theta^2}{2} < \cos(\theta) < 1 - \frac{\theta^2}{2} + \frac{\theta^4}{24}, \\ \frac{\theta^2}{2} - 1 > -\cos(\theta) > -1 + \frac{\theta^2}{2} - \frac{\theta^4}{24}. \end{aligned} \quad (\text{A.6})$$

These inequalities imply that

$$\begin{aligned} f(z, 1) &< \frac{1 + x^2 + 2x((a^2\theta^2/2) - 1)}{1 + x^2 + 2x((\theta^2/2) - 1 - \theta^4/24)} \\ &= \frac{(1 - x)^2 + xa^2\theta^2}{(1 - x)^2 + 2x((\theta^2/2) - (\theta^4/24))}. \end{aligned} \quad (\text{A.7})$$

For $|\theta| < (1/10)$, we have $(\theta^2/2) - (\theta^4/24) > 0$ and

$$\begin{aligned} f(z, 1) &< \frac{(1 - x)^2 + xa^2\theta^2}{(1 - x)^2 + 2x((\theta^2/2) - (\theta^4/24))} \\ &< 1 + \frac{a^2}{2((1/2) - (\theta^2/24))}. \end{aligned} \quad (\text{A.8})$$

If $|\theta| < (1/10)$, then $(1/2) - (\theta^2/24) > (1199/2400)$, and it follows that $f(z, 1) < 1 + (1200a^2/1199)$. Thus, $\sqrt{f(z, 1)} < \sqrt{1 + (1200a^2/1199)}$. Now, suppose that $z_0 \neq 0$ or 1. Choose $\varepsilon > 0$ so that if $|(z/z_0) - 1| < \varepsilon$, then $\arg(z/z_0) < (1/10)$. Since $f(z, z_0) = f(z/z_0, 1)$, we get $\sqrt{f(z, z_0)} < \sqrt{1 + (1200a^2/1199)}$ for $|z - z_0| < \varepsilon|z_0|$. After including the case $z_0 = 0$, we have

$$M = \max \left\{ 1, \sqrt{1 + \frac{1200a^2}{1199}} \right\} = \sqrt{1 + \frac{1200a^2}{1199}}. \quad (\text{A.9})$$

Also,

(A1) For any $z \in \mathbb{C}$, there is $\varepsilon > 0$ so that $|l_a(z) - z| < \varepsilon$

(A2) For any $z = xe^{i\theta} \in \mathbb{C}$, we have

$$|l_a(z)| = \frac{1}{2 + (2\sqrt{2}a)^2}x < \frac{1}{2 + 2\sqrt{2}a^2}x = \frac{1}{2 + 2\sqrt{2}a^2}|z|. \quad (\text{A.10})$$

Choosing $c = 1/(2 + 2\sqrt{2}a^2)$, $0 < cM = (1/(2 + 2\sqrt{2}a^2))\sqrt{1 + (1200a^2/1199)} < 1$ and $|l_a(z)| < c|z|$. Thus, all conditions of Theorem 1 hold and $z = 0$ is the unique fixed point of l_a . Moreover, for each $z \in \mathbb{C}$, $\lim_{n \rightarrow \infty} l_a^n(z) = 0$.

Example A.2. Let l_a ($a \geq 2$) be as in Example A.1. and $\Delta : C([0, 1], \mathbb{C}) \rightarrow C([0, 1], \mathbb{C})$ be given as

$$\Delta x(s) = l_{s+2}(x(s)), \quad s \in [0, 1]. \quad (\text{A.11})$$

Clearly, Δ is a subadditive and strongly subadditive map.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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