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Research Article

Instability of Vertical Constant Through Flows in Binary Mixtures in Porous Media with Large Pores

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A binary mixture saturating a horizontal porous layer, with large pores and uniformly heated from below, is considered. The instability of a vertical fluid motion (throughflow) when the layer is salted by one salt (either from above or from below) is analyzed. Ultimately boundedness of solutions is proved, via the existence of positively invariant and attractive sets (i.e. absorbing sets). The critical Rayleigh numbers at which steady or oscillatory instability occurs are recovered. Sufficient conditions guaranteeing that a secondary steady motion or a secondary oscillatory motion can be observed after the loss of stability are found. When the layer is salted from above, a condition guaranteeing the occurrence of "cold" instability is determined. Finally, the influence of the velocity module on the increasing/decreasing of the instability thresholds is investigated.

1. Introduction

Convection in fluid mixtures saturating porous media has attracted—in the past as nowadays—the attention of many scientists due to its practical applications like, for example, in geothermal energy exploitation, extraction of oil from underground reservoirs, ground-water pollution, underground flows movement, thermal engineering, crystal growth, polymer engineering, and ceramic processing. For this reason, several studies have been addressed to this topic [1–10].

The models describing the fluid motion in porous media are reaction-diffusion dynamical systems of P.D.Es, which, as it is well known, play an important role in the modeling and studying of many phenomena {see, for instance, [11, 12] and references therein}. Several geophysical and technological applications involve nonisothermal flow of fluids through porous media called throughflow (i.e., there is flow across the porous medium and the basic flows nonquiescent) which affects the stability of the system significantly. Precisely, in some situations, a vertical motion (throughflow) is observable in a horizontal porous layer heated from below and the problem to determine until this motion is stable is of fundamental importance especially in applications involving cloud physics, hydrological/geophysical studies, seabed hydrodynamics, subterranean pollution, and many industrial and technological processes {see [1, 13–16] }. This is because, after the loss of stability, a secondary motion arises and this motion can be steady or oscillatory.

The effect of vertical throughflow on convective instability, in either the fluid or the porous layer, has been extensively discussed by several investigators. In the fluid layer, the problem is of interest because of the possibility of controlling the convective instability by adjusting the throughflow [16]. In the porous layer, the "in situ processing of energy resources" such as coal, oil shale, or geothermal energy often involves the throughflow in the porous medium. The importance of buoyancy-driven instability in such systems may become significant when precise processing is required [15]. Besides, the throughflow effect in such situations may be of attention due to the opportunity of controlling the convective instability by regulating the throughflow in accumulation to the gravity. Throughflows play an important role also in the directional solidification of concentrated alloys, in which a mushy zone exists and regarded as a porous layer [13]. As demonstrated in several studies, this problem also has a very strong role in mineralization and ore body formation in fluid-saturated porous rock masses. For example, in [14] a hydrothermal system consisting of a horizontal porous layer with upward throughflow has been considered. This horizontal layer of a fluid-saturated porous medium may undergo constant temperature and overpressure at its top, whereas it may undergo constant vertical temperature gradient and eventually constant injection of mass flux at its bottom. The overpressure can result from the presence of impermeable seals in geophysics, while it can be induced by surface structures or human activities in geoenvironmental engineering. The vertical temperature gradient may be generated by either geothermal sources in geophysics or buried heat-generating waste in geoenvironmental engineering. The eventual mass flux may be generated by dehydration reactions or by compaction of underlying soft layers or reservoirs. In this situation, it has been studied the destabilizing effect caused by the increase in upward throughflow. Rapid developments in modern technologies, during the recent past, have posed challenges in studying convective instability problems in different fluid dynamical systems. In fact the effect of vertical throughflow on the convection has been studied in many cases [13–25] (for example, the effect in a rectangular box is considered in [17]; the effect combined with a magnetic field in [18]; stability analysis when the density is quadratic in temperature in [19]; the effect with an inclined temperature gradient in [20]). The stabilizing effect of rotation and Brinkman law (holding for large pores) on a vertical constant throughflow has been analyzed in [23], while in [24, 25] the stability analysis of a vertical constant throughflow in binary mixtures saturating horizontal porous layers has been performed. In the previous papers, only the stability of a throughflow has been investigated, neglecting the question to determine the critical Rayleigh thermal numbers at which instability sets in and how instability arises.

In the present paper, we will focus on the instability analysis of a vertical constant throughflow in a horizontal porous layer, with large pores, uniformly heated from below and uniformly salted by one chemical either from above or below. In particular we determine the critical Rayleigh thermal numbers at which instability occurs and investigate for the kind of secondary motion arising.

Section 2 deals with the introduction of the mathematical problem. The onset of instability is investigated in Section 3. In particular, the critical Rayleigh thermal numbers at which steady or oscillatory instability can occur are determined and sufficient conditions guaranteeing that—after the loss of stability—a secondary steady or oscillatory motion arises have been determined. Further, in the case of a layer salted from above, the onset of "cold" instability (i.e., instability independent of the temperature gradient) has been investigated. The influence of the module of the vertical throughflow velocity, on the instability thresholds, is analyzed in Section 4. The paper ends with Section 5 in which the obtained results are highlighted.

2. Statement of the Problem

Let us consider a fluid moving mixture saturating a horizontal porous layer of depth d, uniformly heated from below, with large pores, salted by a chemical specie (salt) S either from below or from above. Denoting by $Oxyz$ an orthogonal frame of reference with fundamental unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ ($\mathbf{k}$ pointing vertically upwards), the set of governing equations for the fluid motion in the Darcy-Oberbeck-Boussinesq scheme, according to the Brinkman law, is given by [1]

$$\begin{aligned} \nabla p &= -\frac{\mu_1}{k}\mathbf{v} + \mu_2 \Delta \mathbf{v} - \rho_f g \mathbf{k}, \\ \nabla \cdot \mathbf{v} &= 0, \\ M^{-1} T_{,t} + \mathbf{v} \cdot \nabla T &= K_T \Delta T, \\ \varepsilon^* C_{,t} + \mathbf{v} \cdot \nabla C &= K_C \Delta C, \end{aligned} \tag{1}$$

where

$$\rho_f = \rho_0 \left[1 - \alpha_T (T - T_0) + \alpha_C (C - C_0)\right] \tag{2}$$

is the fluid mixture density and

- p = pressure field, T = temperature field, $\mathbf{v}$ = seepage velocity,
- C = solute concentration field, μ_i ($i = 1, 2$) = viscosity coefficients,
- k = permeability, ρ_0 = reference density, T_0 = reference temperature,
- C_0 = reference solute concentration, $-g\mathbf{k}$ = acceleration gravity,
- α_T = thermal expansion coefficient, α_C = solute expansion coefficient,
- K_T = thermal diffusivity, K_C = solute diffusivity, ε^* = porosity,
- $M = (\rho_0 c_p)_f / (\rho_0 c)_m$ = heat capacity ratio, $(\rho_0 c)_m = (1 - \varepsilon^*)(\rho_0 c)_s + \varepsilon^* (\rho_0 c_p)_f$,
- c_p = specific heat of the fluid at constant pressure, c = specific heat of the solid.

The subscripts s and f refer to the solid matrix and fluid, respectively, and $(\cdot)_{,t} = \partial(\cdot)/\partial t$. To (1) we append the boundary conditions

$$\begin{aligned} T(x, y, 0, t) &= T_0 + \frac{(T_1 - T_2)}{2}, \\ T(x, y, d, t) &= T_0 - \frac{(T_1 - T_2)}{2}, \\ C(x, y, 0, t) &= C_0 + \frac{(C_1 - C_2)}{2}, \\ C(x, y, d, t) &= C_0 - \frac{(C_1 - C_2)}{2}, \end{aligned} \tag{3}$$

where T_1, T_2, C_1, C_2 are positive constants such that $T_1 > T_2$. On considering the following dimensionless variables

$$\mathbf{x}' = \frac{\mathbf{x}}{d},$$

$$t' = M \frac{K_T}{d^2} t,$$

$$\mathbf{v}' = \frac{d}{K_T}\mathbf{v},$$
$$p' = \frac{k\left(p + \rho_0 g z\right)}{\mu_1 K_T},$$
$$T' = \frac{T - T_0}{T_1 - T_2},$$
$$C' = H\frac{C - C_0}{C_1 - C_2}, \tag{4}$$

where $H = \pm 1$ according to the layer is salted from below or above, system (1), omitting all the primes, reduces to

$$\nabla p = -\mathbf{v} + D_a \Delta \mathbf{v} + \left(\mathcal{R}_T T - \frac{\mathcal{R}_S}{Le} C\right)\mathbf{k},$$
$$\nabla \cdot \mathbf{v} = 0,$$
$$T_{,t} + \mathbf{v} \cdot \nabla T = \Delta T, \tag{5}$$
$$\varepsilon C_{,t} + \mathbf{v} \cdot \nabla C = \frac{1}{Le}\Delta C,$$

where

$\varepsilon = \varepsilon^* M$ is the normalized porosity,

$L_e = \dfrac{K_T}{K_C}$ is the Lewis number,

$\mathcal{R}_T = \dfrac{k d \rho_0 \alpha_T g\left(T_1 - T_2\right)}{\mu_1 K_T}$ is the thermal Rayleigh number, (6)

$\mathcal{R}_S = \dfrac{k d \rho_0 \alpha_C g\left|C_1 - C_2\right|}{\mu_1 K_C}$ is the solute Rayleigh number,

$D_a = \dfrac{k \mu_2}{d^2 \mu_1}$ is the Darcy number

and the boundary conditions (3) become

$$T\left(x, y, 0, t\right) = \frac{1}{2},$$
$$T\left(x, y, 1, t\right) = -\frac{1}{2},$$
$$C\left(x, y, 0, t\right) = \frac{H}{2}, \tag{7}$$
$$C\left(x, y, 1, t\right) = -\frac{H}{2}.$$

A vertical constant throughflow solution of (5)-(7) is given by

$$\mathbf{v}^* = Q\mathbf{k}, \quad Q = \text{const},$$
$$T^*\left(z\right) = \frac{e^{Qz}}{1 - e^{Q}} - \frac{e^{Q} + 1}{2\left(1 - e^{Q}\right)},$$
$$C^*\left(z\right) = -\frac{H e^{L_e Q z}}{e^{L_e Q} - 1} + \frac{H\left(e^{L_e Q} + 1\right)}{2\left(e^{L_e Q} - 1\right)},$$
$$p^*\left(z\right) = p_0^* - Qz + \mathcal{R}_T \int_0^z T^*\left(\xi\right) d\xi - \frac{\mathcal{R}_S}{L_e}\int_0^z C^*\left(\xi\right) d\xi, \tag{8}$$

where p_0^* is a constant. Setting

$$\mathbf{u} = \mathbf{v} - \mathbf{v}^*,$$
$$\theta = T - T^*,$$
$$\Gamma = C - C^*, \tag{9}$$
$$\pi = p - p^*,$$

system (5) becomes

$$\nabla \pi = -\mathbf{u} + D_a \Delta \mathbf{u} + \left(\mathcal{R}_T \theta - \frac{\mathcal{R}_S}{L_e}\Gamma\right)\mathbf{k},$$
$$\nabla \cdot \mathbf{u} = 0, \tag{10}$$
$$\theta_{,t} + \mathbf{u} \cdot \nabla \theta = f_1\left(z\right) w - Q\theta_{,z} + \Delta\theta,$$
$$\Gamma_{,t} + \frac{1}{\varepsilon}\mathbf{u} \cdot \nabla \Gamma = H f_2\left(z\right) w - \frac{Q}{\varepsilon}\Gamma_{,z} + \frac{1}{\varepsilon L_e}\Delta\Gamma,$$

where $(\cdot)_{,z} = \partial(\cdot)/\partial z$, $\mathbf{u} = (u, v, w)$, and

$$f_1\left(z\right) = -\frac{dT^*}{dz} = \frac{Q e^{Qz}}{e^{Q} - 1},$$
$$f_2\left(z\right) = -\frac{1}{H}\frac{dC^*}{dz} = \frac{L_e Q e^{L_e Q z}}{e^{L_e Q} - 1}. \tag{11}$$

To (10) we append the initial-boundary conditions

$$\mathbf{u}\left(\mathbf{x}, 0\right) = \mathbf{u}_0\left(\mathbf{x}\right),$$
$$\pi\left(\mathbf{x}, 0\right) = \pi_0\left(\mathbf{x}\right),$$
$$\theta\left(\mathbf{x}, 0\right) = \theta_0\left(\mathbf{x}\right), \tag{12}$$
$$\Gamma\left(\mathbf{x}, 0\right) = \Gamma_0\left(\mathbf{x}\right),$$
$$w = \theta = \Gamma = 0 \quad \text{on } z = 0, 1,$$

with $\nabla \cdot \mathbf{u}_0(\mathbf{x}) = 0$. In the sequel, as usually is done in convection problems in layers, we suppose that

(i) the perturbations $\{\mathbf{u} = (u, v, w), \theta, \Gamma\}$ are periodic in the x and y directions of periods $2\pi/a_x$ and $2\pi/a_y$, respectively;

(ii) $\Omega = [0, 2\pi/a_x] \times [0, 2\pi/a_y] \times [0, 1]$ is the periodicity cell;

(iii) u, v, w, θ, Γ belong to $W^{2,2}(\Omega)$, $\forall t \in \mathbb{R}^+$ and can be expanded in Fourier series, uniformly convergent in Ω, together with all their first derivatives and second spatial derivatives.

Let us denote by

(i) $\|\cdot\|$ the norm in $L^2(\Omega)$;

(ii) $f_+(x) = \max\{0, f(x)\}$, $f_-(x) = \max\{0, -f(x)\}$ where $f : \mathbb{R} \longrightarrow \mathbb{R}$.

We recall that the solutions of (10)-(12) are ultimately bounded since there exist positive invariant and attractive sets (absorbing sets) in the phase space as stated in the following theorem.

Theorem 1. *The set*

$$
\begin{aligned}
\mathrm{S} = \Big\{ & (\mathbf{u}, \theta, \Gamma) \in \left[W^{2,2}(\Omega)\right]^5 : \|(\theta - 1)_+\| + \|(\theta + 1)_-\| < \eta, \\
& \|(\Gamma - H)_+\| + \|(\Gamma + H)_-\| < \eta, \\
& \|\mathbf{u}\| \le \left(\mathscr{R}_T + \frac{\mathscr{R}_S}{L_e}\right)\left[|\Omega|^{1/2} + \eta\right] \Big\},
\end{aligned}
\tag{13}
$$

with $|\Omega|$ measure of Ω, $\eta = \overline{\alpha}\min\{1, 1/\varepsilon L_e\}$, and $\overline{\alpha}(> 0)$ being the positive constant appearing in the Poincaré inequality, is an absorbing set of (10)-(12).

Proof. The proof is obtained by following, step by step, the procedure given in [23, 26]. □

3. Instability Analysis

In this section we study the instability of the throughflow solution (8). To this end, let us neglect the nonlinear terms in (10), and let us denote by $(\nabla\widehat{\pi}, \widehat{\mathbf{u}}, \widehat{\theta}, \widehat{\Gamma})$ the solution of the linearized version of the system (10)-(12). On taking the third component of the double curl of $(10)_1$, one obtains that the linear system governing the evolution of $\{\widehat{w}, \widehat{\theta}, \widehat{\Gamma}\}$ is

$$
\begin{aligned}
\Delta\widehat{w} &= D_a \Delta\Delta\widehat{w} + \Delta_1\left(\mathscr{R}_T\widehat{\theta} - \frac{\mathscr{R}_S}{L_e}\widehat{\Gamma}\right), \\
\nabla\cdot\widehat{\mathbf{u}} &= 0, \\
\widehat{\theta}_{,t} &= f_1(z)\,\widehat{w} - Q\widehat{\theta}_{,z} + \Delta\widehat{\theta}, \\
\widehat{\Gamma}_{,t} &= H\frac{f_2(z)}{\varepsilon}\widehat{w} - \frac{Q}{\varepsilon}\widehat{\Gamma}_{,z} + \frac{1}{\varepsilon L_e}\Delta\widehat{\Gamma}.
\end{aligned}
\tag{14}
$$

under the boundary conditions

$$
\widehat{w} = \widehat{\theta} = \widehat{\Gamma} = 0, \quad \text{on } z = 0, 1. \tag{15}
$$

On looking for solutions of *normal modes* type, in view of periodicity in the x and y directions, one has

$$
\begin{pmatrix} \widehat{w}(x,y,z,t) \\ \widehat{\theta}(x,y,z,t) \\ \widehat{\Gamma}(x,y,z,t) \end{pmatrix} = \begin{pmatrix} \overline{w}(z) \\ \overline{\theta}(z) \\ \overline{\Gamma}(z) \end{pmatrix} e^{-\sigma t + i(a_x x + a_y y)}, \tag{16}
$$

with $\sigma \in \mathbb{C}$. Setting

$$
\begin{aligned}
a^2 &= a_x^2 + a_y^2, \\
D &= \frac{d}{dz},
\end{aligned}
\tag{17}
$$

it easily turns out that $\forall\varphi \in \{\widehat{w}, \widehat{\theta}, \widehat{\Gamma}\}$

$$
\begin{aligned}
\Delta_1\varphi &= -a^2\varphi, \\
\Delta\varphi &= \Delta_1\varphi + D^2\varphi = \left(D^2 - a^2\right)\varphi,
\end{aligned}
\tag{18}
$$

where $\Delta_1 = \partial^2/\partial x^2 + \partial^2/\partial y^2$. Substituting (16) in (14), one obtains

$$
\begin{aligned}
&\left(D^2 - a^2\right)\left[D_a\left(D^2 - a^2\right) - 1\right]\overline{w} - a^2\left(\mathscr{R}_T\overline{\theta} - \frac{\mathscr{R}_S}{L_e}\overline{\Gamma}\right) \\
&\quad = 0, \\
&f_1(z)\,\overline{w} + \left[D^2 - a^2 - QD + \sigma\right]\overline{\theta} = 0, \\
&H\frac{f_2(z)}{\varepsilon}\overline{w} + \left[\frac{D^2 - a^2}{\varepsilon L_e} - \frac{Q}{\varepsilon}D + \sigma\right]\overline{\Gamma} = 0,
\end{aligned}
\tag{19}
$$

under the boundary conditions

$$
\overline{w} = \overline{\theta} = \overline{\Gamma} = 0, \quad \text{on } z = 0, 1. \tag{20}
$$

In order to determine an approximate solution of (19)-(20), let us employ an order-1 Galerkin weighted residuals method [15]. We choose as trial functions (satisfying the boundary conditions (20))

$$
\begin{aligned}
\overline{w} &= A\sin(\pi z), \\
\overline{\theta} &= B\sin(\pi z), \\
\overline{\Gamma} &= C\sin(\pi z),
\end{aligned}
\tag{21}
$$

where A, B, C are constants. Substituting (21) in (19), we obtain three residuals. Making these residuals orthogonal to the trial functions over the range $0 \le z \le 1$, one obtains a system of three linear algebraic equations in the three unknown A, B, C. The vanishing of the determinant of the coefficient matrix, gives the instability threshold

$$
\begin{aligned}
\mathscr{R}_c\left(a^2, \sigma\right) &:= H\frac{-\xi_1 + \sigma}{-\xi_1 + \varepsilon L_e\sigma}\,\frac{4\pi^2 + Q^2}{4\pi^2 + L_e^2 Q^2}\mathscr{R}_S \\
&\quad + \frac{4\pi^2 + Q^2}{4\pi^2}\,\frac{\xi_1\left(\xi_1 - \sigma\right)\left(D_a\xi_1 + 1\right)}{a^2},
\end{aligned}
\tag{22}
$$

with $\xi_1 = a^2 + \pi^2$. When the vertical constant throughflow (8) loses its stability, a secondary motion (steady or oscillatory) arises.

From (16), the critical Rayleigh number at which a secondary steady motion arises (say $R_c^{(s)}$) is obtained on substituting $\sigma = 0$ in (22) and is given by

$$
\begin{aligned}
R_c^{(s)} = \min_{a^2 \in \mathbb{R}^+} \Bigg\{ & H\frac{4\pi^2 + Q^2}{4\pi^2 + L_e^2 Q^2}\mathscr{R}_S \\
& + \frac{\left(4\pi^2 + Q^2\right)}{4\pi^2}\,\frac{\xi_1^2\left(D_a\xi_1 + 1\right)}{a^2} \Bigg\}.
\end{aligned}
\tag{23}
$$

It turns out that (see [2] for details)

$$\min_{a^2\in\mathbb{R}^+} \frac{\xi_1^2 (D_a\xi_1 + 1)}{a^2} = \frac{\left(a_c^2+\pi^2\right)^2 \left[D_a\left(a_c^2+1\right)+1\right]}{a_c^2} := A^*, \tag{24}$$

with

$$a_c^2 = \frac{\sqrt{(D_a\pi^2+1)(9D_a\pi^2+1)} - \left(D_a\pi^2+1\right)}{4D_a}. \tag{25}$$

Hence the critical Rayleigh thermal number at which steady instability can occur is

$$R_c^{(s)} = H\frac{4\pi^2+Q^2}{4\pi^2+L_e^2Q^2}\mathscr{R}_S + \frac{4\pi^2+Q^2}{4\pi^2}A^*. \tag{26}$$

The critical Rayleigh number at which a secondary oscillatory motion arises (say $R_c^{(o)}$) is obtained on substituting $\sigma = i\sigma_1$, ($\sigma_1 \neq 0$) in (22). Setting the imaginary part equal to zero it follows that

$$\sigma_1^2 = -\frac{4\pi^2 a^2}{(1+D_a\xi_1)}\frac{H(1-\varepsilon L_e)\mathscr{R}_S}{\varepsilon^2 L_e^2 (4\pi^2+L_e^2Q^2)} - \frac{\xi_1^2}{\varepsilon^2 L_e^2}. \tag{27}$$

Substituting in the real part of the right-hand side of (22), it follows that

$$R_c^{(o)} = \min_{a^2\in\mathbb{R}^+} \left\{ \frac{H\left(4\pi^2+Q^2\right)}{(4\pi^2+L_e^2Q^2)}\frac{\mathscr{R}_S}{\varepsilon L_e} + \frac{(1+\varepsilon L_e)}{\varepsilon L_e}\frac{\left(4\pi^2+Q^2\right)}{4\pi^2}\frac{\xi_1^2(D_a\xi_1+1)}{a^2}\right\}. \tag{28}$$

Then—in view of (24)—the critical Rayleigh thermal number at which oscillatory instability can occur is

$$R_c^{(o)} = H\frac{\left(4\pi^2+Q^2\right)}{(4\pi^2+L_e^2Q^2)}\frac{\mathscr{R}_S}{\varepsilon L_e} + \frac{(1+\varepsilon L_e)}{\varepsilon L_e}\frac{\left(4\pi^2+Q^2\right)}{4\pi^2}A^*. \tag{29}$$

We remark that (27) excludes a oscillatory motion not only when $H(1-\varepsilon L_e) \geq 0$ but also when

$$H(1-\varepsilon L_e) < 0, \quad \mathscr{R}_S < \frac{\left(4\pi^2+L_e^2Q^2\right)\xi_1^2(1+D_a\xi_1)}{4\pi^2a^2H(\varepsilon L_e-1)}; \tag{30}$$

We have tested numerically that the order-1 Galerkin approximation gives good estimates of the critical Rayleigh thermal numbers if Q is small compared to unity, as it usually happens for problems of this type. Hence, from now on, we will assume that Q is small compared to the unity.

Now, limiting the analysis to the case in which the layer is salted from above ($H = -1$), we investigate for sufficient conditions ensuring the loss of stability of the vertical throughflow, independently of the thermal Rayleigh number, i.e., we look for the onset of "cold convection" [27]. Setting

$$R_S^{(1)} := \frac{\left(4\pi^2+L_e^2Q^2\right)A^*}{4\pi^2}, \quad R_S^{(2)} := \frac{(1+\varepsilon L_e)\left(4\pi^2+L_e^2Q^2\right)A^*}{4\pi^2}, \tag{31}$$

the following theorem holds.

Theorem 2. *In the case $H = -1$, if either*

$$\mathscr{R}_S > R_S^{(1)}, \tag{32}$$

or

$$\mathscr{R}_S > R_S^{(2)}, \tag{33}$$

holds true, then cold instability arises.

Proof. Let us first observe that $\mathscr{R}_T \geq R_c^{(s)}$ or $\mathscr{R}_T \geq R_c^{(o)}$ implies that the throughflow (8) is unstable. Hence the proof follows since either (32) or (33)—guaranteeing, respectively, that the second right-hand side of (26) and (29) is negative—ensures instability independent of $\mathscr{R}_T$.

According to the definition given in [27], the *critical solute Rayleigh number* for the onset of cold convection is $\min\{R_S^{(1)}, R_S^{(2)}\} = R_S^{(1)}$. □

3.1. Onset of Secondary Motions. In order to establish if a secondary steady or oscillatory motion arises when (8) loses its stability, we have to compare the critical Rayleigh thermal numbers for the onset of steady and oscillatory instability. Since the order-1 Galerkin method gives a good approximation of the critical instability thresholds and has to capture the physics of the problem, we assume that the comparison relation between the exact critical instability thresholds is the same of that one between $R_c^{(s)}$ and $R_c^{(o)}$. The following results hold true.

Theorem 3. *If either*

$$H(1-\varepsilon L_e) \geq 0, \tag{34}$$

or

$$H(1-\varepsilon L_e) < 0, \quad \mathscr{R}_S \leq \frac{H\left(4\pi^2+L_e^2Q^2\right)A^*}{4\pi^2(\varepsilon L_e-1)}, \tag{35}$$

then instability occurs via a steady state at $\mathscr{R}_T = R_c^{(s)}$.

Proof. From (26) and (29), it follows that

$$R_c^{(s)} - R_c^{(o)} = H \frac{(\varepsilon L_e - 1)}{\varepsilon L_e} \frac{(4\pi^2 + Q^2)}{(4\pi^2 + L_e^2 Q^2)} \mathscr{R}_S - \frac{4\pi^2 + Q^2}{4\pi^2 \varepsilon L_e} A^*. \tag{36}$$

Each one of conditions (34) or (35) guarantees that $R_c^{(s)} - R_c^{(o)} \leq 0$. □

Theorem 4. *If*

$$H(1 - \varepsilon L_e) < 0,$$
$$\mathscr{R}_S > \frac{H(4\pi^2 + L_e^2 Q^2) A^*}{4\pi^2 (\varepsilon L_e - 1)}, \tag{37}$$

then instability occurs via an oscillatory state at $\mathscr{R}_T = R_c^{(o)}$.

Proof. The proof follows since (37) guarantees that $R_c^{(s)} - R_c^{(o)} > 0$. □

4. Stabilizing/Destabilizing Effect of the Throughflow on the Onset of Instability

In this section we investigate for the influence of Q on the instability thresholds (26) and (29). We say that the throughflow has a *stabilizing effect* if it inhibits the onset of instability (i.e., increases $R_c^{(s)}$ and $R_c^{(o)}$). Vice versa, we say that the throughflow has a *destabilizing effect*.

Lemma 5. *If*

$$H(1 - L_e) \geq 0, \tag{38}$$

or

$$H(1 - L_e) < 0,$$
$$Q^2 \geq \frac{4\pi^2}{L_e^2} \left[\sqrt{\frac{H(L_e^2 - 1)\mathscr{R}_S}{A^*}} - 1 \right], \tag{39}$$

the throughflow has a stabilizing effect on the onset of steady instability. If

$$H(1 - L_e) < 0,$$
$$Q^2 < \frac{4\pi^2}{L_e^2} \left[\sqrt{\frac{H(L_e^2 - 1)\mathscr{R}_S}{A^*}} - 1 \right], \tag{40}$$

the throughflow has a destabilizing effect on the onset of steady instability.

Proof. From (26) it turns out that

$$\frac{\partial R_c^{(s)}}{\partial Q^2} = H\mathscr{R}_S \frac{4\pi^2 (1 - L_e^2)}{(4\pi^2 + L_e^2 Q^2)^2} + \frac{A^*}{4\pi^2}. \tag{41}$$

Hence, each one of (38)-(39) guarantees that $R_c^{(s)}$ is increasing with Q^2, while (40) guarantees that $R_c^{(s)}$ decreases with Q^2. □

Lemma 6. *If (38) or*

$$H(1 - L_e) < 0,$$
$$Q^2 \geq \frac{4\pi^2}{L_e^2} \left[\sqrt{\frac{H(L_e^2 - 1)\mathscr{R}_S}{(1 + \varepsilon L_e) A^*}} - 1 \right], \tag{42}$$

the throughflow has a stabilizing effect on the onset of oscillatory instability. If

$$H(1 - L_e) < 0,$$
$$Q^2 < \frac{4\pi^2}{L_e^2} \left[\sqrt{\frac{H(L_e^2 - 1)\mathscr{R}_S}{(1 + \varepsilon L_e) A^*}} - 1 \right], \tag{43}$$

the throughflow has a destabilizing effect on the onset of oscillatory instability.

Proof. From (29) it turns out that

$$\frac{\partial R_c^{(o)}}{\partial Q^2} = H\mathscr{R}_S \frac{4\pi^2 (1 - L_e^2)}{\varepsilon L_e (4\pi^2 + L_e^2 Q^2)^2} + \frac{(1 + \varepsilon L_e) A^*}{4\pi^2 \varepsilon L_e}. \tag{44}$$

Hence, each one of (38), (42) guarantees that $R_c^{(o)}$ is increasing with Q^2, while (43) guarantees that $R_c^{(o)}$ decreases with Q^2. □

From Lemmas 5–6, the following theorem holds.

Theorem 7. *If (38) or (39) holds, the throughflow has a stabilizing effect. If (43) holds, the throughflow has a destabilizing effect.*

5. Conclusions

In this paper we analyze a vertical fluid motion of a binary mixture saturating a horizontal porous layer with large pores, uniformly heated from below and salted by one salt (either from above or from below). In particular,

(1) the definitely boundedness of solutions (existence of absorbing sets) has been recalled;

(2) the critical Rayleigh numbers at which steady or oscillatory instability occurs have been recovered;

(3) sufficient conditions guaranteeing that a steady or oscillatory secondary motion sets in after the loss of stability have been found.

(4) the onset of "cold" instability, possible only when the layer is salted from above, has been analyzed;

(5) the stabilizing/destabilizing effect of the vertical throughflow has been investigated.

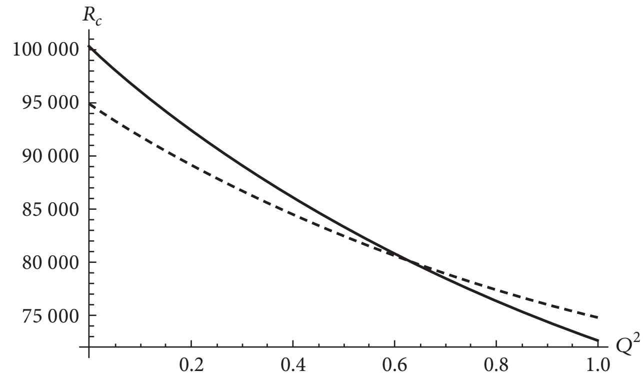

Figure 1: Behavior of $R_c^{(s)}$ and R_c^0 as functions of Q^2. Comparison between $R_c^{(s)}$ (solid line) and R_c^0 (dashed line) as functions of Q^2 with the other parameters set as {H=1, L_e=4.8, R_S=80000, D_a=3.2, ε=0.28}.

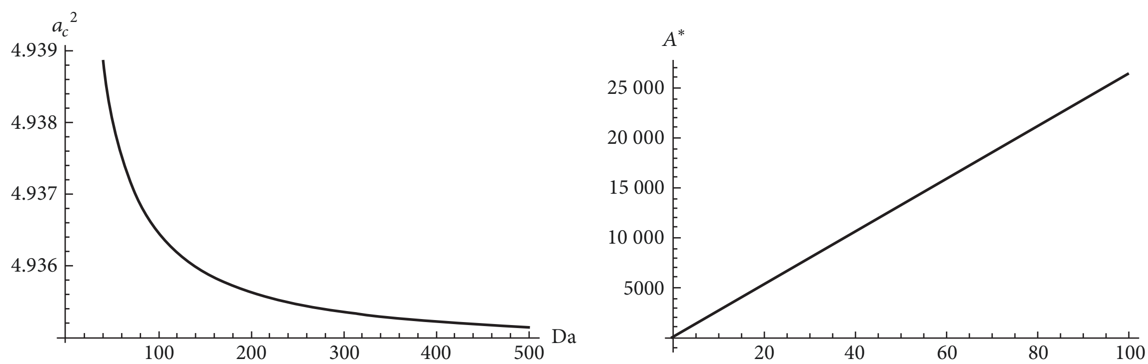

Figure 2: Behavior of a_c^2 and A^*. Left: asymptotic behavior of a_c^2 given by (25) with respect to D_a. Right: asymptotic behavior of A^* given by (24) with respect to D_a.

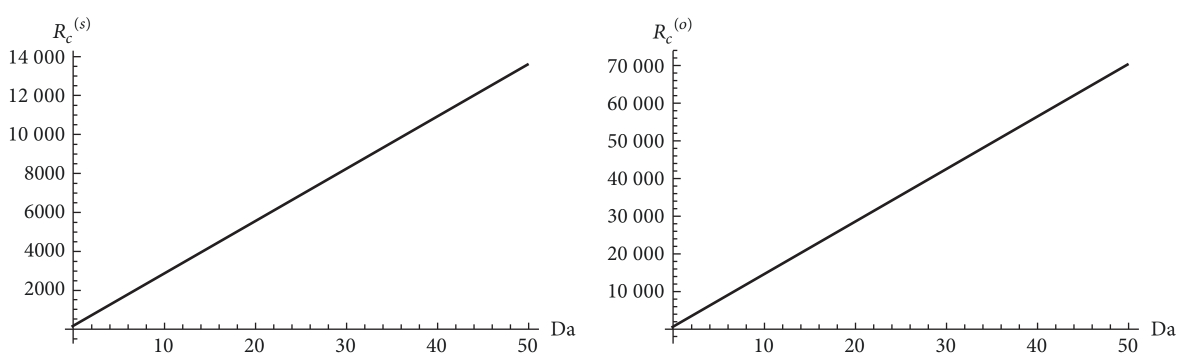

Figure 3: Behavior of $R_c^{(s)}$ and R_c^0 as functions of Q^2. Left: asymptotic behavior of $R_c^{(s)}$ given by (26) with respect to D_a. Right: asymptotic behavior of $R_c^{(o)}$ given by (29) with respect to D_a. The other parameters are set as {$H = 1$, L_e=1.2, R_S=100, $Q^2 = 0.8$, ε=0.2}.

We conclude this section by showing some numerical simulations on the performed analysis. Figure 1 shows the behavior of $R_c^{(s)}$ (solid line) and R_c^0 (dashed line) as functions of Q^2 with the other parameters set as {H=1, L_e=4.8, R_S=80000, D_a=3.2, ε=0.28}. With this choice $a_c^2 \simeq 4.99$, $A^* \simeq 20263.10$. According to (43), $R_c^{(s)}$ and $R_c^{(o)}$ decrease with Q^2. From Figure 1 it turns out that there exists a critical value of Q^2, $Q_c^2 \simeq 0.6137$, such that the motion arising when the throughflow is no longer observable is oscillatory or steady according to $Q^2 < Q_c^2$ or $Q^2 \geq Q_c^2$, respectively. This result is in agreement with the results performed in Section 3 since $Q^2 \geq Q_c^2$ and $Q^2 < Q_c^2$ guarantee, respectively, that (35) and (37) hold. Finally, in order to investigate the influence of the Darcy number (and hence of the Brinkman viscosity), let us observe that the critical wave number a_c^2 given in (25) is a decreasing function of D_a, while A^* given in (24) is an increasing function of D_a (see Figure 2). Hence, as one is expected, $R_c^{(s)}$ and $R_c^{(o)}$, respectively, given by (26) and (29), are increasing functions of D_a; i.e., the Brinkman viscosity has a stabilizing effect (see Figure 3).

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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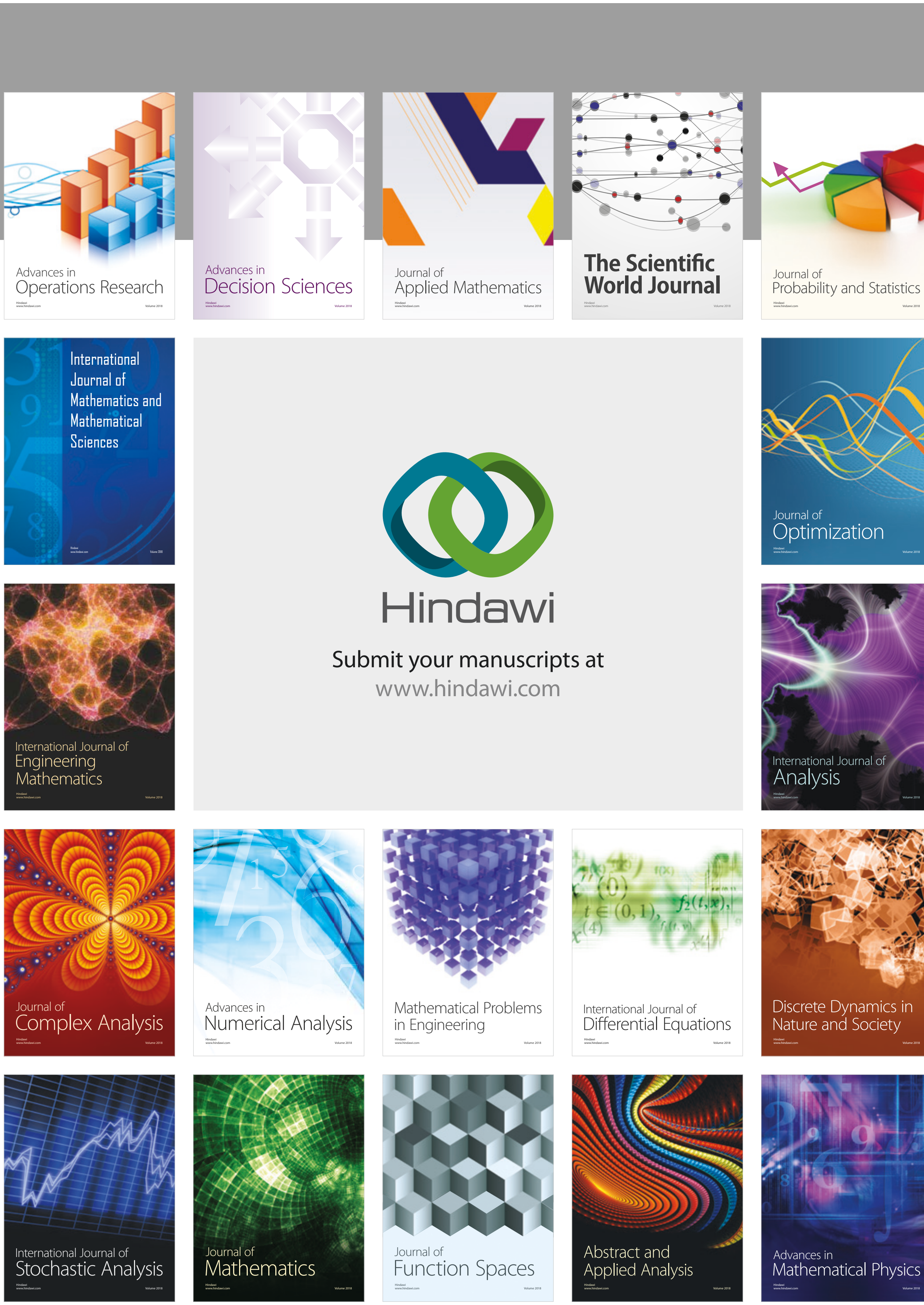
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