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Research Article

Algebraic Formulation for Moderately Thick Elastic Frames, Beams, Trusses, and Grillages within Timoshenko Theory

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The paper is dedicated to the algebraic formulation of elastic frame equations. The obtained set of equations describe deformations of moderately thick frames made of both compressible and incompressible bars, grillages of rigid or pin-joined connections, and trusses. Plane as well as space structures are presented. The paper is an extension of the article of T. Lewiński written in 2001 related to thin bars. Algebraic equations with diagonal constitutive matrix are original and suitable for various engineering applications and for educational purposes.

1. Introduction

Bar theories in the linear elastic range are widely used in the design and modeling of various engineering structures. Different analytical and numerical methods are used to calculate boundary problems. The use of the finite element method has definitely dominated for many years, but it is based on the approximation of displacement fields [1, 2]. The direct algebraic formulation in which the matrix form of the solving system of displacement equations is the product of three matrices with diagonal constitutive matrix [3, 4] is a specific mathematical problem within the theory of frames. This form is used, among others, in topological optimization [4, 5], analysis of systems with large uncertainties [6], the use of convex sets techniques [7], plastic range of the analysis [8], and shakedown of elastic-plastic structures [9]. Potential applications of the formulation as well as of the present paper can be easily extended for the gridwork and framework methods in plate theories [10], including moderately thick plates. In the direct formulation, systems of equations are built for the entire bar structure taking into account the boundary conditions. In the literature on the subject, the algebraic formulation of truss theory is commonly known and used [11–13]. Less known is the work of Lewiński [3] who generalized the truss formulation for any flat or spatial bar structure within the framework of Bernoulli-Euler's thin bar theory. The equations are recently reminded in the monograph of Michel structures [4], important from the point of view of topological optimization in which diagonal constitutive matrix is essential.

The aim of this work is to generalize the formulation [4] into the theory of moderately thick frames within the Timoshenko theory [14, 15], while maintaining the diagonal form of the matrix of elasticity coefficients. According to the authors' best knowledge, such a formulation is not known in the literature. The significance of the proposed generalization lies in that it allows for a wide extension of the application of the theory to the often occurring members of medium thickness. The present paper presents formulation for frames, beams, grillages, and trusses, flat and spatial. A variational equilibrium equation in the field of kinematically permissible displacement fields is used. Each group of equations is built for the whole structure, not for single bars, which avoids the procedures of building local matrices and their aggregation to a global form. A special case of derived equations is the formulation for thin frames given by Lewiński [3, 4] and algebraic equations of the truss theory [3, 4, 14, 15]. The key advantage of the presented formulation is that the obtained stiffness matrices are identical to those of the finite element method; however, their process of preparation does

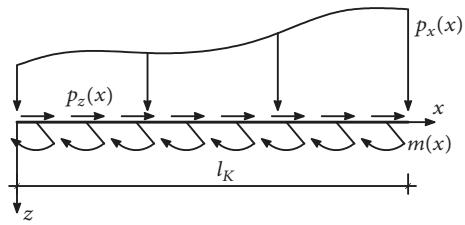

Figure 1: Bar in a local coordinate system xyz with the loading.

not require approximation of the displacement fields and the diagonal constitutive matrix is applied. It is crucial for some advance algorithms in topology optimization, elastic-plastic analysis (including shakedown), and sophisticated techniques of large uncertainities and convex sets algebra used in structural mechanics. The didactic values of the algebraic formulation for moderately thick bars are also worth emphasizing.

An important extension of the analysis can be the initial prestress of the frameworks [16–18]. If the prestress of the structure is realized according to the truss analogy, the algebraic formulation of equations described in [19] is possible to apply. The geometric stiffness matrices can be defined as a multiplication of three matrices with a diagonal matrix of self-equilibrated forces in the middle [19]. The problem of initial internal forces of the members lies beyond the scope of this paper.

The detailed derivation of algebraic formulas is given for flat frames with compressible rods within the Timoshenko theory. Next, it is shown that the proposed theory at the border crossing leads to the formulation given by Lewiński in [3]. The next chapters outline the following: plane and space trusses, plane frames made of incompressible Timoshenko bars, rigid joints grillages, pin joints grillages, and space frames. The last chapter is dedicated to summarize and conclude the presented formulation.

2. Plane Elastic Frames of Compressible Timoshenko Bars

Let us consider a frame made of e straight and prismatic bars lying in the X-Z plane of global cartesian coordinate system XYZ. l_K $(K = 1, 2, \ldots, e)$ represents length of K-th bar and EA_K, H_K, EI_K its axial, transversal, and flexural stiffnesses. The bar is subjected to the loading $p_x(x)$, $p_z(x)$ and $m(x)$, respectively (see. Figure 1). The deformation of a single bar in local cartesian coordinate system xyz is described with axial displacement $u_K(x)$, transversal displacement $w_K(x)$, and rotation $\varphi_K(x)$, where x complies with the axis of K-th bar and $0 \leqslant x \leqslant l_K$.

The unknown internal forces: axial $N_K(x)$, transverse $T_K(x)$, and bending moment $M_K(x)$ are correlated to the strains: normal $\varepsilon_K(x)$, transverse shear $\gamma_K(x)$, and curvature $\kappa_K(x)$ by

$$\begin{aligned} N_K(x) &= EA_K \varepsilon_K(x), \\ T_K(x) &= H_K \gamma_K(x), \\ M_K(x) &= EI_K \kappa_K(x), \end{aligned} \tag{1}$$

while the strains are given by

$$\begin{aligned} \varepsilon_K(x) &= \frac{du_K}{dx}, \\ \gamma_K(x) &= \frac{dw_K}{dx} - \varphi_K(x), \\ \kappa_K(x) &= -\frac{d\varphi_K}{dx}. \end{aligned} \tag{2}$$

The equilibrium equations in local coordinate system are given by the differential equations:

$$\begin{aligned} \frac{dN_K}{dx} + p_x(x) &= 0, \\ \frac{dT_K}{dx} + p_z(x) &= 0, \\ T_K(x) + m(x) &= \frac{dM_K}{dx} \end{aligned} \tag{3}$$

or in the variational form

$$\begin{aligned} &\int_0^{l_K} \left[N_K(x)\,\overline{\varepsilon}(x) + T_K(x)\,\overline{\gamma}(x) + M_K(x)\,\overline{\kappa}(x)\right] dx \\ &= \int_0^{l_K} \left[p_x(x)\,\overline{u}(x) + p_z(x)\,\overline{w}(x) - m(x)\,\overline{\varphi}(x)\right] dx \\ &\quad + \left[N_K(x)\,\overline{u}(x) + T_K(x)\,\overline{w}(x) + M_K(x)\,\overline{\beta}(x)\right]_0^{l_K}, \end{aligned} \tag{4}$$

where $[f]_0^l = f(l) - f(0)$ and overlines $\overline{(\cdot)}$ indicate the trial fields. The relation between virtual deformations $(\overline{\varepsilon}, \overline{\gamma}, \overline{\kappa})$ and displacements $(\overline{u}, \overline{w}, \overline{\varphi})$ is defined by (2). Further considerations assume that kinematic boundary conditions are homogeneous with preserving all given connections in the frame joints. Then variational equilibrium equation of the frame has the following form:

$$\begin{aligned} &\int_S \left[N(x)\,\overline{\varepsilon}(x) + T(x)\,\overline{\gamma}(x) + M(x)\,\overline{\kappa}(x)\right] dS \\ &= \int_S \left[p_x(x)\,\overline{u}(x) + p_z(x)\,\overline{w}(x) - m(x)\,\overline{\varphi}(x)\right] dS, \end{aligned} \tag{5}$$

where S is the coordinate which runs through all of the frame bars and $0 \leqslant S \leqslant \sum_{K=1}^{e} l_K$.

Substitution of (2) into (1) and further into (3), with the use of dimensionless coordinate $\xi = x/l_K$, leads to the equations which describe displacement functions $u_K(\xi)$, $w_K(\xi)$, $\varphi_K(\xi)$ of the bar in local coordinate system ξyz:

$$\begin{gathered}\frac{d^2 u_K}{d\xi^2} = \frac{p_x(\xi) l_K^2}{EA_K}, \\ \frac{1}{l_K}\frac{dw_K}{d\xi} + \rho_K \frac{d^2\varphi_K}{d\xi^2} - \varphi_K(\xi) = \frac{m(\xi)}{H_K}, \\ \frac{d^2 w_K}{d\xi^2} - l_K \frac{d\varphi_K}{d\xi} = -\frac{p_z(\xi) l_K^2}{H_K},\end{gathered} \tag{6}$$

where

$$\varrho_K = \frac{EI_K}{H_K l_K^2}. \tag{7}$$

The solution of homogeneous system of (6) has the following form:

$$\begin{aligned} u_K(\xi) &= A_0 + A_1\xi, \\ w_K(\xi) &= B_0 + B_1\xi + B_2\xi^2 + B_3\xi^3, \\ \varphi_K(\xi) &= \frac{1}{l}\left[B_1 + 2B_2\xi + B_3\left(6\rho_K + 3\xi^2\right)\right]. \end{aligned} \tag{8}$$

The local coordinate system allows defining the bars ends. According to denotation proposed by Lewiński in [3], the values with ${}^*(\cdot)$ are related to the left end and the $(\cdot)^*$ with the right. The assumptions enable obtaining displacement fields (8) depending on displacements of the left $({}^*u_K, {}^*w_K, {}^*\varphi_K)$ and the right $(u_K^*, w_K^*, \varphi_K^*)$ bar end, which leads to the following form:

$$\begin{aligned} u_K(\xi) &= {}^*u_K(1-\xi) + u_K^*\xi, \\ w_K(\xi) &= \mu_K\Big[{}^*w_K(1-\xi)\left(2\varrho_K + \frac{1}{6}\left(-2\xi^2 + \xi + 1\right)\right) \\ &\quad + w_K^*\xi\left(2\varrho_K + \frac{1}{6}\left(3\xi - 2\xi^2\right)\right) \\ &\quad + {}^*\varphi_K l_K\left(\xi - \xi^2\right)\left(\varrho_K + \frac{1}{6}(1-\xi)\right) \\ &\quad + \varphi_K^* l_K\left(\xi^2 - \xi\right)\left(\varrho_K + \frac{1}{6}\xi\right)\Big], \\ \varphi_K(\xi) &= \mu_K\Big[{}^*w_K\frac{\left(\xi^2 - \xi\right)}{l_K} - w_K^*\frac{\left(\xi^2 - \xi\right)}{l_K} \\ &\quad + {}^*\varphi_K(\xi - 1)\left(\frac{1}{6}(3\xi - 1) - 2\varrho_K\right) \\ &\quad + \varphi_K^*\xi\left(\frac{1}{6}(3\xi - 2) + 2\varrho_K\right)\Big]. \end{aligned} \tag{9}$$

Then the deformations (2) are given by

$$\begin{aligned} \varepsilon_K(\xi) &= \frac{\Delta_K}{l_K}, \\ \kappa_K(\xi) &= \frac{{}^*\chi_K}{l_K}\,{}^*f(\xi) + \frac{\chi_K^*}{l_K} f^*(\xi), \\ \gamma_K(\xi) &= -\left({}^*\chi_K + \chi_K^*\right)\mu_K\varrho_K, \end{aligned} \tag{10}$$

wherein it is additionally denoted

$$\begin{aligned} {}^*f(\xi) &= \mu_K\left[\frac{1}{3}(2 - 3\xi) + 2\varrho_K\right], \\ f^*(\xi) &= \mu_K\left[\frac{1}{3}(1 - 3\xi) - 2\varrho_K\right], \\ \mu_K &= \frac{6}{12\varrho_K + 1} \end{aligned} \tag{11}$$

and the following quantities represents the deformations of the bar:

$$\begin{aligned} \Delta_K &= u_K^* - {}^*u_K, \\ {}^*\chi_K &= {}^*\varphi_K - \psi_K, \\ \chi_K^* &= \varphi_K^* - \psi_K, \end{aligned} \tag{12}$$

with $\psi_K = (w_K^* - {}^*w_K)/l_K$ which is the slope of the K-th bar.

Let us consider a frame with s possible displacements and rotation of the nodes (called degrees of freedom), which are collected in the vector $\mathbf{q} = (q_1, \ldots, q_s)$. Then the relation between bars ends displacements in the local coordinate system and the displacements of frame nodes in global coordinate system is expressed by the allocation matrices:

$$\begin{aligned} {}^*u_K &= \sum_{j=1}^{s} {}^*A_{Kj}^{(u)} q_j, \\ {}^*w_K &= \sum_{j=1}^{s} {}^*A_{Kj}^{(w)} q_j, \\ {}^*\varphi_K &= \sum_{j=1}^{s} {}^*A_{Kj}^{(\varphi)} q_j, \\ u_K^* &= \sum_{j=1}^{s} A_{Kj}^{*(u)} q_j, \\ w_K^* &= \sum_{j=1}^{s} A_{Kj}^{*(w)} q_j, \\ \varphi_K^* &= \sum_{j=1}^{s} A_{Kj}^{*(\varphi)} q_j. \end{aligned} \tag{13}$$

Therefore relations between bars deformations (12) in the local coordinate system and displacements and rotations $\mathbf{q}$ in the global coordinate system are given by

$$\begin{aligned} \Delta_K &= \sum_{j=1}^{s} B_{Kj} q_j, \\ {}^*\chi_K &= \sum_{j=1}^{s} {}^*B_{Kj} q_j, \\ \chi_K^* &= \sum_{j=1}^{s} B_{Kj}^* q_j \end{aligned} \tag{14}$$

or in the matrix form

$$\begin{aligned} \boldsymbol{\Delta} &= \mathbf{B}\mathbf{q}, \\ {}^*\boldsymbol{\chi} &= {}^*\mathbf{B}\mathbf{q}, \\ \boldsymbol{\chi}^* &= \mathbf{B}^*\mathbf{q}, \end{aligned} \tag{15}$$

where $\mathbf{B}$, ${}^*\mathbf{B}$, $\mathbf{B}^*$ are defined by allocation matrices (13).

With the denotations (14) and with the use of dimensionless coordinate ξ the left-hand side of equilibrium equation (5) takes the following form:

$$\begin{aligned} &\int_S \left[N(x)\,\overline{\varepsilon}(x) + T(x)\,\overline{\gamma}(x) + M(x)\,\overline{\kappa}(x)\right] dS \\ &\quad = \sum_{K=1}^{e} \left(\mathbb{N}_K \overline{\Delta}_K + {}^*\mathbb{M}_K\, {}^*\overline{\chi}_K + \mathbb{M}_K^* \overline{\chi}_K^*\right), \end{aligned} \tag{16}$$

where

$$\begin{aligned} \mathbb{N}_K &= \int_0^1 N_K(\xi)\, d\xi, \\ {}^*\mathbb{M}_K &= \int_0^1 \left[{}^*f(\xi)\, M_K(\xi) - \mu_K \varrho_K l_K T_K(\xi)\right] d\xi, \\ \mathbb{M}_K^* &= \int_0^1 \left[f^*(\xi)\, M_K(\xi) - \mu_K \varrho_K l_K T_K(\xi)\right] d\xi. \end{aligned} \tag{17}$$

Substitution of (14) into (16) results in the left-hand side of equilibrium equation in the matrix form:

$$\begin{aligned} &\int_S \left[N(x)\,\overline{\varepsilon}(x) + T(x)\,\overline{\gamma}(x) + M(x)\,\overline{\kappa}(x)\right] dS \\ &\quad = \overline{\mathbf{q}}^T \left[\mathbf{B}^\top \mathbf{N} + \left({}^*\mathbf{B}\right)^\top {}^*\mathbf{M} + \left(\mathbf{B}^*\right)^\top \mathbf{M}^*\right] \end{aligned} \tag{18}$$

with the vectors

$$\begin{aligned} \mathbf{N} &= \left[\mathbb{N}_1 \ldots \mathbb{N}_e\right]^\top, \\ {}^*\mathbf{M} &= \left[{}^*\mathbb{M}_1 \ldots {}^*\mathbb{M}_e\right]^\top, \\ \mathbf{M}^* &= \left[\mathbb{M}_1^* \ldots \mathbb{M}_e^*\right]^\top. \end{aligned} \tag{19}$$

Right-hand side of (5) expressed with parameters ${}^*u_K$, ${}^*w_K$, ${}^*\varphi_K$, u_K^*, w_K^*, φ_K^* and consideration of (13) leads to the following form:

$$\begin{aligned} &\int_0^{l_K} \left[p_x(x)\,\overline{u}(x) + p_z(x)\,\overline{w}(x) - m(x)\,\overline{\varphi}(x)\right] dx \\ &\quad = \overline{\mathbf{q}}^\top \mathbf{Q}, \end{aligned} \tag{20}$$

where $\mathbf{Q} = (Q_i)$ and Q_i is the work of loads $p_x(x)$, $p_z(x)$, $m(x)$ on the virtual displacements $\overline{u}^{(i)}(x)$, $\overline{w}^{(i)}(x)$, $\overline{\varphi}^{(i)}(x)$ corresponding to virtual displacement field of the frame where $\overline{q_i} = 0$, $\overline{q_j} \neq 0$ for $i \neq j$, $i = 1, 2, \ldots, s$, $j = 1, 2, \ldots, s$.

Comparison of (18) and (20) leads to the following equality:

$$\mathbf{B}^\top \mathbf{N} + \left({}^*\mathbf{B}\right)^\top {}^*\mathbf{M} + \left(\mathbf{B}^*\right)^\top \mathbf{M}^* = \mathbf{Q} \tag{21}$$

When K-th bar is loaded on the span the internal forces can be decomposed on the part dependent on the displacements of the nodes $\mathbf{q}$ and values $N_K^0(\xi)$, $T_K^0(\xi)$, $M_K^0(\xi)$, which are the forces imposed by external loading applied to frame when $q_j = 0$, $j = 1, 2, \ldots, s$. Then

$$\begin{aligned} N_K(\xi) &= \frac{EA_K}{l_K} \Delta_K + N_K^0(\xi), \\ T_K(\xi) &= -H_K \left({}^*\chi_K + \chi_K^*\right) \mu_K \varrho_K + T_K^0(\xi), \\ M_K(\xi) &= \frac{EI_K}{l_K} \left[{}^*\chi_K\, {}^*f(\xi) + \chi_K^* f^*(\xi)\right] + M_K^0(\xi). \end{aligned} \tag{22}$$

It is possible to prove that

$$\begin{aligned} &\int_0^1 N_K^0(\xi)\, d\xi = 0, \\ &\int_0^1 \left[{}^*f(\xi)\, M_K^0(\xi) - \mu_K \varrho_K l_K T_K^0(\xi)\right] d\xi = 0, \\ &\int_0^1 \left[f^*(\xi)\, M_K^0(\xi) - \mu_K \varrho_K l_K T_K^0(\xi)\right] d\xi = 0, \end{aligned} \tag{23}$$

and hence the substitution of (22) into (17) leads to the constitutive equations of the frame:

$$\begin{aligned} \mathbb{N}_K &= \frac{EA_K}{l_K} \Delta_K, \\ {}^*\mathbb{M}_K &= \frac{2EI_K \mu_k}{6l_K} \left[\left(2 + 6\varrho_K\right) {}^*\chi_K + \left(1 - 6\varrho_K\right) \chi_K^*\right], \\ \mathbb{M}_K^* &= \frac{2EI_K \mu_K}{6l_K} \left[\left(1 - 6\varrho_K\right) {}^*\chi_K + \left(2 + 6\varrho_K\right) \chi_K^*\right] \end{aligned} \tag{24}$$

or in the matrix form

$$\begin{aligned} \mathbf{N} &= \mathbf{E}\boldsymbol{\Delta}, \\ {}^*\mathbf{M} &= \mathbf{D}_1\, {}^*\boldsymbol{\chi} + \mathbf{D}_2 \boldsymbol{\chi}^*, \\ \mathbf{M}^* &= \mathbf{D}_2\, {}^*\boldsymbol{\chi} + \mathbf{D}_1 \boldsymbol{\chi}^* \end{aligned} \tag{25}$$

with diagonal $e \times e$ constitutive matrices

$$\mathbf{E} = \operatorname{diag}\left(\frac{EA_K}{l_K}\right),$$
$$\mathbf{D}_1 = \operatorname{diag}\left(\frac{2EI_K}{l_K}\frac{\mu_K(1+3\varrho_K)}{3}\right), \tag{26}$$
$$\mathbf{D}_2 = \operatorname{diag}\left(\frac{2EI_K}{l_K}\frac{\mu_K(1-6\varrho_K)}{6}\right).$$

The values of $\mathbb{N}_K$, ${}^*\mathbb{M}_K$, and $(-\mathbb{M}_K^*)$ are the axial force, left-end, and right-end bending moment, respectively, and are imposed by displacements ${}^*u_K$, ${}^*w_K$, ${}^*\varphi_K$, u_K^*, w_K^*, φ_K^*. The following relations are valid $\mathbb{N}_K = N_K(0) = N_K(1) = N_K(\xi)$, ${}^*\mathbb{M}_K = M_K(0)$, $\mathbb{M}_K^* = -M_K(1)$ with $N_K^0(\xi) = 0$, $T_K^0(\xi) = 0$, $M_K^0(\xi) = 0$.

Substitution of (14) into (24) and further into (21) leads to the system of equations in matrix form:

$$\mathbf{Kq} = \mathbf{Q} \tag{27}$$

with the stiffness matrix

$$\mathbf{K} = \mathbf{B}^\top\mathbf{EB} + ({}^*\mathbf{B})^\top\mathbf{D}_1{}^*\mathbf{B} + ({}^*\mathbf{B})^\top\mathbf{D}_2\mathbf{B}^* + (\mathbf{B}^*)^\top\mathbf{D}_2{}^*\mathbf{B} + (\mathbf{B}^*)^\top\mathbf{D}_1\mathbf{B}^*. \tag{28}$$

It is worth noting that it is possible to reformulate above considerations so the integrals (17) depend on single deformations $\boldsymbol{\chi}^s$ and $\boldsymbol{\chi}^a$. For this purpose, the decomposition is proposed

$${}^*\boldsymbol{\chi} = \boldsymbol{\chi}^s + \boldsymbol{\chi}^a,$$
$$\boldsymbol{\chi}^* = \boldsymbol{\chi}^s - \boldsymbol{\chi}^a, \tag{29}$$

where

$$\boldsymbol{\chi}^s = \mathbf{B}^s\mathbf{q},$$
$$\boldsymbol{\chi}^a = \mathbf{B}^a\mathbf{q}, \tag{30}$$

Then the deformations (10) are given by

$$\varepsilon_K(\xi) = \frac{\Delta_K}{l_K},$$
$$\kappa_K(\xi) = \frac{\chi_K^s}{l_K}f^s(\xi) + 2\frac{\chi_K^a}{l_K}, \tag{31}$$
$$\gamma_K(\xi) = -2\mu_K\varrho_K\chi_K^s,$$

in which

$$f^s(\xi) = \mu_K(1-2\xi). \tag{32}$$

Such denotations allows writing (16) in the following form:

$$\int_S \left[N(x)\overline{\varepsilon}(x) + T(x)\overline{\gamma}(x) + M(x)\overline{\kappa}(x)\right]dS = \sum_{K=1}^{e}\left(\mathbb{N}_K\overline{\Delta}_K + \mathbb{M}^s\overline{\chi}_K^s + \mathbb{M}_K^a\overline{\chi}_K^a\right), \tag{33}$$

where

$$\mathbb{N}_K = \int_0^1 N_K(\xi)\,d\xi,$$
$$\mathbb{M}_K^s = \int_0^1 \left[f^s(\xi)M_K(\xi) - 2\mu_K\varrho_K l_K T_K(\xi)\right]d\xi, \tag{34}$$
$$\mathbb{M}_K^a = 2\int_0^1 M_K(\xi)\,d\xi.$$

which, after the consideration of (1) and (31), leads to the following formulae:

$$\mathbb{N}_K = \frac{EA_K}{l_K}\Delta_K,$$
$$\mathbb{M}_K^s = \frac{2EI_K\mu_K}{l_K}\chi_K^s, \tag{35}$$
$$\mathbb{M}_K^a = \frac{4EI_K}{l_K}\chi_K^a$$

or in the matrix form

$$\mathbf{N} = \mathbf{E}\boldsymbol{\Delta},$$
$$\mathbf{M}^s = \mathbf{D}^s\boldsymbol{\chi}^s, \tag{36}$$
$$\mathbf{M}^a = \mathbf{D}^a\boldsymbol{\chi}^a,$$

with diagonal constitutive matrices

$$\mathbf{E} = \operatorname{diag}\left(\frac{EA_K}{l_K}\right),$$
$$\mathbf{D}^s = \operatorname{diag}\left(\frac{2EI_K\mu_K}{l_K}\right), \tag{37}$$
$$\mathbf{D}^a = \operatorname{diag}\left(\frac{4EI_K}{l_K}\right).$$

It is possible to prove that integrals analogous to (23)

$$\int_0^1 N_K^0(\xi)\,d\xi = 0,$$
$$\int_0^1 \left[f^s(\xi)M_K^0(\xi) - 2\mu_K\varrho_K l_K T_K^0(\xi)\right]d\xi = 0, \tag{38}$$
$$\int_0^1 M_K^0(\xi)\,d\xi = 0$$

are equal to zero.

Values $\mathbb{M}_K^s$ and $\mathbb{M}_K^a$ can be interpreted as a difference and a sum of the end bending moments imposed by deflections ${}^*u_K$, ${}^*w_K$, ${}^*\varphi_K$, u_K^*, w_K^*, φ_K^*. It is $\mathbb{M}_K^s = M_K(0) - M_K(1)$ and $\mathbb{M}_K^a = M_K(0) + M_K(1)$ or inversely: $M_K(0) = (\mathbb{M}_K^a + \mathbb{M}_K^s)/2$, $M_K(1) = (\mathbb{M}_K^a - \mathbb{M}_K^s)/2$ with the assumption that $T_K^0(\xi) = 0$, $M_K^0(\xi) = 0$.

The above considerations lead to the system of equations (27) with the stiffness matrix given by the following formula:

$$\mathbf{K} = \mathbf{B}^\top\mathbf{EB} + (\mathbf{B}^s)^\top\mathbf{D}^s\mathbf{B}^s + (\mathbf{B}^a)^\top\mathbf{D}^a\mathbf{B}^a. \tag{39}$$

Equations (39) and (37) have more accessible form than (28) and (26); however, the construction of matrices $\mathbf{B}^s$ and $\mathbf{B}^a$ can cause slightly more difficulties because

$$\chi_K^s = \frac{1}{2}{}^*\varphi_K + \frac{1}{2}\varphi_K^* - \psi_K,$$
$$\chi_K^a = \frac{1}{2}{}^*\varphi_K - \frac{1}{2}\varphi_K^*. \tag{40}$$

Let $\mathbf{b}_P$, $\mathbf{b}_P^s$, $\mathbf{b}_P^a$ be the P-th rows of matrices $\mathbf{B}$, $\mathbf{B}^s$, $\mathbf{B}^a$, respectively. Then stiffness matrix (39) can be written as

$$\mathbf{K} = \sum_{P=1}^{e}\left[\frac{EA_P}{l_P}\mathbf{b}_P \otimes \mathbf{b}_P + \frac{2EI_P}{l_P}\left(\mu_P \mathbf{b}_P^s \otimes \mathbf{b}_P^s + 2\mathbf{b}_P^a \otimes \mathbf{b}_P^a\right)\right], \tag{41}$$

where $\mathbf{a} \otimes \mathbf{b} = [a_i b_j]$ is the dyadic product of vectors $\mathbf{a}$ and $\mathbf{b}$. This kind of decomposition of matrix $\mathbf{K}$ can be successfully used in optimisation [4] or during the uncertainty analysis with the use of convex sets [6].

3. Plane Frames Made of Compressible Euler-Bernoulli Bars

Let the bars have rectangular cross section and h_K be the height of K-th bar. The equations of frames made of Timoshenko bars become the equations of Euler-Bernoulli frames when $l_K \gg h_K$, so when the ratio $h/l \longrightarrow 0$. Then the limits of ϱ_K and μ_K equal 0 and 6, respectively; hence,

$$\mathbf{E} = \operatorname{diag}\left(\frac{EA_K}{l_K}\right),$$
$$\mathbf{D}_1 = \operatorname{diag}\left(\frac{4EI_K}{l_K}\right) = 2\mathbf{D}, \tag{42}$$
$$\mathbf{D}_2 = \operatorname{diag}\left(\frac{2EI_K}{l_K}\right) = \mathbf{D},$$

and the constitutive relations are reduced to the following:

$$\mathbf{N} = \mathbf{E}\boldsymbol{\Delta},$$
$${}^*\mathbf{M} = \mathbf{D}\left(2{}^*\boldsymbol{\chi} + \boldsymbol{\chi}^*\right), \tag{43}$$
$$\mathbf{M}^* = \mathbf{D}\left({}^*\boldsymbol{\chi} + 2\boldsymbol{\chi}^*\right).$$

The deformations (15) and matrices $\mathbf{B}$, ${}^*\mathbf{B}$, $\mathbf{B}^*$ remain unchanged. The stiffness matrix of the plane frame made of Euler-Bernoulli bars has the following form:

$$\mathbf{K} = \mathbf{B}^\top\mathbf{E}\mathbf{B} + 2\left({}^*\mathbf{B}\right)^\top \mathbf{D}^*\mathbf{B} + \left({}^*\mathbf{B}\right)^\top \mathbf{D}\mathbf{B}^* + \left(\mathbf{B}^*\right)^\top \mathbf{D}^*\mathbf{B} + 2\left(\mathbf{B}^*\right)^\top \mathbf{D}\mathbf{B}^*, \tag{44}$$

which is consistent with the matrix proposed by [3].

4. Plane and Space Trusses

Equations of plane trusses were obtained by the omission of bending moments, transverse forces, and the stiffness EI/l and H. Then (21), (25), and (15) have the form $\mathbf{B}^\top\mathbf{N} = \mathbf{Q}$, $\mathbf{N} = \mathbf{E}\boldsymbol{\Delta}$, $\boldsymbol{\Delta} = \mathbf{B}\mathbf{q}$, respectively, and stiffness matrix is given by $\mathbf{K} = \mathbf{B}^\top\mathbf{E}\mathbf{B}$. The relations are well known in the literature [3, 5, 8, 9, 11]. The equations of space frames have the same form.

5. Plane Elastic Frames Made of Incompressible Timoshenko Bars

If the bars are incompressible the axial stiffness of K-th bar $EA_K \longrightarrow \infty$. Then ${}^*u_K = u_K^* = u_K$ with the allocation matrix

$$u_K = \sum_{j=1}^{s} A_{Kj}^{(u)} q_j \tag{45}$$

and $\Delta_K = 0$. The independent degrees of freedom are still denoted by (q_j), yet they have new interpretation: displacements of the bar ends are described by the slope ψ_K instead of displacements of the nodes. Then (21) is reduced to

$$\left(\mathbf{B}^a\right)^\top \mathbf{M}^a + \left(\mathbf{B}^s\right)^\top \mathbf{M}^s = \mathbf{Q}, \tag{46}$$

with the vector $\mathbf{Q}$, which interpretation changes according to the interpretation of $\mathbf{q}$. The constitutive relations of bending remain unchanged, given by $(35)_{2,3}$ with diagonal constitutive matrices $(37)_{2,3}$. The stiffness matrix is given by

$$\mathbf{K} = \left(\mathbf{B}^s\right)^\top \mathbf{D}^s\mathbf{B}^s + \left(\mathbf{B}^a\right)^\top \mathbf{D}^a\mathbf{B}^a. \tag{47}$$

It should be noted that matrices $\mathbf{B}^s$, $\mathbf{B}^a$ have different form compared to those in frames made of compressible bars, because of different vector $\mathbf{q}$.

6. Rigid Joints Grillages Made of Timoshenko Bars

Let us consider a grillage made of e straight and prismatic bars lying in the X-Y plane of cartesian coordinate system XYZ. The values $M(x)$, $T(x)$, $\kappa(x)$, $\gamma(x)$, $w(x)$, $\varphi(x)$, EI, H introduced in the previous section remain their interpretation. GC_K is the torsional stiffness of K-th bar and $\phi_K(x)$ the angle of torsion. The connections between bars are rigid. The nodes are capable of transfer bending and torsion. Grillages are loaded in Z direction; hence $\varepsilon_K(x) = 0$. The constitutive relations of single bars are given by

$$\mathfrak{m}_K(x) = GC_K\tau_K(x),$$
$$T_K(x) = H_K\gamma_K(x), \tag{48}$$
$$M_K(x) = EI_K\kappa_K(x),$$

and strains

$$\tau_K(x) = \frac{d\phi_K}{dx},$$
$$\gamma_K(x) = \frac{dw_K}{dx} - \varphi_K(x), \tag{49}$$
$$\kappa_K(x) = -\frac{d\varphi_K}{dx}.$$

The equilibrium equations are given by the differential equations:

$$\frac{d\mathfrak{m}_K}{dx} = 0,$$
$$\frac{dT_K}{dx} + p_z(x) = 0, \tag{50}$$
$$T_K(x) + m(x) = \frac{dM_K}{dx}.$$

Then the variational equilibrium equation of the grillage has the following form:

$$\int_S \left(\mathfrak{m}(x)\overline{\tau}(x) + T(x)\overline{\gamma}(x) + M(x)\overline{\kappa}(x)\right) dS = \int_S \left(p_z(x)\overline{w}(x) - m(x)\overline{\varphi}(x)\right) dS, \tag{51}$$

Likewise for truss state the function of the angle of torsion is defined by second=order differential equation expressed in the term of dimensionless coordinate ξ:

$$\frac{d^2\phi_K(\xi)}{d\xi^2} = 0, \tag{52}$$

which leads to the function

$$\phi_K(\xi) = {}^*\phi_K(1-\xi) + \phi_K^*\xi, \tag{53}$$

when angles of torsion of left and right bars end are denoted by $({}^*\phi_K, \phi_K^*)$.

It follows that torsional strain is constant on each bar and can be expressed as

$$\tau_K(\xi) = \frac{\theta_K}{l_K},$$
$$\theta_K = \phi_K^* - {}^*\phi_K. \tag{54}$$

Thus it can be written as

$$\int_S \mathfrak{m}(x)\overline{\tau}(x)\,dS = \sum_{K=1}^{e} \mathfrak{m}_K\overline{\theta}_K \tag{55}$$

and

$$\mathfrak{m}_K = \frac{GC_K}{l_K}\theta_K. \tag{56}$$

Likewise in (14) the relation can be written as follows:

$$\theta_K = \sum_{j=1}^{s} \mathfrak{B}_{Kj} q_j, \tag{57}$$

so (55) takes the following form:

$$\int_S \mathfrak{m}(x)\overline{\tau}(x)\,dS = \mathbf{q}^{\mathsf{T}}\left(\mathfrak{B}^{\mathsf{T}}\mathfrak{m}\right). \tag{58}$$

The final equilibrium equation has the following form:

$$\mathfrak{B}^{\mathsf{T}}\mathfrak{m} + \left(\mathbf{B}^s\right)^{\mathsf{T}}\mathbf{M}^s + \left(\mathbf{B}^a\right)^{\mathsf{T}}\mathbf{M}^a = \mathbf{Q}. \tag{59}$$

By analogy to (25) constitutive relations can be expressed as

$$\mathfrak{m} = \mathbf{G}\boldsymbol{\theta},$$
$$\mathbf{M}^s = \mathbf{D}^s\boldsymbol{\chi}^s, \tag{60}$$
$$\mathbf{M}^a = \mathbf{D}^a\boldsymbol{\chi}^a,$$

with the matrix $\mathbf{G} = \operatorname{diag}(GC_K/l_K)$. Then the system of equations has the form (27) with stiffness matrix:

$$\mathbf{K} = \mathfrak{B}^{\mathsf{T}}\mathbf{G}\mathfrak{B} + \left(\mathbf{B}^s\right)^{\mathsf{T}}\mathbf{D}^s\mathbf{B}^s + \left(\mathbf{B}^a\right)^{\mathsf{T}}\mathbf{D}^a\mathbf{B}^a. \tag{61}$$

7. Pin Joints Grillages Made of Timoshenko Bars

Let the only one mutual degree of freedom for bars be the displacement perpendicular to grillage plane. Then the bars interact in nodes only with vertical force and nodes do not transfer torsion; hence $\varepsilon_K(x) = 0$ and $\phi_K(x) = 0$. The equilibrium equation can be obtained by removal of $\mathfrak{B}^{\mathsf{T}}\mathfrak{m}$ from (59) and the stiffness matrix by removal of $\mathfrak{B}^{\mathsf{T}}\mathbf{G}\mathfrak{B}$ from matrix (61), obtaining equations identical to (46) and (47), respectively.

8. Space Frames Made of Timoshenko Bars

In the present section, the algebraic form of space frame equations is introduced. It is the generalisation of the form presented in section dedicated to plane frames.

Let the local coordinate system xyz be assigned to each bar. The x axis is compatible with bars axis. The values

$$\mathbf{M}^s,\ \mathbf{M}^a,\ \boldsymbol{\chi}^s,\ \boldsymbol{\chi}^a,\ \mathbf{B}^s,\ \mathbf{B}^a,\ \mathbf{D}^s,\ \mathbf{D}^a,\ \mu_K \tag{62}$$

concern bending in x - z plane, while

$$\widehat{\mathbf{M}}^s,\ \widehat{\mathbf{M}}^a,\ \widehat{\boldsymbol{\chi}}^s,\ \widehat{\boldsymbol{\chi}}^a,\ \widehat{\mathbf{B}}^s,\ \widehat{\mathbf{B}}^a,\ \widehat{\mathbf{D}}^s,\ \widehat{\mathbf{D}}^a,\ \widehat{\mu}_K \tag{63}$$

in x - y plane. Then equilibrium equation is given by

$$\mathbf{B}^{\mathsf{T}}\mathbf{N} + \mathfrak{B}^{\mathsf{T}}\mathfrak{m} + \left(\mathbf{B}^s\right)^{\mathsf{T}}\mathbf{M}^s + \left(\mathbf{B}^a\right)^{\mathsf{T}}\mathbf{M}^a + \left(\widehat{\mathbf{B}}^s\right)^{\mathsf{T}}\widehat{\mathbf{M}}^s + \left(\widehat{\mathbf{B}}^a\right)^{\mathsf{T}}\widehat{\mathbf{M}}^a = \mathbf{Q} \tag{64}$$

and constitutive equations are the following:

$$\begin{aligned}\mathbf{N} &= \mathbf{E}\boldsymbol{\Delta},\\ \mathfrak{m} &= \mathbf{G}\boldsymbol{\theta},\\ \mathbf{M}^s &= \mathbf{D}^s\boldsymbol{\chi}^s,\\ \mathbf{M}^a &= \mathbf{D}^a\boldsymbol{\chi}^a,\\ \widehat{\mathbf{M}}^s &= \widehat{\mathbf{D}}^s\widehat{\boldsymbol{\chi}}^s,\\ \widehat{\mathbf{M}}^a &= \widehat{\mathbf{D}}^a\widehat{\boldsymbol{\chi}}^a.\end{aligned} \tag{65}$$

The above reasoning leads to a system of equations (27) with the stiffness matrix:

$$\begin{aligned}\mathbf{K} = {} & \mathbf{B}^{\mathsf{T}}\mathbf{E}\mathbf{B} + \mathfrak{B}^{\mathsf{T}}\mathbf{G}\mathfrak{B} + \left(\mathbf{B}^s\right)^{\mathsf{T}}\mathbf{D}^s\mathbf{B}^s + \left(\mathbf{B}^a\right)^{\mathsf{T}}\mathbf{D}^a\mathbf{B}^a\\ & + \left(\widehat{\mathbf{B}}^s\right)^{\mathsf{T}}\widehat{\mathbf{D}}^s\widehat{\mathbf{B}}^s + \left(\widehat{\mathbf{B}}^a\right)^{\mathsf{T}}\widehat{\mathbf{D}}^a\widehat{\mathbf{B}}^a.\end{aligned} \tag{66}$$

Likewise for plane frames it is possible to decompose stiffness matrix with the use of dyadic product. Let $\mathbf{b}_P$, $\mathfrak{b}_P$, $\mathbf{b}_P^s$, $\mathbf{b}_P^a$, $\widehat{\mathbf{b}}_P^s$, $\widehat{\mathbf{b}}_P^a$ be P-th rows of matrices $\mathbf{B}$, $\mathfrak{B}$, $\mathbf{B}^s$, $\mathbf{B}^a$, $\widehat{\mathbf{B}}^s$, $\widehat{\mathbf{B}}^a$, respectively. Then stiffness matrix (66) can be written as

$$\begin{aligned}\mathbf{K} = \sum_{P=1}^{e} \Bigg[& \frac{EA_P}{l_P}\mathbf{b}_P \otimes \mathbf{b}_P + \frac{GC_P}{l_P}\mathfrak{b}_P \otimes \mathfrak{b}_P\\ & + \frac{2EI_P}{l_P}\left(\mu_P\mathbf{b}_P^s \otimes \mathbf{b}_P^s + 2\mathbf{b}_P^a \otimes \mathbf{b}_P^a\right)\\ & + \frac{2\widehat{EI}_P}{l_P}\left(\widehat{\mu}_P\widehat{\mathbf{b}}_P^s \otimes \widehat{\mathbf{b}}_P^s + 2\widehat{\mathbf{b}}_P^a \otimes \widehat{\mathbf{b}}_P^a\right)\Bigg],\end{aligned} \tag{67}$$

where $\mathbf{a} \otimes \mathbf{b} = [a_i b_j]$ is the dyadic product of vectors $\mathbf{a}$ and $\mathbf{b}$.

9. Conclusions

Derivation of algebraic formulas for flat as well as space frames, trusses, and grillages with compressible and incompressible rods within the Timoshenko theory is formulated in the present paper. It is shown that the proposed theory at the border crossing leads to the formulation given by Lewiński in [3]. The following features and advantages of the method can be stressed:

(i) The method is direct and does not require approximation of unknown displacement fields and accidental stresses.

(ii) Frame deformation measures are defined in a natural way.

(iii) Node numbering is not required for structure analysis. All one need to do is define the numbers of rods and degrees of freedom.

(iv) Expression of the stiffness matrices as a sum of dyadic products is original.

(v) The method is easy to algorithmize. It is recommended to use packages for symbolic calculations like *Mathematica* or *Maple*.

(vi) The generalization of considerations [3] into frames composed of bars of medium thickness greatly expands the range of applications of the theory.

(vii) Formulation with diagonal constitutive matrix can be used in topological optimization, plastic range of the analysis, shakedown of elastic-plastic structures, and the analysis taking into account large uncertainty of parameters, using convex set techniques.

(viii) Potential applications of the formulation can be easily extended for the gridwork and framework methods in moderately thick plate theories.

(ix) The formulation can be successfully used for educational purposes.

According to the present authors' best knowledge, the formulation according to Timoshenko's theory is not known in the literature.

Data Availability

All data used to support finding of this study are included within the article.

Disclosure

The research was performed as part of the employment of the authors in Warsaw University of Technology.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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