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Research Article

An Analytical Approach to Wet Cooling Towers Based on Functional Analysis

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An analytical solution for computing the temperature distribution of air and water over the height through the cooling tower is so complex that finding the exact solution takes too much time. The purpose of this paper is to present efficient and accurate analytical expressions for the heat and mass transfer model in cooling towers. Based on the method of functional analysis, we derived an analytical solution for temperature distribution of water and air by using the method of solving linear differential equations. The error estimation, the existence, and uniqueness of the solution are given by using Banach contraction mapping theorem. The basic equation of the model on the basis of the additional assumptions on the cooling tower is solved, and the outlet parameters are also obtained.

1. Introduction

The cooling tower role is of cooling the circulating water for repeated use (as shown in Figure 1). With the development of industry, the demand for the cooling tower is on the increase. Especially in the chemical industry, refrigeration, air conditioning, textile, power plant, and water-shortage area, cooling towers have been widely used in the circulating water supply systems [1]. Fast and accurate calculation of the heat and mass transfer process in the cooling tower is the basis and theoretical guidance for its optimization control. In the thermal calculation for cooling towers, due to the nonlinear relationship between the saturated air enthalpy and the temperature, the two mainly used methods are numerical simulation and artificial neural network (black-box) modeling.

The basic principle of the cooling tower was first proposed by Walker in 1923, and then Merkel developed the further theory. Merkel [2] introduced the concept of enthalpy in his paper, and he unified the heat and mass transfer into enthalpy and proposed the enthalpy difference model. At present, most scholars are using the Merkel enthalpy difference model to calculate the performance of cooling towers. Baker and Shryock [3] analyzed the performance curve and the packing characteristic curve of the cooling tower, established the working point of the cooling tower, and applied this method to calculate the cross-flow cooling tower. Fisenko et al. [4] established the four-variable model. Compared with the Merkel enthalpy model, the four-variable model considered the evaporative loss of water. So the accuracy was improved, but it also increased the solving difficulty. Whiller [5] elaborated another method to derive the formula for the characteristic parameters, but this method is not very useful for engineering design.

The Poppe method proposed in the 1970s does not use Merkel's simplified hypothesis and considers evaporative water. Predictions from this model are in good agreement with the actual cooling tower test results. In addition, the Poppe method can accurately predict the moisture content of outlet air [6, 7]. This fact can be used to design a hybrid cooling tower [8]. Sutherland [9] abandons the hypothesis of Merkel and makes a rigorous derivation of the heat and mass transfer process in the cooling tower. The calculation results show that the outlet temperature is reduced by 5%~15%, compared with the Merkel enthalpy model. Based on the theory of heat and mass transfer and the concept of heat

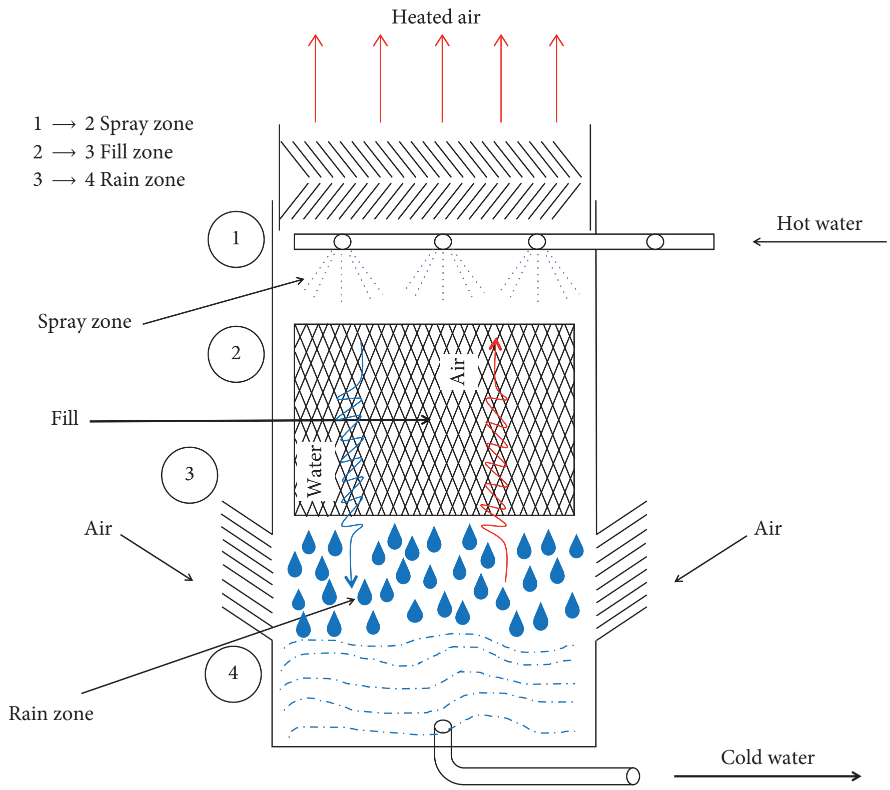

Figure 1: Schematic of a counter flow wet cooling tower.

exchanger, Jaber and Webb [10] put forward the number of heat transfer unit (ε-NTU) model, which provides another method for the calculation of the cooling tower. It is worth noting that the model, like the Merkel enthalpy model, ignores the effect of water evaporation.

Gan [11, 12] put forward several mathematical models. These models take into account the mass, momentum, and energy transfer simultaneously. These models are based on the Merkel theory and its modified form, and they can analyze the thermal processes of different cooling towers. They also carried out numerical simulation analysis of closed cooling towers and obtained an earlier relatively mature simulation experience.

In addition, many authors carried out two-dimensional and three-dimensional numerical calculation of filling area, rain, and spray area [13, 14]. These simulating methods are so complicated that solving the equations is difficult. Some authors introduced the artificial intelligence model to calculate the thermodynamic performance of the cooling tower [15–17]. All kinds of artificial intelligence models belong to "black-box operation," and they are unable to study the internal mechanism of the cooling system. It is difficult to reflect the heat and mass transfer mechanism of the cooling system clearly.

To sum up, the commonly used simple mathematical models for cooling towers mainly include the Merkel enthalpy difference model, efficiency heat transfer unit number (ε-NTU) model, three-variable model, and four-variable model. The Merkel equation of the enthalpy difference model is simple. It is convenient to calculate and easy to understand, so the physical concept is widely used; the ε-NTU model refers to the model of heat exchanger design, which is convenient to design and calculate cooling towers, but the calculation is relatively complex; the physical concept of the three-variable model is easy to understand. The heat transfer process of air and water is analyzed in detail; but these three models do not consider the effects of water evaporation on the outlet water temperature. The four-variable model considers the influence of water quantity caused by water evaporation on the outlet water temperature. But the model has more equations and the calculation is more complex. Although all of the numerical simulations play a huge role in much of this field, very little has been done on the mathematical treatment of the distributed parameter problems associated with cooling towers. Therefore, in view of the accuracy and applicability of the calculation, this paper intends to build the differential equation of cooling towers and obtain its analytical solution.

The aim of this study is to develop analytical expressions through theoretical analysis for the cooling towers. The coupled heat and mass transfer equations will be solved by using the method of functional analysis. The exact solution of the state of air and water on each profile will be obtained, and the error estimation will be given too. Based on the additional assumptions, the basic equation model will be solved to obtain the outlet parameters and efficiency formula for the cooling tower. The model accommodates direct and quick calculation of air, liquid, and interface temperature profiles and moisture content of air along the vertical length of the tower. The model will provide theoretical basis for future research.

2. Heat and Mass Transfer Analysis of Cooling Towers

The model of coupled heat and mass transfer is based on the following three assumptions:

(1) Thin-film model, the heat and mass transfer takes place on the surface layer of saturated air between the air and water, and the saturated air temperature is equal to the water temperature, so the air moisture content value on the saturated air layer is the monodrome function of the water temperature

(2) The effect of mass transfer on heat transfer is negligible

(3) Lewis factor is assumed to be unity, i.e., $h_{\mathrm{M}} = h_{\mathrm{H}}/c_{\mathrm{pa}}$

For a cooling tower, it is assumed that the cross section area is A and the height is L, as shown in Figure 2.

The mass transfer at the air-water interface due to the difference in vapour concentration in the discrete height dl is

$$-dm_{\mathrm{w}} = m_{\mathrm{a}} dw = h_{\mathrm{M}} (w_{\mathrm{sw}} - w_{\mathrm{a}}) A_{\mathrm{M}} \mathrm{dl}, \quad (1)$$

where m_{w} is the water mass flow rate, kg/s; m_{a} is the air mass flow rate, kg/s; h_{M} is the mass transfer coefficient driven by the difference of vapour concentration, kg/$(\mathrm{m}^2 \cdot \mathrm{s})$; w_{a} is the humidity ratio of moist air, kg/kg; w_{sw} is the humidity ratio of saturated moist air at water temperature, kg/kg; A_{M} is the mass transfer area per unit height, m^2/m; and dl is the discrete height, m.

The sensible heat transmitted from the saturated air at the air-water interface to the main air in the discrete height is

$$dQ_{\mathrm{H}} = m_{\mathrm{a}} c_{\mathrm{p}} dt_{\mathrm{a}} = h_{\mathrm{H}} (t_w - t_a) A_{\mathrm{H}} \mathrm{dl}, \quad (2)$$

where c_{p} is the specific heat capacity at constant pressure, J/$(\mathrm{kg} \cdot \mathrm{K})$; h_{H} is the convective heat transfer coefficient between air and water, $\mathrm{W/m^2 \cdot K}$; A_{H} is the heat transfer area per unit height, m^2/m; t_{a} is the air temperature, K; and t_{w} is the water temperature, K.

The latent heat transmitted by the saturated air from the gas-water interface to the mainstream air in the discrete height is

$$dQ_{\mathrm{M}} = i h_{\mathrm{M}} (w_{\mathrm{sw}} - w_{\mathrm{a}}) A_{\mathrm{M}} \mathrm{dl}, \quad (3)$$

where i is the specific enthalpy, J/kg;

The total heat transmitted to the air is

$$m_{\mathrm{a}} \left(c_{\mathrm{pa}} dt_{\mathrm{a}} + i\, dw \right) = \left[h_{\mathrm{H}} (t_{\mathrm{w}} - t_{\mathrm{a}}) A_{\mathrm{H}} + i h_{\mathrm{M}} (w_{\mathrm{sw}} - w_{\mathrm{a}}) A_{\mathrm{M}} \right] \mathrm{dl}. \quad (4)$$

The heat loss of water in the cooling tower is

$$dQ_{\mathrm{w}} = (m_{\mathrm{w}} dt_{\mathrm{w}} + t_{\mathrm{w}} dm_{\mathrm{w}}) c_{\mathrm{w}}, \quad (5)$$

where c_{w} is the specific heat capacity of water, J/$(\mathrm{kg} \cdot \mathrm{K})$.

3. Exact Solution of Heat and Mass Transfer Equations of Cooling Tower Based on Functional Analysis

Because the basic equations of the heat and mass transfer in the cooling tower are three-variable ordinary differential equations, it is difficult to be solved by the usual method. In this paper, the exact solution of the air and water state in each section of the cooling tower is obtained by using the method of functional analysis, which provides a theoretical basis for future research.

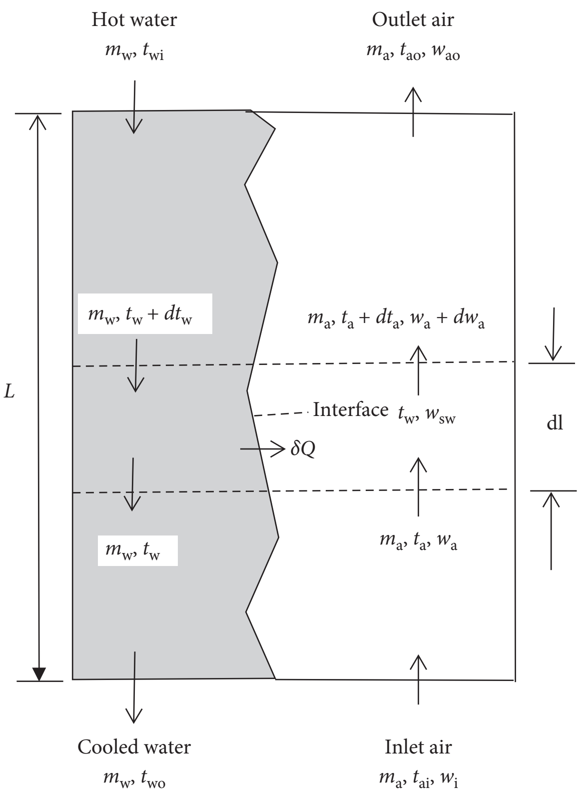

Figure 2: A control segment of cooling tower.

3.1. Methodology for the Exact Solution of the Heat and Mass Transfer Equations [18].

It can be obtained from equations (2)–(5) that

$$\begin{cases} \dfrac{dt_{\mathrm{w}}(l)}{\mathrm{dl}} - (kK_1 + K_2) t_{\mathrm{w}}(l) = -K_1 w_{\mathrm{a}}(l) - K_2 t_{\mathrm{a}}(l), \\ \\ \dfrac{dw_{\mathrm{a}}(l)}{\mathrm{dl}} + K_3 w_{\mathrm{a}}(l) = K_3 (w_{\mathrm{sw}}(l)), \\ \\ \dfrac{dt_{\mathrm{a}}(l)}{\mathrm{dl}} + K_4 t_{\mathrm{a}}(l) = K_4 t_{\mathrm{w}}(l), \end{cases} \quad (6)$$

where $K_1 = h_{\mathrm{M}} A_{\mathrm{M}} i / (m_{\mathrm{w}} c_{\mathrm{w}})$, $K_2 = h_{\mathrm{H}} A_{\mathrm{H}} / (m_{\mathrm{w}} c_{\mathrm{w}})$, $K_3 = h_{\mathrm{M}} A_{\mathrm{M}} / m_{\mathrm{a}}$, and $K_4 = h_{\mathrm{H}} A_{\mathrm{H}} / m_{\mathrm{a}} c_{\mathrm{pa}}$.

Boundary conditions: $t_{\mathrm{w}}(0) = t_{\mathrm{wo}}$, $w_{\mathrm{a}}(0) = w_{\mathrm{i}}$, $t_{\mathrm{a}}(0) = t_{\mathrm{ai}}$, and $w_{\mathrm{sw}} = kt_{\mathrm{d}}$.

All the coefficients and boundary conditions are positive.

As w_{sw} is the function of t_{w}, it can be written as $w_{\mathrm{sw}} = kt_{\mathrm{w}}$.

Let the three equations in (6) be multiplied by $e^{-(kK_1+K_2)l}$, $e^{K_3 l}$, and $e^{K_4 l}$ on both sides, respectively. Then, integrate the three Eqs. (a), (b), and (c), and utilize the boundary conditions, then the following equation group is obtained:

$$t_{\mathrm{w}}(l)=t_{\mathrm{wo}}e^{(kK_1+K_2)l}-\int_0^l e^{(kK_1+K_2)(l-s)}(K_1w_{\mathrm{a}}(s)+K_2t_{\mathrm{a}}(s))\mathrm{d}s, \quad (7a)$$

$$w_{\mathrm{a}}(l)=w_ie^{-K_3l}+\int_0^l e^{K_3(s-l)}K_3kt_{\mathrm{w}}(s)\mathrm{d}s, \quad (7b)$$

$$t_{\mathrm{a}}(l)=t_{\mathrm{ai}}e^{-K_4l}+\int_0^l e^{K_4(s-l)}K_4t_{\mathrm{w}}(s)\mathrm{d}s. \quad (7c)$$

It is obvious that the ordinary differential equation group (6) is equivalent to the integral equation group (7a)–(7c).

Let $x=t_{\mathrm{w}}$, $y=w_{\mathrm{a}}$, and $z=t_{\mathrm{a}}$, then

$$\begin{cases} x=A_1(x,y,z),\\ y=A_2(x,y,z),\\ z=A_3(x,y,z),\end{cases} \quad (8)$$

where

$$A_1(x,y,z)=t_{\mathrm{wo}}e^{(K_1k+K_2)l}-\int_0^l e^{(K_1k+K_2)(l-s)}(K_1y(s)+K_2z(s))\mathrm{d}s, \quad (9)$$

$$A_2(x,y,z)=w_ie^{-K_3l}+\int_0^l e^{K_3(s-l)}K_3kx(s)\mathrm{d}s, \quad (10)$$

$$A_3(x,y,z)=t_{\mathrm{ai}}e^{-K_4l}+\int_0^l e^{K_4(s-l)}K_4x(s)\mathrm{d}s. \quad (11)$$

As $Av=(A_1v,A_2v,A_3v)$, $v=(x,y,z)$, and $\|v\|=\|x\|+\|y\|+\|z\|$,

where $x(s),y(s),z(s)\in C[0,L]$,

$\|x\|=\max\exp(-Ms)|x(s)|_{0\le s\le L}$, and $M>N=\max\{kK_1+K_2+K_3+K_4,K_1+K_3\}$.

For product spaces $\overline{C}[0,L]=C[0,L]\times C[0,L]\times C[0,L]$, it is a *Banach* space [19].

According to *Banach's* contraction mapping principle, operator A has a unique fixed point $\overline{v}(l)=(\overline{x}(l),\overline{y}(l),\overline{z}(l))$ in product space $\overline{C}[0,L]$. Iterate at a point (x_0,y_0,z_0) in $\overline{C}[0,L]$, we obtain that $x_{n+1}=A_1(x_n,y_n,z_n)$, $y_{n+1}=A_2(x_n,y_n,z_n)$, and $z_{n+1}=A_3(x_n,y_n,z_n)$ $n=0,1,2,\ldots$.

The only solution for (6) converged, that is, $\overline{x}(l)=\lim_{n\longrightarrow\infty}x_n(l)$, $\overline{y}(l)=\lim_{n\longrightarrow\infty}y_n(l)$, and $\overline{z}(1)=\lim_{n\longrightarrow\infty}z_n(l)$.

$$\overline{x}(l)=a+\sum_{n=1}^{\infty}a_nl^n,\quad a_n=\frac{1}{n}[(kK_1+K_2)a_{n-1}-K_1b_{n-1}-K_2c_{n-1}], \quad (12a)$$

$$\overline{y}(l)=b+\sum_{n=1}^{\infty}b_nl^n,\quad b_n=\frac{1}{n}K_3(ka_{n-1}-b_{n-1}), \quad (12b)$$

$$\overline{z}(l)=c+\sum_{n=1}^{\infty}c_nl^n,\quad c_n=\frac{1}{n}K_4(a_{n-1}-c_{n-1}), \quad (12c)$$

$$\begin{aligned} a_0&=a,\\ b_0&=b,\\ c_0&=c.\end{aligned} \quad (12d)$$

And the error estimation is

$$\begin{cases} \|\overline{x}-x_n\|\le\left(\dfrac{\alpha^n}{(1-\alpha)}\right)\|A_1v_0-x_0\|,\\ \|\overline{y}-y_n\|\le\left(\dfrac{\alpha^n}{(1-\alpha)}\right)\|A_1v_0-y_0\|,\\ \|\overline{z}-z_n\|\le\left(\dfrac{\alpha^n}{(1-\alpha)}\right)\|A_1v_0-z_0\|,\end{cases} \quad (13)$$

where $\alpha=N/M$.

The proof detail of the exact solution and error estimation is given in the Appendix.

Equations (12a)–(12d) and (13) are exact solutions and error estimation, respectively. The coefficients K_1, K_2, K_3, and K_4 are related to many factors, such as the types of cooling tower, spraying way, filling types, the droplet size of water, and the mass flow rate of air and water. When n is finite, the approximate solution and the error estimation can be calculated, which could provide a powerful criterion for future experimental data or numerical solution.

4. Analytical Example

To simplify the analysis of a convective heat and mass transfer at the air-water interface, we assume that the Lewis factor is equal to unity $h_{\mathrm{M}}=h_{\mathrm{H}}/c_{\mathrm{pa}}$ and $A_{\mathrm{H}}=A_{\mathrm{M}}$. Equation (6) is converted to

$$\frac{dt_{\mathrm{w}}}{\mathrm{dl}}=K_1w_{\mathrm{w}}+K_2t_{\mathrm{w}}-K_1w_{\mathrm{a}}-K_2t_{\mathrm{a}}, \quad (14a)$$

$$\frac{dw_{\mathrm{a}}}{\mathrm{dl}}=K_3(w_{\mathrm{w}}-w_{\mathrm{a}}), \quad (14b)$$

$$\frac{dt_{\mathrm{a}}}{\mathrm{dl}}=K_3(t_{\mathrm{w}}-t_{\mathrm{a}}). \quad (14c)$$

In the above equation group, w_{w} is the function of t_{w}. Referring to the physical parameter table in the range of working temperature variation, the formula relationship is determined by the difference calculation as follows:

$$w_{\mathrm{w}} = \beta_0 + \beta_1 t_{\mathrm{w}} + \beta_2 t_{\mathrm{w}}^2. \tag{15}$$

Substituting (15) into (14a)–(14c), we get

$$\begin{cases} \dfrac{dt_{\mathrm{w}}}{\mathrm{dl}} = K_1\beta_0 + (K_1\beta_1 + K_2)t_{\mathrm{w}} - K_1 w_{\mathrm{a}} - K_2 t_{\mathrm{a}}, \\ \dfrac{dw_{\mathrm{a}}}{\mathrm{dl}} = K_3\left(\beta_0 + \beta_1 t_{\mathrm{w}} + \beta_2 t_{\mathrm{w}}^2 - w_{\mathrm{a}}\right), \\ \dfrac{dt_{\mathrm{a}}}{\mathrm{dl}} = K_3 (t_{\mathrm{w}} - t_{\mathrm{a}}). \end{cases} \tag{16}$$

For convenience, let $x = T_{\mathrm{w}}$, $y = d$, and $z = T$, then the equations group (13) is converted to

$$x' = K_1\beta_0 + (K_1\beta_1 + K_2)x + K_1\beta_2 x^2 - K_1 y - K_2 z, \tag{17a}$$

$$y' = K_3\left(\beta_0 + \beta_1 x + \beta_2 x^2 - y\right), \tag{17b}$$

$$z' = K_3 (x - z). \tag{17c}$$

Equations (14b) and (14c) in are converted to

$$y = \beta_0 + \beta_1 x + \beta_2 x^2 - \frac{1}{K_3} y', \tag{18}$$

$$z = x - \frac{1}{K_3} z'. \tag{19}$$

Substituting (18) and (19) into (17a), and upon arrangement

$$K_3 x' - K_1 y' - K_2 z' = 0. \tag{20}$$

Integrating (21), and let the integral constant be D_1, then we obtain the following equation:

$$K_3 x - K_1 y - K_2 z = D_1. \tag{21}$$

Substituting (21) into (17a), we get

$$x' = K_1\beta_0 + D_1 + (K_1\beta_1 + K_2 - K_3)x + K_1\beta_2 x^2. \tag{22}$$

Let $Q = 4K_1\beta_2 (K_1\beta_0 + D_1) - (K_1\beta_1 + K_2 - K_3)^2$

(1) For $Q > 0$, the following solution of equations (17a)–(17c) is obtained:

$$x_1 (l) = \frac{1}{2K_1\beta_2} \left\{ \sqrt{Q} \tan\left[\frac{\sqrt{Q}}{2} (l + D_2) \right] + K_3 - K_2 - K_1\beta_1 \right\}, \tag{23}$$

where D_2 is the integral constant.

(2) For $Q < 0$, the following two solutions of equations (17a)–(17c) are obtained:

$$x_2 (l) = \frac{K_1\beta_1 + K_2 - K_3 + \sqrt{-Q}}{2K_1\beta_2} \left[\frac{2\sqrt{-Q}}{1 - e^{\sqrt{-Q}\,(l+D_2)}} - 1 \right], \tag{24}$$

$$x_3 (l) = \frac{-1}{2K_1\beta_2} \left\{ \sqrt{Q} \tan\left[\frac{\sqrt{-Q}}{2} (l + D_2) \right] + K_2 - K_3 + K_1\beta_1 \right\}. \tag{25}$$

(3) For $Q = 0$, the following solution of equations (17a)–(17c) is obtained:

$$x_4 (l) = \pm \frac{1}{\sqrt{K_1\beta_2}} \left(\frac{1}{l + D_2} + \sqrt{K_1\beta_0 + D_2} \right). \tag{26}$$

According to (17c), we get

$$z' + K_3 z = K_3 x (l). \tag{27}$$

This is a first-order linear nonhomogeneous ordinary differential equation with respect to an unknown function $z(l)$. Its general solution is

$$z_i (l) = e^{-K_3 l} \left[K_3 \int x_i (l) e^{K_3 l} \mathrm{dl} + D_3 \right], \quad i = 1, 2, 3, 4, \tag{28}$$

where D_3 is the integral constant.

From equation (20), we get

$$y_i (l) = \frac{1}{K_1} \left[K_3 x_i (l) - K_2 z_i (l) - D_1 \right], \quad i = 1, 2, 3, 4. \tag{29}$$

According to equations (23)–(26), (28), and (29), the equation group (17a)–(17c) has four sets of analytical solutions $\{x_i (l), y_i (l), z_i (l)\}$, $i = 1, 2, 3, 4$, in which (23)–(26) are taken in turn for $x_i (l)$ and (29) and (28) are for $y_i (l)$ and $z_i (l)$, respectively.

So far, the analytical solution of the linear equations for the model has been obtained. This corresponds to the different processes in the heat and mass transfer of the cooling tower, which is characterized by the equations. According to various conditions, some parameters can be properly adjusted (such as m_{w}, m_{a}, h_{M}, h_{H}, and so on) to obtain the exact solution of the actual process.

5. Results and Discussion

In order to verify the analytical results, the performance data of the cooling tower tested by Milosavljevic and Heikkilä [19] are used in this paper. The measurement data are shown in Table 1.

The comparison was made under typical operating condition investigated experimentally in [19]. The performance and outlet parameters of the cooling tower are predicted by the analytical method in this paper. The parameter profiles obtained by different methods are demonstrated in Figure 3 that show the profiles of water temperature along the tower.

Table 1: Experimental data at industrial cooling tower [18].

Cooling tower parameter	Experimental values
A	$180\,m^2$
H	0.6 m
t_{wi}	40°C
t_{wo}	29.7°C
t_{ai}	25.5°C
t_{ao}	30.7°C
w_{ai}	0.0190 kg H_2O/kg air
w_{ao}	0.0290 kg H_2O/kg air
m_w	450 kg/s
m_a	650kg/s

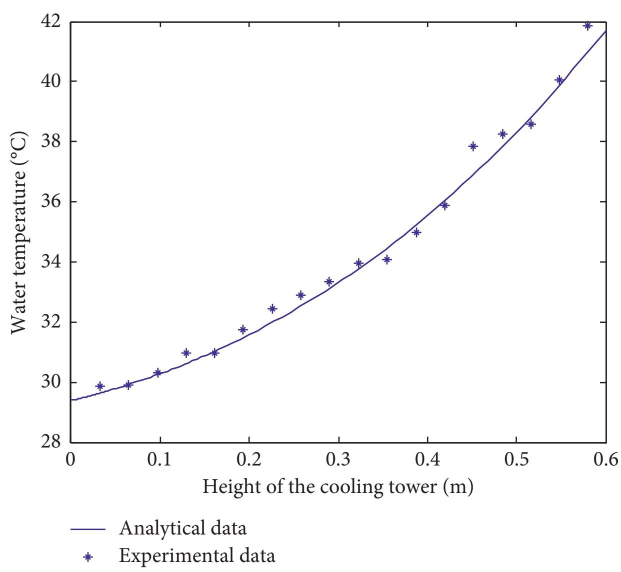

Figure 3: Experimental and analytical comparison of water temperature profiles along the cooling tower.

From the comparison, it is easy to find that the relative errors by the analytical method are generally less than 10% except for two cases in water temperature. In the cases that the inlet water temperatures are high, the errors in the linearization of air saturation humidity at the water surface are great. This leads to increased relative errors by the analytical model.

Based on the above discussion and comparison, the following conclusion can be obtained: the outlet parameters and parameter distributions of cooling towers calculated by means of the analytical solution of this paper are in good agreement with the experimental results. In the typical case, most of the average and maximum relative errors are far less than 10%.

6. Conclusions

Quick and accurate analysis of cooling tower performance is very important in rating and design calculations. This paper has set up an approximate analytic solution of heat and mass transfer in cooling towers. The equations were solved based on the application of the functional analysis. The analytical model also accommodates the direct and quick calculation of gas, liquid, and interface temperature profiles and moisture content of air along the vertical length of the tower. The error estimation, the existence, and the uniqueness of the solution were given by using *Banach* contraction mapping theorem. The basic equation of the model on the basis of the additional assumptions on the cooling tower was solved, and the outlet parameters were also obtained.

Appendix

Mathematical Details with regard to the Derivation of the Exact Solution of the Heat and Mass Transfer Equation

Before introducing the theorem, we present the definition of a contraction mapping.

Definition A.1 (see [20]). Let $(X, \|\cdot\|)$ be a *Banach* space. Then, a map $T\colon X \longrightarrow X$ is called a contraction mapping on X if there exists $k \in [0, 1)$ such that $\|T(x) - T(y)\| \le k\|x - y\|$ for all x, y in X.

Banach fixed-point theorem [19]: let $(X, \|\cdot\|)$ be a *Banach* space with a contraction mapping $T\colon X \longrightarrow X$. Then, T admits a unique fixed-point x^* in X (i.e., $T(x^*) = x^*$). Furthermore, x^* can be found as follows: start with an arbitrary element x_n in X and define a sequence $(x_n)_{n=0,1,2,3,\ldots}$ by $x_{n+1} = T(x_n)$, then $x_n \longrightarrow x^*$. The error estimate formula can be written as $\|x^* - x_n\| \le k^n/1 - k\|x_0 - T(x_0)\|$.

Based on the *Banach* space, the *Banach* fixed-point theorem (also called contraction mapping theorem) is used without proof.

Proof (operator A is the compression of the product space to itself [17]). Let $\overrightarrow{v} = (x, y, z)$ and $A\overrightarrow{v} = (A_1\overrightarrow{v}, A_2\overrightarrow{v}, A_3\overrightarrow{v})$, then the norm of vector $\overrightarrow{v}$ is $\|\overrightarrow{v}\| = \|x\| + \|y\| + \|z\| = \max|x(s)|_{C[0,l]} + \max|y(s)|_{s\in[0,l]} + \max|z(s)|_{s\in[0,l]}$, so continuous vector space $B = C[0, L] \times C[0, L] \times C[0, L]$ is a *Banach* space, and A is a bounded operator on B; the *Banach* compression mapping principle will be used to prove that A has a unique fixed point on B $\overline{v(l)} = (\overline{x(l)}, \overline{y(l)}, \overline{z(l)})$, suppose $v_1 = (x_1(l), y_1(l), z_1(l)) \in B$ and $v_2 = (x_2(l), y_2(l), z_2(l)) \in B$.

$\because Av_j = (A_1v_j, A_2v_j, A_3v_j)$, $j = 1, 2$, $Av_1 = (A_1v_1, A_2v_1, A_3v_1)$, and $Av_2 = (A_1v_2, A_2v_2, A_3v_2)$.

$\|x\| = \max\exp(-Ms)|x(s)|_{0\le s\le L}$,
$M > N = \max\{kK_1 + K_2 + K_3 + K_4, K_1 + K_3\}$,

where

$$\begin{aligned} A_1v_j &= tw_0 e^{(K_1k+k_2)l} - \int_0^l e^{(K_1k+K_2)(l-s)}\big(K_1y_j(s) \\ &\quad + K_2z_j(s)\big)ds, \\ A_2v_j &= w_i e^{-K_3l} + \int_0^l e^{K_3(s-l)}K_3kx_j(s)ds, \\ A_3v_j &= t_{ai}e^{-K_4l} + \int_0^l e^{K_4(s-l)}K_4x_j(s)ds, \end{aligned} \tag{A.1}$$

$$\big|y_2(s) - y_1(s)\big| \le \big\|v_1 - v_2\big\|. \tag{A.2}$$

So

$$\begin{aligned}\left\|A_1 v_1 - A_1 v_2\right\| &= \left\|\int_0^l e^{(K_1 k + K_2)(l-s)}\left[K_1\left(y_2(s) - y_1(s)\right)\right.\right.\\ &\quad \left.\left.+ K_2\left(z_2(s) - z_1(s)\right)\right]\mathrm{d}s\right\|\\ &\le \left\|\int_0^l e^{(K_1 k + K_2)(l-s)}\left[K_1\left\|y_2(s) - y_1(s)\right\|\right.\right.\\ &\quad \left.\left.+ K_2\left\|z_2(s) - z_1(s)\right\|\right]\mathrm{d}s\right\|\\ &\le \left\|\int_0^l e^{(K_1 k + K_2)(l-s)}\left(K_1\left\|v_1 - v_2\right\| + K_2\left\|v_1\right.\right.\right.\\ &\quad \left.\left.\left.- v_2\right\|\right)\mathrm{d}s\right\|\\ &= \left\|\left(K_1 + K_2\right)\left\|v_1 - v_2\right\|\int_0^l e^{(K_1 k + K_2)(l-s)}\mathrm{d}s\right\|\\ &= \left(K_1 + K_2\right)\left\|v_1 - v_2\right\|\frac{e^{(K_1 k + K_2)l} - 1}{K_1 k + K_2},\end{aligned} \tag{A.3}$$

$$\begin{aligned}\left\|A_2 v_1 - A_2 v_2\right\| &= \left\|\int_0^l e^{K_3(s-l)} K_3 k\left[x_1(s) - x_2(s)\right]\mathrm{d}s\right\|\\ &\le K_3 k\left\|v_1 - v_2\right\|\int_0^l e^{K_3(s-l)}\mathrm{d}s\\ &= K_3 k\left\|v_1 - v_2\right\|\frac{1}{K_3}\left(1 - e^{-K_3 l}\right)\\ &= k\left(1 - e^{-K_3 l}\right)\left\|v_1 - v_2\right\|,\end{aligned} \tag{A.4}$$

$$\begin{aligned}\left\|A_3 v_1 - A_3 v_2\right\| &= \left\|\int_0^l e^{K_4(s-l)} K_4\left[x_1(s) - x_2(s)\right]\mathrm{d}s\right\|\\ &\le \left\|v_1 - v_2\right\|\left(1 - e^{-K_4 l}\right),\end{aligned} \tag{A.5}$$

$$\begin{aligned}\left\|A v_1 - A v_2\right\| &= \left\|A_1 v_1 - A_1 v_2\right\| + \left\|A_2 v_1 - A_2 v_2\right\| + \left\|A_3 v_1\right.\\ &\quad \left.- A_3 v_2\right\| \le \left[\frac{K_1 + K_2}{K_1 k + K_2}\left(e^{(K_1 k + K_2)l} - 1\right)\right.\\ &\quad \left.+ k\left(1 - e^{-K_3 l}\right) + \left(1 - e^{-K_4 l}\right)\right]\left\|v_1 - v_2\right\|\\ &= d\left\|v_1 - v_2\right\|,\end{aligned} \tag{A.6}$$

in which $d = K_1 + K_2/K_1 k + K_2\,(e^{(K_1 k + K_2)l} - 1) + k\,(1 - e^{-K_3 l}) + (1 - e^{-K_4 l})$, as $0 < d < 1$, the operator A is contracted on the space B, and the only solution that satisfies $A\overline{v} = \overline{v}$ with the boundary conditions $t_{\mathrm{w}(0)} = t_{\mathrm{wo}}$, $w_{\mathrm{a}(0)} = w_i$, $t_{\mathrm{a}(0)} = t_{\mathrm{ai}}$ exists according to *Banach* contraction mapping theorem. □

Proof (error estimation formula).

$$\overline{y} - v_n = \left(\overline{x}, \overline{y}, \overline{z}\right) - \left(x_n, y_n, z_n\right) = \left(\overline{x} - x_n, \overline{y} - y_n, \overline{z} - z_n\right), \tag{A.7}$$

$$\begin{aligned}A v_0 - v_0 &= \left(A_1 v_0, A_2 v_0, A_3 v_0\right) - \left(x_0, y_0, z_0\right)\\ &= \left(A_1 v_0 - x_0, A_2 v_0 - y_0, A_3 v_0 - z_0\right),\end{aligned} \tag{A.8}$$

$$\left\|\overline{v} - v_n\right\| = \left\|\overline{x} - x_n\right\| + \left\|\overline{y} - y_n\right\| + \left\|\overline{z} - z_n\right\|, \tag{A.9}$$

$$\left\|A v_0 - v_0\right\| = \left\|A_1 v_0 - x_0\right\| + \left\|A_2 v_0 - y_0\right\| + \left\|A_3 v_0 - z_0\right\|. \tag{A.10}$$

According to *Banach* contraction mapping theorem,

$$\left\|\overline{v} - v_n\right\| \le \frac{d^n}{1 - d}\left\|A v_0 - v_0\right\|. \tag{A.11}$$

So

$$\begin{aligned}\left\|\overline{x} - x_n\right\| + \left\|\overline{y} - y_n\right\| + \left\|\overline{z} - z_n\right\| &\le \frac{d^n}{1 - d}\left(\left\|A_1 v_0 - x_0\right\| + \left\|A_2 v_0\right.\right.\\ &\quad \left.\left.- y_0\right\| + \left\|A_3 v_0 - z_0\right\|\right).\end{aligned} \tag{A.12}$$

Therefore, (13) is workable. □

Proof (accurate solution expression of air dry-bulb temperature, vapour concentration, and water temperature). Let $x_0 = a$, $y_0 = b$, and $z_0 = c$, then the analytical solution of equation (12a)–(12d) can be expressed as follows:

$$\begin{cases} x_1(l) = \displaystyle\int_0^1 \left[(kK_1 + K_2)a - K_1 b - K_2 c\right]\mathrm{d}s + a = a_1 l + a,\\ y_1(l) = K_3\displaystyle\int_0^1 (ka - b)\mathrm{d}s + b = b_1 l + b,\\ z_1(l) = K_4\displaystyle\int_0^1 (a - c)\mathrm{d}s + c = c_1 l + c.\end{cases} \tag{A.13}$$

With the definitions of $x_{n-1}(l) = a + a_1 l + \cdots + a_{n-1} l^{n-1}$, $y_{n-1}(l) = b + b_1 l + \cdots + b_{n-1} l^{n-1}$, and $z_{n-1}(l) = c + c_1 l + \cdots + c_{n-1} l^{n-1}$, the following equation is obtained from equation (11):

$$\begin{aligned}x_n(l) &= \int_0^l \left[(kK_1 + K_2)x_{n-1}(s) - K_1 y_{n-1}(s) - K_2 z_n(s)\right]\mathrm{d}s\\ &\quad + a = x_n(l) = \int_0^l \left\{\left[(kK_1 + K_2)a_{n-1} - K_1 b_{n-1}\right.\right.\\ &\quad \left. - K_2 c_n\right]s^{n-1} + \cdots + \left[(kK_1 + K_2)a - K_1 b_1 - K_2 c_1\right]\\ &\quad \left. + \left[(kK_1 + K_2)a - K_1 b - K_2 c\right]\right\}\mathrm{d}s + a = a_n l^n\\ &\quad + \cdots + a_1 l + a.\end{aligned} \tag{A.14}$$

Similarly, expressions for $y_n(l)$ and $z_n(l)$ can be obtained.

We can get by mathematical induction that

$$\begin{aligned} x_n(l) &= a + \sum_{k=1}^{n} a_k l^k, \\ y_n(l) &= b + \sum_{k=1}^{n} b_k l^k, \\ z_n(l) &= c + \sum_{k=1}^{n} c_k l^k, \\ n &= 1, 2, \cdots. \end{aligned} \tag{A.15}$$

In accordance with the *Banach* contraction mapping theorem, $x_n(l)$, $y_n(l)$, and $z_n(l)$ converge to $\overline{x}(l)$, $\overline{y}(l)$, and $\overline{z}(l)$, respectively. Thus, equation (12a)–(12d) is workable, that is,

$$\begin{cases} t_{\mathrm{w}}(l) = a + \sum_{k=1}^{\infty} a_n l^n, & a_n = \dfrac{1}{n}\left[(kK_1 + K_2)a_{n-1} - K_1 b_{n-1} - K_2 c_{n-1}\right], \\ w_{\mathrm{a}}(l) = b + \sum_{k=1}^{\infty} b_n l^n, & b_n = \dfrac{1}{n} K_3 (k a_{n-1} - b_{n-1}), \\ t_{\mathrm{a}}(l) = c + \sum_{k=1}^{\infty} c_n l^n, & c_n = \dfrac{1}{n} K_4 (a_{n-1} - c_{n-1}), \\ a = a_0, b = b_0, c = c_0. \end{cases} \tag{A.16}$$

□

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Additional Points

A new analytic solution method is developed to find the exact solutions for the cooling tower system. A simple differential equation for the counter-flow wet cooling tower is solved analytically taking into consideration the nonlinear dependency of the saturated air enthalpy on temperature. The existence, uniqueness, and error estimation for the heat and mass transfer equations are shown by using *Banach* contraction mapping theorem.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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