Hindawi
Mathematical Problems in Engineering
Volume 2019, Article ID 8631272, 13 pages
https://doi.org/10.1155/2019/8631272

*Research Article*

# Dynamics Analysis of a Stochastic Leslie–Gower Predator-Prey Model with Feedback Controls

**Huailan Ren and Wencai Zhao**

*College of Mathematics and Systems Science, Shandong University of Science and Technology, Qingdao 266590, China*

Correspondence should be addressed to Wencai Zhao; zhaowencai@sdust.edu.cn

Received 9 August 2019; Accepted 6 November 2019; Published 9 December 2019

Academic Editor: Zhen-Lai Han



This study focuses on the investigation of a stochastic Leslie–Gower predator-prey model with feedback controls and Holling type II functional responses. First, the existence and uniqueness of a global positive solution to the system under white noise interference are proved. Second, the conditions for the existence of the system's positive recurrence are established by constructing suitable Lyapunov functions. Additionally, the persistence and extinction of prey and predator in the system are discussed, and the impacts of noise interference and feedback controls on the system are revealed. Finally, we validate the theoretical results by numerical simulations.

## 1. Introduction

The relationship between biological populations is usually expressed and analyzed by differential equations [1–5]. Among numerous studies of biological populations, Leslie [1] presented a classical predator-prey model as follows:

$$\begin{cases} \mathrm{d}N_1(t) = \left[ r_1 N_1(t)\left(1 - \dfrac{N_1(t)}{K}\right) - p(N_1(t))N_2(t) \right]\mathrm{d}t, \\ \mathrm{d}N_2(t) = N_2(t)\left(r_2 - \dfrac{hN_2(t)}{N_1(t)}\right)\mathrm{d}t, \end{cases} \tag{1}$$

with $N_1(0) > 0$ and $N_2(0) > 0$, where $N_1(t)$ and $N_2(t)$ are the density of the prey and the predator at time $t$, respectively. $K$ denotes the carrying capacity of prey, and $r_1$ and $r_2$ are the intrinsic growth rates of the prey and the predator. $p(N_1(t))$ is the functional response of the predator to the prey. $h$ is the measure of food quantity that the prey provides for conversion into one predator birth. The term $(r_2 N_1(t))/h$ is the predator's environmental capacity, and $(hN_2(t))/(N_1(t))$ is called the Leslie–Gower term. Leslie accentuated that the decrease in the number of predators was related to the per capita availability of the prey to be hunted, and the environmental carrying capacity of the predator was proportional to the number of preys.

On the basis of model (1), Aziz-Alaoui and Daher Okiye [6] proposed a modified Leslie–Gower predator-prey model with Holling type II functional responses as follows:

$$\begin{cases} \mathrm{d}N_1(t) = N_1(t)\left[ r_1\left(1 - \dfrac{N_1(t)}{K}\right) - \dfrac{aN_2(t)}{n + N_1(t)} \right]\mathrm{d}t, \\ \mathrm{d}N_2(t) = N_2(t)\left(r_2 - \dfrac{bN_2(t)}{m + N_1(t)}\right)\mathrm{d}t, \end{cases} \tag{2}$$

where $a$ is the maximum of the average predation rate and $b$ is the maximum digestibility rate of predators. $n$ and $m$ are the environment protecting effects of the prey and the predator, respectively. Combined with the Lyapunov function methods, the global stability of the coexisting interior equilibrium point of model (2) was investigated.

As is known to all, ecosystems are inevitably disturbed by the environmental noise [7–15]. Strong environmental disturbances can cause population fluctuations and even extinction. Based on model (2), Ji et al. [16] presented the following Leslie–Gower predator-prey model with stochastic perturbation:

$$\begin{cases} \mathrm{d}N_1(t) = N_1(t)\left[r_1\left(1-\dfrac{N_1(t)}{K}\right)-\dfrac{aN_2(t)}{m+N_1(t)}\right]\mathrm{d}t+\sigma_1 N_1(t)\mathrm{d}B_1(t), \\ \mathrm{d}N_2(t) = N_2(t)\left(r_2-\dfrac{bN_2(t)}{m+N_1(t)}\right)\mathrm{d}t+\sigma_2 N_2(t)\mathrm{d}B_2(t). \end{cases} \tag{3}$$

To simplify the problem, they chose $n = m$ (assuming the predator and prey receive the same protection from the environment) in model (3). Here, $\sigma_i^2\,(i = 1, 2)$ denotes the intensity of white noise, and $B_i(t)\,(i = 1, 2)$ are the standard Brownian motions independent of each other in the complete probability space $(\Omega, \mathscr{F}, \mathscr{F}_{t\geq 0}, \mathbb{P})$. Also, the conditions for the persistence in mean and extinction of the system were provided. In recent years, many research results have been achieved in the study of stochastic ecosystems [17–25]. For instance, Miao et al. [26] verified the stationary distribution of an $n$-species stochastic differential equation model. Lahrouz et al. [27] studied a stochastic Leslie–Gower predator-prey model with regime switching where the effects of white noise and colored noise on the model were considered comprehensively, and the long-term dynamic behavior of the population and sufficient conditions for the existence of stationary distribution were investigated. In [28], a predator-prey model of stochastic Holling type II schemes with Markov switching and prey harvesting was proposed, and they obtained the conditions for the existence of the stationary distribution and got the optimal harvesting strategy.

How to regulate the ecosystem reasonably to ensure its balance and sustainable development is of importance for issue facing mankind [29–33]. Maltby et al. [34] considered ecosystem management to be a control of physical, chemical, and biological processes, which linked the regulation of organisms and their nonliving environment and human activities to create an ideal ecosystem state. In recent years, widely research has been conducted on the biological mathematical models with feedback controls [35–40]. By introducing control variables, Gopalsamy and Weng [41] presented a competition model with time delay and feedback controls following by the discussion on the existence and global attractiveness of the model's positive equilibrium. In [42], a Lotka–Volterra two-species competition model was studied, and the influence of feedback controls on the global stability of the system was explored.

Synthesizing the above literatures, we construct the following stochastic Leslie–Gower predator-prey model with feedback controls:

$$\begin{cases} \mathrm{d}N_1(t) = N_1(t)\left[r_1\left(1-\dfrac{N_1(t)}{K}\right)-\dfrac{aN_2(t)}{m+N_1(t)}-c_1u_1(t)\right]\mathrm{d}t+\sigma_1 N_1(t)\mathrm{d}B_1(t), \\ \mathrm{d}N_2(t) = N_2(t)\left(r_2-\dfrac{bN_2(t)}{m+N_1(t)}-c_2u_2(t)\right)\mathrm{d}t+\sigma_2 N_2(t)\mathrm{d}B_2(t), \\ \mathrm{d}u_1(t) = \left(-f_1u_1(t)+g_1N_1(t)\right)\mathrm{d}t, \\ \mathrm{d}u_2(t) = \left(-f_2u_2(t)+g_2N_2(t)\right)\mathrm{d}t, \end{cases} \tag{4}$$

where $u_1(t)$ and $u_2(t)$ are the feedback control variables and $c_1, c_2, f_1, f_2, g_1$, and $g_2$ are all positive constants from the realistic biological significance.

The rest of this paper is structured as follows: in Section 2, we prove the existence and uniqueness of global positive solutions to system (4). In Section 3, we obtain the conditions for the existence of positive recurrence of the solutions to system (4). In Section 4, we analyze the survival conditions of the system and obtain sufficient conditions for persistence in mean and extinction. In Section 5, we use MATLAB to perform some numerical simulations to supplement and illustrate the result.

## 2. Existence and Uniqueness of Global Positive Solutions

In order to study the dynamic behaviors of system (4), it is necessary to first determine whether there is a global positive solution for any given initial conditions.

**Theorem 1.** *For any given initial value* $(N_1(0), N_2(0), u_1(0), u_2(0)) \in \mathbb{R}_+^4$, *the model (4) has a unique positive solution* $(N_1(t), N_2(t), u_1(t), u_2(t))$ *on* $t \geq 0$ *and the solution will remain in* $\mathbb{R}_+^4$ *with probability one.*

*Proof.* Since the coefficients of the model (4) satisfy the local Lipschitz condition, for any initial value $(N_1(0), N_2(0), u_1(0), u_2(0)) \in \mathbb{R}_+^4$, there exists a unique local solution $(N_1(t), N_2(t), u_1(t), u_2(t)) \in \mathbb{R}_+^4$ on $t \in [0, \tau_e)$, where $\tau_e$ is the explosion time. We want to prove this solution is global, i.e., to show the $\tau_e = \infty$ a.s. Define a number $h_0 \geq 1$ which is large enough to make $N_i(0), u_i(0) \in [(1/h_0), h_0]$ $(i = 1, 2)$. For each integer $h \geq h_0$, we define the stopping time $\tau_h$ as follows:

$$\tau_h = \inf\left\{t \in [0, \tau_e) : \min\{N_1(t), N_2(t), u_1(t), u_2(t)\} \leq \frac{1}{h} \text{ or } \max\{N_1(t), N_2(t), u_1(t), u_2(t)\} \geq h\}\right\}, \tag{5}$$

where the sequence $\tau_h$ is increasing with respect to $h$. Let $\lim_{h \longrightarrow \infty} \tau_h = \tau_\infty$, by the definition of stopping time, $\tau_\infty \leq \tau_e$ a.s. Next, we need to prove $\tau_\infty = \infty$ so that $\tau_e = \infty$. If this statement is false, then there exist constants $T > 0$ and $\varepsilon \in (0, 1)$ such that

$$\mathbb{P}\{\tau_\infty \leq T\} > \varepsilon. \tag{6}$$

Thus, there is an integer $h_1 \geq h_0$ such that

$$\mathbb{P}\{\tau_h \leq T\} \geq \varepsilon, \text{ for all } h \geq h_1. \tag{7}$$

For the solution $(N_1(t), N_2(t), u_1(t), u_2(t))$ of model (4), we define a $C^2$ function $V$: $\mathbb{R}_+^4 \longrightarrow \mathbb{R}_+$ by

$$\begin{aligned} V(N_1, N_2, u_1, u_2) &= g_1(N_1 - 1 - \ln N_1) \\ &\quad + g_2(N_2 - 1 - \ln N_2) + c_1\frac{(u_1 - 1)^2}{2} \\ &\quad + c_2\frac{(u_2 - 1)^2}{2} = V_1 + V_2, \end{aligned} \tag{8}$$

where

$$\begin{aligned} V_1 &= g_1(N_1 - 1 - \ln N_1) + g_2(N_2 - 1 - \ln N_2), \\ V_2 &= c_1\frac{(u_1 - 1)^2}{2} + c_2\frac{(u_2 - 1)^2}{2}. \end{aligned} \tag{9}$$

By using the generalized Itô formula, we get

$$\mathrm{d}V = LV\mathrm{d}t + g_1(N_1 - 1)\sigma_1\mathrm{d}B_1(t) + g_2(N_2 - 1)\sigma_2\mathrm{d}B_2(t), \tag{10}$$

where

$$\begin{aligned} LV &= LV_1 + LV_2, \\ LV_1 &= g_1(N_1 - 1)\left[r_1\left(1 - \frac{N_1}{K}\right) - \frac{aN_2}{m + N_1} - c_1u_1\right] + \frac{g_1\sigma_1^2}{2} + g_2(N_2 - 1)\left(r_2 - \frac{bN_2}{m + N_1} - c_2u_2\right) + \frac{g_2\sigma_2^2}{2} \\ &= g_1r_1N_1 - \frac{g_1r_1}{K}N_1^2 - \frac{g_1aN_1N_2}{m + N_1} - g_1c_1u_1N_1 - g_1r_1 + \frac{g_1r_1}{K}N_1 + \frac{g_1aN_2}{m + N_1} + g_1c_1u_1 + \frac{g_1\sigma_1^2}{2} + g_2r_2N_2 \\ &\quad - \frac{g_2bN_2^2}{m + N_1} - g_2c_2u_2N_2 - g_2r_2 + \frac{g_2bN_2}{m + N_1} + g_2c_2u_2 + \frac{g_2\sigma_2^2}{2} \\ &\leq -\frac{g_1r_1}{K}N_1^2 + \left(g_1r_1 + \frac{g_1r_1}{K}\right)N_1 + g_1c_1u_1 + \left(\frac{g_1a + g_2b}{m} + g_2r_2\right)N_2 - g_1c_1u_1N_1 - g_2c_2u_2N_2 + g_2c_2u_2 + \frac{g_1\sigma_1^2}{2} + \frac{g_2\sigma_2^2}{2}, \\ LV_2 &= c_1(u_1 - 1)(-f_1u_1 + g_1N_1) + c_2(u_2 - 1)(-f_2u_2 + g_2N_2) \\ &= -f_1c_1u_1^2 + f_1c_1u_1 + c_1g_1u_1N_1 - c_1g_1N_1 - c_2f_2u_2^2 + f_2c_2u_2 + c_2g_2u_2N_2 - c_2g_2N_2 \\ &\leq -f_1c_1u_1^2 + f_1c_1u_1 + c_1g_1u_1N_1 - c_2f_2u_2^2 + f_2c_2u_2 + c_2g_2u_2N_2, \\ LV &\leq \max\left\{-\frac{g_1r_1}{K}N_1^2 + \left(g_1r_1 + \frac{g_1r_1}{K}\right)N_1\right\} + \max\left\{-f_1c_1u_1^2 + (g_1c_1 + c_1f_1)u_1\right\} \\ &\quad + \max\left\{-c_2f_2u_2^2 + (g_2c_2 + f_2c_2)u_2\right\} + \left(\frac{g_1a + g_2b}{m} + g_2r_2\right)N_2 + \frac{g_1\sigma_1^2}{2} + \frac{g_2\sigma_2^2}{2} \\ &= M + \left(\frac{g_1a + g_2b}{m} + g_2r_2\right)N_2 + \frac{g_1\sigma_1^2}{2} + \frac{g_2\sigma_2^2}{2}, \end{aligned} \tag{11}$$

where

$$\begin{aligned} M = {} & \max\left\{-\frac{g_1 r_1}{K}N_1^2 + \left(g_1 r_1 + \frac{g_1 r_1}{K}\right)N_1\right\} \\ & + \max\left\{-f_1 c_1 u_1^2 + (g_1 c_1 + c_1 f_1)u_1\right\} \\ & + \max\left\{-c_2 f_2 u_2^2 + (g_2 c_2 + f_2 c_2)u_2\right\}. \end{aligned} \tag{12}$$

Applying the Itô formula to (4), one can get

$$\begin{aligned} N_2(t) = {} & N_2(0)e^{r_2 t} - \int_0^t \frac{bN_2^2(s)}{m + N_1(s)} e^{r_2(t-s)} \mathrm{d}s \\ & - \int_0^t c_2 u_2(s) N_2(s) e^{r_2(t-s)} \mathrm{d}s \\ & + \sigma_2 \int_0^t e^{r_2(t-s)} N_2(s) \mathrm{d}B_2(s) \\ \le {} & N_2(0)e^{r_2 t} + \sigma_2 \int_0^t e^{r_2(t-s)} N_2(s) \mathrm{d}B_2(s). \end{aligned} \tag{13}$$

Therefore,

$$\begin{aligned} \mathrm{d}V \le {} & \left[M + \left(\frac{g_1 a + g_2 b}{m} + g_2 r_2\right)\left(N_2(0)e^{r_2 t}\right.\right. \\ & \left.\left. + \sigma_2 \int_0^t e^{r_2(t-s)} N_2(s) \mathrm{d}B_2(s)\right) + \frac{g_1\sigma_1^2}{2} + \frac{g_2\sigma_2^2}{2}\right]\mathrm{d}t \\ & + g_1(N_1 - 1)\sigma_1 \mathrm{d}B_1(t) + g_2(N_2 - 1)\sigma_2 \mathrm{d}B_2(t). \end{aligned} \tag{14}$$

Integrating both sides of (14) from 0 to $\tau_h \wedge T$ and taking the expectation, we obtain

$$\begin{aligned} & EV(N_1(\tau_h \wedge T), N_2(\tau_h \wedge T), u_1(\tau_h \wedge T), u_2(\tau_h \wedge T)) \\ & \le V(N_1(0), N_2(0), u_1(0), u_2(0)) \\ & \quad + \left(M + \frac{g_1\sigma_1^2}{2} + \frac{g_2\sigma_2^2}{2}\right)E(\tau_h \wedge T) \\ & \quad + \left(\frac{g_1 a + g_2 b}{m} + g_2 r_2\right)N_2(0)\int_0^{\tau_h \wedge T} E\left(e^{r_2 t}\right)\mathrm{d}t \\ & \le V(N_1(0), N_2(0), u_1(0), u_2(0)) + \left(M + \frac{g_1\sigma_1^2 + g_2\sigma_2^2}{2}\right)T \\ & \quad + \frac{N_2(0)}{r_2}\left(\frac{g_1 a + g_2 b}{m} + g_2 r_2\right)e^{r_2 T}. \end{aligned} \tag{15}$$

For all $h \ge h_1$, let $\Omega_h = \{\tau_h \le T\}$, we have $\mathbb{P}(\Omega_h) \ge \varepsilon$. Note that, for any $\omega \in \Omega_h$, at least one of $N_1(\tau_h, \omega), N_2(\tau_h, \omega)$, $u_1(\tau_h, \omega)$, or $u_2(\tau_h, \omega)$ equals $1/h$ or $h$, and then

$$\begin{aligned} & V(N_1(\tau_h, \omega), N_2(\tau_h, \omega), u_1(\tau_h, \omega), u_2(\tau_h, \omega)) \\ & \ge g(h - 1 - \ln h) \wedge \left(\frac{1}{h} - 1 + \ln h\right), \end{aligned} \tag{16}$$

where

$$g = \min\{g_1, g_2\}. \tag{17}$$

Combining inequality (15), which implies that

$$\begin{aligned} & V(N_1(0), N_2(0), u_1(0), u_2(0)) + \left(M + \frac{g_1\sigma_1^2}{2} + \frac{g_2\sigma_2^2}{2}\right)T \\ & + \frac{1}{r_2}N_2(0)\left(\frac{g_1 a + g_2 b}{m} + g_2 r_2\right)e^{r_2 T} \\ & \ge E\left[1_{\Omega_h}(\omega)V(N_1(\tau_h, \omega), N_2(\tau_h, \omega), u_1(\tau_h, \omega), u_2(\tau_h, \omega))\right] \\ & \ge g(h - 1 - \ln h) \wedge \left(\frac{1}{h} - 1 + \ln h\right)\mathbb{P}(\Omega_h) \\ & \ge \varepsilon g(h - 1 - \ln h) \wedge \left(\frac{1}{h} - 1 + \ln h\right), \end{aligned} \tag{18}$$

where $1_{\Omega_h}$ is the indicator function of $\Omega_h$. As $h \longrightarrow \infty$, there is

$$\begin{aligned} \infty > {} & V(N_1(0), N_2(0), u_1(0), u_2(0)) \\ & + \left(M + \frac{g_1\sigma_1^2}{2} + \frac{g_2\sigma_2^2}{2}\right)T \\ & + \frac{1}{r_2}N_2(0)\left(\frac{g_1 a + g_2 b}{m} + g_2 r_2\right)e^{r_2 T} = \infty. \end{aligned} \tag{19}$$

Obviously, this is contradictory. The existence and uniqueness of global positive solutions are proved. □

## 3. Existence of Positive Recurrence

For the stochastic model with feedback controls (4), we are primarily concerned with the long-term dynamic behavior of the population. For example, under what conditions can the path from one point return to a given region for a limited time? This requires us to discuss the positive recurrence of the solution $(N_1(t), N_2(t), u_1(t), u_2(t))$ of model (4).

Let $X(t)$ be a regular time-homogeneous Markov process in $E_l$ (the $l$-dimensional Euclidean space) of the stochastic equation:

$$\mathrm{d}X(t) = b(X)\mathrm{d}t + \sum_{r=1}^{l} g_r(X)\mathrm{d}B_r(t). \tag{20}$$

The diffusion matrix is defined as follows:

$$\begin{aligned} \mathbf{A}(x) &= \left(a_{ij}(x)\right), \\ a_{ij}(x) &= \sum_{r=1}^{l} g_r^i(x) g_r^j(x). \end{aligned} \tag{21}$$

*Definition 1* (see [43]). The Markov process $X(t)$ is said to be positive recurrent with a bounded domain $U \subset E_l$, if $E(\tau_x) < \infty$ for any $x \notin U$, here $\tau_x$ is the time when the trajectory from $x$ arrives at $U$, namely, $\tau_x = \inf\{t : X(t) \in U\}$.

**Lemma 1** (see [43]). *Suppose that the process $X(t)$ almost surely exits from each bounded domain in a finite time. Then, a sufficient condition for positive recurrence is that there exists a nonnegative function $V(t,x)$ in the domain $\{t > 0\} \times U^c$ such that*

$$V_R = \inf_{t>0,|x|\geq R} V(t,x) \longrightarrow \infty, \quad \text{as } R \longrightarrow \infty, \qquad LV \leq 0. \tag{22}$$

First, we consider the equilibrium point of the corresponding deterministic system. From model (4), we set

$$\begin{cases} r_1\left(1 - \dfrac{N_1}{K}\right) - \dfrac{aN_2}{m+N_1} - c_1u_1 = 0, \\ r_2 - \dfrac{bN_2}{m+N_1} - c_2u_2 = 0, \\ -f_1u_1 + g_1N_1 = 0, \\ -f_2u_2 + g_2N_2 = 0. \end{cases} \tag{23}$$

By calculating (23), the following equation is obtained:

$$\begin{aligned} u_1 &= \frac{g_1N_1}{f_1}, \\ u_2 &= \frac{g_2N_2}{f_2}, \\ N_2 &= \frac{r_2f_2(m+N_1)}{f_2b + c_2g_2(m+N_1)}, \\ & c_2g_2(c_1g_1K + r_1f_1)N_1^2 \\ &\quad + [(f_2b + c_2g_2m)(c_1g_1K + r_1f_1) \\ &\quad - f_1r_1c_2g_2K]N_1 \\ &\quad + ar_2f_1f_2K - mf_1r_1c_2g_2K - f_1r_1f_2bK = 0. \end{aligned} \tag{24}$$

Let

$$\xi = (f_2b + c_2g_2m)(c_1g_1K + r_1f_1) - r_1f_1c_2g_2K. \tag{25}$$

If $\xi$ satisfies

$$\Delta = \xi^2 - 4c_2g_2(c_1g_1K + r_1f_1) \cdot (ar_2f_1f_2K - mr_1f_1c_2g_2K - r_1f_1f_2bK) \geq 0, \tag{26}$$

and

$$ar_2f_2 < r_1(bf_2 + mc_2g_2), \tag{27}$$

then there exists a unique positive solution $(N_1^*, N_2^*, u_1^*, u_2^*)$ to equations (23). That is, the deterministic system corresponding to the stochastic model (4) has a unique positive equilibrium point.

**Theorem 2.** *Let conditions (26) and (27) hold. If the hypothesis*

$$\begin{aligned} &(H1)\ g_i = c_i\,(i = 1,2) \\ &(H2)\ \frac{r_1m}{K} > \frac{a^2}{b}, a = \frac{bN_2^*}{m+N_1^*} \\ &(H3)\ \delta < \min\left\{\left(\frac{r_1m}{K} - \frac{a^2}{b}\right)\right. \\ &\qquad \cdot \left(N_1^* + \frac{N_1^*\sigma_1^2 + N_2^*\sigma_2^2}{4((r_1m/K) - (a^2/b))}\right)^2, \\ &\qquad \left. bN_2^{*2}, mf_1u_1^{*2}, mf_2u_2^{*2}\right\} \\ &\delta = \frac{(N_1^*\sigma_1^2 + N_2^*\sigma_2^2)^2}{16((r_1m/K) - (a^2/b))} \\ &\qquad + \frac{1}{2}(N_1^*\sigma_1^2 + N_2^*\sigma_2^2)(N_1^* + m), \end{aligned} \tag{28}$$

*holds, then the solution to model (4) is positive recurrence with respect to the elliptic domain $U$.*

*Proof.* Let $(N_1(t), N_2(t), u_1(t), u_2(t))$ be a solution to (4) for any given initial value.

$(N_1(0), N_2(0), u_1(0), u_2(0)) \in R_+^4$. Define a $C^2$ function

$$\begin{aligned} V &= \left(N_1 - N_1^* - N_1^* \ln \frac{N_1}{N_1^*}\right) + \left(N_2 - N_2^* - N_2^* \ln \frac{N_2}{N_2^*}\right) \\ &\quad + \frac{(u_1 - u_1^*)^2}{2} + \frac{(u_2 - u_2^*)^2}{2} \\ &= V_1 + V_2 + V_3, \end{aligned} \tag{29}$$

where

$$\begin{aligned} V_1 &= N_1 - N_1^* - N_1^* \ln \frac{N_1}{N_1^*}, \\ V_2 &= N_2 - N_2^* - N_2^* \ln \frac{N_2}{N_2^*}, \\ V_3 &= \frac{(u_1 - u_1^*)^2}{2} + \frac{(u_2 - u_2^*)^2}{2}. \end{aligned} \tag{30}$$

Obviously, $V_R = \inf_{t>0,|x|\geq R} V(t,x) \longrightarrow \infty$ as $R \longrightarrow \infty$, where $x = (N_1, N_2, u_1, u_2)^T$. By using the Itô formula, we get

$$
\begin{aligned}
LV_1 &= (N_1 - N_1^*)\left(r_1 - r_1\frac{N_1}{K} - \frac{aN_2}{m+N_1} - c_1u_1\right) + \frac{N_1^*\sigma_1^2}{2} \\
&= (N_1 - N_1^*)\left(\frac{r_1}{K}N_1^* + \frac{aN_2^*}{m+N_1^*} + c_1u_1^* - r_1\frac{N_1}{K} - \frac{aN_2}{m+N_1} - c_1u_1\right) + \frac{N_1^*\sigma_1^2}{2} \\
&= -\frac{r_1}{K}(N_1 - N_1^*)^2 + \left(\frac{aN_2^*(N_1 - N_1^*)}{(m+N_1^*)(m+N_1)} - \frac{a(N_2 - N_2^*)}{m+N_1}\right)(N_1 - N_1^*) - c_1(u_1 - u_1^*)(N_1 - N_1^*) + \frac{N_1^*\sigma_1^2}{2} \\
&= -\frac{r_1}{K}(N_1 - N_1^*)^2 + \frac{aN_2^*(N_1 - N_1^*)^2}{(m+N_1^*)(m+N_1)} - \frac{a(N_2 - N_2^*)(N_1 - N_1^*)}{m+N_1} - c_1(u_1 - u_1^*)(N_1 - N_1^*) + \frac{N_1^*\sigma_1^2}{2}, \\
LV_2 &= (N_2 - N_2^*)\left(\frac{bN_2^*}{m+N_1^*} + c_2u_2^* - \frac{bN_2}{m+N_1} - c_2u_2\right) + \frac{N_2^*\sigma_2^2}{2} \\
&= (N_2 - N_2^*)\left[\frac{bN_2^*(N_1 - N_1^*)}{(m+N_1^*)(m+N_1)} - \frac{b(N_2 - N_2^*)}{m+N_1} - c_2(u_2 - u_2^*)\right] + \frac{N_2^*\sigma_2^2}{2} \\
&= \frac{bN_2^*(N_1 - N_1^*)(N_2 - N_2^*)}{(m+N_1^*)(m+N_1)} - \frac{b(N_2 - N_2^*)^2}{m+N_1} - c_2(u_2 - u_2^*)(N_2 - N_2^*) + \frac{N_2^*\sigma_2^2}{2}, \\
LV_3 &= (u_1 - u_1^*)(-f_1u_1 + g_1N_1) + (u_2 - u_2^*)(-f_2u_2 + g_2N_2) \\
&= (u_1 - u_1^*)(-f_1u_1 + g_1N_1 + f_1u_1^* - g_1N_1^*) + (u_2 - u_2^*)(-f_2u_2 + g_2N_2 + f_2u_2^* - g_2N_2^*) \\
&= (u_1 - u_1^*)[-f_1(u_1 - u_1^*) + g_1(N_1 - N_1^*)] + (u_2 - u_2^*)[-f_2(u_2 - u_2^*) + g_2(N_2 - N_2^*)] \\
&= -f_1(u_1 - u_1^*)^2 + g_1(N_1 - N_1^*)(u_1 - u_1^*) - f_2(u_2 - u_2^*)^2 + g_2(u_2 - u_2^*)(N_2 - N_2^*).
\end{aligned}
\tag{31}
$$

Therefore,

$$
\begin{aligned}
LV &= -\frac{r_1}{K}(N_1 - N_1^*)^2 + \frac{aN_2^*(N_1 - N_1^*)^2}{(m+N_1^*)(m+N_1)} - \frac{a(N_2 - N_2^*)(N_1 - N_1^*)}{m+N_1} - c_1(u_1 - u_1^*)(N_1 - N_1^*) \\
&\quad + \frac{N_1^*\sigma_1^2}{2} + \frac{bN_2^*(N_1 - N_1^*)(N_2 - N_2^*)}{(m+N_1^*)(m+N_1)} - \frac{b(N_2 - N_2^*)^2}{m+N_1} - c_2(u_2 - u_2^*)(N_2 - N_2^*) + \frac{N_2^*\sigma_2^2}{2} \\
&\quad - f_1(u_1 - u_1^*)^2 + g_1(N_1 - N_1^*)(u_1 - u_1^*) - f_2(u_2 - u_2^*)^2 + g_2(u_2 - u_2^*)(N_2 - N_2^*) \\
&= -\frac{r_1/K(m+N_1) - (aN_2^*/m+N_1^*)}{m+N_1}(N_1 - N_1^*)^2 + \left(\frac{bN_2^*}{m+N_1^*} - a\right)\frac{(N_2 - N_2^*)(N_1 - N_1^*)}{m+N_1} \\
&\quad + (g_1 - c_1)(u_1 - u_1^*)(N_1 - N_1^*) + (g_2 - c_2)(u_2 - u_2^*)(N_2 - N_2^*) - \frac{b(N_2 - N_2^*)^2}{m+N_1} \\
&\quad - f_1(u_1 - u_1^*)^2 - f_2(u_2 - u_2^*)^2 + \frac{N_1^*\sigma_1^2}{2} + \frac{N_2^*\sigma_2^2}{2}.
\end{aligned}
\tag{32}
$$

From the hypotheses $(H1)$ and $(H2)$, we get

$$LV \le -\frac{((r_1/K)m-(a^2/b))}{m+N_1}(N_1-N_1^*)^2-\frac{b(N_2-N_2^*)^2}{m+N_1} - f_1(u_1-u_1^*)^2-f_2(u_2-u_2^*)^2+\frac{N_1^*\sigma_1^2}{2}+\frac{N_2^*\sigma_2^2}{2}. \tag{33}$$

It can be written as

$$\begin{aligned}(m+N_1)LV &\le -\left(\frac{r_1m}{K}-\frac{a^2}{b}\right)(N_1-N_1^*)^2-b(N_2-N_2^*)^2-(m+N_1)f_1(u_1-u_1^*)^2\\ &\quad -(m+N_1)f_2(u_2-u_2^*)^2+\frac{N_1^*\sigma_1^2+N_2^*\sigma_2^2}{2}(m+N_1)\\ &\le -\left(\frac{r_1m}{K}-\frac{a^2}{b}\right)(N_1-N_1^*)^2-b(N_2-N_2^*)^2-mf_1(u_1-u_1^*)^2-mf_2(u_2-u_2^*)^2+\frac{N_1^*\sigma_1^2+N_2^*\sigma_2^2}{2}(m+N_1)\\ &= -\left(\frac{r_1m}{K}-\frac{a^2}{b}\right)\left[N_1-N_1^*-\frac{N_1^*\sigma_1^2+N_2^*\sigma_2^2}{4((r_1m/K)-(a^2/b))}\right]^2-b(N_2-N_2^*)^2-mf_1(u_1-u_1^*)^2\\ &\quad -mf_2(u_2-u_2^*)^2+\frac{(N_1^*\sigma_1^2+N_2^*\sigma_2^2)^2}{16((r_1m/K)-(a^2/b))}+\frac{1}{2}(N_1^*\sigma_1^2+N_2^*\sigma_2^2)(N_1^*+m).\end{aligned} \tag{34}$$

Obviously, the ellipsoid

$$-\left(\frac{r_1m}{K}-\frac{a^2}{b}\right)\left[N_1-N_1^*-\frac{N_1^*\sigma_1^2+N_2^*\sigma_2^2}{4((r_1m/K)-(a^2/b))}\right]^2 - b(N_2-N_2^*)^2-mf_1(u_1-u_1^*)^2-mf_2(u_2-u_2^*)^2+\delta\ge 0, \tag{35}$$

lies entirely in $R_+^4$. For any $x\in E_l/U$, where $U$ is a neighborhood of the ellipsoid, we have $LV\le -M$ ($M$ is a positive constant).

From Lemma 1, we obtain the existence of positive recurrence to model (4). Because the diffusion matrix $A(x)$ of system (4) does not satisfy the uniformly elliptical condition in $U$, we only prove that the system is positive recurrent but does not obtain the ergodic property. □

## 4. Persistence and Extinction

In the above section, we obtained the existence conditions for the positive recurrence of system (4). In this section, we discuss the persistence and extinction of species of the stochastic model (4) with feedback controls.

For convenience, let us define some notations as follows:

$$\Delta_1=\frac{(r_1f_1-c_1g_1K)(r_1-(\sigma_1^2/2))}{f_1r_1},\qquad \Delta_2=\frac{(bf_2-c_2g_2m)(r_2-(\sigma_2^2/2))}{bf_2}. \tag{36}$$

**Theorem 3.** *For any given initial condition $(N_1(0),N_2(0),u_1(0),u_2(0))\in R_+^4$, the solution $(N_1(t),N_2(t),u_1(t),u_2(t))$ to model (4) have the following properties:*

*(i) When $r_1-(\sigma_1^2/2)<0$ and $r_2-(\sigma_2^2/2)<0$, $N_1$ and $N_2$ are extinct, and $u_1$ and $u_2$ are also extinct*

*(ii) When $r_1-(\sigma_1^2/2)>0$, $r_2-(\sigma_2^2/2)<0$, and $\Delta_1>0$, $N_2$ is extinct, and $N_1$ and $u_1$ are persistent in mean*

*(iii) When $r_1-(\sigma_1^2/2)<0$, $r_2-(\sigma_2^2/2)>0$, and $\Delta_2>0$, $N_1$ is extinct, and $N_2$ and $u_2$ are persistent in mean.*

*Proof.* Applying the Itô formula to system (4), we obtain

$$\ln N_1(t)-\ln N_1(0)=\left(r_1-\frac{\sigma_1^2}{2}\right)t -\frac{r_1}{K}\int_0^t N_1(s)\mathrm{d}s-a\int_0^t\frac{N_2(s)}{m+N_1(s)} -c_1\int_0^t u_1(s)\mathrm{d}s+\int_0^t\sigma_1\mathrm{d}B_1(s), \tag{37}$$

$$\ln N_2(t)-\ln N_2(0)=\left(r_2-\frac{\sigma_2^2}{2}\right)t-b\int_0^t\frac{N_2(s)}{m+N_1(s)}\mathrm{d}s -c_2\int_0^t u_2(s)\mathrm{d}s+\int_0^t\sigma_2\mathrm{d}B_2(s). \tag{38}$$

Dividing both sides of (37) and (38) by $t$, we get

$$\begin{aligned} t^{-1}\ln N_1(t) &= t^{-1}\ln N_1(0) + \left(r_1 - \frac{\sigma_1^2}{2}\right) - t^{-1}\frac{r_1}{K}\int_0^t N_1(s)\mathrm{d}s \\ &\quad - t^{-1}a\int_0^t \frac{N_2(s)}{m+N_1(s)}\mathrm{d}s - t^{-1}c_1\int_0^t u_1(s)\mathrm{d}s \\ &\quad + t^{-1}M_1(t), \end{aligned} \tag{39}$$

$$\begin{aligned} t^{-1}\ln N_2(t) &= t^{-1}\ln N_2(0) + \left(r_2 - \frac{\sigma_2^2}{2}\right) \\ &\quad - t^{-1}b\int_0^t \frac{N_2(s)}{m+N_1(s)}\mathrm{d}s \\ &\quad - t^{-1}c_2\int_0^t u_2(s)\mathrm{d}s + t^{-1}M_2(t), \end{aligned} \tag{40}$$

where

$$\begin{aligned} M_1(t) &= \int_0^t \sigma_1 \mathrm{d}B_1(s), \\ M_2(t) &= \int_0^t \sigma_2 \mathrm{d}B_2(s). \end{aligned} \tag{41}$$

Based on the strong law of large number for martingales, we obtain

$$\begin{aligned} \lim_{t\longrightarrow\infty}\frac{M_1(t)}{t} &= 0, \\ \lim_{t\longrightarrow\infty}\frac{M_2(t)}{t} &= 0 \quad \text{a.s.} \end{aligned} \tag{42}$$

□

*Case 1.* According to formula (39), we gain

$$\begin{aligned} t^{-1}\ln N_1(t) &\le t^{-1}\ln N_1(0) + \left(r_1 - \frac{\sigma_1^2}{2}\right) \\ &\quad - t^{-1}\frac{r_1}{K}\int_0^t N_1(s)\mathrm{d}s + t^{-1}M_1(t). \end{aligned} \tag{43}$$

By Lemma 2.3 in [44], when $r_1 - (\sigma_1^2/2) < 0$,

$$\lim_{t\longrightarrow\infty} N_1(t) = 0 \quad \text{a.s.} \tag{44}$$

According to the third equation of system (4), we can acquire

$$\lim_{t\longrightarrow\infty} u_1(t) = 0 \quad \text{a.s.} \tag{45}$$

Similarly, by equation (40), when $r_2 - (\sigma_2^2/2) < 0$, there is

$$\lim_{t\longrightarrow\infty} N_2(t) = 0 \quad \text{a.s.} \tag{46}$$

Furthermore, from the fourth equation of model (4), we can obtain

$$\lim_{t\longrightarrow\infty} u_2(t) = 0 \quad \text{a.s.} \tag{47}$$

*Case 2.* We suppose that $Z_1(t)$ is a solution to the following equation:

$$\begin{aligned} \mathrm{d}Z_1(t) &= Z_1(t)\left(r_1 - \frac{r_1 Z_1(t)}{K} - \frac{aN_2(t)}{m+Z_1(t)}\right)\mathrm{d}t \\ &\quad + \sigma_1 Z_1(t)\mathrm{d}B_1(t), \end{aligned} \tag{48}$$

with $Z_1(0) = N_1(0) > 0$. According to the comparison theorem for the stochastic equation, for $t \ge 0$, we get

$$Z_1(t) \ge N_1(t), \quad \text{a.s.} \tag{49}$$

From the condition $r_2 - (\sigma_2^2/2) < 0$, we can get $\lim_{t\longrightarrow\infty} N_2(t) = 0$. For arbitrary small $\varepsilon > 0$, we have $N_2(t) < \varepsilon$ for $t > T$(T is a constant) and

$$\begin{aligned} &Z_1(t)\left(r_1 - \frac{r_1 Z_1(t)}{K} - \frac{a\varepsilon}{m}\right)\mathrm{d}t + \sigma_1 Z_1(t)\mathrm{d}B_1(t) \\ &\quad \le \mathrm{d}Z_1(t) \le Z_1(t)\left(r_1 - \frac{r_1 Z_1(t)}{K}\right)\mathrm{d}t + \sigma_1 Z_1(t)\mathrm{d}B_1(t). \end{aligned} \tag{50}$$

By $r_1 - (\sigma_1^2/2) > 0$, we have

$$\begin{aligned} \liminf_{t\longrightarrow\infty} t^{-1}\int\int_0^t Z_1(s)\mathrm{d}s &\ge \frac{(r_1 - (\sigma_1^2/2) - (a/m)\varepsilon)K}{r_1}, \\ \limsup_{t\longrightarrow\infty} t^{-1}\int_0^t Z_1(s)\mathrm{d}s &\le \frac{(r_1 - (\sigma_1^2/2))K}{r_1} \quad \text{a.s.} \end{aligned} \tag{51}$$

For the arbitrariness of $\varepsilon$, we get

$$\begin{aligned} \limsup_{t\longrightarrow\infty} t^{-1}\int_0^t N_1(s)\mathrm{d}s &\le \lim_{t\longrightarrow\infty} t^{-1}\int_0^t Z_1(s)\mathrm{d}s \\ &= \frac{(r_1 - (\sigma_1^2/2))K}{r_1} \quad \text{a.s.} \end{aligned} \tag{52}$$

According to the third equation of system (4), we can derive

$$\limsup_{t\longrightarrow\infty} t^{-1}\int_0^t u_1(s)\mathrm{d}s \le \frac{g_1 K(r_1 - (\sigma_1^2/2))}{f_1 r_1} \text{ a.s.} \tag{53}$$

Substituting (53) into equation (39) leads to

$$\begin{aligned} t^{-1}\ln N_1(t) &\ge t^{-1}\ln N_1(0) + \left(r_1 - \frac{\sigma_1^2}{2}\right) - t^{-1}\frac{r_1}{K}\int_0^t N_1(s)\mathrm{d}s \\ &\quad - \frac{c_1 g_1 K(r_1 - (\sigma_1^2/2))}{f_1 r_1} - \frac{a}{m}\varepsilon + t^{-1}M_1(t), \\ &= t^{-1}\ln N_1(0) + \frac{(r_1 f_1 - c_1 g_1 K)(r_1 - (\sigma_1^2/2))}{f_1 r_1} \\ &\quad - t^{-1}\frac{r_1}{K}\int_0^t N_1(s)\mathrm{d}s - \frac{a}{m}\varepsilon + t^{-1}M_1(t). \end{aligned} \tag{54}$$

When

$$\Delta_1 = \frac{(r_1 f_1 - c_1 g_1 K)(r_1 - (\sigma_1^2/2))}{f_1 r_1} > 0, \tag{55}$$

we acquire

$$\liminf_{t\longrightarrow\infty} t^{-1}\int_0^t N_1(s)\mathrm{d}s \geq \frac{\Delta_1 K}{r_1} \quad \text{a.s.} \tag{56}$$

Considering the third equation of (4) again, we can see

$$\liminf_{t\longrightarrow\infty} t^{-1}\int_0^t u_1(s)\mathrm{d}s \geq \frac{\Delta_1 K g_1}{r_1 f_1} > 0 \quad \text{a.s.} \tag{57}$$

That is, $N_1$ and $u_1$ are persistent in mean.

*Case 3.* Let $Z_2(t)$ be a solution of the following equation:

$$\begin{aligned}
\mathrm{d}Z_2(t) &= Z_2(t)\left(r_2 - \frac{bZ_2(t)}{m+N_1(t)}\right)\mathrm{d}t + \sigma_2 Z_2(t)\mathrm{d}B_2(t) \\
&= Z_2(t)\left(r_2 - \frac{bZ_2(t)(m+N_1(t))}{m(m+N_1(t))} + \frac{bN_1(t)Z_2(t)}{m(m+N_1(t))}\right)\mathrm{d}t \\
&\quad + \sigma_2 Z_2(t)B_2(t) \\
&= Z_2(t)\left(r_2 - \frac{b}{m}Z_2(t) + \frac{bN_1(t)Z_2(t)}{m(m+N_1(t))}\right)\mathrm{d}t \\
&\quad + \sigma_2 Z_2(t)\mathrm{d}B_2(t),
\end{aligned} \tag{58}$$

with $Z_2(0) = N_2(0) > 0$. Using the comparison theorem for the stochastic equation, we get that, for $t \geq 0$,

$$Z_2(t) \geq N_2(t) \quad \text{a.s.} \tag{59}$$

By the sure reason as the proof of Case (i), we know that $\lim_{t\longrightarrow+\infty} N_1(t) = 0$ holds when $r_1 - (\sigma_1^2/2) < 0$. Hence for arbitrary small $\varepsilon > 0$, there is $(N_1(t)/m + N_1(t)) \leq \varepsilon$ for $t \geq t_0$. Thus, we can obtain

$$\begin{aligned}
&Z_2(t)\left(r_2 - \frac{bZ_2(t)}{m}\right)\mathrm{d}t + \sigma_2 Z_2(t)\mathrm{d}B_2(t) \leq \mathrm{d}Z_2(t) \\
&\leq Z_2(t)\left(r_2 - \frac{b(1-\varepsilon)Z_2(t)}{m}\right)\mathrm{d}t + \sigma_2 Z_2(t)\mathrm{d}B_2(t).
\end{aligned} \tag{60}$$

Since $r_2 - (\sigma_2^2/2) > 0$, then we have

$$\begin{aligned}
&\liminf_{t\longrightarrow\infty} t^{-1}\int_0^t Z_2(s)\mathrm{d}s \geq \frac{r_2 - (\sigma_2^2/2)}{b/m}, \\
&\limsup_{t\longrightarrow\infty} t^{-1}\int_0^t Z_2(s)\mathrm{d}s \leq \frac{r_2 - (\sigma_2^2/2)}{(b(1-\varepsilon))/m} \quad \text{a.s.}
\end{aligned} \tag{61}$$

From the arbitrariness of $\varepsilon$, one can get

$$\begin{aligned}
\limsup_{t\longrightarrow\infty} t^{-1}\int_0^t N_2(s)\mathrm{d}s &\leq \lim_{t\longrightarrow\infty} t^{-1}\int_0^t Z_2(s)\mathrm{d}s \\
&= \frac{m(r_2 - (\sigma_2^2/2))}{b} \quad \text{a.s.}
\end{aligned} \tag{62}$$

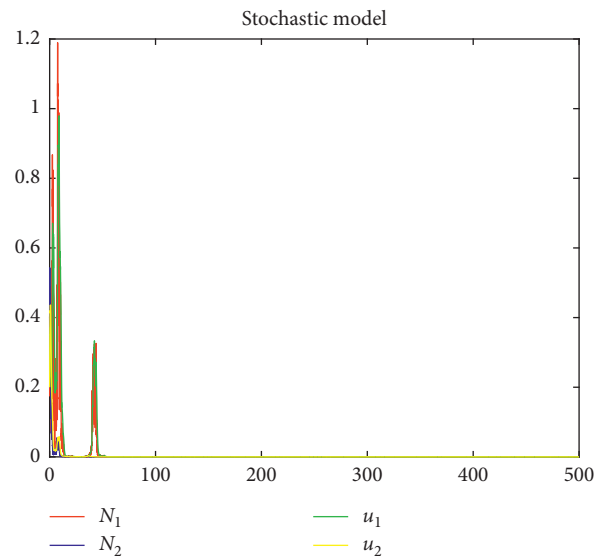


Figure 1: For populations $N_1$ and $N_2$ and feedback control variables $u_1$ and $u_2$, we choose the parameters as $f_1 = 2$, $f_2 = 2$, $g_1 = 3$, $g_2 = 3$, $c_1 = 0.3$, $c_2 = 0.3$, $\sigma_1 = 1.1$, and $\sigma_2 = 0.75$ and the initial value $(N_1(0), N_2(0), u_1(0), u_2(0)) = (0.4, 0.4, 0.2, 0.2)$.

Then, we have from the fourth equation in system (4)

$$\limsup_{t\longrightarrow\infty} t^{-1}\int_0^t u_2(s)\mathrm{d}s \leq \frac{g_2 m(r_2 - (\sigma_2^2/2))}{bf_2} \quad \text{a.s.} \tag{63}$$

Combining (40) with (63) leads to

$$\begin{aligned}
t^{-1}\ln N_2(t) &\geq t^{-1}\ln N_2(0) + \left(r_2 - \frac{\sigma_2^2}{2}\right) - t^{-1}\frac{b}{m}\int_0^t N_2(s)ds \\
&\quad - \frac{c_2 g_2 m(r_2 - (\sigma_2^2/2))}{bf_2} + t^{-1}M_2(t).
\end{aligned} \tag{64}$$

Considering $\Delta_2 = ((bf_2 - c_2 g_2 m)(r_2 - (\sigma_2^2/2)))/bf_2 > 0$, we can conclude

$$\liminf_{t\longrightarrow\infty} t^{-1}\int_0^t N_2(s)\mathrm{d}s \geq \frac{\Delta_2 m}{b} > 0 \quad \text{a.s.} \tag{65}$$

Combined with the fourth equation of system (4), we can obtain

$$\liminf t^{-1}\int_0^t u_2(s)\mathrm{d}s \geq \frac{\Delta_2 g_2 m}{bf_2} > 0 \quad \text{a.s.} \tag{66}$$

This completes the proof. □

## 5. Numerical Simulations

In this section, we use MATLAB to perform some numerical simulations to illustrate our main theoretical results by choosing the fixed parameters as follows: $a = 0.7$, $b = 0.7$, $m = 4$, $K = 0.5$, $r_1 = 0.4$, and $r_2 = 0.2$.

In Figure 1, we choose $f_1 = 2$, $f_2 = 2$, $g_1 = 3$, $g_2 = 3$, $c_1 = 0.3$, $c_2 = 0.3$, $\sigma_1 = 1.1$, and $\sigma_2 = 0.75$. And we calculate

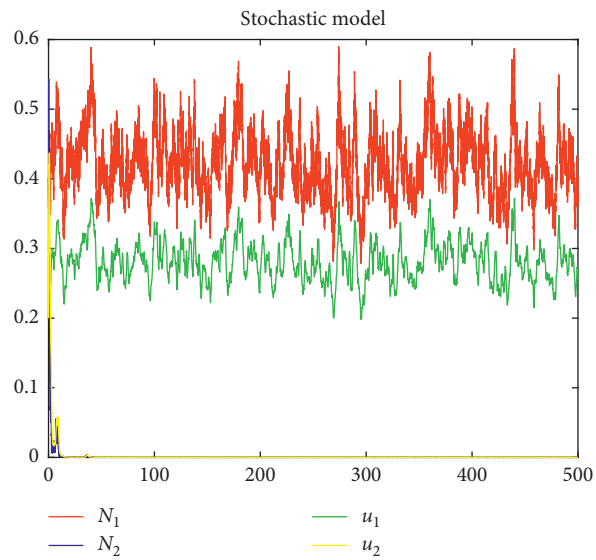


Figure 2: For populations $N_1$ and $N_2$ and feedback control variables $u_1$ and $u_2$, we choose the parameters as $f_1 = 3$, $f_2 = 2$, $g_1 = 2$, $g_2 = 3$, $c_1 = 0.2$, $c_2 = 0.3$, $\sigma_1 = 0.1$, and $\sigma_2 = 0.75$ and the initial value $(N_1(0), N_2(0), u_1(0), u_2(0)) = (0.4, 0.4, 0.2, 0.2)$.

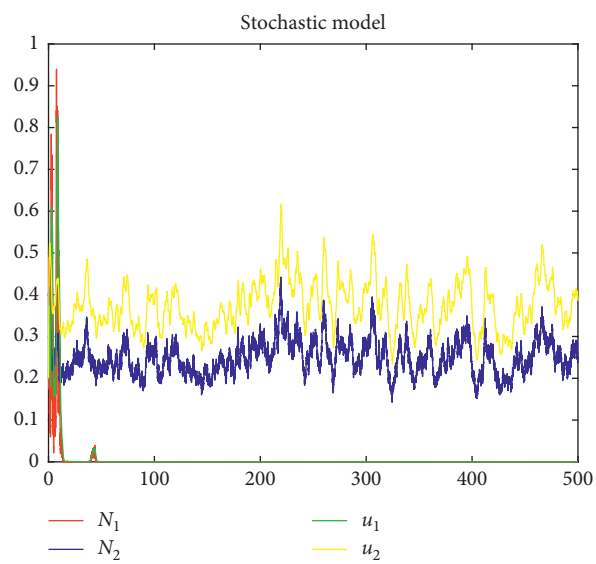


Figure 3: For populations $N_1$ and $N_2$ and feedback control variables $u_1$ and $u_2$, we choose the parameters as $f_1 = 2$, $f_2 = 3$, $g_1 = 3$, $g_2 = 2$, $c_1 = 0.3$, $c_2 = 0.2$, $\sigma_1 = 1.1$, and $\sigma_2 = 0.1$ and the initial value $(N_1(0), N_2(0), u_1(0), u_2(0)) = (0.4, 0.4, 0.2, 0.2)$.

that $r_1 - (\sigma_1^2/2) = -0.205 < 0$ and $r_2 - (\sigma_2^2/2) = -0.081 < 0$; in view of (i) of Theorem 3, the populations $N_1$ and $N_2$ are extinct, and the feedback control variables $u_1$ and $u_2$ are also extinct.

In Figure 2, we choose $f_1 = 3$, $f_2 = 2$, $g_1 = 2$, $g_2 = 3$, $c_1 = 0.2$, $c_2 = 0.3$, $\sigma_1 = 0.1$, and $\sigma_2 = 0.75$, where $r_1 - (\sigma_1^2/2) = 0.395 > 0$ and $r_2 - (\sigma_2^2/2) = -0.081 < 0$ and $\Delta_1 = 0.329 > 0$. By (ii) of Theorem 3, the population $N_2$ is extinct, and the population $N_1$ and the feedback control variable $u_1$ are persistent.

In Figure 3, we choose $f_1 = 2$, $f_2 = 3$, $g_1 = 3$, $g_2 = 2$, $c_1 = 0.3$, $c_2 = 0.2$, $\sigma_1 = 1.1$, and $\sigma_2 = 0.1$, where we have $r_2 - (\sigma_2^2/2) = 0.195 > 0$, $r_1 - (\sigma_1^2/2) = -0.205 < 0$, and $\Delta_2 = 0.121 > 0$. According to (iii) in Theorem 3, the population $N_1$ is extinct, and the population $N_2$ and the feedback control variable $u_2$ are persistent.

In Figure 4, we choose $f_1 = 2$, $f_2 = 2$, $g_1 = 3$, $g_2 = 3$, $c_1 = 0.3$, $c_2 = 0.3$, $\sigma_1 = 0.1$, and $\sigma_2 = 0.1$. Calculated by the MATLAB program, which satisfy the conditions $\Delta = \xi^2 - 4c_2g_2(c_1g_1K + r_1f_1)(ar_2f_1f_2K - mr_1f_1c_2g_2K - r_1f_1f_2b$

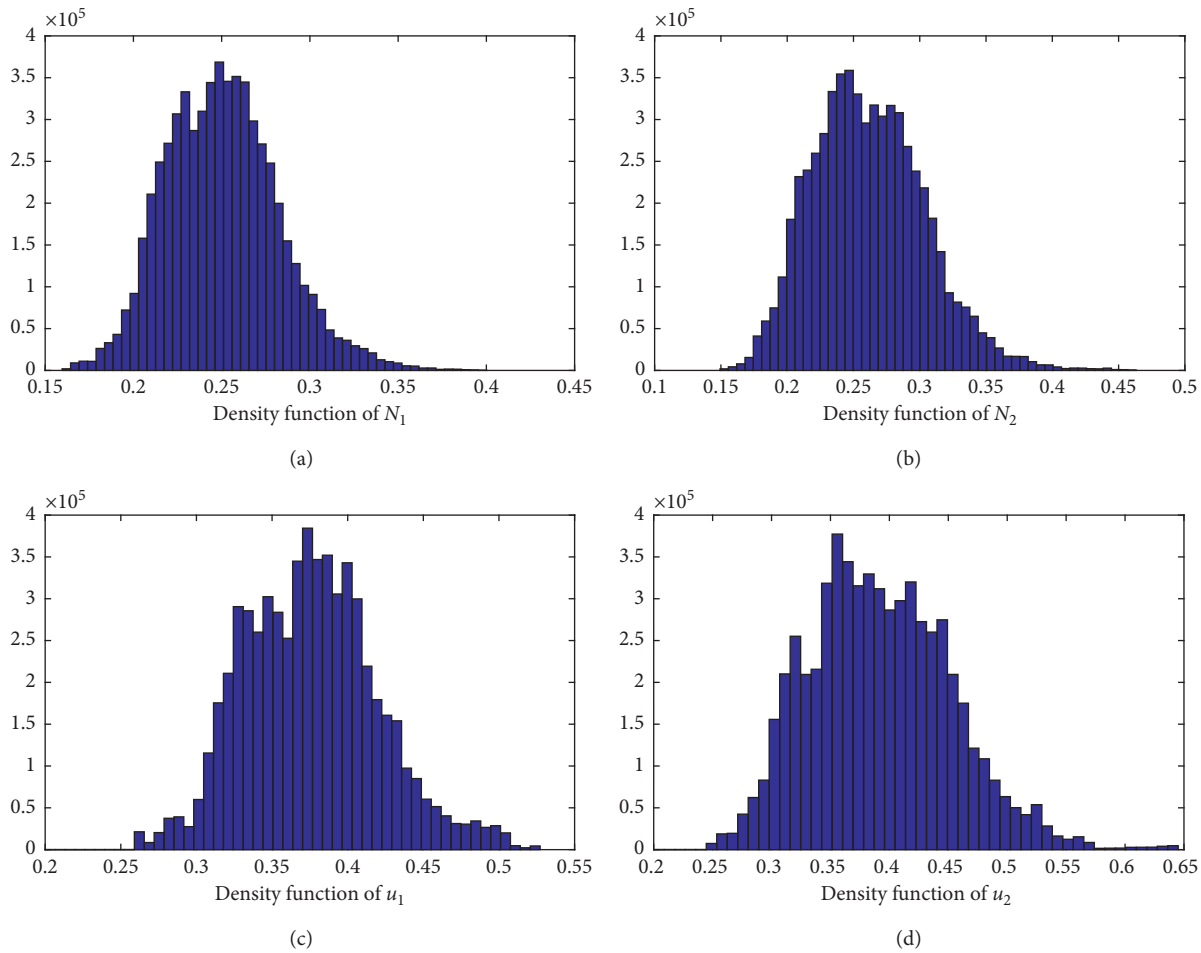


Figure 4: For populations $N_1$ and $N_2$ and feedback control variables $u_1$ and $u_2$, we choose the parameters as $f_1 = 2$, $f_2 = 2$, $g_1 = 3$, $g_2 = 3$, $c_1 = 0.3$, $c_2 = 0.3$, $\sigma_1 = 0.1$, and $\sigma_2 = 0.1$ and the initial value $(N_1(0), N_2(0), u_1(0), u_2(0)) = (0.4, 0.4, 0.2, 0.2)$.

$K) \geq 0$, $ar_2 f_2 < r_1(bf_2 + mc_2 g_2)$, and $(H1)$–(H3) of Theorem 2, we can get that the solution of system (4) is positive recurrence. And from Figure 4, we can find that $N_1$ and $N_2$ are persistent in mean.

## 6. Conclusions

This paper presents a stochastic Leslie–Gower predator-prey model with feedback controls and Holling type II functional response. Firstly, we prove the existence and uniqueness of global positive solution. Then, we discuss the existence of positive recurrence of system (4) by constructing appropriate Lyapunov functions. Moreover, we obtain the sufficient conditions for the persistence and extinction of populations in Theorem 3. The results imply that the white noise intensity has effects on dynamic behaviors of the system, and the larger white noise intensity will lead to the extinction of the biological population. In addition, the persistence and extinction of the populations will also be affected by the value of the feedback control variable coefficients.

## Data Availability

Data sharing is not applicable to this article as all data sets are hypothetical during the current study.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

## Acknowledgments

This work was supported by the Shandong Provincial Natural Science Foundation of China (no. ZR2019MA003) and SDUST Research Fund (no. 2014TDJH102).

## References

[1] P. H. Leslie, "Some further notes on the use of matrices in population mathematics," *Biometrika*, vol. 35, no. 3/4, pp. 213–245, 1948.

[2] S.-B. Hsu and T.-W. Huang, "Global stability for a class of predator-prey systems," *SIAM Journal on Applied Mathematics*, vol. 55, no. 3, pp. 763–783, 1995.

[3] T. Zhang, T. Xu, J. Wang, Y. Song, and Z. Jiang, "Geometrical analysis of a pest management model in food-limited environments with nonlinear impulsive state feedback control," *Journal of Applied Analysis & Computation*, vol. 9, no. 6, pp. 2261–2277, 2019.

[4] M. Chi and W. Zhao, "Dynamical analysis of two-microorganism and single nutrient stochastic chemostat model with monod-haldane response function," *Complexity*, vol. 2019, Article ID 8719067, 13 pages, 2019.

[5] Y.-Y. Li, X.-L. Zhuo, and F.-X. Zhang, "Multiperiodicity to a certain delayed predator-prey model," *Qualitative Theory of Dynamical Systems*, vol. 18, no. 3, pp. 793–811, 2019.

[6] M. A. Aziz-Alaoui and M. Daher Okiye, "Boundedness and global stability for a predator-prey model with modified Leslie-Gower and Holling-type II schemes," *Applied Mathematics Letters*, vol. 16, no. 7, pp. 1069–1075, 2003.

[7] F. Zhu, X. Meng, and T. Zhang, "Optimal harvesting of a competitive n-species stochastic model with delayed diffusions," *Mathematical Biosciences and Engineering*, vol. 16, no. 3, pp. 1554–1574, 2019.

[8] Y. Li and X. Meng, "Dynamics of an impulsive stochastic nonautonomous chemostat model with two different growth rates in a polluted environment," *Discrete Dynamics in Nature and Society*, vol. 2019, Article ID 5498569, 15 pages, 2019.

[9] S. Zhang, X. Meng, T. Feng, and T. Zhang, "Dynamics analysis and numerical simulations of a stochastic non-autonomous predator-prey system with impulsive effects," *Nonlinear Analysis: Hybrid Systems*, vol. 26, pp. 19–37, 2017.

[10] F. Bian, W. Zhao, Y. Song, and R. Yue, "Dynamical analysis of a class of prey-predator model with Beddington-DeAngelis functional response, stochastic perturbation, and impulsive toxicant input," *Complexity*, vol. 2017, Article ID 3742197, 18 pages, 2017.

[11] T. Feng, Z. Qiu, X. Meng, and L. Rong, "Analysis of a stochastic HIV-1 infection model with degenerate diffusion," *Applied Mathematics and Computation*, vol. 348, pp. 437–455, 2019.

[12] T. Ma, X. Meng, and Z. Chang, "Dynamics and optimal harvesting control for a stochastic one-predator-two-prey time delay system with jumps," *Complexity*, vol. 2019, Article ID 5342031, 19 pages, 2019.

[13] T. Feng, Z. Qiu, and X. Meng, "Analysis of a stochastic recovery-relapse epidemic model with periodic parameters and media coverage," *Journal of Applied Analysis and Computation*, vol. 9, no. 3, pp. 1007–1021, 2019.

[14] W. Zhao, J. Liu, M. Chi, and F. Bian, "Dynamics analysis of stochastic epidemic models with standard incidence," *Advances in Difference Equations*, vol. 2019, Article ID 22, 2019.

[15] W. Zhang, X. Meng, and Y. Dong, "Periodic solution and ergodic stationary distribution of stochastic SIRI epidemic systems with nonlinear perturbations," *Journal of Systems Science and Complexity*, vol. 32, no. 4, pp. 1104–1124, 2019.

[16] C. Ji, D. Jiang, and N. Shi, "Analysis of a predator-prey model with modified Leslie-Gower and Holling-type II schemes with stochastic perturbation," *Journal of Mathematical Analysis and Applications*, vol. 359, no. 2, pp. 482–498, 2009.

[17] X. Li, A. Gray, D. Jiang, and X. Mao, "Sufficient and necessary conditions of stochastic permanence and extinction for stochastic logistic populations under regime switching," *Journal of Mathematical Analysis and Applications*, vol. 376, no. 1, pp. 11–28, 2011.

[18] M. Liu and K. Wang, "Survival analysis of stochastic single-species population models in polluted environments," *Ecological Modelling*, vol. 220, no. 9-10, pp. 1347–1357, 2009.

[19] T. Feng, Z. Qiu, and X. Meng, "Dynamics of a stochastic hepatitis C virus system with host immunity," *Discrete & Continuous Dynamical Systems-B*, vol. 24, no. 12, pp. 6367–6385, 2019.

[20] T. Zhang, N. Gao, T. Wang et al., "Global dynamics of a model for treating microorganisms in sewage by periodically adding microbial flocculants," *Mathematical Biosciences and Engineering*, vol. 17, no. 1, pp. 179–201, 2020.

[21] Q. Liu and Q. Chen, "Analysis of a stochastic delay predator-prey system with jumps in a polluted environment," *Applied Mathematics and Computation*, vol. 242, pp. 90–100, 2014.

[22] Q. Liu, D. Jiang, T. Hayat, and A. Alsaedi, "Dynamics of a stochastic predator-prey model with stage structure for predator and Holling type II functional response," *Journal of Nonlinear Science*, vol. 28, no. 3, pp. 1151–1187, 2018.

[23] N. Gao, Y. Song, X. Wang, and J. Liu, "Dynamics of a stochastic SIS epidemic model with nonlinear incidence rates," *Advances in Difference Equations*, vol. 2019, Article ID 41, 2019.

[24] Y. Song, A. Miao, T. Zhang, X. Wang, and J. Liu, "Extinction and persistence of a stochastic SIRS epidemic model with saturated incidence rate and transfer from infectious to susceptible," *Advances in Difference Equations*, vol. 2018, Article ID 293, 2018.

[25] A. Miao, T. Zhang, J. Zhang, and C. Wang, "Dynamics of a stochastic SIR model with both horizontal and vertical transmission," *Journal of Applied Analysis and Computation*, vol. 8, no. 4, pp. 1108–1121, 2018.

[26] X. Mao, "Stationary distribution of stochastic population systems," *Systems & Control Letters*, vol. 60, no. 6, pp. 398–405, 2011.

[27] A. Lahrouz, A. Settati, and P. S. Mandal, "Dynamics of a switching diffusion modified Leslie-Gower predator-prey system with Beddington-DeAngelis functional response," *Nonlinear Dynamics*, vol. 85, no. 2, pp. 853–870, 2016.

[28] M. Liu, "Dynamics of a stochastic regime-switching predator-prey model with modified Leslie-Gower Holling-type II schemes and prey harvesting," *Nonlinear Dynamics*, vol. 96, no. 1, pp. 417–442, 2019.

[29] T. Zhang, X. Liu, X. Meng, and T. Zhang, "Spatio-temporal dynamics near the steady state of a planktonic system," *Computers & Mathematics with Applications*, vol. 75, no. 12, pp. 4490–4504, 2018.

[30] W. Wang, W. Ma, and Z. Feng, "Dynamics of reaction-diffusion equations for modeling CD4+ T cells decline with general infection mechanism and distinct dispersal rates," *Nonlinear Analysis: Real World Applications*, vol. 51, 2020.

[31] W. Lv, F. Wang, and Y. Li, "Adaptive finite-time tracking control for nonlinear systems with unmodeled dynamics using neural networks," *Advances in Difference Equations*, vol. 2018, Article ID 159, 2018.

[32] Z. Jiang, X. H. Bi, T. Q. Zhang et al., "Global hopf bifurcation of a delayed phytoplankton-zooplankton system considering toxin producing effect and delay dependent coefficient," *Mathematical Biosciences and Engineering*, vol. 16, no. 5, pp. 3807–3829, 2019.

[33] T. Zhang, T. Zhang, and X. Meng, "Stability analysis of a chemostat model with maintenance energy," *Applied Mathematics Letters*, vol. 68, pp. 1–7, 2017.

[34] E. Maltby, M. Holdgate, M. Acreman, and A. G. Weir, "Ecosystem management: questions for science and society,"

in *Ecosystem Management*, Royal Holloway University of London, Egham, UK, 1999.

[35] J. Yang and Y. Tan, "Effects of pesticide dose on Holling II predator-prey model with feedback control," *Journal of Biological Dynamics*, vol. 12, no. 1, pp. 527–550, 2018.

[36] Y.-H. Fan and L.-L. Wang, "Global asymptotical stability of a logistic model with feedback control," *Nonlinear Analysis: Real World Applications*, vol. 11, no. 4, pp. 2686–2697, 2010.

[37] F. Chen, X. Gong, L. Pu, and Z. Miao, "Dynamic behaviors of a Lotka-Volterra predator-prey system with feedback controls," *Journal of Biomathematics*, vol. 30, no. 2, pp. 328–332, 2015.

[38] J. Chen, T. Zhang, Z. Zhang et al., "Stability and output feedback control for singular markovian jump delayed systems," *Mathematical Control & Related Fields*, vol. 8, no. 2, pp. 475–490, 2018.

[39] W. Lv and F. Wang, "Adaptive tracking control for a class of uncertain nonlinear systems with infinite number of actuator failures using neural networks," *Advances in Difference Equations*, vol. 2017, Article ID 374, 2017.

[40] Y. Li, Y. Li, Y. Liu, and H. Cheng, "Stability analysis and control optimization of a prey-predator model with linear feedback control," *Discrete Dynamics in Nature and Society*, vol. 2018, Article ID 4945728, 12 pages, 2018.

[41] K. Gopalsamy and P. Weng, "Global attractivity in a competition system with feedback controls," *Computers & Mathematics with Applications*, vol. 45, no. 4-5, pp. 665–676, 2003.

[42] Z. Li, M. Han, and F. Chen, "Influence of feedback controls on an autonomous lotka-volterra competitive system with infinite delays," *Nonlinear Analysis: Real World Applications*, vol. 14, no. 1, pp. 402–413, 2013.

[43] R. Khasminskii, "Stochastic stability of differential equations," *vol. 66 of Stochastic Modelling and Applied Probability*, Springer-Verlag, Berlin, Germany, 2nd edition, 2012.

[44] M. Chi and W. Zhao, "Dynamical analysis of multi-nutrient and single microorganism chemostat model in a polluted environment," *Advances in Difference Equations*, vol. 2018, Article ID 120, 2018.