Hindawi
Mathematical Problems in Engineering
Volume 2019, Article ID 8792497, 14 pages
https://doi.org/10.1155/2019/8792497

Research Article

Global Dynamics of SIRS Model with No Full Immunity on Semidirected Networks

Jiawei Huo, Yimin Li, and Jing Hua

Faculty of Science, Jiangsu University, Zhenjiang 212013, China

Correspondence should be addressed to Yimin Li; llym@ujs.edu.cn

Received 21 June 2019; Accepted 3 October 2019; Published 23 October 2019

Academic Editor: Chris Goodrich

In this paper, an epidemic model with no full immunity is analyzed on semidirected networks. Directed networks led into previous scale-free networks, and we consider that some infectious diseases do not have full immunity. So we use strong self-protection instead of immunity and establish a semidirected network infectious disease model without full immunity. The basic reproduction number R_0 is calculated. If $R_0 < 1$, the disease-free equilibrium E^0 is locally and globally asymptotically stable. And the endemic equilibrium E^* is globally asymptotically stable in some condition. A large number of simulation results in this paper verify the correctness of the above conclusions and provide a solution for controlling disease transmission in the future.

1. Introduction

Based on mathematical models, especially the research on infectious diseases based on dynamics models has a history over 100 years. In 1873–1894, P D EN'KO established a model of modern mathematical infectious diseases [1]. In 1927, Kermack and McKendrick studied the SIR compartmental model after studying the Black Death and plague [2] and proposed the SIS compartmental model in 1932. Based on the compartmental model, threshold theory to distinguish whether epidemics exist was proposed [3]. After that, especially in the past 30 years, biomathematicians established and studied various infectious disease dynamic models for infectious disease compartmental models [4–6].

In recent years, with the rapid development of complex networks, many practical problems can be abstracted into complex network models for research. The infectious disease model is also applied to the scale-free network of complex networks. In 2001, Pastor-Satorras and Vespignani used the average field theory to study the SIS infectious disease model on the general network, applied it to the scale-free network, and proved that the scale-free network does not have a threshold under the appropriate parameters [7]. In the same year, May and Lloyd gave the basic regeneration number of the general network in the scale-free SIR model [8]. In 2002, Pastor-Satorras and Vespignani studied the transmission threshold of finite-scale network-free infectious diseases and proposed consistent immunity and optimized immunity for SIS infectious disease models [9]. In 2004, Liu et al. considered the dynamic network model of birth-death infectious disease with the static network [10]. In the same year, Hayashi et al. proposed the SIR virus propagation model under the linear growth scale-free network [11]. With the deep understanding of the network, this symmetry hypothesis is often not correct in the process of propagation research, such as mother-to-child transmission, virus transmission on computer networks, and information dissemination are dissymmetric. A large number of networks in nature involve directed networks [12–14]. Li et al. established a directed network propagation model [15]. In a real network, the contact between nodes is not all symmetric and asymmetric, but a situation of directed and undirected coexistence. Sharkey et al. established pair-level approximations to the spatiotemporal dynamics of epidemics on asymmetric contact networks [16]. Meyers et al. distinguished from previous work by using the probabilistic parent function method to study semidirected network propagation problems [17]. Zhang et al. studied the SIS model of semidirected networks and gave detailed dynamic analysis [18]. In many communication processes, not all communication processes gain immunity, such as the spread of mobile viruses [19], as do many diseases. However, although the node has not obtained immunity, it is less likely to be transmitted than before. To research this propagation, this paper establishes a new propagation model on a semidirected network. This article replaces immunity with strong self-protection, and the expected conclusion is closer to the actual situation. And Liu et al. [20] and Huang [21] has researched this propagation in the scale-free network but not on semidirected networks. This paper set up this propagation model on the semidirected network. This is a new propagation model. This study wants to use the model to find the methods and final range of the propagation.

Based on the aforementioned research results, this paper uses the idea of semidirected network and self-protection to establish a semidirected network epidemic model without full immunity, which can be used for the transmission of some certain diseases or computer viruses or mobile viruses. It may help to prevent the propagation of diseases in our daily living networks. And this study wants to research the methods of the propagation and the range affected by diseases. This paper systematically analyzes the dynamic properties of the model. The basic regeneration number and equilibrium point expression are calculated, and the stability of the equilibrium point is studied. The correctness of the result is proved by a large number of simulations, which can provide suggestions for the control of propagation.

2. Model Description

In this section, an epidemic model with no full immunity on semidirected complex networks is described. In the semidirected network, each node sends a directed or undirected connection to other nodes. The connection status of the node can be expressed by $(i, j; n)$, where i represents the in-degree, j represents the out-degree, and n represents the undirected degree. In the semidirected network, it is assumed that the disease propagates only along the out-degree edge and the undirected edge. The undirected degree and in-degree of each node indicate the possibility that the node will be propagated. The undirected degree and out-degree of each node indicate the possibility that the node propagates the disease to others. We use M_{in}, M_{out}, and M_{u} to represent the max in-degree, the max out-degree, and the max undirected degree, respectively. $N_{i,j;n}$ represents the number of nodes with in-degree i, out-degree j, and undirected degree n. $P(i, j; n)$ represents the possibility that the node with in-degree i, out-degree j, and undirected degree n is selected randomly.

The states of node in the propagation model are divided into S (susceptible), I (infected), and R (strong self-protection) states. Not all diseases will equip immune after healing, so this article introduces a strong self-protection state instead of immune status. The nodes which in state S or R may be infected by disease. And the probability of infection is represented by α_{S} and α_{R}. In order to reflect the self-protection of state R, we guarantee $\alpha_{\mathrm{S}} > \alpha_{\mathrm{R}}$. Using τ_{d} and τ_{u}, respectively, to indicate the probability the infected node will transmit disease to adjacent noninfected nodes through the directed and undirected edges. The susceptible nodes are transformed into strong self-protection nodes by the probability γ due to the influence of the surrounding strong self-protection nodes. The strong self-protection nodes are also transformed into susceptible nodes by probability η because it is not infected by disease for a long time. The recovery rate of the infected nodes is β. The number of susceptible nodes, infected nodes, and strong self-protection nodes with degree $(i, j; n)$ is recorded as $S_{(i,j;n)}$, $I_{(i,j;n)}$, and $R_{(i,j;n)}$, respectively. So $S_{(i,j;n)} + I_{(i,j;n)} + R_{(i,j;n)} = N_{(i,j;n)}$. $N_{(i,j;n)}$ is constant. We can express it by the relative density method. $S_{(i,j;n)}(t) + I_{(i,j;n)}(t) + R_{(i,j;n)}(t) = 1$. In order to reduce the length of the equation, we use $S_{(i,j;n)}$, $I_{(i,j;n)}$, and $R_{(i,j;n)}$ to replace $S_{(i,j;n)}(t)$, $I_{(i,j;n)}(t)$, and $R_{(i,j;n)}(t)$. Based on this, we can get the following dynamic model of the semidirected network:

$$\begin{aligned}
\frac{\mathrm{d}S_{(i,j;n)}}{\mathrm{d}t} &= -\alpha_{\mathrm{S}}\left(\tau_{\mathrm{d}} i S_{(i,j;n)}\Theta^{\mathrm{d}} + \tau_{\mathrm{u}} n S_{(i,j;n)}\Theta^{\mathrm{u}}\right) \\
&\quad + \eta R_{(i,j;n)} - \gamma S_{(i,j;n)}, \\
\frac{\mathrm{d}I_{(i,j;n)}}{\mathrm{d}t} &= \alpha_{\mathrm{S}}\left(\tau_{\mathrm{d}} i S_{(i,j;n)}\Theta^{\mathrm{d}} + \tau_{\mathrm{u}} n S_{(i,j;n)}\Theta^{\mathrm{u}}\right) \\
&\quad + \alpha_{\mathrm{R}}\left(\tau_{\mathrm{d}} i R_{(i,j;n)}\Theta^{\mathrm{d}} + \tau_{\mathrm{u}} n R_{(i,j;n)}\Theta^{\mathrm{u}}\right) - \beta I_{(i,j;n)}, \\
\frac{\mathrm{d}R_{(i,j;n)}}{\mathrm{d}t} &= -\alpha_{\mathrm{R}}\left(\tau_{\mathrm{d}} i R_{(i,j;n)}\Theta^{\mathrm{d}} + \tau_{\mathrm{u}} n R_{(i,j;n)}\Theta^{\mathrm{u}}\right) \\
&\quad - \eta R_{(i,j;n)} + \gamma S_{(i,j;n)} + \beta I_{(i,j;n)},
\end{aligned} \tag{1}$$

where $0 \le i \le M_{\mathrm{in}}$, $0 \le j \le M_{\mathrm{out}}$, $0 \le n \le M_{\mathrm{u}}$, Θ^{d} indicates the probability that the node in the network is connected to infected nodes through in-degree, and Θ^{u} indicates the probability that the node in the network is connected to infected nodes through undirected degree. Their mathematical expressions are as follows:

$$\begin{aligned}
\Theta^{\mathrm{d}} &= \frac{\sum jP(i, j; n) I_{(i,j;n)}(t)}{\langle d_{\mathrm{out}} \rangle}, \\
\Theta^{\mathrm{u}} &= \frac{\sum nP(i, j; n) I_{(i,j;n)}(t)}{\langle d_{\mathrm{u}} \rangle},
\end{aligned} \tag{2}$$

where $\mathrm{d}_{\mathrm{out}} = \sum jP(i, j; n)$, $d_{\mathrm{u}} = \sum nP(i, j; n)$, and $d_{\mathrm{in}} = \sum iP(i, j; n)$. Because this network is a semidirected network, $d_{\mathrm{out}} = d_{\mathrm{in}}$ is correct. The in-degree of the node corresponds to the out-degree of the other node.

3. Positive Invariant Set

Lemma 1. *System (1) has the positive invariant set:*

$$\Omega = \left\{ \left(S_{(1,1;1)}, I_{(1,1;1)}, S_{(1,1;2)}, I_{(1,1;2)}, \ldots, S_{(1,1;n)}, I_{(1,1;n)}, \ldots, S_{(M_{\text{in}},M_{\text{out}},M_{\text{u}})}, I_{(M_{\text{in}},M_{\text{out}},M_{\text{u}})} \right) \in R_{+} : S_{(i,j;n)} + I_{(i,j;n)} \le 1 \right\}. \quad (3)$$

Proof. Rewrite the above invariant set into the following form:

$$\Omega^{*} = \left\{ \left(z_1, z_2, \ldots, z_{2(i+j+n)} \right) \in R_{+} : z_{k+l+m} + z_{2(i+j+n)-k-l-m} \le 1 \right\}. \quad (4)$$

The boundary of Ω_{*} consists of the following three kinds of hyperplanes:

$$\begin{aligned} V_a &= \left\{ z \in \Omega^{*} \mid z_a = 0 \right\}, \\ H_b &= \left\{ z \in \Omega^{*} \mid z_b = 0 \right\}, \\ Q_a &= \left\{ z \in \Omega^{*} \mid z_a + z_b = 1 \right\}, \end{aligned} \quad (5)$$

where $a + b = 2(M_{\text{in}} + M_{\text{out}} + M_{\text{u}})$, $a, b > 0$ which have

$$\begin{aligned} \phi_{\text{a}} &= \left\{ 0, \ldots, 0, \overset{a}{-1}, 0, \ldots, 0 \right\}, \\ \varphi_{\text{b}} &= \left\{ 0, \ldots, 0, \overset{b}{-1}, 0, \ldots, 0 \right\}, \\ \psi_{\text{a}} &= \left\{ 0, \ldots, 0, \overset{a}{1}, 0, \ldots, \overset{b}{1}, \ldots, 0 \right\}, \end{aligned} \quad (6)$$

as their outer normal vectors, respectively.

Next, consider system (1), for $0 \le i \le M_{\text{in}}$, $0 \le j \le M_{\text{out}}$, and $0 \le n \le M_{\text{u}}$, calculations yield

$$\begin{aligned} \left(\frac{\mathrm{d}z}{\mathrm{d}t}\Big|_{z \in V_a, \phi_a} \right) &= -\eta \left(1 - z_b \right) < 0, \\ \left(\frac{\mathrm{d}z}{\mathrm{d}t}\Big|_{z \in H_b, \varphi_b} \right) &= -(\alpha_{\text{S}} - \alpha_{\text{R}}) z_b \left(\tau_{\text{d}} i \Theta^{\text{d}} + \tau_{\text{u}} m \Theta^{\text{u}} \right) \\ &\quad - \alpha_{\text{R}} \left(\tau_{\text{d}} i \Theta^{\text{d}} + \tau_{\text{u}} m \Theta^{\text{u}} \right) < 0, \\ \left(\frac{\mathrm{d}z}{\mathrm{d}t}\Big|_{z \in V_a, \phi_a} \right) &= -\gamma z_a - \beta z_b < 0. \end{aligned} \quad (7)$$

Consider a system $\mathrm{d}x/\mathrm{d}t = f(x)$ which is defined at least in a compact set C. Then, C is invariant if for every point y on ∂C (the boundary of C), the vector $f(y)$ is tangent to or pointing into C [22, 23]. According to this theorem, we can obtain the conclusion that Ω is positively invariant. □

4. Equilibria and Basic Reproduction Number

Obviously, system (1) has a disease-free equilibrium $E^{0}_{(i,j;n)} = (\eta/(\eta + r), 0, r/(\eta + r))$ for $0 \le i \le M_{\text{in}}$, $0 \le j \le M_{\text{out}}$, $0 \le n \le M_{\text{u}}$. So, the basic reproduction number R_0 can be calculated. Basic reproduction number is the expected number of secondary cases produced by a typical infected individual during its entire period of infectiousness in a completely susceptible population [24]. It can be clearly seen that system (1) is a closed system. According to the calculation method given in [25, 26], $R_0 = \rho(FV^{-1})$. After calculation, it can be concluded as follows:

$$V = \begin{bmatrix} -\beta & 0 & \cdots & 0 \\ 0 & -\beta & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -\beta \end{bmatrix} = -\beta E. \quad (8)$$

So, $V^{-1} = -(1/\beta)E$. The matrix F can be expressed in the following form:

$$F = \begin{bmatrix} F^{(1,1,1)}_{(1,1,1)} & \cdots & F^{(1,1,d_{\text{u}})}_{(1,1,1)} & \cdots & F^{(d_{\text{in}},d_{\text{out}},1)}_{(1,1,1)} & \cdots & F^{(d_{\text{in}},d_{\text{out}},d_{\text{u}})}_{(1,1,1)} \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ F^{(1,1,1)}_{(1,1,d_{\text{u}})} & \cdots & F^{(1,1,d_{\text{u}})}_{(1,1,d_{\text{u}})} & \cdots & F^{(d_{\text{in}},d_{\text{out}},1)}_{(1,1,d_{\text{u}})} & \cdots & F^{(d_{\text{in}},d_{\text{out}},d_{\text{u}})}_{(1,1,d_{\text{u}})} \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ F^{(1,1,1)}_{(d_{\text{in}},d_{\text{out}},1)} & \cdots & F^{(1,1,d_{\text{u}})}_{(d_{\text{in}},d_{\text{out}},1)} & \cdots & F^{(d_{\text{in}},d_{\text{out}},1)}_{(d_{\text{in}},d_{\text{out}},1)} & \cdots & F^{(d_{\text{in}},d_{\text{out}},d_{\text{u}})}_{(d_{\text{in}},d_{\text{out}},1)} \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ F^{(1,1,1)}_{(d_{\text{in}},d_{\text{out}},d_{\text{u}})} & \cdots & F^{(1,1,d_{\text{u}})}_{(d_{\text{in}},d_{\text{out}},d_{\text{u}})} & \cdots & F^{(d_{\text{in}},d_{\text{out}},1)}_{(d_{\text{in}},d_{\text{out}},d_{\text{u}})} & \cdots & F^{(d_{\text{in}},d_{\text{out}},d_{\text{u}})}_{(d_{\text{in}},d_{\text{out}},d_{\text{u}})} \end{bmatrix}, \quad (9)$$

where

$$F_{(k,l;m)}^{(i,j,n)} = \alpha_S\tau_d k\frac{\eta}{\eta+\gamma}\frac{jP(i,j;n)}{\langle d_{out}\rangle} + \alpha_S\tau_u m\frac{\eta}{\eta+\gamma}\frac{nP(i,j;n)}{\langle d_u\rangle} + \alpha_R\tau_d k\frac{\gamma}{\eta+\gamma}\frac{jP(i,j;n)}{\langle d_{out}\rangle} + \alpha_R\tau_u m\frac{\gamma}{\eta+\gamma}\frac{nP(i,j;n)}{\langle d_u\rangle}$$
$$= \frac{\alpha_S\eta+\alpha_R\gamma}{\eta+\gamma}\tau_d k\frac{jP(i,j;n)}{\langle d_{out}\rangle} + \frac{\alpha_S\eta+\alpha_R\gamma}{\eta+\gamma}\tau_u m\frac{nP(i,j;n)}{\langle d_u\rangle}$$
$$= \frac{\alpha_S\eta+\alpha_R\gamma}{\eta+\gamma}\left(\tau_d k\frac{jP(i,j;n)}{\langle d_{out}\rangle} + \tau_u m\frac{nP(i,j;n)}{\langle d_u\rangle}\right). \tag{10}$$

Bring the above results to the original matrix. $FV^{-1} = \left\{-(1/\beta)F_{(k,l;m)}^{(i,j,n)}\right\}$. Next, we calculate its spectral radius. After calculation, it is found that the matrix has a similar matrix $F_S = (\alpha_S\eta+\alpha_R\gamma)/(\eta+\gamma)\begin{bmatrix} F^* & 0 \\ 0 & 0\end{bmatrix}$. The F^* is

$$F^* = \begin{bmatrix} \tau_u\frac{\langle d_u^2\rangle}{\langle d_u\rangle} & \tau_u\frac{\langle d_{in}d_u\rangle}{\langle d_u\rangle} \\ \tau_d\frac{\langle d_{out}d_u\rangle}{\langle d_{out}\rangle} & \tau_d\frac{\langle d_{out}d_{in}\rangle}{\langle d_{out}\rangle}\end{bmatrix}, \tag{11}$$

where $\langle d_{in}d_{out}\rangle = \sum_{i,j,n} ijP(i,j;n)$, $\langle d_{in}d_u\rangle = \sum_{i,j,n} inP(i,j;n)$, and $\langle d_u d_{out}\rangle = \sum_{i,j,n} jnP(i,j;n)$. The characteristic polynomial of matrix F^* is

$$\lambda^2 - \left(\tau_d\frac{\langle d_{in}d_{out}\rangle}{\langle d_{out}\rangle} + \tau_u\frac{\langle d_u^2\rangle}{\langle d_u\rangle}\right)\lambda + \frac{\tau_u\tau_d}{\langle d_{out}\rangle\langle d_u\rangle}\left(\langle d_{in}d_{out}\rangle\langle d_u^2\rangle - \langle d_{in}d_u\rangle\langle d_u d_{out}\rangle\right) = 0. \tag{12}$$

The discriminant of the equation is

$$\Delta = \left(\tau_d\frac{\langle d_{in}d_{out}\rangle}{\langle d_{out}\rangle} + \tau_u\frac{\langle d_u^2\rangle}{\langle d_u\rangle}\right)^2 - 4\frac{\tau_u\tau_d}{\langle d_{out}\rangle\langle d_u\rangle}\left(\langle d_{in}d_{out}\rangle\langle d_u^2\rangle - \langle d_{in}d_u\rangle\langle d_u d_{out}\rangle\right)$$
$$= \left(\tau_d\frac{\langle d_{in}d_{out}\rangle}{\langle d_{out}\rangle} - \tau_u\frac{\langle d_u^2\rangle}{\langle d_u\rangle}\right)^2 + 4\tau_u\tau_d\frac{\langle d_{in}d_u\rangle\langle d_u d_{out}\rangle}{\langle d_{out}\rangle\langle d_u\rangle} > 0. \tag{13}$$

So, the matrix F^* has two unequal real roots. And the basic reproduction number of the model is

$$R_0 = \frac{\alpha_S\eta+\alpha_R\gamma}{2\beta(\eta+\gamma)}\left(\tau_d\frac{\langle d_{in}d_{out}\rangle}{\langle d_{out}\rangle} + \tau_u\frac{\langle d_u^2\rangle}{\langle d_u\rangle} + \sqrt{\Delta}\right), \tag{14}$$

where $\Delta = (\tau_d(\langle d_{in}d_{out}\rangle/\langle d_{out}\rangle) - \tau_u(\langle d_u^2\rangle/\langle d_u\rangle))^2 + 4\tau_u\tau_d((\langle d_{in}d_u\rangle\langle d_u d_{out}\rangle)/\langle d_{out}\rangle\langle d_u\rangle)$.

Lemma 2. *If $R_0 > 1$, then exists a endemic equilibrium $E^*(S^*_{(i,j;n)}, I^*_{(i,j;n)}, R^*_{(i,j;n)})$, where*

$$S^*_{(i,j;n)} = \frac{\eta\left(1-I^*_{(i,j;n)}\right)}{\gamma+\eta+\alpha_S\left(\tau_d i\Theta^{*d}+\tau_u n\Theta^{*u}\right)},$$
$$R^*_{(i,j;n)} = \frac{(\beta-\gamma)I^*_{(i,j;n)}+\gamma}{\gamma+\eta+\alpha_R\left(\tau_d i\Theta^{*d}+\tau_u n\Theta^{*u}\right)},$$
$$I^*_{(i,j;n)} = 1 - \frac{\beta}{\beta+\alpha_R\left(\tau_d i\Theta^{*d}+\tau_u n\Theta^{*u}\right)+\left(\left(\eta(\alpha_S-\alpha_R)\left(\tau_d i\Theta^{*d}+\tau_u n\Theta^{*u}\right)\right)/\left(\gamma+\eta+\alpha_S\left(\tau_d i\Theta^{*d}+\tau_u n\Theta^{*u}\right)\right)\right)},$$
$$\Theta^{*d} = \frac{1}{\langle d_{out}\rangle}\sum jP(i,j;n)I^*_{(i,j;n)},\quad \Theta^{*u} = \frac{1}{\langle d_u\rangle}\sum nP(i,j;n)I^*_{(i,j;n)}. \tag{15}$$

5. Dynamical Analysis of the Model

Theorem 1. *If $R_0 < 1$, the disease-free equilibrium E^0 of system (1) is stable. If $R_0 > 1$, E^0 is unstable.*

Theorem 2. *If $R_0 < 1$, the disease-free equilibrium E^0 of system (1) is globally asymptotically stable.*

Proof. In the feasible region, we consider the first equation of system (1):

$$\frac{dS_{(i,j;n)}(t)}{dt} = -\alpha_S\left(\tau_d iS_{(i,j;n)}(t)\Theta^d + \tau_u nS_{(i,j;n)}(t)\Theta^u\right) + \eta R_{(i,j;n)}(t) - \gamma S_{(i,j;n)}(t). \tag{16}$$

Obviously, $((dS_{(i,j;n)})/dt) \le \eta - (\eta+\gamma)S_{(i,j;n)}$. So, we can obtain $\lim_{t\to\infty}\sup S_{(i,j;n)} \le (\eta/(\eta+\gamma)) = S^0_{(i,j;n)}$ such that $S_{(i,j;n)} \le S^0_{(i,j;n)} + \xi_1$. Next, we analyze the second equation of system (1). When $t > T_1$,

$$\begin{aligned}
\frac{\mathrm{d}I_{(i,j;n)}}{\mathrm{d}t} &\le (\alpha_S-\alpha_R)\left(\tau_d i\Theta^d+\tau_u n\Theta^u\right)\left(S^0_{(i,j;n)}+\xi_1\right)-\beta I_{(i,j;n)}+\alpha_R\left(\tau_d i\Theta^d+\tau_u n\Theta^u\right)\left(1-I_{(i,j;n)}\right)\\
&\le (\alpha_S-\alpha_R)\left(\tau_d i\Theta^d+\tau_u n\Theta^u\right)\left(S^0_{(i,j;n)}+\xi_1\right)-\beta I_{(i,j;n)}+\alpha_R\left(\tau_d i\Theta^d+\tau_u n\Theta^u\right),\\
\frac{\mathrm{d}\left(\Theta^d+\Theta^u\right)}{\mathrm{d}t} &= \frac{\mathrm{d}\Theta^d}{\mathrm{d}t}+\frac{\mathrm{d}\Theta^u}{\mathrm{d}t}\\
&= \frac{1}{\langle d_{out}\rangle}\sum jP(i,j;n)\frac{\mathrm{d}I_{(i,j;n)}}{\mathrm{d}t}+\frac{1}{\langle d_u\rangle}\sum nP(i,j;n)\frac{\mathrm{d}I_{(i,j;n)}}{\mathrm{d}t}\\
&= \sum\left(\frac{j}{\langle d_{out}\rangle}+\frac{n}{\langle d_u\rangle}\right)P(i,j;n)\frac{\mathrm{d}I_{(i,j;n)}}{\mathrm{d}t}\\
&= \sum\left(\frac{j}{\langle d_{out}\rangle}+\frac{n}{\langle d_u\rangle}\right)P(i,j;n)\left[(\alpha_S-\alpha_R)\left(\tau_d i\Theta^d+\tau_u n\Theta^u\right)\left(S^0_{(i,j;n)}+\xi_1\right)\right]\\
&\quad+\sum\left(\frac{j}{\langle d_{out}\rangle}+\frac{n}{\langle d_u\rangle}\right)P(i,j;n)\left[\alpha_R\left(\tau_d i\Theta^d+\tau_u n\Theta^u\right)-\beta I_{(i,j;n)}\right]\\
&= \sum\left(\frac{j}{\langle d_{out}\rangle}+\frac{n}{\langle d_u\rangle}\right)P(i,j;n)(\alpha_S-\alpha_R)\left(\tau_d i\Theta^d+\tau_u n\Theta^u\right)\left(S^0_{(i,j;n)}+\xi_1\right)\\
&\quad+\sum\left(\frac{j}{\langle d_{out}\rangle}+\frac{n}{\langle d_u\rangle}\right)P(i,j;n)\alpha_R\left(\tau_d i\Theta^d+\tau_u n\Theta^u\right)-\sum\left(\frac{j}{\langle d_{out}\rangle}+\frac{n}{\langle d_u\rangle}\right)P(i,j;n)\beta I_{(i,j;n)}\\
&= \sum\left(\frac{j}{\langle d_{out}\rangle}+\frac{n}{\langle d_u\rangle}\right)P(i,j;n)\left[(\alpha_S-\alpha_R)\left(S^0_{(i,j;n)}+\xi_1\right)+\alpha_R\right]\left(\tau_d i\Theta^d+\tau_u n\Theta^u\right)-2\beta\left(\Theta^d+\Theta^u\right).
\end{aligned} \tag{17}$$

In this, $\sum((j/\langle d_{out}\rangle)+(n/\langle d_u\rangle))P(i,j;n)[(\alpha_S-\alpha_R)(S^0_{(i,j;n)}+\xi_1)+\alpha_R](\tau_d i\Theta^d+\tau_u n\Theta^u)$ can be divided into the following four parts:

The first part:

$$\begin{aligned}
&\Theta^d\tau_d\sum\frac{ij}{\langle d_{out}\rangle}P(i,j;n)\left[(\alpha_S-\alpha_R)\left(S^0_{(i,j;n)}+\xi_1\right)+\alpha_R\right]\\
&= \Theta^d\tau_d\sum\frac{ij}{\langle d_{out}\rangle}P(i,j;n)\left[(\alpha_S-\alpha_R)\frac{\eta}{\eta+\gamma}+\alpha_R+(\alpha_S-\alpha_R)\xi_1\right]\\
&= \Theta^d\tau_d\sum\frac{ij}{\langle d_{out}\rangle}P(i,j;n)\left[\frac{\eta}{\eta+\gamma}\alpha_S+\frac{\gamma}{\eta+\gamma}\alpha_R+(\alpha_S-\alpha_R)\xi_1\right]\\
&= \Theta^d\frac{\eta\alpha_S+\gamma\alpha_R}{\eta+\gamma}\tau_d\sum\frac{ij}{\langle d_{out}\rangle}P(i,j;n)+\varepsilon_1\\
&= \Theta^d\frac{\eta\alpha_S+\gamma\alpha_R}{\eta+\gamma}\tau_d\frac{\langle d_{in}d_{out}\rangle}{\langle d_{out}\rangle}+\varepsilon_1.
\end{aligned} \tag{18}$$

The second part:

$$\begin{aligned}
&\Theta^u\tau_u\sum\frac{n^2}{\langle d_u\rangle}P(i,j;n)\left[(\alpha_S-\alpha_R)\left(S^0_{(i,j;n)}+\xi_1\right)+\alpha_R\right]\\
&= \Theta^u\tau_u\sum\frac{n^2}{\langle d_u\rangle}P(i,j;n)\left[\frac{\eta\alpha_S+\gamma\alpha_R}{\eta+\gamma}+(\alpha_S-\alpha_R)\xi_1\right]\\
&= \Theta^u\frac{\eta\alpha_S+\gamma\alpha_R}{\eta+\gamma}\tau_u\frac{\langle d_u^2\rangle}{\langle d_u\rangle}+\varepsilon_2.
\end{aligned} \tag{19}$$

The third part:

$$\begin{aligned}
&\Theta^d\tau_d\sum\frac{in}{\langle d_u\rangle}P(i,j;n)\left[(\alpha_S-\alpha_R)\left(S^0_{(i,j;n)}+\xi_1\right)+\alpha_R\right]\\
&= \Theta^d\tau_d\sum\frac{in}{\langle d_u\rangle}P(i,j;n)\left[\frac{\eta\alpha_S+\gamma\alpha_R}{\eta+\gamma}+(\alpha_S-\alpha_R)\xi_1\right]\\
&= \Theta^d\frac{\eta\alpha_S+\gamma\alpha_R}{\eta+\gamma}\tau_d\frac{\langle d_{in}d_u\rangle}{\langle d_u\rangle}+\varepsilon_3.
\end{aligned} \tag{20}$$

The fourth part:

$$\begin{aligned}
&\Theta^{u}\tau_{u}\sum\frac{jn}{\langle d_{u}\rangle}P(i,j;n)\left[(\alpha_{S}-\alpha_{R})\left(S^{0}_{(i,j;n)}+\xi_{1}\right)+\alpha_{R}\right]\\
&\quad=\Theta^{u}\tau_{u}\sum\frac{jn}{\langle d_{u}\rangle}P(i,j;n)\left[\frac{\eta\alpha_{S}+\gamma\alpha_{R}}{\eta+\gamma}+(\alpha_{S}-\alpha_{R})\xi_{1}\right]\\
&\quad=\Theta^{u}\frac{\eta\alpha_{S}+\gamma\alpha_{R}}{\eta+\gamma}\tau_{u}\frac{\langle d_{out}d_{u}\rangle}{\langle d_{out}\rangle}+\varepsilon_{4}.
\end{aligned}\tag{21}$$

So,

$$\begin{aligned}
\frac{d\left(\Theta^{d}+\Theta^{u}\right)}{dt}&=\frac{d\Theta^{d}}{dt}+\frac{d\Theta^{u}}{dt}\\
&=\frac{\eta\alpha_{S}+\gamma\alpha_{R}}{\eta+\gamma}\left(\Theta^{d}\tau_{d}\frac{\langle d_{in}d_{out}\rangle}{\langle d_{out}\rangle}+\Theta^{u}\tau_{u}\frac{\langle d_{u}^{2}\rangle}{\langle d_{u}\rangle}\right.\\
&\quad\left.+\Theta^{d}\tau_{d}\frac{\langle d_{in}d_{u}\rangle}{\langle d_{u}\rangle}+\Theta^{u}\tau_{u}\frac{\langle d_{out}d_{u}\rangle}{\langle d_{out}\rangle}+\varepsilon\right)\\
&\quad-2\beta\left(\Theta^{d}+\Theta^{u}\right)\\
&=2\beta\left(\Theta^{d}+\Theta^{u}\right)\left[R_{0}+\varepsilon^{*}-1\right],
\end{aligned}\tag{22}$$

where $\varepsilon^{*}=\varepsilon_{1}+\varepsilon_{2}+\varepsilon_{3}+\varepsilon_{4}$. In order to prove that the last step of the above formula is established, it proves that $(\Theta^{d}+\Theta^{u})R_{0}-4\,\text{parts}=0$ is established. Next, we prove it.

$$\begin{aligned}
&\left(\Theta^{d}+\Theta^{u}\right)R_{0}-4\,\text{parts}\\
&\quad=\frac{\eta\alpha_{S}+\gamma\alpha_{R}}{\eta+\gamma}\left[\left(\Theta^{d}+\Theta^{u}\right)\left(\tau_{d}\frac{\langle d_{in}d_{out}\rangle}{\langle d_{out}\rangle}+\tau_{u}\frac{\langle d_{u}^{2}\rangle}{\langle d_{u}\rangle}+\sqrt{\Delta}\right)\right.\\
&\qquad-\Theta^{d}\tau_{d}\frac{\langle d_{in}d_{out}\rangle}{\langle d_{out}\rangle}-\Theta^{u}\tau_{u}\frac{\langle d_{u}^{2}\rangle}{\langle d_{u}\rangle}-\Theta^{d}\tau_{d}\frac{\langle d_{in}d_{u}\rangle}{\langle d_{u}\rangle}\\
&\qquad\left.-\Theta^{u}\tau_{u}\frac{\langle d_{out}d_{u}\rangle}{\langle d_{out}\rangle}\right]\\
&\quad=\Theta^{u}\tau_{d}\frac{\langle d_{in}d_{out}\rangle}{\langle d_{out}\rangle}+\Theta^{d}\tau_{u}\frac{\langle d_{u}^{2}\rangle}{\langle d_{u}\rangle}+\left(\Theta^{d}+\Theta^{u}\right)\sqrt{\Delta}\\
&\qquad-\Theta^{d}\tau_{d}\frac{\langle d_{in}d_{u}\rangle}{\langle d_{u}\rangle}-\Theta^{u}\tau_{u}\frac{\langle d_{out}d_{u}\rangle}{\langle d_{out}\rangle}.
\end{aligned}\tag{23}$$

To prove that the formula is 0. That is, satisfy

$$\begin{aligned}
&\left(\Theta^{u}\tau_{d}\frac{\langle d_{in}d_{out}\rangle}{\langle d_{out}\rangle}+\Theta^{d}\tau_{u}\frac{\langle d_{u}^{2}\rangle}{\langle d_{u}\rangle}-\Theta^{d}\tau_{d}\frac{\langle d_{in}d_{u}\rangle}{\langle d_{u}\rangle}\right.\\
&\quad\left.-\Theta^{u}\tau_{u}\frac{\langle d_{out}d_{u}\rangle}{\langle d_{out}\rangle}\right)^{2}=\left(\Theta^{d}+\Theta^{u}\right)^{2}\Delta.
\end{aligned}\tag{24}$$

After simplification, we can obtain

$$\begin{aligned}
&\left[\left(\langle d_{in}d_{out}\rangle\langle d_{u}\rangle\tau_{d}-\langle d_{out}\rangle\langle d_{u}^{2}\rangle\tau_{u}\right)^{2}\right.\\
&\quad\left.-\left(\langle d_{in}d_{u}\rangle\langle d_{out}\rangle\tau_{d}-\langle d_{u}d_{out}\rangle\langle d_{u}\rangle\tau_{u}\right)\right](*)=0.
\end{aligned}\tag{25}$$

According to abovementioned equation,

$$\begin{aligned}
&\left(\langle d_{in}d_{out}\rangle\langle d_{u}\rangle\tau_{d}-\langle d_{out}\rangle\langle d_{u}^{2}\rangle\tau_{u}\right)^{2}\\
&\quad-\left(\langle d_{in}d_{u}\rangle\langle d_{out}\rangle\tau_{d}-\langle d_{u}d_{out}\rangle\langle d_{u}\rangle\tau_{u}\right)=0.
\end{aligned}\tag{26}$$

So $(d(\Theta^{d}+\Theta^{u}))/dt\le 2\beta(\Theta^{d}+\Theta^{u})[R_{0}+\varepsilon^{*}-1]$ is correct. It guarantees that if $R_{0}<1$, $((d(\Theta^{d}+\Theta^{u}))/dt)<0$. Hence, $\lim_{t\longrightarrow+\infty}(\Theta^{d}+\Theta^{u})=0$ and $\lim_{t\longrightarrow+\infty}I_{(i,j;n)}=0$ for all $1\le i\le M_{in}$, $1\le i\le M_{out}$, and $1\le n\le M_{u}$. For any ξ_{2}, there exist $T_{2}>0$ such that $0\le I_{(i,j;n)}\le\xi_{2}$. According to system (1), we can obtain

$$\frac{dS_{(i,j;n)}}{dt}\ge\eta-(\eta+\gamma)S_{(i,j;n)}-\left(\eta+\alpha_{S}\left(\tau_{d}iS_{(i,j;n)}\Theta^{d}+\tau_{u}nS_{(i,j;n)}\Theta^{u}\right)\right)\xi_{2}.\tag{27}$$

Setting $\xi_{2}\longrightarrow 0$, then $(dS_{(i,j;n)}/dt)\ge\eta-(\eta+\gamma)S_{(i,j;n)}$. So, $\lim_{t\longrightarrow\infty}\sup S_{(i,j;n)}\ge(\eta/(\eta+\gamma))=S^{0}_{(i,j;n)}$. And, we have obtained $\lim_{t\longrightarrow\infty}\sup S_{(i,j;n)}\le\eta/(\eta+\gamma)=S^{0}_{(i,j;n)}$. According to $\lim_{t\longrightarrow\infty}S_{(i,j;n)}=(\eta/(\eta+\gamma))=S^{0}_{(i,j;n)}$ and $\lim_{t\longrightarrow\infty}I_{(i,j;n)}=0=I^{0}_{(i,j;n)}$ for $0\le i\le M_{in}$, $0\le i\le M_{out}$, and $0\le n\le M_{u}$, the disease-free equilibrium E^{0} is globally asymptotically stable. □

Theorem 3. *If $R_{0}>1$, the endemic equilibrium E^{*} of system (1) is globally asymptotically stable provided that $2\sqrt{\gamma(\eta+\beta)}>\gamma+\eta-\beta$.*

Proof. Let us consider the following four positive definite functions, which is defined along a solution of system (1):

$$\begin{aligned}
V_{S(i,j;n)}&=\frac{1}{2}\left(S_{(i,j;n)}-S^{*}_{(i,j;n)}\right)^{2},\\
V_{R(i,j;n)}&=\frac{1}{2}\left(R_{(i,j;n)}-R^{*}_{(i,j;n)}\right)^{2},\\
V_{\Theta^{d}}&=\Theta^{d}-\Theta^{*d}-\Theta^{*d}\ln\frac{\Theta^{d}}{\Theta^{*d}},\\
V_{\Theta^{u}}&=\Theta^{u}-\Theta^{*u}-\Theta^{*u}\ln\frac{\Theta^{u}}{\Theta^{*u}}.
\end{aligned}\tag{28}$$

The following solves the function for the differential form:

$$\begin{aligned}
\frac{\mathrm{d}V_{\mathrm{S}(i,j;n)}}{\mathrm{d}t} &= \left(S_{(i,j;n)} - S^{*}_{(i,j;n)}\right)\frac{\mathrm{d}S_{(i,j;n)}}{\mathrm{d}t} \\
&= \left(S_{(i,j;n)} - S^{*}_{(i,j;n)}\right)\left[-\alpha_{\mathrm{S}}\left(\tau_{\mathrm{d}} i S_{(i,j;n)}\Theta^{\mathrm{d}} + \tau_{\mathrm{u}} n S_{(i,j;n)}\Theta^{\mathrm{u}}\right) + \eta R_{(i,j;n)} - \gamma S_{(i,j;n)}\right] \\
&= \left(S_{(i,j;n)} - S^{*}_{(i,j;n)}\right)\left[-\alpha_{\mathrm{S}}\tau_{\mathrm{d}} i\left(S_{(i,j;n)}\Theta^{\mathrm{d}} - S^{*}_{(i,j;n)}\Theta^{*\mathrm{d}}\right) - \alpha_{\mathrm{S}}\tau_{\mathrm{u}} n\left(S_{(i,j;n)}\Theta^{\mathrm{u}} - S^{*}_{(i,j;n)}\Theta^{*\mathrm{u}}\right) + \eta\left(R_{(i,j;n)} - R^{*}_{(i,j;n)}\right) - \gamma\left(S_{(i,j;n)} - S^{*}_{(i,j;n)}\right)\right] \\
&= \gamma\left(S_{(i,j;n)} - S^{*}_{(i,j;n)}\right)^{2} + \eta\left(R_{(i,j;n)} - R^{*}_{(i,j;n)}\right)\left(S_{(i,j;n)} - S^{*}_{(i,j;n)}\right) - \alpha_{\mathrm{S}}\tau_{\mathrm{d}} i\Theta^{\mathrm{d}}\left(S_{(i,j;n)} - S^{*}_{(i,j;n)}\right)^{2} \\
&\quad - \alpha_{\mathrm{S}}\tau_{\mathrm{u}} n\Theta^{\mathrm{u}}\left(S_{(i,j;n)} - S^{*}_{(i,j;n)}\right)^{2} - \alpha_{\mathrm{S}}\tau_{\mathrm{d}} i S^{*}_{(i,j;n)}\left(S_{(i,j;n)} - S^{*}_{(i,j;n)}\right)\left(\Theta^{\mathrm{d}} - \Theta^{*\mathrm{d}}\right) \\
&\quad - \alpha_{\mathrm{S}}\tau_{\mathrm{u}} n S^{*}_{(i,j;n)}\left(S_{(i,j;n)} - S^{*}_{(i,j;n)}\right)\left(\Theta^{\mathrm{u}} - \Theta^{*\mathrm{u}}\right), \\
\frac{\mathrm{d}V_{\mathrm{R}(i,j;n)}}{\mathrm{d}t} &= \left(R_{(i,j;n)} - R^{*}_{(i,j;n)}\right)\frac{\mathrm{d}R_{(i,j;n)}}{\mathrm{d}t} \\
&= \left(R_{(i,j;n)} - R^{*}_{(i,j;n)}\right)\left[-\alpha_{\mathrm{R}}\left(\tau_{\mathrm{d}} k R_{(i,j;n)}\Theta^{\mathrm{d}} + \tau_{\mathrm{u}} m R_{(i,j;n)}\Theta^{\mathrm{u}}\right) - \eta R_{(i,j;n)} + \gamma S_{(i,j;n)} + \beta I_{(i,j;n)}\right] \\
&= \left(R_{(i,j;n)} - R^{*}_{(i,j;n)}\right)\left[-\alpha_{\mathrm{R}}\left(\tau_{\mathrm{d}} i R_{(i,j;n)}\Theta^{\mathrm{d}} + \tau_{\mathrm{u}} n R_{(i,j;n)}\Theta^{\mathrm{u}}\right) - (\eta+\beta)R_{(i,j;n)} + (\gamma-\beta)S_{(i,j;n)} + \beta\right] \\
&= \left(R_{(i,j;n)} - R^{*}_{(i,j;n)}\right) - \Big[\alpha_{\mathrm{R}}\tau_{\mathrm{d}} i\left(R_{(i,j;n)}\Theta^{\mathrm{d}} - R^{*}_{(i,j;n)}\Theta^{*\mathrm{d}}\right) - \alpha_{\mathrm{R}}\tau_{\mathrm{u}} n\left(R_{(i,j;n)}\Theta^{\mathrm{u}} - R^{*}_{(i,j;n)}\Theta^{*\mathrm{u}}\right) - (\eta+\beta)\left(R_{(i,j;n)} - R^{*}_{(i,j;n)}\right) \\
&\quad + (\gamma-\beta)\left(S_{(i,j;n)} - S^{*}_{(i,j;n)}\right)\Big] \\
&= -(\eta+\beta)\left(R_{(i,j;n)} - R^{*}_{(i,j;n)}\right)^{2} + (\gamma-\beta)\left(S_{(i,j;n)} - S^{*}_{(i,j;n)}\right)\left(R_{(i,j;n)} - R^{*}_{(i,j;n)}\right) - \alpha_{\mathrm{R}}\tau_{\mathrm{d}} i\Theta^{\mathrm{d}}\left(R_{(i,j;n)} - R^{*}_{(i,j;n)}\right)^{2} \\
&\quad - \alpha_{\mathrm{R}}\tau_{\mathrm{u}} n\Theta^{\mathrm{u}}\left(R_{(i,j;n)} - R^{*}_{(i,j;n)}\right)^{2} - \alpha_{\mathrm{R}}\tau_{\mathrm{d}} i R^{*}_{(i,j;n)}\left(R_{(i,j;n)} - R^{*}_{(i,j;n)}\right)\left(\Theta^{\mathrm{d}} - \Theta^{*\mathrm{d}}\right) \\
&\quad - \alpha_{\mathrm{R}}\tau_{\mathrm{u}} n R^{*}_{(i,j;n)}\left(R_{(i,j;n)} - R^{*}_{(i,j;n)}\right)\left(\Theta^{\mathrm{u}} - \Theta^{*\mathrm{u}}\right), \\
\frac{\mathrm{d}V_{\Theta^{\mathrm{d}}}}{\mathrm{d}t} &= \frac{\Theta^{\mathrm{d}} - \Theta^{*\mathrm{d}}}{\Theta^{\mathrm{d}}}\frac{\mathrm{d}\Theta^{\mathrm{d}}}{\mathrm{d}t} \\
&= \frac{\Theta^{\mathrm{d}} - \Theta^{*\mathrm{d}}}{\Theta^{\mathrm{d}}}\frac{1}{\langle d_{\mathrm{out}}\rangle}\sum jP(i,j;n)\alpha_{\mathrm{S}}\left(\tau_{\mathrm{d}} i S_{(i,j;n)}\Theta^{\mathrm{d}} + \tau_{\mathrm{u}} n S_{(i,j;n)}\Theta^{\mathrm{u}}\right) \\
&\quad + \frac{\Theta^{\mathrm{d}} - \Theta^{*\mathrm{d}}}{\Theta^{\mathrm{d}}}\frac{1}{\langle d_{\mathrm{out}}\rangle}\sum jP(i,j;n)\left[\alpha_{\mathrm{R}}\left(\tau_{\mathrm{d}} i R_{(i,j;n)}\Theta^{\mathrm{d}} + \tau_{\mathrm{u}} n R_{(i,j;n)}\Theta^{\mathrm{u}}\right) - \beta I_{(i,j;n)}\right] \\
&= \frac{\Theta^{\mathrm{d}} - \Theta^{*\mathrm{d}}}{\Theta^{\mathrm{d}}}\frac{1}{\langle d_{\mathrm{out}}\rangle}\sum jP(i,j;n)\alpha_{\mathrm{S}}\left(\tau_{\mathrm{d}} i S_{(i,j;n)}\Theta^{\mathrm{d}} + \tau_{\mathrm{u}} n S_{(i,j;n)}\Theta^{\mathrm{u}}\right) \\
&\quad + \frac{\Theta^{\mathrm{d}} - \Theta^{*\mathrm{d}}}{\Theta^{\mathrm{d}}}\left[\frac{1}{\langle d_{\mathrm{out}}\rangle}\sum jP(i,j;n)\alpha_{\mathrm{R}}\left(\tau_{\mathrm{d}} i R_{(i,j;n)}\Theta^{\mathrm{d}} + \tau_{\mathrm{u}} n R_{(i,j;n)}\Theta^{\mathrm{u}}\right) - \beta\Theta^{\mathrm{d}}\right] \\
&= \left(\Theta^{\mathrm{d}} - \Theta^{*\mathrm{d}}\right)\frac{1}{\langle d_{\mathrm{out}}\rangle}\sum jP(i,j;n)\alpha_{\mathrm{S}}\left(\tau_{\mathrm{d}} i S_{(i,j;n)} + \tau_{\mathrm{u}} n S_{(i,j;n)}\frac{\Theta^{\mathrm{u}}}{\Theta^{\mathrm{d}}}\right) \\
&\quad + \left(\Theta^{\mathrm{d}} - \Theta^{*\mathrm{d}}\right)\left[\frac{1}{\langle d_{\mathrm{out}}\rangle}\sum jP(i,j;n)\alpha_{\mathrm{R}}\left(\tau_{\mathrm{d}} i R_{(i,j;n)} + \tau_{\mathrm{u}} n R_{(i,j;n)}\frac{\Theta^{\mathrm{u}}}{\Theta^{\mathrm{d}}}\right) - \beta\right] \\
&= \frac{1}{\langle d_{\mathrm{out}}\rangle}\sum jP(i,j;n)\Bigg[\alpha_{\mathrm{S}}\tau_{\mathrm{d}} i\left(S_{(i,j;n)} - S^{*}_{(i,j;n)}\right)\left(\Theta^{\mathrm{d}} - \Theta^{*\mathrm{d}}\right) + \alpha_{\mathrm{S}}\tau_{\mathrm{u}} n\left(S_{(i,j;n)}\frac{\Theta^{\mathrm{u}}}{\Theta^{\mathrm{d}}} - S^{*}_{(i,j;n)}\frac{\Theta^{*\mathrm{u}}}{\Theta^{*\mathrm{d}}}\left(\Theta^{\mathrm{d}} - \Theta^{*\mathrm{d}}\right)\right) \\
&\quad + \alpha_{\mathrm{S}}\tau_{\mathrm{d}} i\left(R_{(i,j;n)} - R^{*}_{(i,j;n)}\right)\left(\Theta^{\mathrm{d}} - \Theta^{*\mathrm{d}}\right) + \alpha_{\mathrm{S}}\tau_{\mathrm{u}} n\left(R_{(i,j;n)}\frac{\Theta^{\mathrm{u}}}{\Theta^{\mathrm{d}}} - R^{*}_{(i,j;n)}\frac{\Theta^{*\mathrm{u}}}{\Theta^{*\mathrm{d}}}\right)\left(\Theta^{\mathrm{d}} - \Theta^{*\mathrm{d}}\right)\Bigg].
\end{aligned} \tag{29}$$

For the same reason,

$$\frac{dV_{\Theta^{u}}}{dt}=\frac{1}{\langle d_{u}\rangle}\sum nP(i,j;n)\left[\alpha_{S}\tau_{d}i\left(S_{(i,j;n)}\frac{\Theta^{d}}{\Theta^{u}}-S^{*}_{(i,j;n)}\frac{\Theta^{*d}}{\Theta^{*u}}\right)\left(\Theta^{u}-\Theta^{*u}\right)+\alpha_{S}\tau_{u}n\left(S_{(i,j;n)}-S^{*}_{(i,j;n)}\right)\left(\Theta^{u}-\Theta^{*u}\right)\right.$$
$$\left.+\alpha_{S}\tau_{d}i\left(R_{(i,j;n)}\frac{\Theta^{d}}{\Theta^{u}}-R^{*}_{(i,j;n)}\frac{\Theta^{*d}}{\Theta^{*u}}\right)+\alpha_{S}\tau_{u}n\left(R_{(i,j;n)}-R^{*}_{(i,j;n)}\right)\left(\Theta^{u}-\Theta^{*u}\right)\right]. \tag{30}$$

Let us consider the following Lyapunov function:

$$V=\sum\left(\frac{j}{\langle d_{out}\rangle}+\frac{n}{\langle d_{u}\rangle}\right)V_{s(i,j;n)}+\sum\left(\frac{j}{\langle d_{out}\rangle}+\frac{n}{\langle d_{u}\rangle}\right)V_{S(i,j;n)}+V_{\Theta^{d}}+V_{\Theta^{u}},$$
$$\frac{dV}{dt}=\sum\left(\frac{j}{\langle d_{out}\rangle}+\frac{n}{\langle d_{u}\rangle}\right)\left[\gamma\left(S_{(i,j;n)}-S^{*}_{(i,j;n)}\right)^{2}+\eta\left(R_{(i,j;n)}-R^{*}_{(i,j;n)}\right)\left(S_{(i,j;n)}-S^{*}_{(i,j;n)}\right)\right]$$
$$-\sum\left(\frac{j}{\langle d_{out}\rangle}+\frac{n}{\langle d_{u}\rangle}\right)\left[\alpha_{S}\tau_{d}i\Theta^{d}\left(S_{(i,j;n)}-S^{*}_{(i,j;n)}\right)^{2}+\alpha_{S}\tau_{u}n\Theta^{u}\left(S_{(i,j;n)}-S^{*}_{(i,j;n)}\right)^{2}\right]$$
$$+\sum\left(\frac{j}{\langle d_{out}\rangle}+\frac{n}{\langle d_{u}\rangle}\right)\left[-(\eta+\beta)\left(R_{(i,j;n)}-R^{*}_{(i,j;n)}\right)^{2}+(\gamma-\beta)\left(S_{(i,j;n)}-S^{*}_{(i,j;n)}\right)\left(R_{(i,j;n)}-R^{*}_{(i,j;n)}\right)\right]$$
$$-\sum\left(\frac{j}{\langle d_{out}\rangle}+\frac{n}{\langle d_{u}\rangle}\right)\left[\alpha_{R}\tau_{d}i\Theta^{d}\left(R_{(i,j;n)}-R^{*}_{(i,j;n)}\right)^{2}+\alpha_{R}\tau_{u}n\Theta^{u}\left(R_{(i,j;n)}-R^{*}_{(i,j;n)}\right)^{2}\right]+A, \tag{31}$$

where

$$A=\frac{1}{\langle d_{out}\rangle}\sum jP(i,j;n)\alpha_{S}\tau_{u}n\left(S_{(i,j;n)}\frac{\Theta^{u}}{\Theta^{d}}-S^{*}_{(i,j;n)}\frac{\Theta^{*u}}{\Theta^{*d}}\right)\left(\Theta^{d}-\Theta^{*d}\right)$$
$$+\frac{1}{\langle d_{out}\rangle}\sum jP(i,j;n)\alpha_{R}\tau_{u}n\left(R_{(i,j;n)}\frac{\Theta^{u}}{\Theta^{d}}-R^{*}_{(i,j;n)}\frac{\Theta^{*u}}{\Theta^{*d}}\right)\left(\Theta^{d}-\Theta^{*d}\right)$$
$$+\frac{1}{\langle d_{u}\rangle}\sum nP(i,j;n)\alpha_{S}\tau_{d}i\left(S_{(i,j;n)}\frac{\Theta^{d}}{\Theta^{u}}-S^{*}_{(i,j;n)}\frac{\Theta^{*d}}{\Theta^{*u}}\right)\left(\Theta^{u}-\Theta^{*u}\right)$$
$$+\frac{1}{\langle d_{u}\rangle}\sum nP(i,j;n)\alpha_{R}\tau_{d}i\left(R_{(i,j;n)}\frac{\Theta^{d}}{\Theta^{u}}-R^{*}_{(i,j;n)}\frac{\Theta^{*d}}{\Theta^{*u}}\right)\left(\Theta^{u}-\Theta^{*u}\right)$$
$$-\frac{1}{\langle d_{out}\rangle}\sum jP(i,j;n)\alpha_{S}\tau_{u}n\left(S_{(i,j;n)}-S^{*}_{(i,j;n)}\right)\left(\Theta^{u}-\Theta^{*u}\right)$$
$$-\frac{1}{\langle d_{out}\rangle}\sum jP(i,j;n)\alpha_{R}\tau_{u}n\left(R_{(i,j;n)}-R^{*}_{(i,j;n)}\right)\left(\Theta^{u}-\Theta^{*u}\right)$$
$$-\frac{1}{\langle d_{u}\rangle}\sum nP(i,j;n)\alpha_{S}\tau_{d}i\left(S_{(i,j;n)}-S^{*}_{(i,j;n)}\right)\left(\Theta^{d}-\Theta^{*d}\right)$$
$$-\frac{1}{\langle d_{u}\rangle}\sum nP(i,j;n)\alpha_{R}\tau_{d}i\left(R_{(i,j;n)}-R^{*}_{(i,j;n)}\right)\left(\Theta^{d}-\Theta^{*d}\right). \tag{32}$$

We simplify formula A, and we combine the first and fifth items to get the following results:

$$\begin{aligned} A_1 &= \frac{1}{\langle d_{\text{out}} \rangle} \sum jP(i,j;n)\alpha_S \tau_u n \left(S_{(i,j;n)} \frac{\Theta^u}{\Theta^d} - S^*_{(i,j;n)} \frac{\Theta^{*u}}{\Theta^{*d}} \right) \left(\Theta^d - \Theta^{*d} \right) \\ &\quad - \frac{1}{\langle d_{\text{out}} \rangle} \sum jP(i,j;n)\alpha_S \tau_u n \left(S_{(i,j;n)} - S^*_{(i,j;n)} \right) \left(\Theta^u - \Theta^{*u} \right) \\ &= \frac{1}{\langle d_{\text{out}} \rangle} \sum jP(i,j;n)\alpha_S \tau_u n \left(S^*_{(i,j;n)} \Theta^u + S_{(i,j;n)} \Theta^{*u} - S^*_{(i,j;n)} \frac{\Theta^{*u}}{\Theta^{*d}} \Theta^d - S_{(i,j;n)} \frac{\Theta^u}{\Theta^d} \Theta^{*d} \right) \\ &= \frac{1}{\langle d_{\text{out}} \rangle} \sum jP(i,j;n)\alpha_S \tau_u n \frac{1}{\Theta^d \Theta^{*d}} \left(\Theta^d \Theta^{*u} - \Theta^u \Theta^{*d} \right) \left(S_{(i,j;n)} \Theta^{*d} - S^*_{(i,j;n)} \Theta^d \right). \end{aligned} \tag{33}$$

According to the definition of the model, it is not difficult to find $\Theta^d \Theta^{*u} - \Theta^u \Theta^{*d} = 0$. So, $A_1 = 0$. $A = 0$ is correct for the same reason. So

$$\begin{aligned} \frac{dV}{dt} &\le \sum \left(\frac{j}{\langle d_{\text{out}} \rangle} + \frac{n}{\langle d_u \rangle} \right) \left[-\gamma \left(S_{(i,j;n)} - S^*_{(i,j;n)} \right)^2 + \eta \left(R_{(i,j;n)} - R^*_{(i,j;n)} \right) \left(S_{(i,j;n)} - S^*_{(i,j;n)} \right) \right] \\ &\quad + \sum \left(\frac{j}{\langle d_{\text{out}} \rangle} + \frac{n}{\langle d_u \rangle} \right) \left[-(\eta + \beta) \left(R_{(i,j;n)} - R^*_{(i,j;n)} \right)^2 + (\gamma - \beta) \left(S_{(i,j;n)} - S^*_{(i,j;n)} \right) \left(R_{(i,j;n)} - R^*_{(i,j;n)} \right) \right] \\ &= \sum \left(\frac{j}{\langle d_{\text{out}} \rangle} + \frac{n}{\langle d_u \rangle} \right) \left[-\gamma \left(S_{(i,j;n)} - S^*_{(i,j;n)} \right)^2 + (\eta + \gamma - \beta) \left(R_{(i,j;n)} - R^*_{(i,j;n)} \right) \left(S_{(i,j;n)} - S^*_{(i,j;n)} \right) \right. \\ &\quad \left. - (\eta + \beta) \left(R_{(i,j;n)} - R^*_{(i,j;n)} \right)^2 \right]. \end{aligned} \tag{34}$$

When, $2\sqrt{\gamma(\eta+\beta)} > \gamma + \eta - \beta$, $V' = 0$ if and only if $S_{(i,j;n)} = S^*_{(i,j;n)}$, $I_{(i,j;n)} = I^*_{(i,j;n)}$, and $R_{(i,j;n)} = R^*_{(i,j;n)}$. According to the Lyapunov theorem [27] and the LaSalle's invariant principal [28], we can conclude that the endemic equilibrium E^* of system (1) is globally asymptotically stable. □

6. Numerical Simulation and Analysis

In this section, we give some numerical simulation and analysis to verify the theorems which is obtained in Sections 4 and 5. All the simulations are based on the semidirected networks. The way of building the semidirected networks is as follows: First, we build a scale-free network. We randomly select the connected edges between nodes to become directed, and randomly specify the direct of connection to ensure the generality of the network. To fit the actual situation, the number of the undirected connection is much more than the directed connection. None of the nodes in this numerical simulation are isolated nodes. The number of all nodes on the semidirected network is 500.

For the case that if $R_0 < 1$, Theorem 1 and Theorem 2 show that the disease-free E^0 is locally and globally asymptotically stable. We will make use of the semidirected network to verify the correctness of it.

Example 1. The parameters in system (1) are taken as $\alpha_S = 0.2$, $\alpha_R = 0.1$, $\tau_d = 0.02$, $\tau_u = 0.03$, $\gamma = 0.1$, $\beta = 0.1$, and $\eta = 0.3$. The semidirected network that is built randomly has the following property. $\langle d_{\text{in}} \rangle = 1.536$, $\langle d_{\text{out}} \rangle = 1.536$, $\langle d_u \rangle = 4.78$, $\langle d_{\text{in}} d_{\text{out}} \rangle = 2.680$, $\langle d_u^2 \rangle = 43.492$, $\langle d_u d_{\text{out}} \rangle = 10.282$, and $\langle d_{\text{in}} d_u \rangle = 12.936$. So the basic reproduction number $R_0 = 0.546 < 1$. The disease-free equilibrium E^0 is globally asymptotically stable. Therefore, the diseases will disappear in the semidirected network.

From Figure 1 it is clear that nodes with degree (2, 10; 12) will recover from diseases. And all of the nodes will recover from the diseases. This is to say the entire semidirected network will not be affected by the disease.

Next, we observe an evolution of two different kinds of nodes. The degree of one kind node is (2, 6; 7). It is a kind of node with fewer connections. Another kind of nodes with

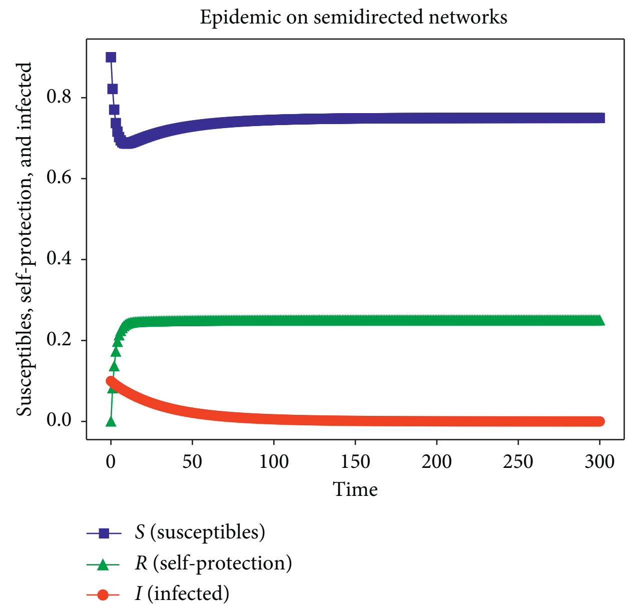

FIGURE 1: Evolution of nodes with degree (2, 10; 12) on semidirected networks.

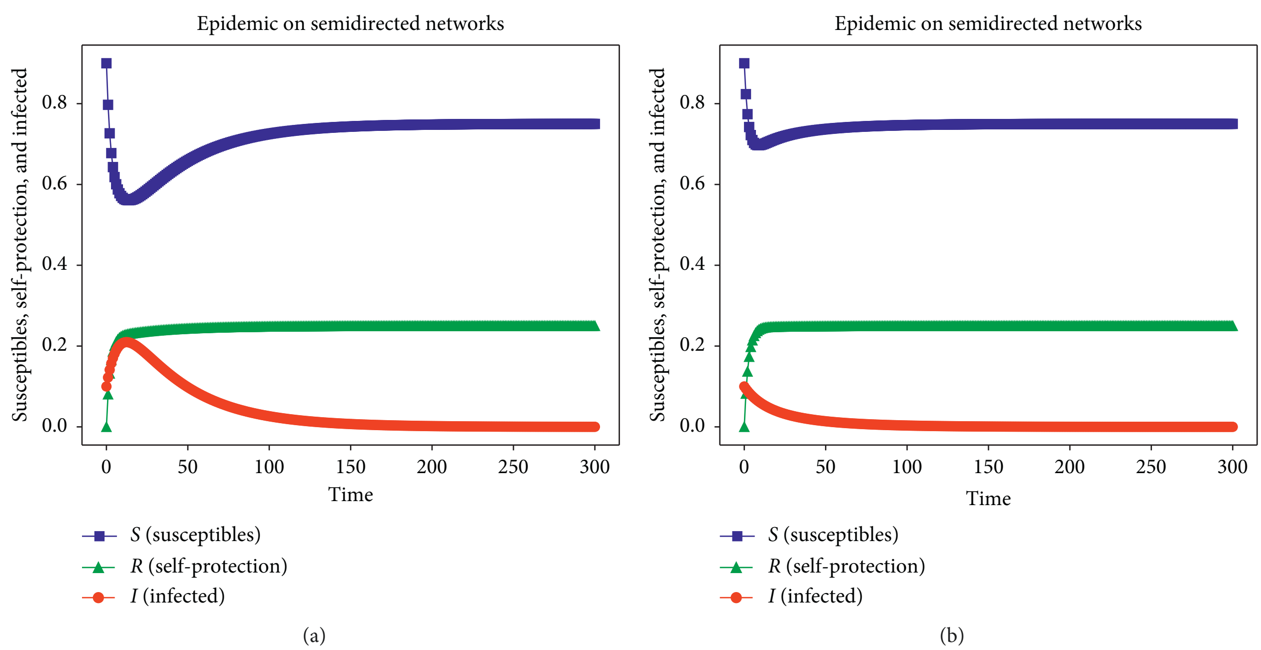

FIGURE 2: Evolution of the nodes of different degrees.

degree (8, 22; 46) is the centre node in the semidirected network.

The first figure in Figure 2 is the evolution of the centre node with degree (8, 22; 46), and the second is the node with degree (2, 6; 7). It is easy to find that the node with a different degree will reach the same situation. It fits the conclusion that all nodes will reach the disease-free equilibrium $E^0_{(i,j;n)} = (\eta/(\eta+r), 0, r/(\eta+r))$ for $0 \le i \le M_{\text{in}}$, $0 \le j \le M_{\text{out}}$, and $0 \le n \le M_{\text{u}}$. It is clear that the node with bigger degree has a greater magnitude of change than the node with fewer degree. The infectious rate of the node with degree (8, 22; 46) increases at the beginning, and the centre node is more susceptible to disease transmission. The disease propagates in the seminetwork mainly through nodes with bigger degree because they have enough edges which connect to the whole network. Finally, it becomes 0. And the disease disappears on the semidirected networks. When disease disappears, nodes in state S and state R will meet a balance according to the stability of the two-dimensional system. And in Figure 3, we will show the infectious evolution of the whole nodes. It meets our conclusion. The node with small degree in state I will disappear not increase like the nodes

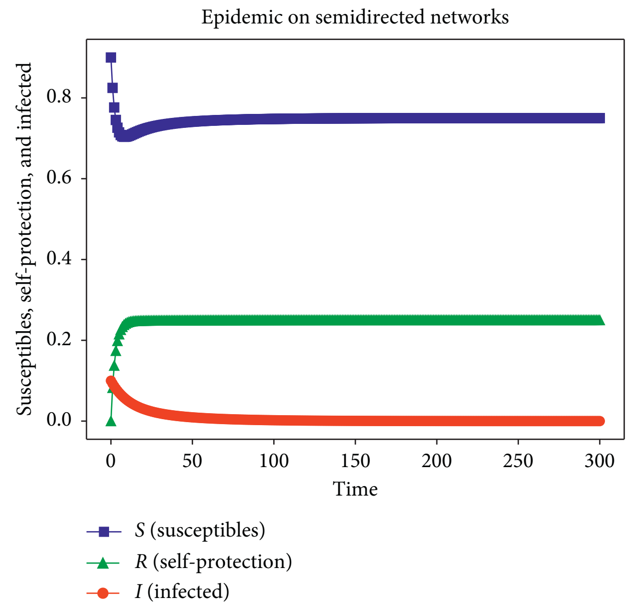

Figure 3: Evolution of the whole nodes.

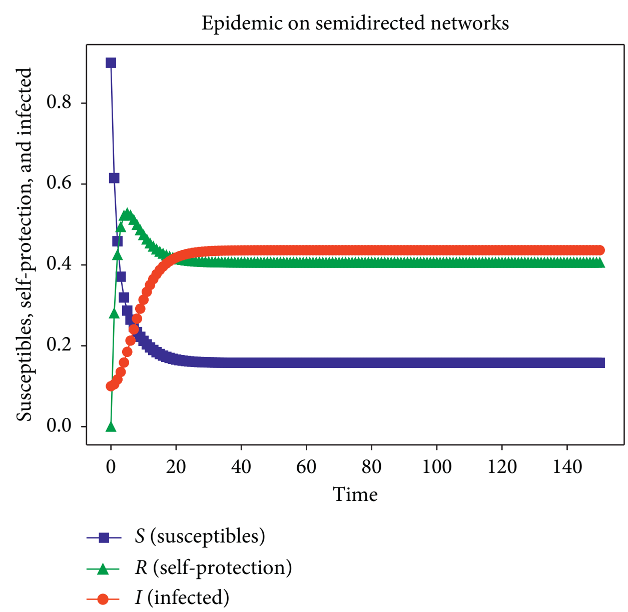

Figure 4: Evolution of nodes with degree (1, 2; 5) on semidirected networks.

with big degree in the beginning. All the nodes will not be affected by the disease. So the disease disappears.

Example 2. The parameters in system (1) are taken as $\alpha_S = 0.2$, $\alpha_R = 0.1$, $\tau_d = 0.15$, $\tau_u = 0.2$, $\gamma = 0.4$, $\beta = 0.2$, and $\eta = 0.2$. The semidirected network that is rebuilt randomly has the following property: $\langle d_{in} \rangle = 1.536$, $\langle d_{out} \rangle = 1.536$, $\langle d_u \rangle = 4.78$, $\langle d_{in} d_{out} \rangle = 2.680$, $\langle d_u^2 \rangle = 43.492$, $\langle d_u d_{out} \rangle = 10.282$, and $\langle d_{in} d_u \rangle = 12.936$. So the basic reproduction number $R_0 = 1.430 > 1$. The epidemic equilibrium E^* is globally asymptotically stable. And the disease will exist on the semidirected networks forever, and the number of infectious nodes is invariable.

From Figure 4, we can find that the node with degree will be in a balanced state. The number of nodes in the infected state is invariable. All the nodes are in the dynamic equilibrium state. The number of nodes in the self-protection state increases in the beginning, and it will fall down after experiencing the highest peak, because the infectious diseases have not spread in the network and a large number of the nodes are in the state S. Almost all nodes with different degrees have such a trend.

Next, we compare three kinds of nodes with different degrees. One is (8, 22; 32) and another is (3, 8; 9). One node has a large degree, and it belongs to the central node. Other has the middle degree on the semidirected network. The second one is the node with degree (1, 2; 5). The results are shown in Figure 5. It has the smaller degree, and it belongs to the edge node.

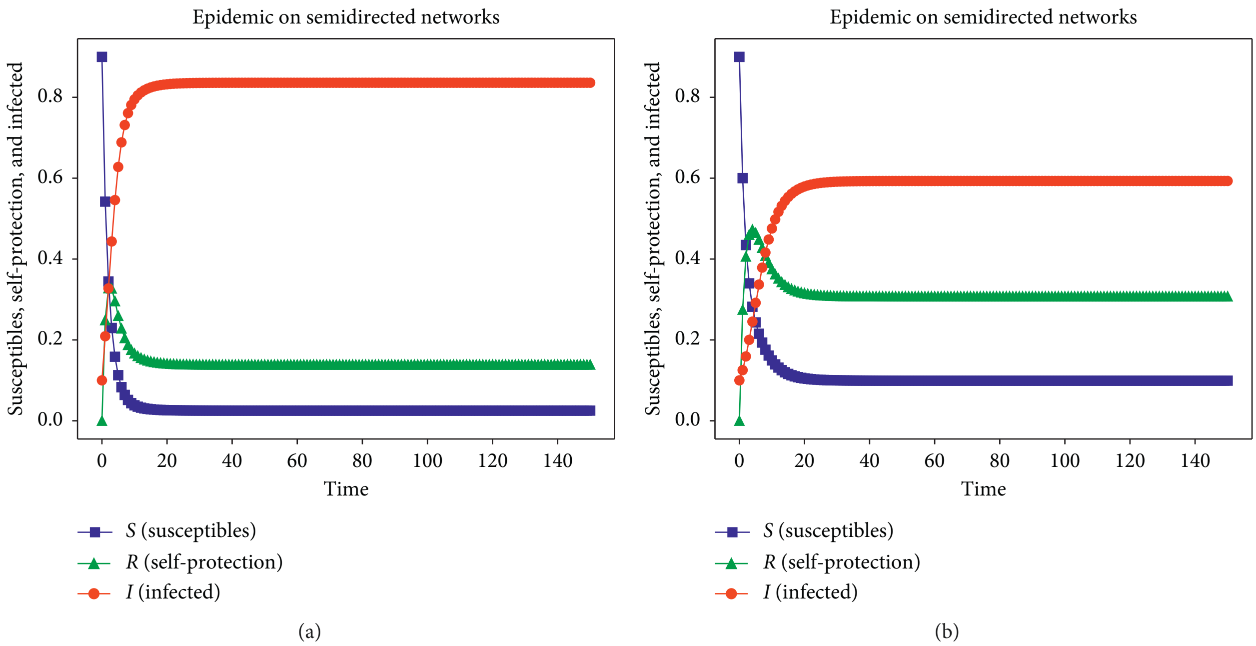

Figure 5: Evolution of the nodes of different degrees.

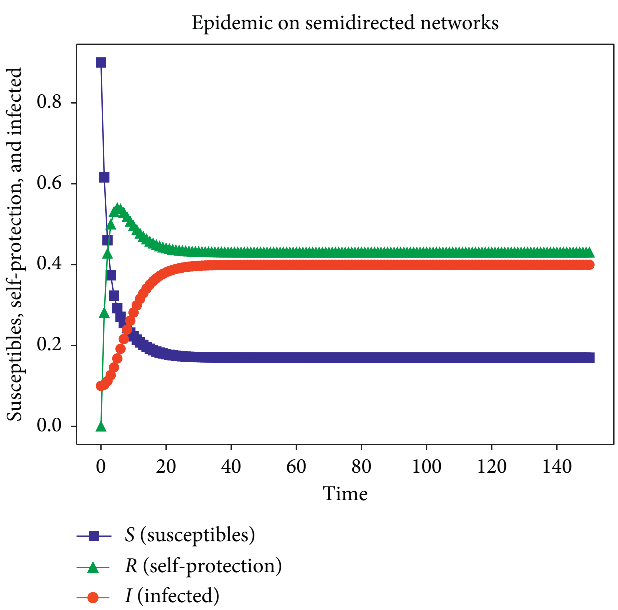

Figure 6: Evolution of the whole nodes.

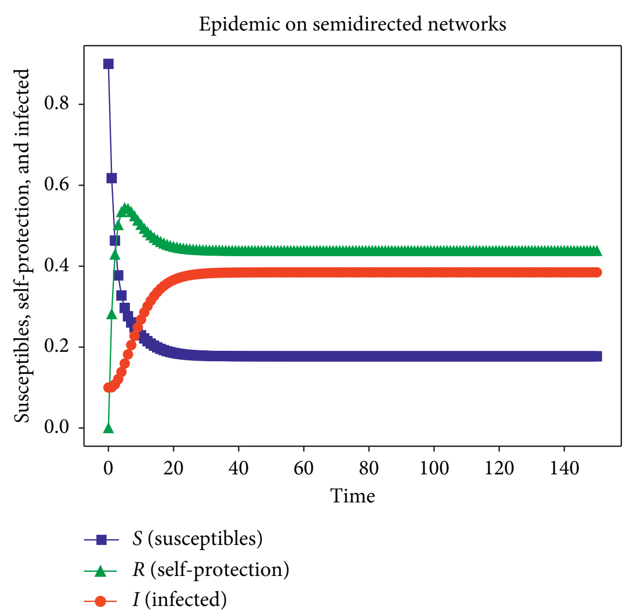

Figure 7: Evolution of the whole nodes (1000 nodes).

For three different degrees of nodes, the trend of change is almost the same. The difference is that the value of the final balance point is different. It is clear that the node with great degree will have the higher rate of the infected in the end. The result can be seen in expression of $I^*_{i,j;n}$ due to its strong connectivity. Higher degree may result in increase in the value of $I^*_{i,j;n}$ increases. The greater the degree of the node, the greater the probability of being infected. It means that if we control the rate of the infected nodes with bigger degree, the rate of the infected nodes with smaller degree will decrease. In this way, we can control the disease propagation in the semidirected network. As long as we control the nodes with high degrees, we can control the spread of the entire network. In Figure 6, we will give evolution of the whole nodes. In this figure, $I = \sum P(i, j; n) I_{(i,j;n)}$, $S = \sum P(i, j; n) S_{(i,j;n)}$, and $R = \sum P(i, j; n) R_{(i,j;n)}$.

It is clear that the nodes on the semidirected networks reach equilibrium after a period of time. Its changing trace is similar to the figures given earlier. Disease will exist in the network forever. The state of the node reaches dynamic

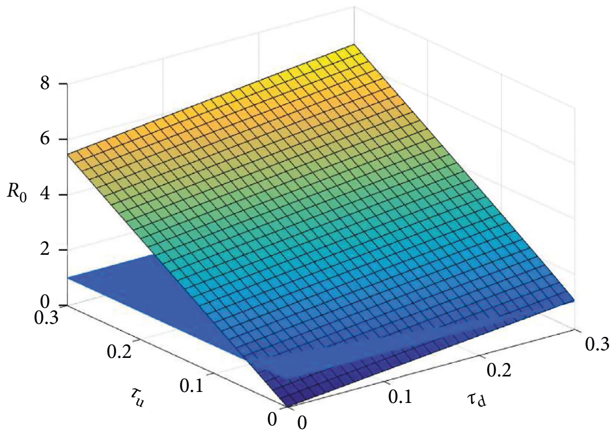

FIGURE 8: R_0 under different τ_d and τ_u.

balance. Next, we rebuild the semidirected network with 1000 nodes, and the parameters are unchanged. The simulation is shown in Figure 7.

There is no significant change between Figures 6 and 7. It means that the increase of the node number affects the final result, unless changing the method of connection. Next, we present Figure 8 to show R_0 under different τ_d and τ_u. It meets that the semidirected networks which we build has large undirected connection and few directed connection.

7. Conclusion and Discussion

In this paper, we study the epidemic model with no full immunity on semidirected networks. In this model, we use self-protection to replace the immunity which is used in the past. Based on this model, the basic reproduction number R_0 can be used to control infectious diseases. We mainly analyze stability of disease-free equilibrium E^0 and endemic equilibrium E^*. If $R_0 < 1$, the disease-free equilibrium E^0 is locally and globally asymptotically stable. It has been proved. And endemic equilibrium E^* is globally asymptotically stable under some condition. It means that the infectious diseases will exist on the semidirected networks in a long time.

According to the basic reproduction number R_0 that we calculated, the propagation of diseases is affected by many factors. And, it is also affected by the degree distribution of the networks. We can consider more about the recovery η. If the recovery is increasing, the diseases can be controlled effectively.

When we prove the globally asymptotically stable, we give some condition on it. We cannot investigate the globally asymptotical stability only when $R_0 > 1$. We hope to improve in the future.

Data Availability

No data were used in the manuscript.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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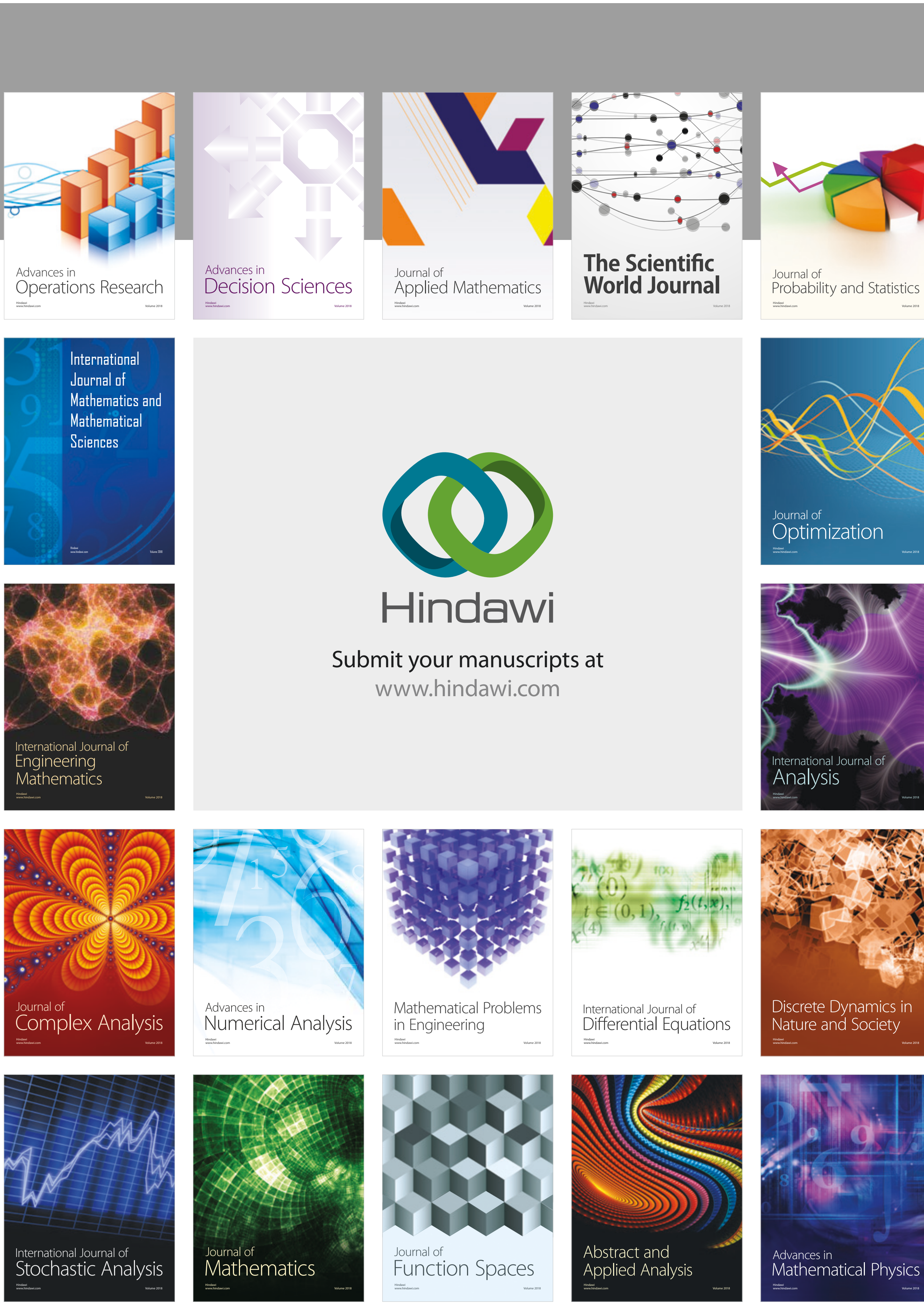

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