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Research Article

On Novel Nonhomogeneous Multivariable Grey Forecasting Model NHMGM

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A novel nonhomogeneous multivariable grey forecasting model termed NHMGM$(1, m, k^p, c)$ is proposed in this paper for use in nonhomogeneous multivariable exponential data sequences. The NHMGM$(1, m, k^p, c)$ model is able to reflect the nonlinear relation of the data sequences in the system, and it is proved that many classic grey forecasting models can be derived from NHMGM$(1, m, k^p, c)$ model. Parameters of the novel model are obtained by using least square method, and the time response function is given. A numerical example is presented to show the effectiveness of the proposed model, six different grey forecasting models are built for modeling, and two popular accuracy criteria (ARPE and MAPE) are adopted to test the reliability of the novel model. The example demonstrates that NHMGM-2 model provides favorable performance compared with the other five grey models. Additionally, the multiplication transformation properties of NHMGM$(1, m, k^p, c)$ are systematically analysed, which establish a theoretical foundation for further applications of the model.

1. Introduction

Grey system theory has been adopted to various aspects of fields including energy, environment, industry, and so on [1–3]. The grey forecasting model is one of the most widely exploited techniques in forecasting field and develops greatly since it was proposed by Deng [4]. Compared with qualitative theory of knowing the system structure [5, 6], the grey forecasting model shows advantages in dealing with partially known and partially unknown information, and it makes more contribution to uncertainty problems. Chen and Huang [7] studied necessary and sufficient conditions for GM$(1, 1)$, Ye et al. [8] constructed a Grey-Markov forecasting model, and Wu et al. [9] put forward a fractional-order grey forecasting model. Scholars have always worked diligently to enrich the research and application of grey forecasting models, and some researchers have combined intelligent techniques with grey forecasting model to form hybrid grey models [10–13]. For example, Wang and Hsu [12] combined grey theory and genetic algorithms to forecast the output trends of high technology industry in Taiwan and obtained encouraging results. The mentioned studies improve simulative and predictive precision in a certain extent; however, these studies are based on the hypothesis that original data sequence is in accord with homogeneous index trend rather than nonhomogeneous index trend.

The other researchers were concerned with the nonhomogeneous data principle to improve the model [14–16]. Xie et al. [14] investigated NDGM model based on pure nonhomogeneous index sequence. Cui et al. [15] proposed a novel grey model NGM$(1, 1, k)$ in order to solve the nonhomogeneous exponential data sequence and laid the foundation on the studies of nonhomogeneous grey models. The single variable nonhomogeneous grey forecasting model optimized by Cui et al. [15] is a useful way to deal with the nonhomogeneous data and attracts considerable interest of researchers. Ma et al. [16] utilized the kernel method to build a novel kernel regularized nonhomogeneous grey model abbreviated as KRNGM, and the results showed that KRNGM model outperformed the existing grey prediction models. All those studies indicate that the nonhomogeneous data sequence occupies an important part in grey forecasting, which motivates us to explore the nonhomogeneous multivariable grey forecasting model.

The most commonly used multivariable grey forecasting MGM$(1,m)$ model is proposed by Zhai et al. [17], which can uniformly describe each variable from viewpoint of system analysis, reflected the interactional relation of variables, and performed preferable prediction accuracy for modeling and forecasting in multiple variable system. MGM$(1,m)$ model attracts many researchers attention and has been successfully applied in various fields [18–22]. Dai et al. [18] investigated MGM$(1,m)$ model with optimized background value, and evidence of experiment results demonstrated that the optimized MGM$(1,m)$ model had higher forecasting accuracy for monotone sequences and oscillation sequences. Zou [19] applied a step by step new information modeling method to construct new information background value of multivariable nonequidistance grey model, and the novel model can be used to nonequal interval time series. Guo et al. [22] extended MGM$(1,m)$ model to predict engineering settlement deformation, which further expanded the application of multivariable grey model.

From the above analysis we know that most scholars only optimized the model from the view of modeling parameters to better fit data sequences with grey exponential law but ignored the nonhomogeneous multivariable data sequences. It is inevitable leading to errors if we forecast by MGM$(1,m)$ model while the data sequences are not in accord with homogeneous index trend. In this work, we put forward a novel multivariable grey forecasting model named NHMGM$(1,m,k^p,c)$ to handle the nonhomogeneous multivariable exponential data sequences. The novel NHMGM$(1,m,k^p,c)$ model is able to reflect the nonlinear relation of the data sequences in the system and makes it able to achieve better simulation and prediction performance. In order to compare the superiority of the proposed model, a numerical example is utilized to validate the simulation accuracy, six different grey forecasting models are built for modeling, and two accuracy criteria are adopted to test the accuracy. The example demonstrates that NHMGM-2 model is superior to NMGM proposed in [20], NMGM model is superior to MGM$(1,m)$ model discussed in [17], and MGM$(1,m)$ model is superior to single variable grey models GM(1, 1) and NGM$(1,1,k)$. In a word, the novel NHMGM-2 model provides excellent performance compared with traditional classic grey models and presents advantages of dealing with nonhomogeneous multivariable exponential data sequences.

Exploring properties of parameters is also a unique perspective to utilize the model proficiently [23–26]. Li [23] investigated parameters nature of GM(1, 1) model after multiplication transformation and set off the hot spot on researching impact of multiplication transformation to parameters of the model. The multiplication transformation properties of the novel NHMGM$(1,m,k^p,c)$ model indicate that parameters of the transformed model have relation to the amount of multiplication transformation, and we cannot apply different data transformations to simplify the modeling process. Hence, it is interesting to study multiplication transformation properties of the novel model NHMGM$(1,m,k^p,c)$.

This study proposes a novel nonhomogeneous multivariable grey forecasting model NHMGM$(1,m,k^p,c)$ and discusses its properties. The remainder of the paper is organized as follows. A novel nonhomogeneous multivariable grey forecasting model NHMGM$(1,m,k^p,c)$ and its derived models are presented in Section 2. The multiplication transformation properties of NHMGM$(1,m,k^p,c)$ are studied in Section 3. An illustrative example is given to demonstrate the practicality of the novel model in Section 4. Section 5 discusses different forms of NHMGM$(1,m,k^p,c)$ model and further studies. Some conclusions are summarised in Section 6.

2. Grey NHMGM$(1,m,k^p,c)$ Model

In this section, modeling mechanism and prediction functions of the novel NHMGM$(1,m,k^p,c)$ model are presented.

Definition 1. Let the original data matrix be $X^{(0)} = (X_1^{(0)}, X_2^{(0)}, \ldots, X_m^{(0)})^T$, where $X_j^{(0)}$ is

$$X_j^{(0)} = \left(x_j^{(0)}(1), x_j^{(0)}(2), \ldots, x_j^{(0)}(n)\right), \quad j = 1,2,\ldots,m. \tag{1}$$

The data matrix $X^{(1)} = (X_1^{(1)}, X_2^{(1)}, \ldots, X_m^{(1)})^T$ is said to be the first-order accumulated generation (1-AGO) matrix of $X^{(0)}$, where

$$\begin{gathered} X_j^{(1)} = \left(x_j^{(1)}(1), x_j^{(1)}(2), \ldots, x_j^{(1)}(n)\right), \quad j = 1,2,\ldots,m, \\ x_j^{(1)}(k) = \sum_{s=1}^{k} x_j^{(0)}(s), \quad k = 1,2,\ldots,n. \end{gathered} \tag{2}$$

The adjacent neighbour average matrix $Z^{(1)} = (Z_1^{(1)}, Z_2^{(1)}, \ldots, Z_m^{(1)})^T$ is said to be the background value of the model, where

$$\begin{gathered} Z_j^{(1)} = \left(z_j^{(1)}(2), \ldots, z_j^{(1)}(n)\right), \quad j = 1,2,\ldots,m, \\ z_j^{(1)}(k) = 0.5\left(x_j^{(1)}(k) + x_j^{(1)}(k-1)\right), \\ j = 1,2,\ldots,m, \; k = 2,3,\ldots,n. \end{gathered} \tag{3}$$

Definition 2. Assume that $X^{(0)}$ is a nonnegative original data matrix, $X^{(1)}$ is 1-AGO of $X^{(0)}$, and $Z^{(1)}$ is the adjacent neighbour average matrix. The whitenization differential equations of the novel nonhomogeneous multivariable grey forecasting model abbreviated NHMGM$(1, m, k^p, c)$ are defined as follows:

$$\begin{aligned}
\frac{dx_1^{(1)}(t)}{dt} &= \gamma_{11}x_1^{(1)}(t) + \gamma_{12}x_2^{(1)}(t) + \cdots + \gamma_{1m}x_m^{(1)}(t) \\
&\quad + \alpha_1 t^p + \beta_1, \\
\frac{dx_2^{(1)}(t)}{dt} &= \gamma_{21}x_1^{(1)}(t) + \gamma_{22}x_2^{(1)}(t) + \cdots + \gamma_{2m}x_m^{(1)}(t) \\
&\quad + \alpha_2 t^p + \beta_2, \\
&\vdots \\
\frac{dx_m^{(1)}(t)}{dt} &= \gamma_{m1}x_1^{(1)}(t) + \gamma_{m2}x_2^{(1)}(t) + \cdots + \gamma_{mm}x_m^{(1)}(t) \\
&\quad + \alpha_m t^p + \beta_m,
\end{aligned} \tag{4}$$

where $p \geq 0$. We denote the notation for convenience

$$\Gamma = \begin{pmatrix} \gamma_{11} & \gamma_{12} & \cdots & \gamma_{1m} \\ \gamma_{21} & \gamma_{22} & \cdots & \gamma_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{m1} & \gamma_{m2} & \cdots & \gamma_{mm} \end{pmatrix},$$
$$\alpha = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_m \end{pmatrix}, \tag{5}$$
$$\beta = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_m \end{pmatrix}.$$

Therefore, (4) can be written in matrix form, which is

$$\frac{dX^{(1)}(t)}{dt} = \Gamma X^{(1)}(t) + \alpha t^p + \beta. \tag{6}$$

Consequently, the differential equation

$$\frac{dX^{(1)}(t)}{dt} = \Gamma Z^{(1)}(t) + \alpha t^p + \beta \tag{7}$$

is called the original form of nonhomogeneous multivariable grey forecasting NHMGM$(1, m, k^p, c)$ model. From (7), we deduce that the discrete form of NHMGM$(1, m, k^p, c)$ model is

$$x_j^{(0)}(k) = \sum_{l=1}^{m} \gamma_{jl} z_l^{(1)}(k) + \alpha_j k^p + \beta_j, \quad j = 1, 2, \ldots, m. \tag{8}$$

The novel model NHMGM$(1, m, k^p, c)$ contains a nonlinear term αk^p, and αk^p is named the nonlinear correction term in (8). The nonlinear term αk^p can reflect the nonhomogeneous data sequences in the restored function, and the restored values of original data sequences can be adjusted through their coefficients αk^p and β. Therefore, NHMGM$(1, m, k^p, c)$ can deal with the nonlinear relation of data sequences and makes it able to achieve better simulation and prediction performance.

In what follows, we present the parameters of NHMGM$(1, m, k^p, c)$ model, discuss the derived models, and give the time response functions of models.

Theorem 3. *Assume that $X^{(0)}$ is a nonnegative original data matrix, $X^{(1)}$ is 1-AGO of $X^{(0)}$, and $Z^{(1)}$ is the adjacent neighbour average matrix. The parameters Γ, α, and β are defined in (5). Then*

$$\begin{pmatrix} \widehat{\Gamma}' \\ \widehat{\alpha}' \\ \widehat{\beta}' \end{pmatrix} = \left(M^T M\right)^{-1} M^T \left(N_1, N_2, \ldots, N_m\right), \tag{9}$$

where

$$M = \begin{pmatrix} z_1^{(1)}(2) & z_2^{(1)}(2) & \cdots & z_m^{(1)}(2) & 2^p & 1 \\ z_1^{(1)}(3) & z_2^{(1)}(3) & \cdots & z_m^{(1)}(3) & 3^p & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ z_1^{(1)}(n) & z_2^{(1)}(n) & \cdots & z_m^{(1)}(n) & n^p & 1 \end{pmatrix},$$
$$N_j = \left(x_j^{(0)}(2), x_j^{(0)}(3), \ldots, x_j^{(0)}(n)\right)^T, \quad j = 1, 2, \ldots, m. \tag{10}$$

The novel nonhomogeneous multivariable grey forecasting model NHMGM$(1, m, k^p, c)$ is the extension of traditional MGM$(1, m)$ model, and many grey forecasting models can be derived from NHMGM$(1, m, k^p, c)$. For example, NMGM$(1, m, k^p)$ studied in [20] can be derived from NHMGM$(1, m, k^p, c)$ while c=0, and MGM$(1, m)$ model can also be derived from NHMGM$(1, m, k^p, c)$ model when p=0. In the following, we denote NHMGM$(1, m, k^2, c)$ model as NHMGM-2 while p=2 in NHMGM$(1, m, k^p, c)$ and denote NHMGM$(1, m, k, c)$ model as NHMGM-1 while p=1 in NHMGM$(1, m, k^p, c)$. We give the time response functions of NHMGM-2, NHMGM-1, NHMGM$(1, m, k^p, c)$, and some corollaries.

Theorem 4. *Assume that $X^{(0)}$, $X^{(1)}$, and $Z^{(1)}$ are defined as in Definition 1. The parameters $\widehat{\Gamma}, \widehat{\alpha}$, and $\widehat{\beta}$ are obtained by Theorem 3. Then the following assertions hold.*

(1) *The time response function of NHMGM-2 model is*

$$\widehat{X}^{(1)}(k)=e^{\widehat{\Gamma}(k-1)}\left(X^{(1)}(1)+\widehat{\Gamma}^{-1}\left(\widehat{\alpha}+\widehat{\beta}\right)+2\left(\widehat{\Gamma}^{-1}\right)^{2}\widehat{\alpha}+2\left(\widehat{\Gamma}^{-1}\right)^{3}\widehat{\alpha}\right)-\left(\widehat{\Gamma}^{-1}\widehat{\alpha}k^{2}+2\left(\widehat{\Gamma}^{-1}\right)^{2}\widehat{\alpha}k+2\left(\widehat{\Gamma}^{-1}\right)^{3}\widehat{\alpha}+\widehat{\Gamma}^{-1}\widehat{\beta}\right),\quad k\geq 2. \tag{11}$$

(2) *The restored value of* $\widehat{X}^{(0)}(k)$ *is*

$$\widehat{X}^{(0)}(k)=\left(e^{\widehat{\Gamma}(k-1)}-e^{\widehat{\Gamma}(k-2)}\right)\left(X^{(1)}(1)+\widehat{\Gamma}^{-1}\left(\widehat{\alpha}+\widehat{\beta}\right)+2\left(\widehat{\Gamma}^{-1}\right)^{2}\widehat{\alpha}+2\left(\widehat{\Gamma}^{-1}\right)^{3}\widehat{\alpha}\right)-2k\widehat{\Gamma}^{-1}\widehat{\alpha}+\widehat{\Gamma}^{-1}\widehat{\alpha},\quad k\geq 2. \tag{12}$$

Proof. (1) From (6), we deduce that the whitenization differential equation of NHMGM-2 model is

$$\frac{dX^{(1)}(t)}{dt}-\widehat{\Gamma}X^{(1)}(t)=\widehat{\alpha}t^{2}+\widehat{\beta}. \tag{13}$$

Multiplying (13) by $e^{-\widehat{\Gamma}t}$, we obtain

$$e^{-\widehat{\Gamma}t}\frac{dX^{(1)}(t)}{dt}-e^{-\widehat{\Gamma}t}\widehat{\Gamma}X^{(1)}(t)=e^{-\widehat{\Gamma}t}\left(\widehat{\alpha}t^{2}+\widehat{\beta}\right), \tag{14}$$

which yields that

$$\frac{d\left(e^{-\widehat{\Gamma}t}X^{(1)}(t)\right)}{dt}=e^{-\widehat{\Gamma}t}\left(\widehat{\alpha}t^{2}+\widehat{\beta}\right). \tag{15}$$

Integrating (15) from t_0 to t implies that

$$e^{-\widehat{\Gamma}t}X^{(1)}(t)-e^{-\widehat{\Gamma}t_0}X^{(1)}(t_0)=e^{-\widehat{\Gamma}t_0}\left(\widehat{\Gamma}^{-1}\widehat{\alpha}t_0^{2}+2\left(\widehat{\Gamma}^{-1}\right)^{2}\widehat{\alpha}t_0+2\left(\widehat{\Gamma}^{-1}\right)^{3}\widehat{\alpha}+\widehat{\Gamma}^{-1}\widehat{\beta}\right)-e^{-\widehat{\Gamma}t}\left(\widehat{\Gamma}^{-1}\widehat{\alpha}t^{2}+2\left(\widehat{\Gamma}^{-1}\right)^{2}\widehat{\alpha}t+2\left(\widehat{\Gamma}^{-1}\right)^{3}\widehat{\alpha}+\widehat{\Gamma}^{-1}\widehat{\beta}\right). \tag{16}$$

Multiplying (16) by $e^{\widehat{\Gamma}t}$ and setting $t_0=1$, we have

$$X^{(1)}(t)=e^{\widehat{\Gamma}(t-1)}\left(X^{(1)}(1)+\widehat{\Gamma}^{-1}\left(\widehat{\alpha}+\widehat{\beta}\right)+2\left(\widehat{\Gamma}^{-1}\right)^{2}\widehat{\alpha}+2\left(\widehat{\Gamma}^{-1}\right)^{3}\widehat{\alpha}\right)-\left(\widehat{\Gamma}^{-1}\widehat{\alpha}t^{2}+2\left(\widehat{\Gamma}^{-1}\right)^{2}\widehat{\alpha}t+2\left(\widehat{\Gamma}^{-1}\right)^{3}\widehat{\alpha}+\widehat{\Gamma}^{-1}\widehat{\beta}\right). \tag{17}$$

Letting $t=k$ in (17), then (11) can be obtained.

(2) The restored data can be deduced from 1-AGO and hence we omit it. □

Property 5. The NHMGM-2 model can simulate and forecast the nonhomogeneous multivariable exponential data such as $X(t)=Ae^{Bt}+Ct+D$.

Theorem 6. *Assume that* $X^{(0)}$, $X^{(1)}$, *and* $Z^{(1)}$ *are defined as in Definition 1. The parameters* $\widehat{\Gamma},\widehat{\alpha}$, *and* $\widehat{\beta}$ *are obtained by Theorem 3. Then the following assertions hold.*

(1) *The time response function of NHMGM-1 model is*

$$\widehat{X}^{(1)}(k)=e^{\widehat{\Gamma}(k-1)}\left(X^{(1)}(1)+\widehat{\Gamma}^{-1}\left(\widehat{\alpha}+\widehat{\beta}\right)+\left(\widehat{\Gamma}^{-1}\right)^{2}\widehat{\alpha}\right)-\left(\widehat{\Gamma}^{-1}\widehat{\alpha}k+\left(\widehat{\Gamma}^{-1}\right)^{2}\widehat{\alpha}+\widehat{\Gamma}^{-1}\widehat{\beta}\right),\quad k\geq 2. \tag{18}$$

(2) *The restored value of* $\widehat{X}^{(0)}(k)$ *is*

$$\widehat{X}^{(0)}(k)=\widehat{X}^{(1)}(k)-\widehat{X}^{(1)}(k-1)=\left(e^{\widehat{\Gamma}(k-1)}-e^{\widehat{\Gamma}(k-2)}\right)\cdot\left(X^{(1)}(1)+\widehat{\Gamma}^{-1}\left(\widehat{\alpha}+\widehat{\beta}\right)+\left(\widehat{\Gamma}^{-1}\right)^{2}\widehat{\alpha}\right)-\widehat{\Gamma}^{-1}\widehat{\alpha},\quad k\geq 2. \tag{19}$$

Property 7. The NHMGM-1 model can simulate and forecast the nonhomogeneous multivariable exponential data such as $X(t)=Ae^{Bt}+C$.

Theorem 8. *Assume that* $X^{(0)}$, $X^{(1)}$, *and* $Z^{(1)}$ *are defined as in Definition 1. The parameters* $\widehat{\Gamma},\widehat{\alpha}$, *and* $\widehat{\beta}$ *are obtained by Theorem 3. The time response function of* NHMGM$(1,m,k^{p},c)$ *model is*

$$\widehat{X}^{(1)}(t)=e^{\widehat{\Gamma}(t-t_0)}\left(X^{(1)}(t_0)+\widehat{\Gamma}^{-1}\widehat{\beta}\right)+e^{\widehat{\Gamma}t}\left(\int_{t_0}^{t}t^{p}e^{-\widehat{\Gamma}t}dt\right)\widehat{\alpha}-\widehat{\Gamma}^{-1}\widehat{\beta}. \tag{20}$$

In order to compare the forecasting performance of different grey models, we give the time response functions of NMGM$(1,m,k)$ [20] and MGM$(1,m)$ [17] for the convenience of the reader.

Corollary 9. *Assume that* $X^{(0)}$, $X^{(1)}$, *and* $Z^{(1)}$ *are defined as in Definition 1. The parameters* $\widehat{\Gamma}$ *and* $\widehat{\alpha}$ *are obtained by Theorem 3. Then the following assertions hold.*

(1) The prediction function of NMGM$(1,m,k)$ *model is*

$$\widehat{X}^{(1)}(k)=e^{\widehat{\Gamma}(k-1)}\left(X^{(1)}(1)+\widehat{\Gamma}^{-1}\alpha+\left(\widehat{\Gamma}^{-1}\right)^{2}\widehat{\alpha}\right)-\left(\widehat{\Gamma}^{-1}\widehat{\alpha}k+\left(\widehat{\Gamma}^{-1}\right)^{2}\widehat{\alpha}\right),\quad k\geq 2. \tag{21}$$

(2) The restored value of $\widehat{X}^{(0)}(k)$ *is*

$$\widehat{X}^{(0)}(k)=\widehat{X}^{(1)}(k)-\widehat{X}^{(1)}(k-1),\quad k\geq 2. \tag{22}$$

Corollary 10. *Suppose that* $X^{(0)}$, $X^{(1)}$, *and* $Z^{(1)}$ *are defined as in Definition 1. The parameters* $\widehat{\Gamma},\widehat{\alpha}$, *and* $\widehat{\beta}$ *are obtained by Theorem 3. Then the following assertions hold.*

(1) The time response function of MGM$(1,m)$ model is

$$\widehat{X}^{(1)}(k)=e^{\widehat{\Gamma}(k-1)}\left(X^{(1)}(1)+\widehat{\Gamma}^{-1}\left(\widehat{\alpha}+\widehat{\beta}\right)\right)-\widehat{\Gamma}^{-1}\left(\widehat{\alpha}+\widehat{\beta}\right),\quad k\geq 2. \tag{23}$$

(2) The restored data function is

$$\widehat{X}^{(0)}(k)=\left(e^{\widehat{\Gamma}(k-1)}-e^{\widehat{\Gamma}(k-2)}\right)\left(X^{(1)}(1)+\widehat{\Gamma}^{-1}\left(\widehat{\alpha}+\widehat{\beta}\right)\right),\quad k\geq 2. \tag{24}$$

3. Properties of NHMGM$(1,m,k^p,c)$

In order to grasp properties of NHMGM$(1,m,k^p,c)$ model and establish a theoretical foundation for further applications of the model, in this section, we investigate the multiplication transformation properties of NHMGM$(1,m,k^p,c)$ model.

Definition 11. For the nonnegative original data $x(k)$, if σ is a nonnegative constant and $y(k)=\sigma\cdot x(k)$ $(k=1,2,\ldots,n)$, then it is called the multiplication transformation of $x(k)$, where σ is called the amount of multiplication transformation, $x(k)$ is called the original data, and $y(k)$ is termed the multiplication transformation data.

Suppose that $X^{(0)}$ is the original data matrix, $Z^{(1)}$ is the adjacent neighbour average matrix of $X^{(1)}$, and $Y^{(0)}=(Y_1^{(0)},Y_2^{(0)},\ldots,Y_m^{(0)})^T$ is the multiplication transformation data of $X^{(0)}$, where $y_j^{(0)}(k)=\sigma_j x_j^{(0)}(k)$ $(j=1,2,\ldots,m,\ k=1,2,\ldots,n)$. Moreover, assume that $Y^{(1)}=(Y_1^{(1)},Y_2^{(1)},\ldots,Y_m^{(1)})^T$ is 1-AGO of $Y^{(0)}$ and $\overline{Z}^{(1)}=(\overline{Z}_1^{(1)},\overline{Z}_2^{(1)},\ldots,\overline{Z}_m^{(1)})^T$ is the adjacent neighbour average matrix of $Y^{(1)}$. Thus, we deduce that

$$\begin{aligned}y_j^{(1)}(k)&=\sum_{s=1}^{k}y_j^{(0)}(s)=\sum_{s=1}^{k}\sigma_j x_j^{(0)}(s)=\sigma_j\sum_{s=1}^{k}x_j^{(0)}(s)\\&=\sigma_j x_j^{(1)}(k),\\&\qquad j=1,2,\ldots,m,\ k=1,2,\ldots,n.\end{aligned} \tag{25}$$

Hence, we obtain

$$\begin{aligned}\overline{z}_j^{(1)}(k)&=0.5\left(y_j^{(1)}(k)+y_j^{(1)}(k-1)\right)\\&=0.5\left(\sigma_j x_j^{(1)}(k)+\sigma_j x_j^{(1)}(k-1)\right)\\&=\sigma_j z_j^{(1)}(k),\quad j=1,2,\ldots,m,\ k=2,\ldots,n.\end{aligned} \tag{26}$$

Theorem 12. *Assume that $X^{(0)}$ is the nonnegative original data matrix, $Z^{(1)}$ is the adjacent neighbour average matrix of $X^{(1)}$, and $Y^{(0)}$ is the multiplication transformation data matrix of $X^{(0)}$, where $y_j^{(0)}(k)=\sigma_j x_j^{(0)}(k)$ $(j=1,2,\ldots,m,\ k=1,2,\ldots,n)$. Furthermore, suppose that $Y^{(1)}$ is 1-AGO of $Y^{(0)}$ and $\overline{Z}^{(1)}$ is the adjacent neighbour average matrix of $Y^{(1)}$. If we construct a NHMGM$(1,m,k^p,c)$ model by the multiplication transformation data matrix $Y^{(1)}=(Y_1^{(1)},Y_2^{(1)},\ldots,Y_m^{(1)})^T$, then parameters $\widehat{\widehat{\Gamma}},\widehat{\widehat{\alpha}}$, and $\widehat{\widehat{\beta}}$ of the transformed NHMGM$(1,m,k^p,c)$ model are*

$$\begin{pmatrix}\widehat{\widehat{\Gamma}}'\\ \widehat{\widehat{\alpha}}'\\ \widehat{\widehat{\beta}}'\end{pmatrix}=\left(\overline{M}^T\overline{M}\right)^{-1}\overline{M}^T\left(\overline{N}_1,\overline{N}_2,\ldots,\overline{N}_m\right), \tag{27}$$

where

$$\overline{M}=\begin{pmatrix}\overline{z}_1^{(1)}(2) & \overline{z}_2^{(1)}(2) & \ldots & \overline{z}_m^{(1)}(2) & 2^p & 1\\ \overline{z}_1^{(1)}(3) & \overline{z}_2^{(1)}(3) & \ldots & \overline{z}_m^{(1)}(3) & 3^p & 1\\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ \overline{z}_1^{(1)}(n) & \overline{z}_2^{(1)}(n) & \ldots & \overline{z}_m^{(1)}(n) & n^p & 1\end{pmatrix}, \tag{28}$$

$$\overline{N}_j=\left(y_j^{(0)}(2),y_j^{(0)}(3),\ldots,y_j^{(0)}(n)\right)^T,\quad j=1,2,\ldots,m. \tag{29}$$

In what follows, we investigate multiplication transformation properties of NHMGM$(1,m,k^p,c)$ model.

Theorem 13. *Suppose that $X^{(0)},Z^{(1)},Y^{(0)},Y^{(1)}$, and $\overline{Z}^{(1)}$ are defined as Theorem 12. Assume that $\widehat{\Gamma},\widehat{\alpha}$, and $\widehat{\beta}$ are the parameters of NHMGM$(1,m,k^p,c)$ model constructed by the original data matrix $X^{(1)}=(X_1^{(1)},X_2^{(1)},\ldots,X_m^{(1)})^T$. Moreover, if $\widehat{\widehat{\Gamma}},\widehat{\widehat{\alpha}}$, and $\widehat{\widehat{\beta}}$ are the parameters of transformed NHMGM$(1,m,k^p,c)$ model constructed by the multiplication transformation data matrix $Y^{(1)}=(Y_1^{(1)},Y_2^{(1)},\ldots,Y_m^{(1)})^T$, then the parameters have the following properties:*

$$\begin{aligned}\widehat{\widehat{\gamma}}_{ij}&=\frac{\sigma_i}{\sigma_j}\widehat{\gamma}_{ij},\quad 1\leq i\leq m,\ 1\leq j\leq m,\\ \widehat{\widehat{\alpha}}_i&=\sigma_i\widehat{\alpha}_i,\quad 1\leq i\leq m,\\ \widehat{\widehat{\beta}}_i&=\sigma_i\widehat{\beta}_i,\quad 1\leq i\leq m.\end{aligned} \tag{30}$$

Proof. By (27) in Theorem 12, we obtain

$$\begin{pmatrix}\widehat{\widehat{\Gamma}}'\\ \widehat{\widehat{\alpha}}'\\ \widehat{\widehat{\beta}}'\end{pmatrix}=\left(\overline{M}^T\overline{M}\right)^{-1}\overline{M}^T\left(\overline{N}_1,\overline{N}_2,\ldots,\overline{N}_m\right), \tag{31}$$

where $\overline{M}$ and $\overline{N}_j$ $(j=1,2,\ldots,m)$ are defined as in (28) and (29), respectively. From the definition $\overline{M}$ in (28), we have

$$\overline{M}^T\overline{M} = \begin{pmatrix} \sum_{k=2}^{n} \sigma_1^2 \left(z_1^{(1)}(k)\right)^2 & \cdots & \sum_{k=2}^{n} \sigma_1\sigma_m z_1^{(1)}(k)\, z_m^{(1)}(k) & \sum_{k=2}^{n} k^p \sigma_1 z_1^{(1)}(k) & \sum_{k=2}^{n} \sigma_1 z_1^{(1)}(k) \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ \sum_{k=2}^{n} \sigma_1\sigma_m z_1^{(1)}(k)\, z_m^{(1)}(k) & \cdots & \sum_{k=2}^{n} \sigma_m^2 \left(z_m^{(1)}(k)\right)^2 & \sum_{k=2}^{n} k^p \sigma_m z_m^{(1)}(k) & \sum_{k=2}^{n} \sigma_m z_m^{(1)}(k) \\ \sum_{k=2}^{n} k^p \sigma_1 z_1^{(1)}(k) & \cdots & \sum_{k=2}^{n} k^p \sigma_m z_m^{(1)}(k) & \sum_{k=2}^{n} k^{2p} & \sum_{k=2}^{n} k^p \\ \sum_{k=2}^{n} \sigma_1 z_1^{(1)}(k) & \cdots & \sum_{k=2}^{n} \sigma_m z_m^{(1)}(k) & \sum_{k=2}^{n} k^p & n-1 \end{pmatrix}. \quad (32)$$

It is clear that $\overline{M}^T\overline{M}$ is a symmetric matrix. Set $\sigma = \sigma_1\sigma_2\cdots\sigma_m$. The property of the determinant yields

$$\begin{aligned} \det\left(\overline{M}^T\overline{M}\right) &= \left(\sigma_1\sigma_2\cdots\sigma_m\right)^2 \det\left(M^T M\right) \\ &= \sigma^2 \det\left(M^T M\right). \end{aligned} \quad (33)$$

From the definition of matrix inverse, we have

$$\left(\overline{M}^T\overline{M}\right)^{-1} = \frac{1}{\det\left(\overline{M}^T\overline{M}\right)} \left(\overline{M}^T\overline{M}\right)^{*}. \quad (34)$$

Let $\overline{Q}_{ij}$ be the algebraic cofactor of $\overline{M}^T\overline{M}$ and Q_{ij} be the algebraic cofactor of $M^T M$. We discuss the relation of $\overline{Q}_{ij}$ and Q_{ij} in 4 cases.

Case 1. If $1 \le i \le m$ and $1 \le j \le m$, then

$$\overline{Q}_{ij} = (-1)^{(i+j)} \det \begin{pmatrix} \sum_{k=2}^{n} \left(\sigma_1 z_1^{(1)}(k)\right)^2 & \cdots & \sum_{k=2}^{n} \sigma_1\sigma_{j-1} z_1^{(1)}(k)\, z_{j-1}^{(1)}(k) & \sum_{k=2}^{n} \sigma_1\sigma_{j+1} z_1^{(1)}(k)\, z_{j+1}^{(1)}(k) & \cdots & \sum_{k=2}^{n} \sigma_1 z_1^{(1)}(k) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \sum_{k=2}^{n} \sigma_1\sigma_{i-1} z_{i-1}^{(1)}(k)\, z_1^{(1)}(k) & \cdots & \sum_{k=2}^{n} \sigma_{i-1}\sigma_{j-1} z_{i-1}^{(1)}(k)\, z_{j-1}^{(1)}(k) & \sum_{k=2}^{n} \sigma_{i-1}\sigma_{j+1} z_{i-1}^{(1)}(k)\, z_{j+1}^{(1)}(k) & \cdots & \sum_{k=2}^{n} \sigma_{i-1} z_{i-1}^{(1)}(k) \\ \sum_{k=2}^{n} \sigma_1\sigma_{i+1} z_{i+1}^{(1)}(k)\, z_1^{(1)}(k) & \cdots & \sum_{k=2}^{n} \sigma_{i+1}\sigma_{j-1} z_{i+1}^{(1)}(k)\, z_{j-1}^{(1)}(k) & \sum_{k=2}^{n} \sigma_{i+1}\sigma_{j+1} z_{i+1}^{(1)}(k)\, z_{j+1}^{(1)}(k) & \cdots & \sum_{k=2}^{n} \sigma_{i+1} z_{i+1}^{(1)}(k) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \sum_{k=2}^{n} \sigma_1 z_1^{(1)}(k) & \cdots & \sum_{k=2}^{n} \sigma_{j-1} z_{j-1}^{(1)}(k) & \sum_{k=2}^{n} \sigma_{j+1} z_{j+1}^{(1)}(k) & \cdots & n-1 \end{pmatrix} = \frac{\sigma^2}{\sigma_i\sigma_j} Q_{ij}. \quad (35)$$

Case 2. If $1 \le i \le m$ and $m+1 \le j \le m+2$, then

$$\overline{Q}_{ij} = \frac{\sigma^2}{\sigma_i} Q_{ij}. \quad (36)$$

Case 3. If $m+1 \le i \le m+2$ and $1 \le j \le m$, then

$$\overline{Q}_{ij} = \frac{\sigma^2}{\sigma_j} Q_{ij}. \quad (37)$$

Case 4. If $m+1 \le i \le m+2$ and $m+1 \le j \le m+2$, then

$$\overline{Q}_{ij} = \sigma^2 Q_{ij}. \quad (38)$$

On the basis of 4 cases above, we deduce that $\overline{Q}_{ij}$ and Q_{ij} have the following relation:

$$\overline{Q}_{ij} = \begin{cases} \dfrac{\sigma^2}{\sigma_i \sigma_j} Q_{ij}, & 1 \le i \le m,\ 1 \le j \le m, \\ \dfrac{\sigma^2}{\sigma_i} Q_{ij}, & 1 \le i \le m,\ m+1 \le j \le m+2, \\ \dfrac{\sigma^2}{\sigma_j} Q_{ij}, & m+1 \le i \le m+2,\ 1 \le j \le m, \\ \sigma^2 Q_{ij}, & m+1 \le i \le m+2,\ m+1 \le j \le m+2. \end{cases} \quad (39)$$

Combining (34) and (39), we conclude that

$$\left(\overline{M}^T \overline{M}\right)^{-1} \overline{M}^T = \begin{pmatrix} \dfrac{\sum_{j=1}^m Q_{1j} z_j^{(1)}(2) + 2^p Q_{1,m+1} + Q_{1,m+2}}{\sigma_1} & \cdots & \dfrac{\sum_{j=1}^m Q_{1j} z_j^{(1)}(n) + n^p Q_{1,m+1} + Q_{1,m+2}}{\sigma_1} \\ \vdots & \ddots & \vdots \\ \dfrac{\sum_{j=1}^m Q_{m,j} z_j^{(1)}(2) + 2^p Q_{m,m+1} + Q_{m,m+2}}{\sigma_m} & \cdots & \dfrac{\sum_{j=1}^m Q_{m,j} z_j^{(1)}(n) + n^p Q_{m,m+1} + Q_{m,m+2}}{\sigma_m} \\ \sum_{j=1}^m Q_{m+1,j} z_j^{(1)}(2) + 2^p Q_{m+1,m+1} + Q_{m+1,m+2} & \cdots & \sum_{j=1}^m Q_{m+1,j} z_j^{(1)}(n) + n^p Q_{m+1,m+1} + Q_{m+1,m+2} \\ \sum_{j=1}^m Q_{m+2,j} z_j^{(1)}(2) + 2^p Q_{m+2,m+1} + Q_{m+2,m+2} & \cdots & \sum_{j=1}^m Q_{m+2,j} z_j^{(1)}(n) + n^p Q_{m+2,m+1} + Q_{m+2,m+2} \end{pmatrix}. \quad (40)$$

It follows that

$$\begin{pmatrix} \hat{\hat{\gamma}}_{11} & \hat{\hat{\gamma}}_{21} & \cdots & \hat{\hat{\gamma}}_{m1} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{\hat{\gamma}}_{1m} & \hat{\hat{\gamma}}_{2m} & \cdots & \hat{\hat{\gamma}}_{mm} \\ \hat{\hat{\alpha}}_1 & \hat{\hat{\alpha}}_2 & \cdots & \hat{\hat{\alpha}}_m \\ \hat{\hat{\beta}}_1 & \hat{\hat{\beta}}_2 & \cdots & \hat{\hat{\beta}}_m \end{pmatrix} = \begin{pmatrix} \hat{\gamma}_{11} & \dfrac{\sigma_2}{\sigma_1}\hat{\gamma}_{21} & \cdots & \dfrac{\sigma_m}{\sigma_1}\hat{\gamma}_{m1} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\sigma_1}{\sigma_m}\hat{\gamma}_{1m} & \dfrac{\sigma_2}{\sigma_m}\hat{\gamma}_{2m} & \cdots & \hat{\gamma}_{mm} \\ \sigma_1\hat{\alpha}_1 & \sigma_2\hat{\alpha}_2 & \cdots & \sigma_m\hat{\alpha}_m \\ \sigma_1\hat{\beta}_1 & \sigma_2\hat{\beta}_2 & \cdots & \sigma_m\hat{\beta}_m \end{pmatrix}. \quad (41)$$

Hence, (30) holds. □

From Theorem 13 we come to conclusions that parameters of the transformed NHMGM$(1, m, k^p, c)$ model are dependent on the amount of multiplication transformation. If we apply different multiplication transformations to the original data, then parameters of the transformed model $\hat{\hat{\gamma}}_{ij}$ are proportional to σ_i and are inversely proportional to σ_j, and parameters $\hat{\hat{\alpha}}_i$ and $\hat{\hat{\beta}}_i$ are proportional to σ_i. Therefore, it is not suitable to predict by applying different multiplication transformations to original data when constructing a NHMGM$(1, m, k^p, c)$ model.

4. Numerical Example Analysis

We employ a multiple variable nonhomogeneous data example to demonstrate effectiveness of the novel model in this part. To better reflect the nonhomogeneous superiority of NHMGM$(1, m, k^p, c)$ model, this paper chooses NHMGM-2, NHMGM-1, NMGM$(1, m, k)$ studied in [20], the traditional grey prediction model MGM$(1, m)$ discussed in [17], the most commonly used GM$(1, 1)$ model, and the single variable nonhomogeneous grey model NGM$(1, 1, k)$ proposed in [15] to compare the simulation and prediction results.

Example 1. Assume that $X^{(0)} = (X_1^{(0)}, X_2^{(0)})^T$ is a multiple variable nonhomogeneous data matrix. The original value of $X_1^{(0)}$ is $X_1^{(0)} = (x_1^{(0)}(1), x_1^{(0)}(2), \ldots, x_1^{(0)}(6)) =$

$(4.5, 11, 29, 82, 238, 696)$, and the original value of $X_2^{(0)}$ is $X_2^{(0)} = (x_2^{(0)}(1), x_2^{(0)}(2), \ldots, x_2^{(0)}(6)) = (5, 10.5, 28, 80, 235, 651)$. Construct NHMGM-2, NHMGM-1, NMGM$(1, m, k)$, MGM$(1, m)$, GM(1, 1), and NGM$(1, 1, k)$ with $(X_1^{(0)}, X_2^{(0)})^T$ to compare the accuracy of different models.

In order to compare the simulation and forecasting results of six different models, we divide the dataset into two parts, in-sample data from the first to the fifth data and out-of-sample data is the sixth.

By the first to the fifth original data of $X^{(0)}$, we obtain 1-AGO sequence $X^{(1)} = (X_1^{(1)}, X_2^{(1)})^T$, where

$$\begin{aligned} X_1^{(1)} &= \left(x_1^{(1)}(1), x_1^{(1)}(2), \ldots, x_1^{(1)}(5)\right) \\ &= (4.5, 15.5, 44.5, 126.5, 364.5), \\ X_2^{(1)} &= \left(x_2^{(1)}(1), x_2^{(1)}(2), \ldots, x_2^{(1)}(5)\right) \\ &= (5, 15.5, 43.5, 123.5, 358.5). \end{aligned} \tag{42}$$

Then we get the adjacent neighbour average matrix $Z^{(1)} = (Z_1^{(1)}, Z_2^{(1)})^T$, where

$$\begin{aligned} Z_1^{(1)} &= (10, 30, 85.5, 245.5), \\ Z_2^{(1)} &= (10.25, 29.5, 83.5, 241). \end{aligned} \tag{43}$$

In what follows, we construct six different grey forecasting models to compare the simulation and prediction accuracy of the model. The NHMGM-2 model can be constructed as follows:

$$\begin{aligned} \frac{dx_1^{(1)}(t)}{dt} &= 2.9721x_1^{(1)}(t) - 1.9938x_2^{(1)}(t) - 0.6122t^2 \\ &\quad + 4.1644, \\ \frac{dx_2^{(1)}(t)}{dt} &= 1.5595x_1^{(1)}(t) - 0.5688x_2^{(1)}(t) - 0.5481t^2 \\ &\quad + 2.9276. \end{aligned} \tag{44}$$

The NHMGM-1 model can be constructed as

$$\begin{aligned} \frac{dx_1^{(1)}(t)}{dt} &= 1.0087x_1^{(1)}(t) - 0.0233x_2^{(1)}(t) - 1.7259t \\ &\quad + 4.6035, \\ \frac{dx_2^{(1)}(t)}{dt} &= -0.1983x_1^{(1)}(t) + 1.1953x_2^{(1)}(t) - 1.5452t \\ &\quad + 3.3207. \end{aligned} \tag{45}$$

The NMGM$(1, m, k)$ model proposed in [20] is

$$\begin{aligned} \frac{dx_1^{(1)}(t)}{dt} &= -1.5029x_1^{(1)}(t) + 2.5181x_2^{(1)}(t) \\ &\quad + 0.0223t, \\ \frac{dx_2^{(1)}(t)}{dt} &= -2.01x_1^{(1)}(t) + 3.0286x_2^{(1)}(t) - 0.2841t. \end{aligned} \tag{46}$$

The MGM$(1, m)$ model discussed in [17] can be constructed as

$$\begin{aligned} \frac{dx_1^{(1)}(t)}{dt} &= -1.4626x_1^{(1)}(t) + 2.4772x_2^{(1)}(t) + 0.0728, \\ \frac{dx_2^{(1)}(t)}{dt} &= -2.4108x_1^{(1)}(t) + 3.434x_2^{(1)}(t) - 0.7355. \end{aligned} \tag{47}$$

We construct single variable grey forecasting model GM(1, 1) and NGM$(1, 1, k)$ model for X_1 and X_2, respectively. GM(1, 1) model can be constructed as follows:

$$\begin{aligned} \frac{dx_1^{(1)}(t)}{dt} &= 0.9666x_1^{(1)}(t) + 0.345, \\ \frac{dx_2^{(1)}(t)}{dt} &= 0.9756x_2^{(1)}(t) - 0.4676. \end{aligned} \tag{48}$$

The single variable nonhomogeneous grey prediction model NGM$(1, 1, k)$ studied in [15] is established:

$$\begin{aligned} \frac{dx_1^{(1)}(t)}{dt} &= 0.9677x_1^{(1)}(t) + 0.0318t, \\ \frac{dx_2^{(1)}(t)}{dt} &= 0.98x_2^{(1)}(t) - 0.2778t. \end{aligned} \tag{49}$$

We use the absolute relative percent error (ARPE) and mean absolute percentage error (MAPE) to evaluate the accuracy of the model. The absolute relative percent error (ARPE) of the model is

$$\text{ARPE} = \frac{\left|\hat{x}_j^{(0)}(k) - x_j^{(0)}(k)\right|}{x_j^{(0)}(k)} \times 100\%. \tag{50}$$

The mean absolute percentage error (MAPE) is

$$\text{MAPE} = \frac{1}{n-1} \sum_{k=2}^{n} \frac{\left|\hat{x}_j^{(0)}(k) - x_j^{(0)}(k)\right|}{x_j^{(0)}(k)} \times 100\%. \tag{51}$$

In order to find the best fitted model, we compare the actual values with simulated and forecasted values done by six models, and two criteria ARPE and MAPE are employed to evaluate the accuracy of the model. By calculating, we obtain the simulation and prediction values of X_1 and X_2 done by NHMGM-2, NHMGM-1, NMGM$(1, m, k)$, MGM$(1, m)$, GM(1, 1), and NGM$(1, 1, k)$ models. The actual, simulated,

Table 1: Simulated and forecasted values of six different grey forecasting models for X_1.

$X_1^{(0)}$	NHMGM-2	NHMGM-1	NMGM	MGM	GM	NGM
4.5	4.5	4.5	4.5	4.5	4.5	4.5
11	11.44	11.21	9.90	9.51	7.91	7.42
29	29.86	27.14	25.65	23.59	20.80	19.58
82	82.92	69.85	66.68	59.50	54.69	51.59
238	242.03	184.39	176.65	153.02	143.78	135.84
Forecast						
696	732.49	491.58	476.67	401.99	378.02	357.58

Table 2: Simulated and forecasted values of six different grey forecasting models for X_2.

$X_2^{(0)}$	NHMGM-2	NHMGM-1	NMGM	MGM	GM	NGM
5	5	5	5	5	5	5
10.5	10.78	10.58	9.69	9.07	7.47	7.65
28	28.14	25.72	24.96	22.65	19.82	19.92
80	77.83	66.34	65.32	57.57	52.58	52.59
235	225.57	175.33	174.45	149.34	139.49	139.66
Forecast						
651	676.12	467.78	473.88	395.84	370.05	371.63

Table 3: ARPE and MAPE of six different grey forecasting models for X_1(%).

k	NHMGM-2	NHMGM-1	NMGM	MGM	GM	NGM
2	4.00	1.91	10.00	13.55	28.09	32.55
3	2.97	6.41	11.55	18.66	28.28	32.48
4	1.12	14.82	18.68	27.44	33.30	37.09
5	1.69	22.53	25.78	35.71	39.59	42.92
MAPE	2.45	11.42	16.50	23.84	32.32	36.26
6	5.24	29.37	31.51	42.24	45.69	48.62

and forecasted values of X_1 and X_2 are listed in Tables 1 and 2, and ARPE and MAPE of X_1 and X_2 are presented in Tables 3 and 4, respectively.

Table 3 shows that MAPE of X_1 simulated by NHMGM-2 model is 2.45% and the ARPE of forecasted value is 5.24%, which presents the smallest MAPE and ARPE compared with the other five models. The MAPE of NHMGM-1 model for X_1 is 11.42%, which is also better than NMGM, MGM, GM, and NGM models. Table 4 indicates that the MAPE of NHMGM-2 model for X_2 is 2.47% and ARPE of the forecasted value is 3.86%, which demonstrates superior effect for both in-sample data and out-of-sample data. The MAPE of NHMGM-1 model for X_2 is 12.84%, which is more accurate than NMGM, MGM, GM, and NGM models.

Figures 1 and 2 depict the fitting results of the simulated and forecasted curves of X_1 and X_2 done by six different models, and their comparison results of MAPE distribution are presented in Figures 3 and 4. As can be seen from Figures 1 and 2, NHMGM-2 model's curve almost agrees with the actual values of X_1 and X_2, which shows stable and ideal simulation and forecasting results. Figures 1–4 also demonstrate that the simulation results of NHMGM-2 are superior to NHMGM-1 model, the performance of NMGM$(1, m, k)$ model studied in [20] is superior to MGM$(1, m)$ in [17], and MGM is better than GM(1,1) and NGM$(1, 1, k)$ [15] models. It can be seen that NHMGM-2 outperforms the grey models NHMGM-1, NMGM$(1, m, k)$, MGM$(1, m)$, GM(1,1), and NGM$(1, 1, k)$.

As can be seen from the above example, the single variable nonhomogeneous grey prediction model NGM$(1, 1, k)$ provides unsatisfactory simulation and prediction results. On the contrary, NHMGM-2 gives better simulation results and follows the tendency of numerical data sequences compared with the other grey models. The example indicates that NHMGM-2 model is the best model among the six models. Hence, it is necessary and useful to expound nonhomogeneous multivariable grey forecasting models.

From the above analysis we know that NHMGM-2 model markedly promotes the simulation and prediction performance compared with the other grey prediction models. Thus, it can be concluded that NHMGM$(1, m, k^p, c)$ is a highly competitive grey forecasting tool for the nonhomogeneous multivariable exponential data sequences. It can also be concluded that the most appropriate model structure can be chosen according to the data characteristics of the modeling sequences, and the chosen model can better catch the tendency of integral development and individual variation of the original data. Therefore, NHMGM$(1, m, k^p, c)$ model with the flexible structure is a reliable prediction model for predicting the nonhomogeneous multivariable exponential data sequences.

Table 4: ARPE and MAPE of six different grey forecasting models for X_2(%).

k	NHMGM-2	NHMGM-1	NMGM	MGM	GM	NGM
2	2.67	0.76	7.71	13.62	28.86	27.14
3	0.50	8.14	10.86	19.11	29.21	28.86
4	2.71	17.08	18.35	28.04	34.28	34.26
5	4.01	25.39	25.77	36.45	40.64	40.57
MAPE	2.47	12.84	15.67	24.31	33.25	32.71
6	3.86	28.14	27.21	39.20	43.16	42.91

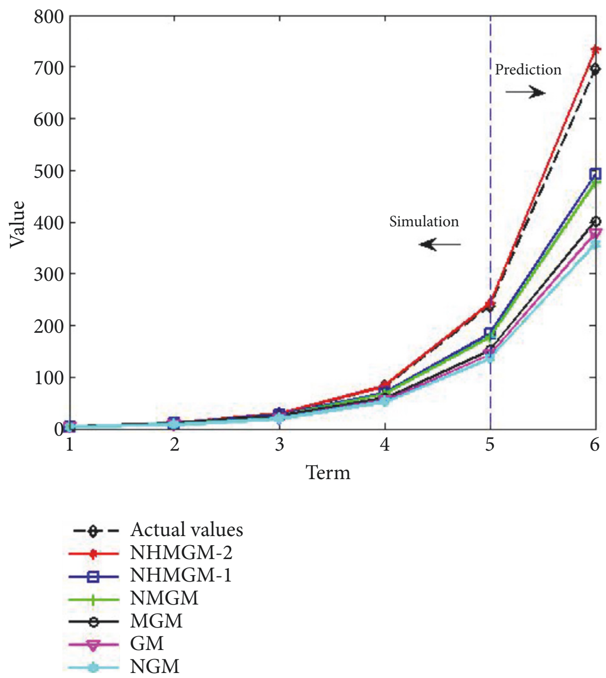

Figure 1: Comparison of actual and simulation values of six different models for X_1.

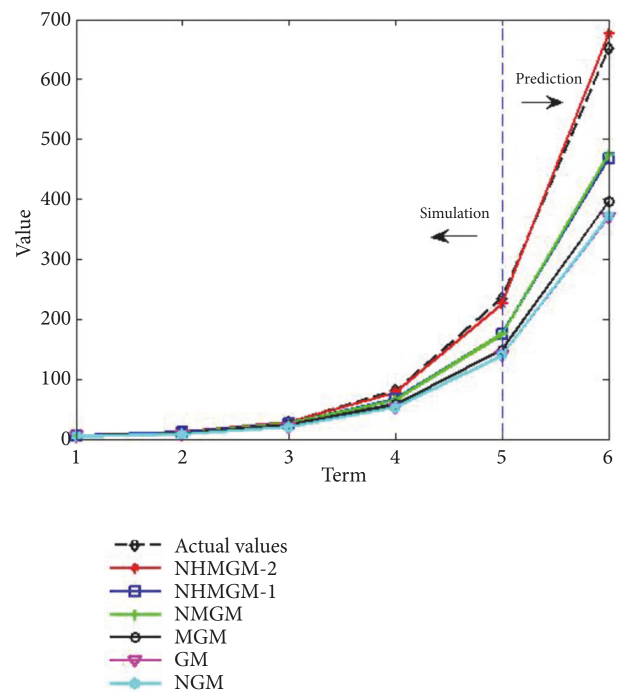

Figure 2: Comparison of actual and simulation values of six different models for X_2.

5. Discussion

In the given example, two criteria ARPE and MAPE are employed to test the accuracy of six different grey forecasting models; NHMGM-2 model gives better simulation and prediction results than those of basic methods, though NHMGM-2 model is a special case of NHMGM$(1, m, k^p, c)$ model. Tables 1–4 and Figures 1–4 all suggest that NHMGM-2 model is superior to NHMGM-1, NMGM$(1, m, k)$, MGM$(1, m)$, GM$(1, 1)$, and NGM$(1, 1, k)$ models.

It is shown that NHMGM$(1, m, k^p, c)$ model proposed in this paper has advantages of flexible structure, and many grey forecasting models can be derived from it. For example, the multivariable NHMGM$(1, m, k^p, c)$ model becomes single variable GM$(1, 1, k^p)$ model when m=1, which was studied in [27]. NHMGM$(1, m, k^p, c)$ model degrades to MGM$(1, m)$ model while p=0, which was researched in [17, 26]. Hence, it is useful to establish a model with flexible structure and investigate its properties for further applications of grey prediction models.

The modeling mechanism of NHMGM$(1, m, k^p, c)$ model and derived models are discussed in this paper; however, this study on NHMGM$(1, m, k^p, c)$ model is not comprehensive. The problem such as finding an ideal algorithm for the optimal p of the model still needs further study; certain intelligent optimization algorithms such as nonlinear programming and particle swarm optimization could be introduced into NHMGM$(1, m, k^p, c)$ model in order to determine the optimal p. The other problems needing to be solved include how to combine self-memory principle, Markov chain, and other optimization techniques with NHMGM$(1, m, k^p, c)$ model for the purpose of further improving the prediction accuracy.

6. Conclusions

The multivariable grey forecasting model has been successfully adopted in many fields; however, the precision

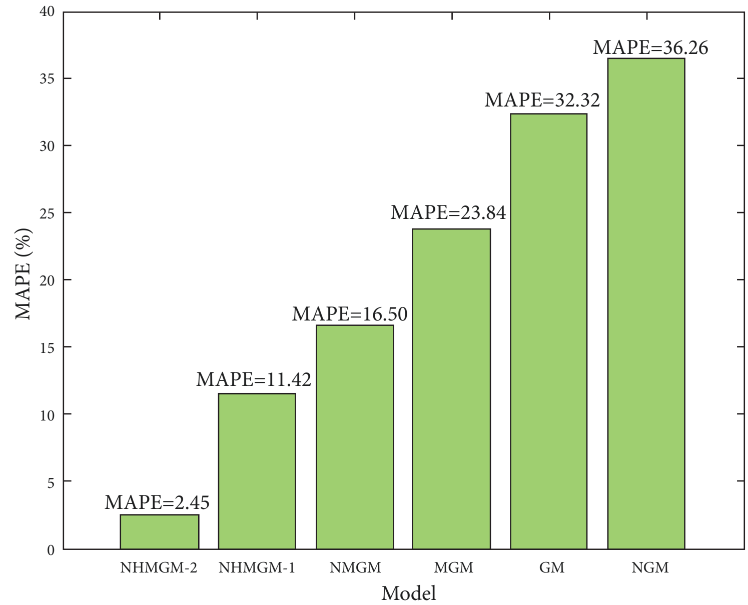

Figure 3: The MAPE distribution of six different models for X_1.

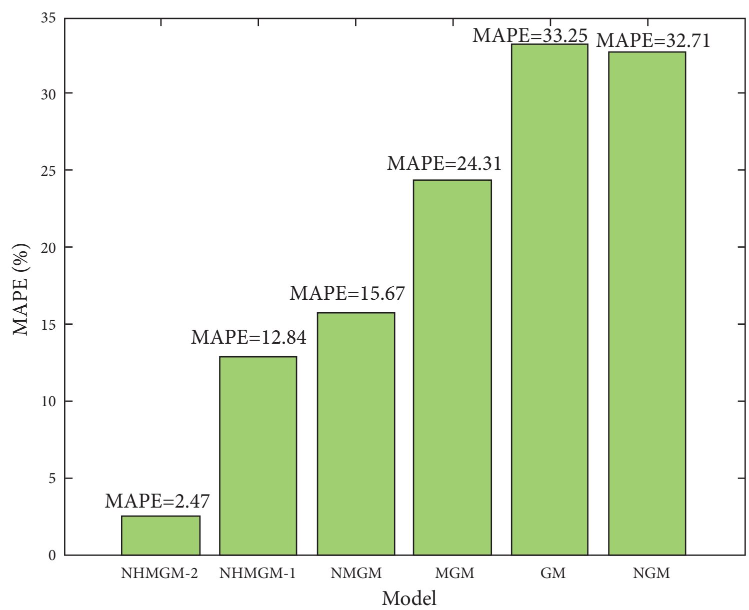

Figure 4: The MAPE distribution of six different models for X_2.

of grey forecasting model is not always satisfactory. Aiming to enhance the prediction accuracy, a novel nonhomogeneous multivariable grey forecasting model named NHMGM$(1, m, k^p, c)$ is proposed in this paper.

This novel nonhomogeneous multivariable grey forecasting NHMGM$(1, m, k^p, c)$ model, based on the multiple variable nonhomogeneous exponential data sequences, is an extension and complement to the existing multivariable grey prediction model MGM$(1, m)$. Parameters of the novel model are obtained by using least square method, the time response function of the novel model is given, and several kinds of grey models derived from NHMGM$(1, m, k^p, c)$ model are discussed. A numerical example is employed to demonstrate the effectiveness of the novel model, six different grey prediction models which contain four multivariable grey models and two single variable grey models are established for modeling, and two popular accuracy test (ARPE and MAPE) are adopted to verify stability of the novel model. The numerical example demonstrates that NHMGM-2 performs a higher simulation and prediction accuracy compared with classic models.

In order to grasp properties of NHMGM$(1, m, k^p, c)$ model, the multiplication transformation properties are investigated. It is proved that parameters $\widehat{\widehat{\Gamma}}, \widehat{\widehat{\alpha}}$, and $\widehat{\widehat{\beta}}$ of the transformed model are dependent on the amount of multiplication transformation, which means that different multiplication transformations can lead to variation of parameters. Therefore, we cannot apply different transformations to construct a transformed NHMGM$(1, m, k^p, c)$ model since parameters depend on the amount of multiplication transformation.

The NHMGM$(1, m, k^p, c)$ model is a type of multivariable grey forecasting model and can be used to simulate and forecast multiple variable nonhomogeneous data sequences. In future studies, the accuracy of NHMGM$(1, m, k^p, c)$ model is expected to be improved by combining other approaches, and it is believed that the novel model will be applied more widely in the application of various fields.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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