Research Article

Existence and Multiplicity of Solutions for Sublinear Schrödinger Equations with Coercive Potentials

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In this study, we consider the following sublinear Schrödinger equations $\begin{cases} -\Delta u + V(x)u = f(x, u), & \text{for } x \in \mathbb{R}^N, \\ u(x) \rightarrow 0, & \text{as } |x| \rightarrow \infty, \end{cases}$ where $f(x, u)$ satisfies some sublinear growth conditions with respect to u and is not required to be integrable with respect to x . Moreover, V is assumed to be coercive to guarantee the compactness of the embedding from working space to $L^p(\mathbb{R}^N)$ for all $p \in [1, 2^*)$. We show that the abovementioned problem admits at least one solution by using the linking theorem, and there are infinitely many solutions when $f(x, u)$ is odd in u by using the variant fountain theorem.

1. Introduction and Main Results

In this study, we consider the following nonlinear Schrödinger equations:

$$\begin{cases} -\Delta u + V(x)u = f(x, u), & \text{for } x \in \mathbb{R}^N, \\ u(x) \rightarrow 0, & \text{as } |x| \rightarrow \infty, \end{cases} \quad (1)$$

where $f(x, u) \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$. The existence and multiplicity of solutions for (1) have attracted much attention of mathematicians. As we know, the existence of solutions for problem (1) is important in finding the solutions for

$$i\hbar \frac{\partial \varphi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \varphi + W(x)\varphi - b(x, |\varphi|)\varphi. \quad (2)$$

With the development of the variational methods in past decades, many mathematicians used this tool to show the existence and multiplicity of solutions for differential equations with superlinear, sublinear, asymptotic linear, and mixed nonlinearities (see [1–25]). In this study, we mainly consider the sublinear case. Many mathematicians have considered this topic with $f(x, t)$ being integrable in x . For example, in 2012, Zhang and Wang [23] assumed V to be coercive and $f(x, t)$ satisfies the following:

(F0) $f(x, t) = \mu h(x)|t|^{\mu-2}t$, where $\mu \in (1, 2)$ is a constant and $h: \mathbb{R}^N \rightarrow \mathbb{R}$ is a positive continuous function such that $h \in L^{2/(2-\mu)}(\mathbb{R}^N)$.

In addition, they obtained infinitely many nontrivial solutions for (1). By using the dual fountain theorem, Bahrouni [1] showed (1) admitted infinitely many solutions when $f(x, t) = a(x)|u|^{p-1}u + b(x)|u|^{q-1}u$, where $0 < q < p < 1$ and a, b satisfy the following conditions:

(F1) $a \in L^\infty(\mathbb{R}^N) \cap L^{2/(1-p)}(\mathbb{R}^N)$ and there exists $\beta > 0$ such that $a(x) < -\beta$ for any $x \in \mathbb{R}^N$.

(F2) $b \in L^{(p+1)/(p-q)}(\mathbb{R}^N)$ and $b > 0$.

Later, Chen and Tang [6] relaxed assumption (F0), and they obtained the following theorem.

Theorem 1. (see [6]) Suppose the following conditions hold

(F3) $V \in C(\mathbb{R}^N, \mathbb{R})$ and $\inf_{\mathbb{R}^N} V(x) > 0$.

(F4) For any $(x, t) \in \mathbb{R}^N \times \mathbb{R}$, there exist $1 < r_1 < r_2 < 2$, $a_1 \in L^{2/(2-r_1)}(\mathbb{R}^N, [0, +\infty))$, and $a_2 \in L^{2/(2-r_2)}(\mathbb{R}^N, [0, +\infty))$ such that

$$|f(x, t)| \leq r_1 a_1(x) |t|^{r_1-1} + r_2 a_2(x) |t|^{r_2-1}, \quad (3)$$

for $(x, t) \in \mathbb{R}^N \times \mathbb{R}$.

(F5) There exist an open set $J \in \mathbb{R}^N$ and $\delta, \eta > 0$, $r_3 \in (1, 2)$ such that $F(x, t) \geq \eta |t|^{r_3}$, $\forall (x, t) \in J \times [-\delta, \delta]$, where $F(x, t) := \int_0^t f(x, s) ds$, $x \in \mathbb{R}^N$, $t \in \mathbb{R}$.

Then, problem (1) possesses at least one nontrivial solution. Moreover, if $f(x, -u) = -f(x, u)$ for all $(x, u) \in \mathbb{R}^N \times \mathbb{R}$, then problem (1) possesses infinitely many nontrivial solutions.

In Theorem 1, V is not required to be coercive, and the authors took advantage of the decay of $a_i(x)$ to show the convergence of the asymptotic critical points sequence. Recently, Zhang et al. [24] introduced some different sublinear conditions of f to obtain infinitely many nontrivial solutions for (1). They assumed the weight functions to be integrable in $L^{2^*/(2^*-\zeta)}(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$ with $\zeta \in (1, 2)$ and $p > (N/2)$.

After then, Hou et al. [7] considered the fractional Schrödinger-type equations with the similar conditions to (F4) and (F5). In the abovementioned studies, the results depend on the integrability of the weight functions. There are still some mathematicians considered the case without any integrable weight functions. In 2009, Benrhouna et al. [3] considered the following problem:

$$\begin{cases} -\Delta u + V(x)u + a(x)|u|^p \operatorname{sgn}(u) = g(x), & \text{for } x \in \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N) \cap L^{p+1}(\mathbb{R}^N), \end{cases} \quad (4)$$

where $V(x) = a(x) = 1$ and $g \in L^2(\mathbb{R}^N) \setminus \{0\}$. They used the minimizing method on a Nehari manifold and concentration-compactness principle due to Lions [10, 11] to show the existence of unique positive solution for (2). Then, Benrhouna [2] improved this result by assuming the following:

(F6) $V \in L^\infty(\mathbb{R}^N)$ and $\lim_{|x| \rightarrow +\infty} V = v_\infty \geq 0$.

(F7) $a \in L^\infty(\mathbb{R}^N)$, $\lim_{|x| \rightarrow +\infty} a = a_\infty > 0$ a.e. $x \in \mathbb{R}^N$.

In 2013, Bao and Han [5] considered (2), where $g \equiv 0$ and V is a sign-changing function while 0 lies in the spectrum of $-\Delta + V(x)$ and obtained infinitely many solutions for this resonant problem. In a recent study, Isernia [8] generalizes the results in [2] to a class of sublinear fractional Schrödinger equations. Motivated by the abovementioned studies, we introduce some new sublinear growth conditions without weight functions to obtain the existence and multiplicity of solutions for (1). First of all, we recall the following coercive condition which is proposed by Wang and Han [14] to show the existence of infinitely many small solutions for a class of Kirchhoff equations with local sublinear nonlinearities by using the symmetric mountain pass theorem.

(V) There are constants $\rho > 0$ and $\alpha > N$ such that for any $b > 0$

$$\lim_{|y| \rightarrow +\infty} \operatorname{meas} \left(\left\{ x \in B_\rho(y) : \frac{V(x)}{|x|^\alpha} \leq b \right\} \right) = 0. \quad (5)$$

Remark 1. Condition (V) is introduced by Wang and Han in [14] to obtain a compactness embedding theorem. In addition, the authors assumed that V possesses the global positive lower bound. Although we do not assume that V is positive in the whole space in this study, the compactness embedding theorem still holds in the same way.

Now, we state our main results.

Theorem 2. Suppose that $V \in C(\mathbb{R}^N, \mathbb{R})$ satisfies (V) and f satisfies the following:

(f₁) There exist $b_1 > 0$ and $1 < \beta < 2$ such that

$$F(x, t) \geq b_1 |t|^\beta, \quad \text{for all } (x, t) \in \mathbb{R}^N \times \mathbb{R}. \quad (6)$$

(f₂) $F(x, 0) = 0$ for all $x \in \mathbb{R}^N$ and there exists $M_0 > 0$ such that

$$|f(x, t)| \leq M_0 (1 + |t|), \quad \text{for all } x \in \mathbb{R}^N \text{ and } t \in \mathbb{R}. \quad (7)$$

(f₃) $\lim_{|t| \rightarrow \infty} ((f(x, t))/t) = 0$ uniformly in x .

(f₄) Letting $\tilde{F}(x, t) = 2F(x, t) - f(x, t)t$, there exist $b_2, r_1 > 0$, and $\mu > 1$ such that

$$\tilde{F}(x, t) \geq b_2 |t|^\mu, \quad \text{for } |t| \geq r_1 \text{ and } x \in \mathbb{R}^N. \quad (8)$$

(f₅) There exists $b_3 > 0$ such that

$$\tilde{F}(x, t) \geq -b_3 |t|, \quad \text{for } |t| \leq r_1 \text{ and } x \in \mathbb{R}^N. \quad (9)$$

Then, there exists at least one nontrivial solution for problem (1).

Theorem 3. Suppose that the conditions of Theorem 2 hold and

(f₆) $\lim_{|t| \rightarrow \infty} \int_{\mathbb{R}^N} F(x, t) dx = +\infty$.

(f₇) $F(x, -t) = F(x, t)$ for any $x \in \mathbb{R}^N$ and $t \in \mathbb{R}$.

Then, there exist infinitely many solutions for problem (1).

Remark 2. In Theorem 2, $F(x, t)$ is sublinear at infinity and can be a sign-changing function. Let

$$F(x, t) = |t|^{5/4} - 3|t|^{3/2} + |t|^{7/4}. \quad (10)$$

Then, $F(x, t)$ satisfies the conditions of Theorem 2, but the conditions in [1–8, 20, 23–25].

2. Preliminaries

First, we show the decomposition of the space. Similar to [15], we denote by $\mathcal{A}$ the self-adjoint extension of the operator $-\Delta + V(x)$ with the domain $D(\mathcal{A}) \subset L^2(\mathbb{R}^N, \mathbb{R})$. Let $\{\mathcal{E}(\vartheta): -\infty < \vartheta < \infty\}$ be the spectral family of $\mathcal{A}$ and $|\mathcal{A}|^{1/2}$ be the square root of $\mathcal{A}$. Define $u = \text{id} - \mathcal{E}(0) - \mathcal{E}(-0)$. u commutes with $\mathcal{A}$, and $\mathcal{A} = u|\mathcal{A}|$ is the polar decomposition of $\mathcal{A}$. Let $E = D(|\mathcal{A}|^{1/2})$ be the domain of $|\mathcal{A}|^{1/2}$. Define the norm on E as

$$(u, w)_o = (|\mathcal{A}|^{1/2}u, |\mathcal{A}|^{1/2}w)_{L^2} + (u, w)_{L^2}, \quad (11)$$

$$\|u\|_o = (u, u)_o^{1/2},$$

where and in the sequel, $(\cdot, \cdot)_{L^2}$ denotes the usual $L^2(\mathbb{R}^N)$ inner product. Then, E is a Hilbert space. The following compact embedding theorem is obtained with the condition (V).

Lemma 1. *Suppose that (V) holds. Then, E is compact embedded into $L^p(\mathbb{R}^N)$ for any $p \in [1, 2^*)$.*

Proof. The proof of this lemma is similar to the proof of Lemma 1 in [14]. $\square$

Remark 3. From Lemma 1, for any $p \in [1, 2^*)$, there exists a constant $\eta_p > 0$ such that

$$\|u\|_p \leq \eta_p \|u\|, \quad \text{for all } u \in E. \quad (12)$$

It follows from Lemma 1 that the spectrum $\sigma(\mathcal{A})$ consists of eigenvalues numbered in $\lambda_1 \leq \lambda_2 \leq \dots \rightarrow \infty$, and a corresponding system of eigenfunctions (e_n) ($\mathcal{A}e_n = \lambda_n e_n$) forms an orthogonal basis in L^2 . Let $n^- = \#\{i | \lambda_i < 0\}$, $n^0 = \#\{i | \lambda_i = 0\}$, and $\bar{n} = n^- + n^0$. Set $E^- = \text{span}\{e_1, \dots, e_{n^-}\}$, $E^0 = \text{span}\{e_{n^-+1}, \dots, e_{\bar{n}}\}$, and $E^+ = \text{span}\{e_{\bar{n}+1}, \dots\}$. Then, $E = E^- \oplus E^0 \oplus E^+$ with

$$(u, w) = (|\mathcal{A}|^{1/2}u, |\mathcal{A}|^{1/2}w)_{L^2} + (u^0, w^0)_{L^2}, \quad (13)$$

$$\|u\| = (u, u)^{1/2},$$

where $u = u^- + u^0 + u^+$ and $w = w^- + w^0 + w^+ \in E = E^- \oplus E^0 \oplus E^+$.

The following lemma can easily be obtained by (f_2) . We omit its proof here.

Lemma 2. *Suppose that (f_2) holds, then for any $\varepsilon > 0$, there exists $d_\varepsilon > 0$ such that*

$$|F(x, t)| \leq 2M_0(|t| + t^2), \quad \text{for all } x \in \mathbb{R}^N \text{ and } t \in \mathbb{R}. \quad (14)$$

Let $I: E \rightarrow \mathbb{R}$ be the corresponding functional defined by

$$I(u) = \frac{1}{2} \left(\|u^+\|^2 - \|u^-\|^2 \right) - \int_{\mathbb{R}^N} F(x, u) dx. \quad (15)$$

It is known that the critical points of I are the solutions for problem (1). The following lemma shows that I is differentiable in E .

Lemma 3. *Suppose the conditions of Theorem 2 hold, then $I \in C^1(E, \mathbb{R})$ and*

$$\begin{aligned} \langle I'(u), v \rangle &= \int_{\mathbb{R}^N} ((u, v) + (V(x)u, v)) dx \\ &\quad - \int_{\mathbb{R}^N} f(x, u)v dx, \quad \forall u, v \in E, \end{aligned} \quad (16)$$

which yields that

$$\langle I'(u), u \rangle = \|u^+\|^2 - \|u^-\|^2 - \int_{\mathbb{R}^N} f(x, u)u dx, \quad \forall u \in E. \quad (17)$$

Proof. By (12), it is easy to see that I is well defined. Subsequently, we show (16) holds. For any $u, v \in E$, $\varepsilon > 0$, and $0 < \theta < 1$, it follows from Lemma 2 and (12) that

$$\begin{aligned} &\int_{\mathbb{R}^N} \max_{s \in [0,1]} |f(x, u + s\theta v)| v dx \\ &\leq (\varepsilon + d_\varepsilon) \int_{\mathbb{R}^N} \max_{s \in [0,1]} (1 + |u + s\theta v|) |v| dx \\ &\leq (\varepsilon + d_\varepsilon) \int_{\mathbb{R}^N} (1 + |u| + |v|) |v| dx \\ &\leq (\varepsilon + d_\varepsilon) (\|v\|_1 + \|u\|_2 \|v\|_2 + \|v\|_2^2) \\ &\leq (\varepsilon + d_\varepsilon) (\eta_1 \|v\| + \eta_2^2 \|u\| \|v\| + \eta_2^2 \|v\|^2) \\ &< \infty. \end{aligned} \quad (18)$$

It follows from the definition of I , (18), the mean value theorem, and the Lebesgue's dominated convergence theorem that there exists $0 < \theta_0 < 1$ such that

$$\begin{aligned} \langle I'(u), v \rangle &= \lim_{s \rightarrow 0^+} \frac{I(u + sv) - I(u)}{s} \\ &= \int_{\mathbb{R}^N} \nabla u \nabla v dx + \int_{\mathbb{R}^N} V(x)uv dx \\ &\quad - \lim_{s \rightarrow 0^+} \frac{1}{s} \int_{\mathbb{R}^N} (F(x, u + sv) - F(x, u)) dx \\ &= \int_{\mathbb{R}^N} \nabla u \nabla v dx + \int_{\mathbb{R}^N} V(x)uv dx \\ &\quad - \lim_{s \rightarrow 0^+} \int_{\mathbb{R}^N} f(x, u + s\theta_0 v) v dx \\ &= \int_{\mathbb{R}^N} \nabla u \nabla v dx + \int_{\mathbb{R}^N} V(x)uv dx \\ &\quad - \int_{\mathbb{R}^N} f(x, u) v dx. \end{aligned} \quad (19)$$

Then, we obtain (16). Subsequently, we show I' is continuous. Let

$$\langle \psi'(u), v \rangle = \int_{\mathbb{R}^N} f(x, u)v dx, \quad \text{for any } v \in E. \quad (20)$$

Assume that $u_n \rightharpoonup u$ in E as $n \rightarrow +\infty$. By Lemma 1, we obtain that

$$u_n \rightarrow u \text{ in } L^p(\mathbb{R}^N), \quad \text{for all } p \in [1, 2^*), \quad (21)$$

which implies that for any $\varepsilon > 0$ and $v \in E$ with $\|v\| = 1$, there exists $T > 0$ such that

$$\begin{aligned} \int_{|x| \geq T} u_n^2 dx &\leq \varepsilon, \\ \int_{|x| \geq T} u^2 dx &\leq \varepsilon, \\ \int_{|x| \geq T} |v| dx &\leq \varepsilon, \\ \int_{|x| \geq T} u_n^2 dx &\leq 2 \left(\int_{|x| \geq T} (u_n - u)^2 dx + \int_{|x| \geq T} u^2 dx \right) \leq 3\varepsilon, \end{aligned} \quad (22)$$

for n large enough. Since $f(x, t)$ is continuous in t , we get

$$\int_{|x| < T} |f(x, u_n) - f(x, u)|^2 dx \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (23)$$

Then, for n large enough, we have

$$\begin{aligned} \sup_{\|v\|=1} |\psi'(u_n)v - \psi'(u)v| &= \sup_{\|v\|=1} \left| \int_{\mathbb{R}^N} (f(x, u_n) - f(x, u))v dx \right| \\ &\leq \sup_{\|v\|=1} \int_{\mathbb{R}^N} |f(x, u_n) - f(x, u)| |v| dx \\ &= \sup_{\|v\|=1} \left(\int_{|x| < T} |f(x, u_n) - f(x, u)| |v| dx \right. \\ &\quad \left. + \int_{|x| \geq T} |f(x, u_n) - f(x, u)| |v| dx \right) \\ &\leq \sup_{\|v\|=1} \left(\eta_2 \left(\int_{|x| < T} |f(x, u_n) - f(x, u)|^2 dx \right)^{1/2} \right. \\ &\quad \left. + M_0 \int_{|x| \geq T} (2 + |u_n| + |u|) |v| dx \right) \\ &\leq (2M_0 + 1)\eta_2\varepsilon + 2M_0\varepsilon, \end{aligned} \quad (24)$$

which yields that $\psi'(u_n)v - \psi'(u)v \rightarrow 0$ as $u_n \rightarrow u$. Hence, ψ' is weak continuous. Therefore, ψ is compact and I' is continuous. The proof of this lemma is finished. $\square$

3. Proof of Theorem 1

In order to apply the link theorem by Rabinowitz (see [6], Theorem 5.29) to obtain the critical points of I , we assume $J = -I$ in the following proof. Obviously, I and J share the same critical points.

Lemma 4. Suppose the conditions of Theorem 1 hold, then J satisfies the (PS) condition.

Proof. Let $\{u_n\} \subset E$ be a (PS) sequence, then there exists a constant $M_1 > 0$ such that

$$\begin{aligned} |J(u_n)| &\leq M_1, \\ \|J'(u_n)\| &\leq M_1. \end{aligned} \quad (25)$$

First, we show that $\{u_n\}$ is bounded in E . Arguing in an indirect way, we assume that $u_n \rightarrow +\infty$ as $n \rightarrow \infty$. It follows from (25), (f_4) and (f_5) that

$$\begin{aligned} M_1(2 + \|u_n\|) &\geq 2J(u_n) - \langle J'(u_n), u_n \rangle \\ &\geq \int_{\mathbb{R}^N} \tilde{F}(x, u_n) dx \\ &\geq -b_3 \int_{|u_n| \leq r_1} |u_n| dx + b_2 \int_{|u_n| \geq r_1} |u_n|^\mu dx. \end{aligned} \quad (26)$$

Let

$$v_n = \begin{cases} u_n, & \text{for } |u_n| \leq r_1, \\ 0, & \text{for } |u_n| \geq r_1, \end{cases} \quad (27)$$

$$w_n = u_n - v_n,$$

then, we have

$$b_2 \|w_n\|_\mu^\mu \leq M_1(2 + \|u_n\|) + b_3 \eta_1 \|u_n\|. \quad (28)$$

Since $E^- \oplus E^0$ is a finite-dimensional subspace of E , there exists $M_2 > 0$ such that

$$\begin{aligned} \|u_n^- + u_n^0\|_2^2 &= \int_{|u_n| \leq r_1} |u_n^- + u_n^0|^2 dx + \int_{|u_n| \geq r_1} |u_n^- + u_n^0|^2 dx \\ &\leq \int_{|u_n| \leq r_1} |u_n^- + u_n^0| |u_n| dx + \int_{|u_n| \geq r_1} |u_n^- + u_n^0| |u_n| dx \\ &\leq r_1 \int_{|u_n| \leq r_1} |u_n^- + u_n^0| dx + \int_{|u_n| \geq r_1} |u_n^- + u_n^0| |w_n| dx \\ &\leq r_1 \|u_n^- + u_n^0\|_1 + \|u_n^- + u_n^0\|_{\mu/(\mu-1)} \|w_n\|_\mu \\ &\leq M_2 \|u_n^- + u_n^0\|_2 (1 + \|w_n\|_\mu). \end{aligned} \quad (29)$$

Moreover, there exist $M_3, M_4 > 0$ such that

$$\begin{aligned} \|u_n^- + u_n^0\| &\leq M_3 \|u_n^- + u_n^0\|_2 \leq M_2 M_3 (1 + \|w_n\|_\mu) \\ &\leq M_4 (1 + \|u_n\|^{1/\mu}), \end{aligned} \quad (30)$$

which implies that

$$\frac{\|u_n^- + u_n^0\|}{\|u_n\|} \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (31)$$

Choosing $0 < s < \min\{1, (\mu/2)\}$, we can deduce from (26) and (12) that

$$\begin{aligned}
\int_{|u_n| \geq r_1} |u_n|^2 dx &= \int_{|u_n| \geq r_1} |u_n|^{2s} |u_n|^{2(1-s)} dx \\
&\leq \left(\int_{|u_n| \geq r_1} |u_n|^\mu dx \right)^{2s/\mu} \\
&\quad \cdot \left(\int_{|u_n| \geq r_1} |u_n|^{(2\mu(1-s))/(\mu-2s)} dx \right)^{(\mu-2s)/\mu} \\
&\leq M_5 \left(1 + \int_{|u_n| \leq r_1} |u_n| dx + \|u_n\| \right)^{2s/\mu} \\
&\quad \cdot \|u_n\|_{(2\mu(1-s))/(\mu-2s)}^{2(1-s)} \\
&\leq M_6 (1 + \|u_n\|)^{2s/\mu} \|u_n\|^{2(1-s)} \\
&\leq M_7 \left(\|u_n\|^{2(1-s)} + \|u_n\|^{2(s+\mu-s\mu)/\mu} \right), \tag{32}
\end{aligned}$$

for some $M_5, M_6, M_7 > 0$, where $\max\{2(1-s), 2(s+\mu-s\mu)/\mu\} < 2$. By (25), we obtain

$$\begin{aligned}
M_1 \|u_n\| &\geq \|J'(u_n)\| (1 + \|u_n^+\|) \\
&\geq \left| \|u_n^+\|^2 - \int_{\mathbb{R}^N} f(x, u_n) u_n^+ dx \right| \tag{33} \\
&\geq \|u_n^+\|^2 - \int_{\mathbb{R}^N} |f(x, u_n)| |u_n^+| dx.
\end{aligned}$$

Then, it follows from (f_2) , (32), and (33) that

$$\begin{aligned}
\|u_n^+\|^2 &\leq \int_{\mathbb{R}^N} |f(x, u_n)| |u_n^+| dx + M_1 \|u_n\| \\
&\leq M_0 \int_{\mathbb{R}^N} (1 + |u_n|) |u_n^+| dx + M_1 \|u_n\| \\
&\leq M_8 \left(\|u_n\| + \int_{|u_n| \leq r_1} |u_n| |u_n^+| dx + \int_{|u_n| \geq r_1} |u_n| |u_n^+| dx \right) \\
&\quad + M_1 \|u_n\| \\
&\leq M_8 \left(\|u_n\| + r_1 \int_{|u_n| \leq r_1} |u_n^+| dx + \left(\int_{|u_n| \geq r_1} |u_n|^2 dx \right)^{1/2} \right. \\
&\quad \cdot \left. \left(\int_{|u_n| \geq r_1} |u_n^+|^2 dx \right)^{1/2} \right) \\
&\leq M_9 \left(\|u_n\| + \|u_n^+\| \left(\|u_n\|^{1-s} + \|u_n\|^{(s+\mu-s\mu)/\mu} \right) \right) \\
&\leq M_9 \left(\|u_n\| + \|u_n\|^{2-s} + \|u_n\|^{(s+2\mu-s\mu)/\mu} \right), \tag{34}
\end{aligned}$$

for some $M_8, M_9 > 0$, which implies that $\|u_n^+\|/\|u_n\| \rightarrow 0$ as $n \rightarrow \infty$. Together with (31), we obtain $1 = \|u_n\|/\|u_n\| \rightarrow 0$ as $n \rightarrow \infty$, which is a contradiction. Hence, u_n is bounded in E . Then, there exists $u_0 \in E$ such that $u_n \rightharpoonup u_0$, which implies that $u_n \rightarrow u_0$ uniformly in $L^p(\mathbb{R}^N)$ for any $p \in [1, 2^*)$. Similar to (4), we deduce that

$$\begin{aligned}
\|u_n^+ + u_0^+\|^2 &= -\langle J'(u_n) - J'(u_0), u_n^+ - u_0^+ \rangle \\
&\quad + \int_{\mathbb{R}^N} (f(x, u_n) - f(x, u_0)) (u_n^+ - u_0^+) dx \\
&\leq \|J'(u_n)\| (\|u_n^+\| + \|u_0^+\|) + \langle J'(u_0), u_n^+ - u_0^+ \rangle \\
&\quad + \int_{\mathbb{R}^N} |f(x, u_n) - f(x, u_0)| |u_n^+ - u_0^+| dx \\
&\leq o(1) + M_0 \int_{\mathbb{R}^N} (2 + |u_n| + |u_0|) |u_n^+ - u_0^+| dx \\
&\leq o(1) + M_0 \left(\|u_n\|_2^2 + \|u_0\|_2^2 \right) \int_{\mathbb{R}^N} |u_n^+ - u_0^+|^2 dx \\
&\quad + 2M_0 \int_{\mathbb{R}^N} |u_n^+ - u_0^+| dx \\
&\rightarrow 0 \text{ as } n \rightarrow \infty, \tag{35}
\end{aligned}$$

which shows that $u_n \rightarrow u_0$ as $n \rightarrow \infty$ since $E^- \oplus E^0$ is of finite dimensions. Hence, J satisfies the (PS) condition. $\square$

Lemma 5. Suppose the conditions of Theorem 1 hold, then for any $m \in \mathbb{N}$, $E_m = E^- \oplus E^0 \oplus \text{span}\{e_{\bar{n}+1}, \dots, e_{\bar{n}+m}\}$, there are $\rho = \rho(m)$ and $\delta = \delta(m) > 0$ such that $J \geq 0$ on $B_\rho \cap E_m$ and $J|_{\partial B_\rho \cap E_m} \geq \delta$, where $B_\rho = \{u \in E : u \leq \rho\}$.

Proof. From (f_1) , for any $u = u^- + u^0 + se$ with $u^- + u^0 \in E^- \oplus E^0$ and $s > 0$, there exists $M_{10} > 0$ such that

$$s^2 = (se, se)_{L^2} = (se, u)_{L^2} \leq \|se\|_{\beta/(\beta-1)} \|u\|_\beta \leq M_{10} s \|u_\beta\|. \tag{36}$$

Since $\dim E_m < +\infty$, for $u \in E_m$, one has

$$\begin{aligned}
J(u) &= \int_{\mathbb{R}^N} F(x, u) dx + \frac{1}{2} \|u^-\|^2 - \frac{1}{2} \|se\|^2 \\
&\geq b_1 \int_{\mathbb{R}^N} |u|^\beta dx - \frac{1}{2} s^2 \\
&\geq M_{11} s^\beta - \frac{1}{2} s^2, \tag{37}
\end{aligned}$$

for some $M_{11} > 0$. Then, we can easily find the desired ρ and δ . $\square$

Lemma 6. Suppose the conditions of Theorem 1 hold, then $J(u) \rightarrow -\infty$ as $u \in E^+$.

Proof. For any $u \in E^+$, from (f_3) , there exists $r_\infty > 0$ such that

$$|F(x, u)| \leq \frac{1}{4\eta_2^2} u^2, \quad \text{for all } x \in \mathbb{R}^N \text{ and } |t| \geq r_\infty, \tag{38}$$

which, together with Lemma 2, implies that

$$\begin{aligned}
J(u) &\leq \int_{\mathbb{R}^N} F(x, u) dx - \frac{1}{2} \|u\|^2 \\
&\leq \frac{1}{4\eta_2^2} \int_{|u| \geq r_\infty} u^2 dx + 2M_0 \int_{|u| \leq r_\infty} (|u| + u^2) dx - \frac{1}{2} \|u\|^2 \\
&\leq \frac{1}{4} \|u\|^2 + 2M_0(1 + r_\infty) \int_{|u| \leq r_\infty} |u| dx - \frac{1}{2} \|u\|^2 \\
&\leq 2M_0\eta_1(1 + r_\infty) \|u\| - \frac{1}{4} \|u\| \longrightarrow -\infty \text{ as } \|u\| \longrightarrow +\infty.
\end{aligned} \tag{39}$$

We finish the proof of this lemma. $\square$

Proof of Theorem 2. If $E^- \oplus E^0 = \{0\}$, we can deduce from Lemma 4–6 that J possesses a minimax-type critical point, which is a solution for (1). If $E^- \oplus E^0 \neq \{0\}$, let $\mathbb{X} = E^- \oplus E^0 \oplus \mathbb{R}e_{\bar{n}+1}$. Similar to the proof of Lemma 5, there exists $s_0 > 0$ small and $\tau > 0$ such that

$$J(u) \geq \tau, \quad \forall u \in S \equiv E^- \oplus E^0 + s_0 e. \tag{40}$$

In addition, by Lemma 6, we can choose $r > s_0$ large enough such that

$$f(u) \leq 0, \quad \forall u \in E^+ \text{ with } \|u\| \geq r. \tag{41}$$

Let $Q = B_r \cap E^+$. Since S and ∂Q link, by the linking theorem (see [6], Theorem 5.29), J has a critical point $\bar{u}$ with $J(\bar{u}) \geq \delta$. $\square$

Proof of Theorem 3. In order to obtain infinitely many solutions for problem (1), we need the variant fountain theorem by Zou in [25]. Let E be a Banach space with the norm $\|\cdot\|$ and $E = \bigoplus_{j \in \mathbb{N}} \overline{X_j}$ with $\dim X_j$ for any $j \in \mathbb{N}$. Set $Y_k = \bigoplus_{j=0}^k X_j$ and $Z_k = \bigoplus_{j=k}^{\infty} \overline{X_j}$. Consider the following C^1 -functional $I_\lambda: E \rightarrow \mathbb{R}$ defined by

$$I_\lambda(u) = A(u) - \lambda B(u), \quad \lambda \in [1, 2]. \tag{42}$$

Lemma 7. (see [25]) Suppose that the functional $I_\lambda(u)$ defined above satisfies the following:

(A₁) I_λ maps bounded sets to bounded sets uniformly for $\lambda \in [1, 2]$. Furthermore, $I_\lambda(-u) = I_\lambda(u)$ for all $(\lambda, u) \in [1, 2] \times E$;

(A₂) $B(u) \geq 0$; $B(u) \rightarrow \infty$ as $\|u\| \rightarrow \infty$ on any finite dimensional subspace of E ;

(A₃) There exists $\rho_k > r_k > 0$ such that

$$\begin{aligned}
a_k(\lambda) &:= \inf_{u \in Z_k, \|u\| = \rho_k} I_\lambda(u) \geq 0 > b_k(\lambda) := \max_{u \in Y_k, \|u\| = r_k} I_\lambda(u) \\
&\quad \text{for } \lambda \in [1, 2]; \\
d_k(\lambda) &:= \inf_{u \in Z_k, \|u\| \leq \rho_k} I_\lambda(u) \rightarrow 0 \text{ as } k \rightarrow \infty \text{ uniformly} \\
&\quad \text{for } \lambda \in [1, 2].
\end{aligned} \tag{43}$$

Then, there exist $\lambda_n \rightarrow 1$, $u(\lambda_n) \in Y_n$ such that $I'_{\lambda_n}(u(\lambda_n)) = 0$, $I(u(\lambda_n)) \rightarrow c_k \in [d_k(2), b_k(1)]$ as $n \rightarrow \infty$. In particular, if $\{u(\lambda_n)\}$ has a convergent subsequence for every k , then I_1 has infinitely many nontrivial critical points $u_k \in E \setminus \{0\}$ satisfying $I_1(u_k) \rightarrow 0^-$ as $k \rightarrow \infty$.

Subsequently, we show I possesses infinitely many critical points when $F(x, t)$ is even in t , which are obtained by the variant fountain theorem. Define $X_j = \text{span}\{e_j\}$ for any $j \in \mathbb{N}$. Let E be decomposed as follows $E = \bigoplus_{j \in \mathbb{N}} X_j$. Set $Y_k = \bigoplus_{j=1}^k X_j$ and $Z_k = \bigoplus_{j=k}^{\infty} \overline{X_j}$. In the following proof, we show that I possesses geometrical structure of the variant fountain theorem. In order to make use of the variant fountain theorem, we set

$$\begin{aligned}
A(u) &= \frac{1}{2} \|u^+\|^2, \\
B(u) &= \frac{1}{2} \|u^-\|^2 + \int_{\mathbb{R}^N} F(x, u) dx,
\end{aligned} \tag{44}$$

$$I_\lambda(u) = A(u) - \lambda B(u).$$

Then, we have the following lemmas.

Lemma 8. There exists a sequence $\rho_k \rightarrow 0^+$ as $k \rightarrow \infty$ such that

$$a_k(\lambda) := \inf_{u \in Z_k, \|u\| = \rho_k} I_\lambda(u) \geq 0. \tag{45}$$

Proof. Set

$$\beta_k(p) := \sup_{u \in Z_k, \|u\| = 1} \|u\|_p, \tag{46}$$

for any $p \in [1, 2^*)$. Since $Z_{k+1} \subset Z_k \subset E^+$ for $k > \bar{n} + 1$, we can see that $0 \leq \beta_{k+1}(p) \leq \beta_k(p)$. So $\beta_k(p) \rightarrow \beta(p)$ as $k \rightarrow \infty$ for some $\beta(p) \geq 0$ for any $p \in [1, 2^*)$. For every $k \geq 0$, there exists $u_k \in Z_k$ such that $\|u_k\| = 1$ and $\|u_k\|_p > \beta_k(p)/2$. Furthermore, for any $v \in E$, we can see that $v = \sum_{i=1}^{+\infty} l_i e_i$, which implies that

$$\begin{aligned}
|\langle u_k, v \rangle| &= \left| \left\langle u_k, \sum_{i=k}^{+\infty} l_i e_i \right\rangle \right| \leq \|u_k\| \cdot \left\| \sum_{i=k}^{+\infty} l_i e_i \right\| \\
&= \sqrt{\sum_{i=k}^{+\infty} l_i^2} \rightarrow 0 \text{ as } k \rightarrow +\infty.
\end{aligned} \tag{47}$$

Then, we have $u_k \rightarrow 0$ as $k \rightarrow \infty$ in E . It follows from Lemma 1 that $u_k \rightarrow 0$ in $L^p(\mathbb{R}^N)$. Hence, we conclude that $\beta(p) = 0$ for any $p \in [1, 2^*)$. For any $u \in Z_k$, by the definitions of I_λ and Lemma 2, we obtain

$$\begin{aligned}
I_\lambda(u) &= \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N} F(x, u) dx \\
&\geq \frac{1}{2}\|u\|^2 - 2M_0 \int_{\mathbb{R}^N} (|u| + u^2) dx \\
&\geq \frac{1}{2}\|u\|^2 - 2M_0(\beta_k(1)\|u\| + \beta_k^2(2)\|u\|^2).
\end{aligned} \quad (48)$$

Let $\rho_k = 16M_0\beta_k(1)$. Obviously, we have $\rho_k \rightarrow 0^+$ as $k \rightarrow \infty$. When k large enough, we can conclude that

$$\begin{aligned}
a_k(\lambda) &:= \inf_{u \in Z_k, \|u\|=\rho_k} I_\lambda(u) \\
&\geq \frac{1}{2}\rho_k^2 - 2M_0(\beta_k(1)\rho_k + \beta_k^2(2)\rho_k^2) \\
&\geq \frac{1}{4}\rho_k^2 - 2M_0\beta_k(1)\rho_k \\
&\geq \frac{1}{8}\rho_k^2 \\
&\geq 0.
\end{aligned} \quad (49)$$

Moreover, it follows from (48) that

$$I_\lambda(u) \geq -2M_0(\beta_k(1)\rho_k + \beta_k^2(2)\rho_k^2). \quad (50)$$

Therefore, when k large enough, we can deduce that

$$0 \geq \inf_{u \in Z_k, \|u\| \leq \rho_k} I_\lambda(u) \geq -2M_0(\beta_k(1)\rho_k + \beta_k^2(2)\rho_k^2). \quad (51)$$

Since $\beta_k(p), \rho_k \rightarrow 0$ as $k \rightarrow \infty$, we have

$$\begin{aligned}
d_k(\lambda) &:= \inf_{u \in Z_k, \|u\| \leq \rho_k} I_\lambda(u) \rightarrow 0 \text{ as } k \rightarrow \infty \text{ uniformly,} \\
&\text{for } \lambda \in [1, 2].
\end{aligned} \quad (52)$$

Then, we obtain our conclusion. $\square$

Lemma 9. *There exists a sequence $0 < r_k < \rho_k$ for all $k \in \mathbb{N}$ such that*

$$b_k(\lambda) := \max_{u \in Y_k, \|u\|=r_k} I_\lambda(u) < 0, \quad \text{for } \lambda \in [1, 2]. \quad (53)$$

Proof. For any $k \in \mathbb{N}$, $\dim Y_k = k$. It follows from (f_1) that

$$\begin{aligned}
I_\lambda(u) &\leq \frac{1}{2}\|u^+\|^2 - \int_{\mathbb{R}^N} F(x, u) dx \\
&\leq \frac{1}{2}\|u\|^2 - b_1\|u\|_\beta^\beta \\
&\leq \frac{1}{2}\|u\|^2 - M_{12}^\beta\|u\|^\beta,
\end{aligned} \quad (54)$$

for some $M_{12} > 0$ and any $\lambda \in [1, 2]$. Let

$$0 < r_k < \min\left\{\rho_k, \left(2M_{12}^\beta\right)^{1/(2-\beta)}\right\}. \quad (55)$$

Then, we obtain that

$$b_k(\lambda) = \max_{u \in Y_k, \|u\|=r_k} I_\lambda(u) < 0, \quad \text{for } \lambda \in [1, 2]. \quad (56)$$

The proof is completed.

It is easy to see condition (A_1) is satisfied in Lemma 4 since $F(x, t)$ is even in t . Similar to Lemma 4 in [13], it can easily be deduced from (f_6) that $B(u) \rightarrow \infty$ as $\|u\| \rightarrow \infty$ on any finite dimensional subspace of E . Then, the condition (A_2) in Lemma 7 is satisfied. From Lemma 8 and Lemma 9, we can see that all the conditions of Lemma 1 are satisfied. Then, there exist $\lambda_n \rightarrow 1$, $u_{\lambda_n} \in Y_n$ such that

$$\begin{aligned}
I'_{\lambda_n}|_{Y_n}(u_{\lambda_n}) &= 0, \\
I_{\lambda_n}(u_{\lambda_n}) &\rightarrow c_k \in [d_k(2), b_k(1)] \text{ as } n \rightarrow \infty.
\end{aligned} \quad (57)$$

Similar to Lemma 4, there exists u_k such that $u_{\lambda_n} \rightarrow u_k$ in E as $n \rightarrow \infty$ for any $k \in \mathbb{N}$. Then, we can deduce that $I = I_1$ possesses infinitely many nontrivial critical points. Therefore, (1) has infinitely many nontrivial solutions. $\square$

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this article.

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