Hindawi
Mathematical Problems in Engineering
Volume 2019, Article ID 9416234, 20 pages
https://doi.org/10.1155/2019/9416234

Research Article

Stochastic Dynamics of Discrete-Time Fuzzy Random BAM Neural Networks with Time Delays

Sufang Han,[1] **Tianwei Zhang**,[2] **and Guoxin Liu**[1]

[1]*School of Mathematics and Statistics, Central South University, Changsha 410075, China*
[2]*City College, Kunming University of Science and Technology, Kunming 650051, China*

Correspondence should be addressed to Tianwei Zhang; zhang@kmust.edu.cn

Received 30 April 2019; Revised 20 July 2019; Accepted 4 August 2019; Published 25 August 2019

Academic Editor: Jose Vicente Salcedo

By using the semidiscrete method of differential equations, a new version of discrete analogue of stochastic fuzzy BAM neural networks was formulated, which gives a more accurate characterization for continuous-time stochastic neural networks than that by the Euler scheme. Firstly, the existence of the $2p$-th mean almost periodic sequence solution of the discrete-time stochastic fuzzy BAM neural networks is investigated with the help of Minkowski inequality, Hölder inequality, and Krasnoselskii's fixed point theorem. Secondly, the $2p$-th moment global exponential stability of the discrete-time stochastic fuzzy BAM neural networks is also studied by using some analytical skills in stochastic theory. Finally, two examples with computer simulations are given to demonstrate that our results are feasible. The main results obtained in this paper are completely new, and the methods used in this paper provide a possible technique to study $2p$-th mean almost periodic sequence solution and $2p$-th moment global exponential stability of semidiscrete stochastic fuzzy models.

1. Introduction

In [1, 2], Kosko introduced the bidirectional associative memory (BAM) neural networks, which have been widely applied in psychophysics, parallel computing, perception, robotics, adaptive pattern recognition, associative memory, image processing pattern recognition, combinatorial optimization, and so on. All of these applications heavily depend on the (almost) periodicity and global exponential stability [3–14]. In the past twenty years, a large, fast growing body of investigations focused on the existence and global exponential stability of the equilibrium point, periodic, and almost periodic solutions of BAM neural networks with time delays in literatures [15–27].

Fuzzy theory was conceived in the 1960s by L. A. Zadeh, and it took about 20 years until the broader use of this theory in practice. Fuzzy technology joined forces with artificial neural networks and genetic algorithms under the title of computational intelligence or soft computing. In recent years, the research on the dynamical behaviours of fuzzy neural networks has attracted much attention (see [28–33]). On the other hand, uncertain models described by the stochastic differential equations have caused great concern since uncertain models have widely applications in practice such as engineering, physics, chemistry, and biology [34–38]. In the actual situations, random factors have consequences on the performance of the neural networks. In neural networks, the connection weights of the neurons depend on certain resistance and capacitance values that include modeling errors or uncertainties during the parameter identification process. Therefore, it is worth studying the dynamical behaviors of fuzzy stochastic neural networks.

The discrete-time neural networks become more important than the continuous-time counterparts when implementing the neural networks in a digital way. In order to investigate the dynamical characteristics with respect to digital signal transmission, it is essential to formulate the discrete analogue of neural networks. Mohamad and Gopalsamy [39, 40] proposed a novel method (i.e.,

semidiscretization technique) in formulating a discrete-time analogue of the continuous-time neural networks, which faithfully preserved the characteristics of their continuous-time counterparts. With the help of the semidiscretization technique [39], many scholars obtained the semidiscrete analogue of the continuous-time neural networks and some meaningful results were gained for the dynamical behaviors of the semidiscrete neural networks, such as periodic solutions, almost periodic solutions and global exponential stability (see [41–47]).

For instance, Huang et al. [44] discussed the almost periodic dynamics of the following semidiscrete cellular neural networks:

$$x_i(k+1) = e^{-a_i(k)}x_i(k) + \frac{1-e^{-a_i(k)}}{a_i(k)} \cdot \left[\sum_{j=1}^{n} b_{ij}(k) f_j\big(x_j(k)\big) + I_i(k)\right], \tag{1}$$

where $k \in \mathbb{Z}$, in which $\mathbb{Z}$ denotes the set of integers, and $i = 1, 2, \ldots, n$.

In [45], Huang et al. considered the following semidiscrete models for a class of general neural networks:

$$x_i(k+1) = e^{-a_i(k)}x_i(k) + \frac{1-e^{-a_i(k)}}{a_i(k)} \cdot \left[\sum_{l=1}^{m}\sum_{j=1}^{n} b_{ijl}(k) f_j\big(x_j\big(k-\tau_{ijl}\big)\big) + I_i(k)\right], \tag{2}$$

where $k \in \mathbb{Z}$ and $i = 1, 2, \ldots, n$. The authors [45] derived the existence of locally exponentially convergent 2^N almost periodic sequence solutions of system (2).

By careful observation, it easily discovers that the disquisitive models in literatures [41–47] are deterministic, such as (1) and (2). Stimulated by this point, it is necessary to consider random factors in the determinant models. Therefore, this paper considers the semidiscrete models for the following stochastic fuzzy BAM neural networks:

$$\begin{cases} dx_i(t) = \Big[-a_i(t)x_i(t) + \sum_{j=1}^{m} b_{ij}(t) f_j\big(y_j(t)\big) + \sum_{j=1}^{m} c_{ij}(t) f_j\big(y_j\big(t-\tau_{ij}(t)\big)\big) + \bigwedge_{j=1}^{m} \alpha_{ij} f_j\big(y_j\big(t-\tau_{ij}(t)\big)\big) \\ \quad + \bigwedge_{j=1}^{m} T_{ij}\mu_j + \bigvee_{j=1}^{m} \beta_{ij} f_j\big(y_j\big(t-\tau_{ij}(t)\big)\big) + \bigvee_{j=1}^{m} S_{ij}\mu_j + I_i(t)\Big]dt + \sum_{j=1}^{m} d_{ij}(t)\sigma_j\big(y_j\big(t-\eta_{ij}(t)\big)\big)dw_j(t), \\ dy_j(t) = \Big[-\tilde{a}_j(t)y_j(t) + \sum_{i=1}^{n} \tilde{b}_{ji}(t)\tilde{f}_i(x_i(t)) + \sum_{i=1}^{n} \tilde{c}_{ji}(t)\tilde{f}_i\big(x_i\big(t-\tilde{\tau}_{ji}(t)\big)\big) + \bigwedge_{i=1}^{n} \tilde{\alpha}_{ji}\tilde{f}_i\big(x_i\big(t-\tilde{\tau}_{ji}(t)\big)\big) \\ \quad + \bigwedge_{i=1}^{n} \tilde{T}_{ji}\tilde{\mu}_i + \bigvee_{i=1}^{n} \tilde{\beta}_{ji}\tilde{f}_i\big(x_i\big(t-\tilde{\tau}_{ji}(t)\big)\big) + \bigvee_{i=1}^{m} \tilde{S}_{ji}\tilde{\mu}_i + J_j(t)\Big]dt + \sum_{i=1}^{n} \tilde{d}_{ji}(t)\tilde{\sigma}_i\big(x_i\big(t-\tilde{\eta}_{ji}(t)\big)\big)d\tilde{w}_i(t). \end{cases} \tag{3}$$

System (3) is composed of two layers, that is, X-layer and Y-layer. $x_i(t)$ denotes the membrane potentials of the set of n neurons in X-layer, and $y_j(t)$ denotes the membrane potentials of the set of m neurons in Y-layer at time t; f_j and $\tilde{f}_i$ represent the measures of activation to its incoming potentials of the unit j from Y-layer and the unit i from X-layer, respectively; b_{ij} corresponds to the synaptic connection weight of the unit j on the unit i, and $\tilde{b}_{ji}$ corresponds to the synaptic connection weight of the unit i on the unit j; I_i and J_j signify the bounded external bias or input; a_i and $\tilde{a}_j$ denote rate with which the ith unit and jth unit will reset their potentials to the resting state in isolation when it is disconnected from the network and external inputs, respectively; α_{ij}, β_{ij}, T_{ij}, S_{ij} $\tilde{\alpha}_{ji}$, $\tilde{\beta}_{ji}$, $\tilde{T}_{ji}$, and $\tilde{S}_{ji}$ are elements of fuzzy feedback MIN, MAX template, fuzzy feed forward MIN, and MAX template, respectively; $\wedge$ and $\vee$ denote the fuzzy AND and fuzzy OR operation, respectively; w_j and $\tilde{w}_i$ are the standard Brownian motions, where $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$. The other coefficients in system (3) are similarly specified.

The main aim of this paper is to investigate the dynamics of the semidiscrete analogue of system (3) by using semidiscretization technique [39] and stochastic theory. The main contributions of this paper are summed up as follows: (1) the semidiscrete analogue is established for stochastic fuzzy BAM neural networks (3); (2) a Volterra additive equation is derived for the solution of the semidiscrete model; (3) the existence of $2p$-th mean almost periodic sequence solutions is obtained; (4) a decision theorem is acquired for the $2p$-th moment global exponential stability; and (5) the methods used in this article can be applied to study the dynamics of other discrete stochastic fuzzy models.

The work of this paper is a continuation of literatures [44–47], and the results in this paper complement the corresponding results in [44–47]. The paper is organized as follows. In Section 2, the discrete analogue of system (3) is established and some useful lemmas are given. In Section 3, we employ Krasnoselskii's fixed point theorem to obtain sufficient conditions for the existence of at least one $2p$-th mean almost periodic sequence solution. In Section 4, we

consider the $2p$-th moment global exponential stability. Two illustrative examples and computer simulations are given in Section 5. In Section 6, the conclusions and future works of this paper are presented.

Throughout this paper, we use the following notations. Let $\mathbb{R}$ denote the set of real numbers. $\mathbb{R}^n$ denotes the n-dimensional real vector space. Let $(\Omega, \mathscr{F}, P)$ be a complete probability space. Denote by $BC(\mathbb{Z}, L^p(\Omega, \mathbb{R}^n))$ the vector space of all bounded continuous functions from $\mathbb{Z}$ to $L^p(\Omega, \mathbb{R}^n)$, where $L^p(\Omega, \mathbb{R}^n)$ denotes the collection of all p-th integrable $\mathbb{R}^n$-valued random variables. Then, $BC(\mathbb{Z}, L^p(\Omega, \mathbb{R}^n))$ is a Banach space with the norm $X_p = \sup_{k\in\mathbb{Z}}|X|_p$, $|X|_p = \max_{1\le i\le n}(E|x_i(k)|^p)^{1/p}$, $\forall X = \{x_i\} := \{x_1, x_2, \ldots, x_n\} \in BC(\mathbb{Z}, L^p(\Omega, \mathbb{R}^n))$, where $p > 1$ and $E(\cdot)$ stands for the expectation operator with respect to the given probability measure P. $[a, b]_{\mathbb{Z}} = [a, b] \cap \mathbb{Z}$, $\forall a, b \in \mathbb{R}$.

2. Model Formulation and Preliminaries

2.1. Discrete Analogue of System (3). Consider the following stochastic functional differential equations:

$$\begin{aligned} dx(t) = {} & -a(t)x(t)dt + F(t, x(t), x(t-\tau(t)))dt \\ & + G(t, x(t), x(t-\eta(t)))dw(t), \end{aligned} \tag{4}$$

which yields the following stochastic functional differential equations with piecewise constant arguments:

$$\begin{aligned} dx(t) = {} & -a([t])x(t)dt + F([t], x([t]), x([t]-\tau([t])))dt \\ & + G([t], x([t]), x([t]-\eta([t])))\Delta w([t])dt, \end{aligned} \tag{5}$$

where $[t]$ denotes the integer part of t. Here, the discrete analogue of the stochastic parts of system (4) is obtained by the Euler scheme, i.e., $dw(t) = \Delta w([t])dt = [w([t]+1) - w([t])]dt$. For $t \in \mathbb{R}$, there exists an integer $k \in \mathbb{Z}$ such that $k \le t < k+1$. Then, the above equation becomes

$$\begin{aligned} dx(t) = {} & -a(k)x(t)dt + F(k, x(k), x(k-\tau(k)))dt \\ & + G(k, x(k), x(k-\eta(k)))\Delta w(k)dt. \end{aligned} \tag{6}$$

Integrating the above equation from k to t and letting $t \longrightarrow k+1$, we achieve the discrete analogue of system (4) as follows:

$$\begin{aligned} x(k+1) = {} & e^{-a(k)}x(k) + \frac{1-e^{-a(k)}}{a(k)}[F(k, x(k), x(k-\tau(k))) \\ & + G(k, x(k), x(k-\eta(k)))\Delta w(k)], \end{aligned} \tag{7}$$

where $k \in \mathbb{Z}$. System (7) is the discrete analogue of system (4) by the semidiscrete method. By a similar discussion as that in system (7), we obtain the semidiscrete analogue of system (3) as follows:

$$\begin{cases} x_i(k+1) = e^{-a_i(k)}x_i(k) + \dfrac{1-e^{-a_i(k)}}{a_i(k)}\Bigg[\displaystyle\sum_{j=1}^{m} b_{ij}(k)f_j\big(y_j(k)\big) + \sum_{j=1}^{m} c_{ij}(k)f_j\big(y_j\big(k-\tau_{ij}(k)\big)\big) + \bigwedge_{j=1}^{m}\alpha_{ij}f_j\big(y_j\big(k-\tau_{ij}(k)\big)\big) \\ \qquad + \displaystyle\bigwedge_{j=1}^{m} T_{ij}\mu_j + \bigvee_{j=1}^{m}\beta_{ij}f_j\big(y_j\big(k-\tau_{ij}(k)\big)\big) + \bigvee_{j=1}^{m} S_{ij}\mu_j + I_i(k) + \sum_{j=1}^{m} d_{ij}(k)\sigma_j\big(y_j\big(k-\eta_{ij}(k)\big)\big)\Delta w_j(k)\Bigg], \\ y_j(k+1) = e^{-\widetilde{a}_j(k)}y_j(k) + \dfrac{1-e^{-\widetilde{a}_j(k)}}{\widetilde{a}_j(k)}\Bigg[\displaystyle\sum_{i=1}^{n}\widetilde{b}_{ji}(k)\widetilde{f}_i(x_i(k)) + \sum_{i=1}^{n}\widetilde{c}_{ji}(k)\widetilde{f}_i\big(x_i\big(k-\widetilde{\tau}_{ji}(k)\big)\big) + \bigwedge_{i=1}^{n}\widetilde{\alpha}_{ji}\widetilde{f}_i\big(x_i\big(k-\widetilde{\tau}_{ji}(k)\big)\big) \\ \qquad + \displaystyle\bigwedge_{i=1}^{n}\widetilde{T}_{ji}\widetilde{\mu}_i + \bigvee_{i=1}^{n}\widetilde{\beta}_{ji}\widetilde{f}_i\big(x_i\big(k-\widetilde{\tau}_{ji}(k)\big)\big) + \bigvee_{i=1}^{n}\widetilde{S}_{ji}\widetilde{\mu}_i + J_j(k) + \sum_{i=1}^{n}\widetilde{d}_{ji}(k)\widetilde{\sigma}_i\big(x_i\big(k-\widetilde{\eta}_{ji}(k)\big)\big)\Delta\widetilde{w}_i(k)\Bigg], \quad k \in \mathbb{Z}, \end{cases} \tag{8}$$

where $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$.

2.2. Volterra Additive Equation for the Solution of System (8).

Lemma 1. *Assume that $X = \{x_i\}$ is a solution of system (8), and then X can be expressed as*

$$\begin{cases} x_i(k) = \prod_{s=k_0}^{k-1} e^{-a_i(s)} x_i(k_0) \\ + \sum_{v=k_0}^{k-1} \prod_{s=v+1}^{k-1} \frac{e^{-a_i(s)}\left[1-e^{-a_i(v)}\right]}{a_i(v)} \Big[\sum_{j=1}^{m} b_{ij}(v) f_j\big(y_j(v)\big) + \sum_{j=1}^{m} c_{ij}(v) f_j\big(y_j(v-\tau_{ij}(v))\big) + \bigwedge_{j=1}^{m} \alpha_{ij} f_j\big(y_j(v-\tau_{ij}(v))\big) \\ \quad + \bigwedge_{j=1}^{m} T_{ij}\mu_j + \bigvee_{j=1}^{m} \beta_{ij} f_j\big(y_j(v-\tau_{ij}(v))\big) + \bigvee_{j=1}^{m} S_{ij}\mu_j + I_i(v) + \sum_{j=1}^{m} d_{ij}(v)\sigma_j\big(y_j(v-\eta_{ij}(v))\big)\Delta w_j(v) \Big], \\ y_j(k) = \prod_{s=k_0}^{k-1} e^{-\tilde{a}_j(s)} y_j(k_0) \\ + \sum_{v=k_0}^{k-1} \prod_{s=v+1}^{k-1} \frac{e^{-\tilde{a}_j(s)}\left[1-e^{-\tilde{a}_j(v)}\right]}{\tilde{a}_j(v)} \Big[\sum_{i=1}^{n} \tilde{b}_{ji}(v) \tilde{f}_i(x_i(v)) + \sum_{i=1}^{n} \tilde{c}_{ji}(v) \tilde{f}_i\big(x_i(v-\tilde{\tau}_{ji}(v))\big) + \bigwedge_{i=1}^{n} \tilde{\alpha}_{ji} \tilde{f}_i\big(x_i(v-\tilde{\tau}_{ji}(v))\big) \\ \quad + \bigwedge_{i=1}^{n} \tilde{T}_{ji}\tilde{\mu}_i + \bigvee_{i=1}^{n} \tilde{\beta}_{ji} \tilde{f}_i\big(x_i(v-\tilde{\tau}_{ji}(v))\big) + \bigvee_{i=1}^{n} \tilde{S}_{ji}\tilde{\mu}_i + J_j(v) + \sum_{i=1}^{n} \tilde{d}_{ji}(v)\tilde{\sigma}_i\big(x_i(v-\tilde{\eta}_{ji}(v))\big)\Delta\tilde{w}_i(v) \Big], \end{cases} \tag{9}$$

where $k_0 \in \mathbb{Z}$, $k \in (k_0, +\infty)_{\mathbb{Z}}$, $i = 1, 2, \ldots, n$, *and* $j = 1, 2, \ldots, m$.

Proof. Let

$$\begin{aligned} F_i(k,x) := & \sum_{j=1}^{m} b_{ij}(k) f_j\big(y_j(k)\big) \\ & + \sum_{j=1}^{m} c_{ij}(k) f_j\big(y_j(k-\tau_{ij}(k))\big) \\ & + \bigwedge_{j=1}^{m} \alpha_{ij} f_j\big(y_j(k-\tau_{ij}(k))\big) + \bigwedge_{j=1}^{m} T_{ij}\mu_j \\ & + \bigvee_{j=1}^{m} \beta_{ij} f_j\big(y_j(k-\tau_{ij}(k))\big) + \bigvee_{j=1}^{m} S_{ij}\mu_j + I_i(k) \\ & + \sum_{j=1}^{m} d_{ij}(k)\sigma_j\big(y_j(k-\eta_{ij}(k))\big)\Delta w_j(k), \\ & k \in \mathbb{Z},\ i = 1, 2, \ldots, n. \end{aligned} \tag{10}$$

By $\Delta[u(k)v(k)] = [\Delta u(k)]v(k) + u(k+1)[\Delta v(k)]$ and the first equation of system (8), it gets

$$\Delta\left[\prod_{s=0}^{k-1} e^{a_i(s)} x_i(k)\right] = \prod_{s=0}^{k} \frac{e^{a_i(s)}\left[1-e^{-a_i(k)}\right]}{a_i(k)} F_i(k,x), \tag{11}$$

where $i = 1, 2, \ldots, n$ and $k \in \mathbb{Z}$. So,

$$\sum_{v=k_0}^{k-1} \Delta\left[\prod_{s=0}^{v-1} e^{a_i(s)} x_i(v)\right] = \sum_{v=k_0}^{k-1} \prod_{s=0}^{v} \frac{e^{a_i(s)}\left[1-e^{-a_i(v)}\right]}{a_i(v)} F_i(v,x) \tag{12}$$

is equivalent to

$$\begin{aligned} \prod_{s=0}^{k-1} e^{a_i(s)} x_i(k) = & \prod_{s=0}^{k_0-1} e^{a_i(s)} x_i(k_0) \\ & + \sum_{v=k_0}^{k-1} \prod_{s=0}^{v} \frac{e^{a_i(s)}\left[1-e^{-a_i(v)}\right]}{a_i(v)} F_i(v,x), \end{aligned} \tag{13}$$

where $i = 1, 2, \ldots, n$ and $k \in \mathbb{Z}$. By the above equations, we can easily derive the first equation of system (9). Similarly, we can obtain the second equation of system (9). This completes the proof. □

2.3. Some Lemmas.

Lemma 2 (Minkowski inequality) [48]. *Assume that* $p \geq 1$, $E|\xi|^p < \infty$, $E|\eta|^p < \infty$, *and then*

$$\left(E|\xi+\eta|^p\right)^{1/p} \leq \left(E|\xi|^p\right)^{1/p} + \left(E|\eta|^p\right)^{1/p}. \tag{14}$$

Lemma 3 (Hölder inequality) [48]. *Assume that* $p > 1$, *and then*

$$\sum_k |a_k b_k| \leq \left[\sum_k |a_k|\right]^{1-(1/p)} \left[\sum_k |a_k||b_k|^p\right]^{1/p}. \tag{15}$$

If $p = 1$, *then* $\sum_k |a_k b_k| \leq (\sum_k |a_k|)(sup_k |b_k|)$.

Lemma 4 [35]. *Suppose that* $g \in L^2([a,b], \mathbb{R})$, *and then*

$$E\left[\sup_{t\in[a,b]} \left|\int_a^t g(s)d\omega(s)\right|^p\right] \leq C_p E\left[\int_a^b |g(t)|^2 dt\right]^{p/2}, \tag{16}$$

where

$$C_p = \begin{cases} \left(\frac{32}{p}\right)^{p/2}, & 0 < p < 2, \\ 4, & p = 2, \\ \left[\frac{p^{p+1}}{2(p-1)^{(p-1)}}\right]^{p/2}, & p > 2. \end{cases} \tag{17}$$

Lemma 5. *Assume that $\{x(k) : k \in \mathbb{Z}\}$ is real-valued stochastic process and $w(k)$ is the standard Brownian motion, and then*

$$E|x(k)\Delta w(k)|^p \le C_p E|x(k)|^p, \quad k \in \mathbb{Z}, \tag{18}$$

where C_p is defined as that in Lemma 4, in which $p > 0$.

Proof. By Lemma 4, it follows that

$$\begin{aligned} E|x(k)\Delta w(k)|^p &= E\left|\int_k^{k+1} x(k)dw(s)\right|^p \le C_p E\left|\int_k^{k+1} x^2(k)ds\right|^{p/2} \\ &\le C_p E|x(k)|^p, \quad k \in \mathbb{Z}. \end{aligned} \tag{19}$$

This completes the proof. □

Lemma 6 [49]. *Suppose $X = \{x_i\}$ and $Y = \{y_i\}$ are two states of system (8), and then we have*

$$\begin{aligned} &\left|\bigwedge_{j=1}^{n} \alpha_{ij} f_j(x_j) - \bigwedge_{j=1}^{n} \alpha_{ij} f_j(y_j)\right| \le \sum_{j=1}^{n} |\alpha_{ij}| |f_j(x_j) - f_j(y_j)|, \\ &\left|\bigvee_{j=1}^{n} \beta_{ij} f_j(x_j) - \bigvee_{j=1}^{n} \beta_{ij} f_j(y_j)\right| \le \sum_{j=1}^{n} |\beta_{ij}| |f_j(x_j) - f_j(y_j)|, \\ &\qquad i = 1, 2, \ldots, n. \end{aligned} \tag{20}$$

Lemma 7 [50]. *Assume that Λ is a closed convex nonempty subset of a Banach space $\mathbb{X}$. Suppose further that $\mathscr{B}$ and $\mathscr{C}$ map Λ into $\mathbb{X}$ such that*

(1) *$x, y \in \Lambda$ implies that $\mathscr{B}x + \mathscr{C}y \in \Lambda$*

(2) *$\mathscr{B}$ is continuous and $\mathscr{B}\Lambda$ is contained in a compact set*

(3) *$\mathscr{C}$ is a contraction mapping*

Then, there exists a $z \in \Lambda$ with $z = \mathscr{B}z + \mathscr{C}z$.

Set $f^u = \sup_{k\in\mathbb{Z}}|f(k)|$ and $f^l = \inf_{k\in\mathbb{Z}}|f(k)|$ for bounded function f defined on $\mathbb{Z}$. Throughout this paper, suppose that the following conditions are satisfied:

(H_1) a_i and $\tilde{a}_j$ are bounded sequences defined on $\mathbb{Z}$ with $a_i^l > 0$ and $\tilde{a}_j^l > 0$, $i = 1, 2, \ldots, n$, and $j = 1, 2, \ldots, m$.

(H_2) There exists several constants L_j^f, L_j^σ, $L_j^{\tilde{f}}$, and $L_j^{\tilde{\sigma}}$ such that

$$\begin{aligned} |f_j(u) - f_j(v)| &\le L_j^f |u - v|, \\ |\sigma_j(u) - \sigma_j(v)| &\le L_j^\sigma |u - v|, \\ |\tilde{f}_i(u) - \tilde{f}_i(v)| &\le L_j^{\tilde{f}} |u - v|, \\ |\tilde{\sigma}_j(u) - \tilde{\sigma}_j(v)| &\le L_j^{\tilde{\sigma}} |u - v|, \end{aligned} \tag{21}$$

for all $u, v \in \mathbb{R}$, $i = 1, 2, \ldots, n$, and $j = 1, 2, \ldots, m$.

3. 2*p*-th Mean Almost Periodic Sequence Solution

Define

$$\begin{aligned} a^u &:= \max_{1\le i\le n} a_i^u, \\ a^l &:= \min_{1\le i\le n} a_i^l, \\ \tilde{a}^u &:= \max_{1\le j\le m} \tilde{a}_j^u, \\ \tilde{a}^l &:= \max_{1\le j\le m} \tilde{a}_j^l, \\ D^* &:= \max_{1\le i\le n} \sum_{j=1}^{m} \left\{b_{ij}^u + c_{ij}^u + |\alpha_{ij}| + |\beta_{ij}|\right\} L_j^f, \\ \tilde{D}^* &:= \max_{1\le j\le m} \sum_{i=1}^{n} \left\{\tilde{b}_{ji}^u + \tilde{c}_{ji}^u + |\tilde{\alpha}_{ji}| + |\tilde{\beta}_{ji}|\right\} L_i^{\tilde{f}}, \\ K^* &:= \max_{1\le i\le n} \sum_{j=1}^{m} d_{ij}^u L_j^\sigma, \\ \tilde{K}^* &:= \max_{1\le j\le m} \sum_{i=1}^{n} \tilde{d}_{ji}^u L_i^{\tilde{\sigma}}, \\ r_{2p} &:= \max\left\{\frac{(1 - e^{-a^u})}{a^l(1 - e^{-a^l})}\left(D^* + K^* C_{2p}^{1/2p}\right), \right. \\ &\qquad \left. \frac{(1 - e^{-\tilde{a}^u})}{\tilde{a}^l(1 - e^{-\tilde{a}^l})}\left(\tilde{D}^* + \tilde{K}^* C_{2p}^{1/2p}\right)\right\}, \\ \beta_{2p} &:= \frac{\alpha_{2p}}{1 - r_{2p}}, \end{aligned} \tag{22}$$

where

$$\alpha_{2p} := \max\left\{e_{2p}, \tilde{e}_{2p}\right\},$$

$$e_{2p} := \frac{\left(1-e^{-a^u}\right)}{a^l\left(1-e^{-a^l}\right)} \max_{1\le i\le n}\left[\sum_{j=1}^{m}\left(b_{ij}^u + c_{ij}^u + \left|\alpha_{ij}\right| + \left|\beta_{ij}\right|\right)\left|f_j(0)\right| + \sum_{j=1}^{m}\left(\left|T_{ij}\right| + \left|S_{ij}\right|\right)\left|\mu_j\right| + I_i^u + \sum_{j=1}^{m} d_{ij}^u \sigma_j(0) C_{2p}^{1/2p}\right],$$

$$\tilde{e}_{2p} := \frac{\left(1-e^{-\tilde{a}^u}\right)}{\tilde{a}^l\left(1-e^{-\tilde{a}^l}\right)} \max_{1\le j\le m}\left[\sum_{i=1}^{n}\left(\tilde{b}_{ji}^u + \tilde{c}_{ji}^u + \left|\tilde{\alpha}_{ji}\right| + \left|\tilde{\beta}_{ji}\right|\right)\left|\tilde{f}_i(0)\right| + \sum_{i=1}^{n}\left(\left|\tilde{T}_{ji}\right| + \left|\tilde{S}_{ji}\right|\right)\left|\tilde{\mu}_i\right| + J_j^u + \sum_{i=1}^{n} \tilde{d}_{ji}^u \tilde{\sigma}_i(0) C_{2p}^{1/2p}\right]. \tag{23}$$

Definition 3.1 [34]. A stochastic process $X \in BC(\mathbb{Z}; L^p(\Omega; \mathbb{R}^n))$ is said to be p-th mean almost periodic sequence if for each $\epsilon > 0$, there exists an integer $l(\epsilon) > 0$ such that each interval of length $l(\epsilon)$ contains an integer ω for which

$$\left|X(k+\omega) - X(k)\right|_p = \max_{1\le i\le n}\left(E\left|x_i(k+\omega) - x_i(k)\right|^p\right)^{1/p} < \epsilon, \quad \forall k \in \mathbb{Z}. \tag{24}$$

A stochastic process X, which is 2-nd mean almost periodic sequence, is called square-mean almost periodic sequence. Like for classical almost periodic functions, the number ω is called an ϵ-translation of X.

Theorem 1. *Assume that all of the coefficients in system (8) are almost periodic sequences, and then (H_1)–(H_2) hold and the following condition is satisfied:*

(H_3) $r_{2p} < 1$, *where* $2p > 1$

Then, there exists a $2p$-th mean almost periodic sequence solution X of system (8) with $\|X\|_{2p} \le \beta_{2p}$.

Proof. Let $\Lambda \subseteq BC(\mathbb{Z}; L^{2p}(\Omega; \mathbb{R}^{m+n}))$ be the collection of all $2p$-th mean almost periodic sequences X satisfying the inequality $\|X\|_{2p} \le \beta_{2p}$.

Define $\Phi X(k) = \mathscr{B}X(k) + \mathscr{C}X(k)$, where

$$\begin{aligned}
\Phi X(k) &= \left\{(\Phi X)_i(k); (\Phi X)_{n+j}(k)\right\} = \left((\Phi X)_1(k), \ldots, (\Phi X)_n(k), (\Phi X)_{n+1}(k), \ldots, (\Phi X)_{n+m}(k)\right)^T,\\
(\Phi X)_i(k) &= (\mathscr{B}X)_i(k) + (\mathscr{C}X)_i(k),\\
(\Phi X)_{n+j}(k) &= (\mathscr{B}X)_{n+j}(k) + (\mathscr{C}X)_{n+j}(k),
\end{aligned} \tag{25}$$

$$\begin{cases}
(\mathscr{B}X)_i(k) = \displaystyle\sum_{v=-\infty}^{k-1}\prod_{s=v+1}^{k-1}\frac{e^{-a_i(s)}\left[1-e^{-a_i(v)}\right]}{a_i(v)}\Bigg[\sum_{j=1}^{m} b_{ij}(v) f_j\left(y_j(v)\right) + \sum_{j=1}^{m} c_{ij}(v) f_j\left(y_j\left(v-\tau_{ij}(v)\right)\right) + \bigwedge_{j=1}^{m}\alpha_{ij} f_j\left(y_j\left(v-\tau_{ij}(v)\right)\right)\\
\qquad + \displaystyle\bigwedge_{j=1}^{m} T_{ij}\mu_j + \bigvee_{j=1}^{m}\beta_{ij} f_j\left(y_j\left(v-\tau_{ij}(v)\right)\right) + \bigvee_{j=1}^{m} S_{ij}\mu_j + I_i(v)\Bigg],\\
(\mathscr{B}X)_{n+j}(k) = \displaystyle\sum_{v=-\infty}^{k-1}\prod_{s=v+1}^{k-1}\frac{e^{-\tilde{a}_j(s)}\left[1-e^{-\tilde{a}_j(v)}\right]}{\tilde{a}_j(v)}\Bigg[\sum_{i=1}^{n}\tilde{b}_{ji}(v)\tilde{f}_i(x_i(v)) + \sum_{i=1}^{n}\tilde{c}_{ji}(v)\tilde{f}_i\left(x_i\left(v-\tilde{\tau}_{ji}(v)\right)\right) + \bigwedge_{i=1}^{n}\tilde{\alpha}_{ji}\tilde{f}_i\left(x_i\left(v-\tilde{\tau}_{ji}(v)\right)\right)\\
\qquad + \displaystyle\bigwedge_{i=1}^{n}\tilde{T}_{ji}\tilde{\mu}_i + \bigvee_{i=1}^{n}\tilde{\beta}_{ji}\tilde{f}_i\left(x_i\left(v-\tilde{\tau}_{ji}(v)\right)\right) + \bigvee_{i=1}^{n}\tilde{S}_{ji}\tilde{\mu}_i + J_j(v)\Bigg],
\end{cases} \tag{26}$$

$$\begin{cases}
(\mathscr{C}X)_i(k) = \displaystyle\sum_{v=\infty}^{k-1}\prod_{s=v+1}^{k-1}\frac{e^{-a_i(s)}\left[1-e^{-a_i(v)}\right]}{a_i(v)}\sum_{j=1}^{m} d_{ij}(v)\sigma_j\left(y_j\left(v-\eta_{ij}(v)\right)\right)\Delta w_j(v),\\
(\mathscr{C}X)_{n+j}(k) = \displaystyle\sum_{v=-\infty}^{k-1}\prod_{s=v+1}^{k-1}\frac{e^{-\tilde{a}_j(s)}\left[1-e^{-\tilde{a}_j(v)}\right]}{\tilde{a}_j(v)}\sum_{i=1}^{n}\tilde{d}_{ji}(v)\tilde{\sigma}_i\left(x_i\left(v-\tilde{\eta}_{ji}(v)\right)\right)\Delta\tilde{w}_i(v),
\end{cases} \tag{27}$$

where $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$, and $k \in \mathbb{Z}$. Let $X^0 = \left\{x_i^0; y_j^0\right\}$ be defined as

$$\begin{cases} x_i^0(k) = \displaystyle\sum_{v=-\infty}^{k-1} \prod_{s=v+1}^{k-1} \frac{e^{-a_i(s)}\left[1 - e^{-a_i(v)}\right]}{a_i(v)} \Bigg[\sum_{j=1}^{m} b_{ij}(v) f_j(0) + \sum_{j=1}^{m} c_{ij}(v) f_j(0) + \bigwedge_{j=1}^{m} \alpha_{ij} f_j(0) + \bigwedge_{j=1}^{m} T_{ij}\mu_j + \bigvee_{j=1}^{m} \beta_{ij} f_j(0) \\ \quad + \displaystyle\bigvee_{j=1}^{m} S_{ij}\mu_j + \sum_{j=1}^{m} d_{ij}(v)\sigma_j(0)\Delta w_j(v) + I_i(v) \Bigg], \\ y_j^0(k) = \displaystyle\sum_{v=-\infty}^{k-1} \prod_{s=v+1}^{k-1} \frac{e^{-\widetilde{a}_j(s)}\left[1 - e^{-\widetilde{a}_j(v)}\right]}{\widetilde{a}_j(v)} \Bigg[\sum_{i=1}^{n} \widetilde{b}_{ji}(v) \widetilde{f}_i(0) + \sum_{i=1}^{n} \widetilde{c}_{ji}(v) \widetilde{f}_i(0) + \bigwedge_{i=1}^{n} \widetilde{\alpha}_{ji} \widetilde{f}_i(0) + \bigwedge_{i=1}^{n} \widetilde{T}_{ji}\widetilde{\mu}_i + \bigvee_{i=1}^{n} \widetilde{\beta}_{ji} \widetilde{f}_i(0) \\ \quad + \displaystyle\bigvee_{i=1}^{n} \widetilde{S}_{ji}\widetilde{\mu}_i + \sum_{i=1}^{n} \widetilde{d}_{ji}(v)\widetilde{\sigma}_i(0)\Delta\widetilde{w}_i(v) + J_j(v) \Bigg], \end{cases} \tag{28}$$

where $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$, and $k \in \mathbb{Z}$. By Minkoswki inequality in Lemma 2 and Hölder inequality in Lemma 3, we have

$$\begin{aligned} \left\{x_i^0\right\}_{2p} \le \max_{1 \le i \le n} \sup_{k \in \mathbb{Z}} \Bigg\{ & \left[E \left| \sum_{v=-\infty}^{k-1} \prod_{s=v+1}^{k-1} \frac{e^{-a_i(s)}\left[1 - e^{-a_i(v)}\right]}{a_i(v)} \sum_{j=1}^{m} \left(b_{ij}(v) f_j(0) + c_{ij}(v) f_j(0) \right) \right|^{2p} \right]^{1/2p} \\ & + \left[E \left| \sum_{v=-\infty}^{k-1} \prod_{s=v+1}^{k-1} \frac{e^{-a_i(s)}\left[1 - e^{-a_i(v)}\right]}{a_i(v)} \left(\bigwedge_{j=1}^{m} \alpha_{ij} + \bigvee_{j=1}^{m} \beta_{ij} \right) f_j(0) \right|^{2p} \right]^{1/2p} \\ & + \left[E \left| \sum_{v=-\infty}^{k-1} \prod_{s=v+1}^{k-1} \frac{e^{-a_i(s)}\left[1 - e^{-a_i(v)}\right]}{a_i(v)} \left(\bigwedge_{j=1}^{m} T_{ij} + \bigvee_{j=1}^{m} S_{ij} \right) \mu_j \right|^{2p} \right]^{1/2p} \\ & + \left[E \left| \sum_{v=-\infty}^{k-1} \prod_{s=v+1}^{k-1} \frac{e^{-a_i(s)}\left[1 - e^{-a_i(v)}\right]}{a_i(v)} \sum_{j=1}^{m} d_{ij}(v)\sigma_j(0)\Delta w_j(v) \right|^{2p} \right]^{1/2p} \\ & + \left[E \left| \sum_{v=-\infty}^{k-1} \prod_{s=v+1}^{k-1} \frac{e^{-a_i(s)}\left[1 - e^{-a_i(v)}\right]}{a_i(v)} I_i(v) \right|^{2p} \right]^{1/2p} \Bigg\}. \end{aligned} \tag{29}$$

From Lemma 6, it gets from the above inequality that

$$\begin{aligned}\left\{x_i^0\right\}_{2p} \le \max_{1\le i\le n}\sup_{k\in\mathbb{Z}} &\left\{\begin{aligned}&\frac{\left(1-e^{-a^u}\right)}{a^l\left(1-e^{-a^l}\right)}\left[\begin{aligned}&\sum_{j=1}^{m}\left(b_{ij}^u\left|f_j(0)\right|+c_{ij}^u\left|f_j(0)\right|\right)+\sum_{j=1}^{m}\left(\left|\alpha_{ij}\right|+\left|\beta_{ij}\right|\right)\left|f_j(0)\right|\\&+\sum_{j=1}^{m}\left(\left|T_{ij}\right|+\left|S_{ij}\right|\right)\left|\mu_j\right|+I_i^u\end{aligned}\right]\\&+\sum_{j=1}^{m}d_{ij}^u\sigma_j(0)\left[\sum_{v=-\infty}^{k-1}\prod_{s=v+1}^{k-1}\frac{e^{-a_i(s)}\left[1-e^{-a_i(v)}\right]}{a_i(v)}\right]^{1-(1/2p)}\\&\times\left[\sum_{v=-\infty}^{k-1}\prod_{s=v+1}^{k-1}\frac{e^{-a_i(s)}\left[1-e^{-a_i(v)}\right]}{a_i(v)}E\left|\Delta w_j(v)\right|^{2p}\right]^{1/2p}\end{aligned}\right\}\\ \le \frac{\left(1-e^{-a^u}\right)}{a^l\left(1-e^{-a^l}\right)}&\max_{1\le i\le n}\left[\sum_{j=1}^{m}\left(b_{ij}^u+c_{ij}^u+\left|\alpha_{ij}\right|+\left|\beta_{ij}\right|\right)\left|f_j(0)\right|+\sum_{j=1}^{m}\left(\left|T_{ij}\right|+\left|S_{ij}\right|\right)\left|\mu_j\right|+I_i^u+\sum_{j=1}^{m}d_{ij}^u\sigma_j(0)C_{2p}^{1/2p}\right]\coloneqq e_{2p}.\end{aligned} \tag{30}$$

Similarly, we have

$$\left\|\left\{y_j^0\right\}\right\|_{2p}\le\frac{\left(1-e^{-\widetilde{a}^u}\right)}{\widetilde{a}^l\left(1-e^{-\widetilde{a}^l}\right)}\max_{1\le j\le m}\left[\sum_{i=1}^{n}\left(\widetilde{b}_{ji}^u+\widetilde{c}_{ji}^u+\left|\widetilde{\alpha}_{ji}\right|+\left|\widetilde{\beta}_{ji}\right|\right)\left|\widetilde{f}_i(0)\right|+\sum_{i=1}^{n}\left(\left|\widetilde{T}_{ji}\right|+\left|\widetilde{S}_{ji}\right|\right)\left|\widetilde{\mu}_i\right|+J_j^u+\sum_{i=1}^{n}\widetilde{d}_{ji}^u\widetilde{\sigma}_i(0)C_{2p}^{1/2p}\right]\coloneqq\widetilde{e}_{2p}. \tag{31}$$

By the above inequalities, it concludes

$$\left\|X^0\right\|_{2p}\le\max\left\{e_{2p},\widetilde{e}_{2p}\right\}\coloneqq\alpha_{2p}. \tag{32}$$

It follows (25)–(27) that

$$\begin{aligned}&\left\|\left\{(\Phi X)_i(k)-x_i^0(k)\right\}_{i=1}^{n}\right\|_{2p}\\&\le\max_{1\le i\le n}\sup_{k\in\mathbb{Z}}\sum_{j=1}^{m}b_{ij}^uL_j^f\left\{E\left[\sum_{v=-\infty}^{k-1}\prod_{s=v+1}^{k-1}\frac{e^{-a_i(s)}\left[1-e^{-a_i(v)}\right]}{a_i(v)}\left|y_j(v)\right|\right]^{2p}\right\}^{1/2p}\\&\quad+\max_{1\le i\le n,1\le j\le m}\sup_{k\in\mathbb{Z}}D_i^{**}\left\{E\left[\sum_{v=-\infty}^{k-1}\prod_{s=v+1}^{k-1}\frac{e^{-a_i(s)}\left[1-e^{-a_i(v)}\right]}{a_i(v)}\left|y_j\left(v-\tau_{ij}(v)\right)\right|\right]^{2p}\right\}^{1/2p}\\&\quad+\max_{1\le i\le n,1\le j\le m}\sup_{k\in\mathbb{Z}}K^*\left\{E\left[\sum_{v=-\infty}^{k-1}\prod_{s=v+1}^{k-1}\frac{e^{-a_i(s)}\left[1-e^{-a_i(v)}\right]}{a_i(v)}\left|y_j\left(v-\eta_{ij}(v)\right)\Delta w_j(v)\right|\right]^{2p}\right\}^{1/2p},\end{aligned} \tag{33}$$

which yields from Lemmas 2 and 3 that

$$\begin{aligned}\left\|\left\{(\Phi X)_i(k)-x_i^0(k)\right\}_{i=1}^n\right\|_{2p} \le & \max_{1\le i\le n}\sup_{k\in\mathbb{Z}}\sum_{j=1}^m b_{ij}^u L_j^f\left\{\left[\sum_{v=-\infty}^{k-1}\prod_{s=v+1}^{k-1}\frac{e^{-a_i(s)}\left[1-e^{-a_i(v)}\right]}{a_i(v)}\right]^{p-1}\times\sum_{v=-\infty}^{k-1}\prod_{s=v+1}^{k-1}\frac{e^{-a_i(s)}\left[1-e^{-a_i(v)}\right]}{a_i(v)}E\left|y_j(v)\right|^{2p}\right\}^{1/2p}\\ &+\max_{1\le i\le n,1\le j\le m}\sup_{k\in\mathbb{Z}}D_i^{**}\left\{\left[\sum_{v=-\infty}^{k-1}\prod_{s=v+1}^{k-1}\frac{e^{-a_i(s)}\left[1-e^{-a_i(v)}\right]}{a_i(v)}\right]^{p-1}\right.\\ &\left.\times\sum_{v=-\infty}^{k-1}\prod_{s=v+1}^{k-1}\frac{e^{-a_i(s)}\left[1-e^{-a_i(v)}\right]}{a_i(v)}E\left|y_j\left(v-\tau_{ij}(v)\right)\right|^{2p}\right\}^{1/2p},\\ &+\max_{1\le i\le n,1\le j\le m}\sup_{k\in\mathbb{Z}}K^*\left\{\left[\sum_{v=-\infty}^{k-1}\prod_{s=v+1}^{k-1}\frac{e^{-a_i(s)}\left[1-e^{-a_i(v)}\right]}{a_i(v)}\right]^{p-1}\right.\\ &\left.\times\sum_{v=-\infty}^{k-1}\prod_{s=v+1}^{k-1}\frac{e^{-a_i(s)}\left[1-e^{-a_i(v)}\right]}{a_i(v)}E\left|y_j\left(v-\eta_{ij}(v)\right)\Delta w_j(v)\right|^{2p}\right\}^{1/2p}\end{aligned} \tag{34}$$

where $D_i^{**}=D^*-\sum_{j=1}^m b_{ij}^u L_j^f$, $i=1,2,\ldots,n$. Applying Lemma 5 to the above inequality, it derives

$$\begin{aligned}\left\|\left\{(\Phi X)_i(k)-x_i^0(k)\right\}_{i=1}^n\right\|_{2p} &\le \frac{\left(1-e^{-a^u}\right)}{a^l\left(1-e^{-a^l}\right)}\left\{D^*+K^*C_{2p}^{1/2p}\right\}\\ &\quad\cdot\|X\|_{2p}\le r_{2p}\|X\|_{2p}\le\frac{r_{2p}\alpha_{2p}}{1-r_{2p}}.\end{aligned} \tag{35}$$

Similarly,

$$\begin{aligned}\left\|\left\{(\Phi X)_{n+j}(k)-y_j^0(k)\right\}_{j=1}^m\right\|_{2p} &\le \frac{\left(1-e^{-\widetilde{a}^u}\right)}{\widetilde{a}^l\left(1-e^{-\widetilde{a}^l}\right)}\\ &\quad\cdot\left\{\widetilde{D}^*+\widetilde{K}^*C_{2p}^{1/2p}\right\}\|X\|_{2p}\\ &\le r_{2p}\|X\|_{2p}\le\frac{r_{2p}\alpha_{2p}}{1-r_{2p}}.\end{aligned} \tag{36}$$

Together with the above inequalities, we obtain

$$\left\|\Phi X-X^0\right\|_{2p}\le\frac{r_{2p}\alpha_{2p}}{1-r_{2p}}. \tag{37}$$

Hence, for $X=\left\{x_i;y_j\right\}\in\Lambda$, it leads from (32) and (37) to

$$\|\Phi X\|_{2p}\le\left\|X^0\right\|_{2p}+\left\|\Phi X-X^0\right\|_{2p}\le\alpha_{2p}+\frac{r_{2p}\alpha_{2p}}{1-r_{2p}}=\frac{\alpha_{2p}}{1-r_{2p}}:=\beta_{2p}. \tag{38}$$

From (38), $\mathscr{B}\Lambda$ is uniformly bounded. Together with the continuity of $\mathscr{B}$, for any bounded sequence $\{\varphi_n\}$ in Λ, we know that there exists a subsequence $\left\{\varphi_{n_k}\right\}$ in Λ such that $\left\{\mathscr{B}(\varphi_{n_k})\right\}$ is convergent in $\mathscr{B}(\Lambda)$. Therefore, $\mathscr{B}$ is compact on Λ. Then, condition (2) of Lemma 7 is satisfied.

The next step is proving condition (1) of Lemma 7. Now, we consist in proving the $2p$-th mean almost periodicity of $\mathscr{B}X(\cdot)$ and $\mathscr{C}X(\cdot)$. Since $X(\cdot)$ is a $2p$-th mean almost periodic sequence and all the coefficients in system (8) are almost periodic sequences, for all $\epsilon>0$ there exists $l_\epsilon>0$ such that every interval of length $l_\epsilon>0$ contains a ω with the property that

$$\begin{aligned}\left[E\left|x_i(k+\omega)-x_i(k)\right|^{2p}\right]^{1/2p}&<\epsilon,\\ \left|a_i(k+\omega)-a_i(k)\right|&<\epsilon,\\ \left|\widetilde{a}_j(k+\omega)-\widetilde{a}_j(k)\right|&<\epsilon,\\ \left|b_{ij}(k+\omega)-b_{ij}(k)\right|&<\epsilon,\\ \left|\widetilde{b}_{ji}(k+\omega)-\widetilde{b}_{ji}(k)\right|&<\epsilon,\\ \left|c_{ij}(k+\omega)-c_{ij}(k)\right|&<\epsilon,\\ \left|\widetilde{c}_{ji}(k+\omega)-\widetilde{c}_{ji}(k)\right|&<\epsilon,\\ \left|d_{ij}(k+\omega)-d_{ij}(k)\right|&<\epsilon,\\ \left|\widetilde{d}_{ji}(k+\omega)-\widetilde{d}_{ji}(k)\right|&<\epsilon,\\ \left|\tau_{ij}(k+\omega)-\tau_{ij}(k)\right|&<\epsilon,\\ \left|\widetilde{\tau}_{ji}(k+\omega)-\widetilde{\tau}_{ji}(k)\right|&<\epsilon,\\ \left|\eta_{ij}(k+\omega)-\eta_{ij}(k)\right|&<\epsilon,\\ \left|\widetilde{\eta}_{ji}(k+\omega)-\widetilde{\eta}_{ji}(k)\right|&<\epsilon,\\ \left|I_i(k+\omega)-I_i(k)\right|&<\epsilon,\\ \left|J_j(k+\omega)-J_j(k)\right|&<\epsilon,\end{aligned} \tag{39}$$

where $i=1,2,\ldots,n$, $j=1,2,\ldots,m$, and $k\in\mathbb{Z}$. By (26), (27), and (H_2), we could easily find a positive constant M such that

$$\begin{cases} \left[E\left|(\mathscr{B}X)_i(k+\omega)-(\mathscr{B}X)_i(k)\right|^{2p}\right]^{1/2p} \le M \max\limits_{1\le i\le n}\sup\limits_{k\in\mathbb{Z}}\left[E\left|x_i(k+\omega)-x_i(k)\right|^{2p}\right]^{1/2p} < M\epsilon, \\ \left[E\left|(\mathscr{B}X)_{n+j}(k+\omega)-(\mathscr{B}X)_{n+j}(k)\right|^{2p}\right]^{1/2p} \le M \max\limits_{1\le j\le m}\sup\limits_{k\in\mathbb{Z}}\left[E\left|y_j(k+\omega)-y_j(k)\right|^{2p}\right]^{1/2p} < M\epsilon, \end{cases} \tag{40}$$

$$\begin{cases} \left[E\left|(\mathscr{C}X)_i(k+\omega)-(\mathscr{C}X)_i(k)\right|^{2p}\right]^{1/2p} \le M \max\limits_{1\le i\le n}\sup\limits_{k\in\mathbb{Z}}\left[E\left|x_i(k+\omega)-x_i(k)\right|^{2p}\right]^{1/2p} < M\epsilon, \\ \left[E\left|(\mathscr{C}X)_{n+j}(k+\omega)-(\mathscr{C}X)_{n+j}(k)\right|^{2p}\right]^{1/2p} \le M \max\limits_{1\le j\le m}\sup\limits_{k\in\mathbb{Z}}\left[E\left|y_j(k+\omega)-y_j(k)\right|^{2p}\right]^{1/2p} < M\epsilon, \end{cases} \tag{41}$$

where $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$, and $k \in \mathbb{Z}$. From (40) and (41), $\mathscr{B}X(\cdot)$ and $\mathscr{C}X(\cdot)$ are $2p$-th mean almost periodic processes. Further, by (38), it is easy to obtain that $\mathscr{B}X + \mathscr{C}Y \in \Lambda$, $\forall X, Y \in \Lambda$. Then, condition (1) of Lemma 7 holds.

Finally, $\forall X = \{x_i; y_j\}$, $X^* = \{x_i^*; y_j^*\} \in \Lambda$, from (27), it yields

$$\begin{aligned} \left\|\{(\mathscr{C}X)_i(k)-(\mathscr{C}X^*)_i(k)\}_{i=1}^n\right\|_{2p} &\le \frac{\left[1-e^{-a^u}\right]}{a^l}\max_{1\le i\le n}\sup_{k\in\mathbb{Z}}\left\{E\left[\sum_{v=-\infty}^{k-1}\prod_{s=v+1}^{k-1}e^{-a^l}\times\sum_{j=1}^{m}d_{ij}(v)\Big(\sigma_j\big(x_j\big(v-\eta_{ij}(v)\big)\big)\right.\right. \\ &\quad \left.\left.-\sigma_j\big(y_j\big(v-\eta_{ij}(v)\big)\big)\Big)\Delta w_j(v)\right]^{2p}\right\} \\ &\le \frac{\left[1-e^{-a^u}\right]}{a^l}\max_{1\le i\le n,1\le j\le m}\sup_{k\in\mathbb{Z}}K^*\left\{\left[\sum_{v=-\infty}^{k-1}\prod_{s=v+1}^{k-1}e^{-a^l}\right]^{2p-1}\times\sum_{v=-\infty}^{k-1}\prod_{s=v+1}^{k-1}e^{-a^l}\right. \\ &\quad \left.\cdot E\left|\left[x_j\big(v-\eta_{ij}(v)\big)-y_j\big(v-\eta_{ij}(v)\big)\right]\Delta w_j(v)\right|^{2p}\right\} \\ &\le \frac{K^*C_{2p}^{1/2p}\left(1-e^{-a^u}\right)}{a^l\left(1-e^{-a^l}\right)}\|X-Y\|_{2p} \\ &\le r_{2p}\|X-Y\|_{2p}, \end{aligned} \tag{42}$$

similarly,

$$\begin{aligned} &\left\|\left\{(\mathscr{C}X)_{n+j}(k)-(\mathscr{C}X^*)_{n+j}(k)\right\}_{j=1}^m\right\|_{2p} \\ &\le \frac{\widetilde{K}^*C_{2p}^{1/2p}\left(1-e^{-\widetilde{a}^u}\right)}{\widetilde{a}^l\left(1-e^{-\widetilde{a}^l}\right)}\|X-Y\|_{2p} \\ &\le r_{2p}\|X-Y\|_{2p}. \end{aligned} \tag{43}$$

From the above inequalities, it leads

$$\|\mathscr{C}X-\mathscr{C}Y\|_{2p} \le r_{2p}\|X-Y\|_{2p}. \tag{44}$$

In view of (H_3), $\mathscr{C}$ is a contraction mapping. Hence, condition (3) of Lemma 7 is satisfied. Therefore, all the conditions in Lemma 7 hold. By Lemma 7, system (8) has a $2p$-th mean almost periodic sequence solution. This completes the proof. □

According to Theorem 1, we can easily obtain the following theorem.

Theorem 2. *Assume that all conditions in Theorem 1 hold and all coefficients in system (8) are periodic sequences, and then system (8) admits at least one $2p$-th mean periodic sequence solution.*

Remark 1. In view of the definition of r_{2p} in Theorem 1, a_i and $\widetilde{a}_j$ in system (8) are bigger, then the probability of $r_{2p} < 1$ is bigger, where $i = 1, 2, \ldots, n$, and $j = 1, 2, \ldots, m$. The parameters T_{ij}, S_{ij}, μ_{ij}, $\widetilde{T}_{ji}$, $\widetilde{S}_{ji}$, $\widetilde{\mu}_{ji}$, I_i, and J_j have no effect on the value of r_{2p}, but they have effects on the boundedness of almost periodic sequence solutions of system (8), where $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$. In order to derive the existence of almost periodic sequence solutions of system (8), we would better choose bigger a_i and $\widetilde{a}_j$, and the other parameters should be smaller except T_{ij}, S_{ij}, μ_{ij}, $\widetilde{T}_{ji}$, $\widetilde{S}_{ji}$, $\widetilde{\mu}_{ji}$, I_i, and J_j, where $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$.

4. $2p$-th Moment Global Exponential Stability

Suppose that $X = \{x_i; y_j\}$ with initial value $\varphi = \{\varphi_i; \psi_j\}$ and $X^* = \{x_i^*; y_j^*\}$ with initial value $\varphi^* = \{\varphi_i^*; \psi_j^*\}$ are arbitrary two solutions of system (8). For convenience, let

$$\gamma_{2p} = \max_{1\le i\le n, 1\le j\le m} \sup_{s\in[-\mu_0,0]_{\mathbb{Z}}} \left\{ \left(E\left|\varphi_i(s)-\varphi_i^*(s)\right|^{2p}\right)^{1/2p}, \left(E\left|\psi_j(s)-\psi_j^*(s)\right|^{2p}\right)^{1/2p} \right\},$$
$$\mu_0 = \max_{(i,j)}\left\{\tau_{ij}^u, \eta_{ij}^u, \tilde{\tau}_{ji}^u, \tilde{\eta}_{ji}^u\right\}. \tag{45}$$

Definition 4.1 [35]. System (8) is said to be $2p$-th moment global exponential stability if there are positive constants k_0, M, and λ such that

$$\left|X(k)-X^*(k)\right|_{2p} = \max_{1\le i\le n, 1\le j\le m}\left\{\left(E\left|x_i(k)-x_i^*(k)\right|^{2p}\right)^{1/2p}, \left(E\left|y_j(k)-y_j^*(k)\right|^{2p}\right)^{1/2p}\right\} < M\gamma_{2p}e^{-\lambda k}, \tag{46}$$

where $\forall k > k_0$, $k\in\mathbb{Z}$. The 2-nd moment global exponential stability is called square-mean global exponential stability.

Theorem 3. *Assume that (H_1)–(H_3) hold, and then system (8) is $2p$-th moment globally exponentially stable.*

Proof. From Lemmas 1 and 6, it follows that

$$\begin{aligned}
\left|x_i(k)-x_i^*(k)\right| &\le \prod_{s=0}^{k-1} e^{-a_i(s)}\left|\varphi_i(0)-\varphi_i^*(0)\right| + \frac{\left(1-e^{-a^u}\right)}{a^l}\sum_{v=0}^{k-1}\prod_{s=v+1}^{k-1} e^{-a_i(s)}\sum_{j=1}^{m} \\
&\cdot\left\{\begin{array}{c} b_{ij}^u L_j^f\left|y_j(v)-y_j^*(v)\right| \\ +\left(c_{ij}^u+\left|\alpha_{ij}\right|+\left|\beta_{ij}\right|\right)L_j^f\left|y_j\left(v-\tau_{ij}(v)\right)-y_j^*\left(v-\tau_{ij}(v)\right)\right| \\ +d_{ij}^u L_j^\sigma\left|y_j\left(v-\eta_{ij}(v)\right)-y_j^*\left(v-\eta_{ij}(v)\right)\right|\left|\Delta w_j(v)\right| \end{array}\right\} \\
&\le e^{-a^l k}\left|\varphi_i(0)-\varphi_i^*(0)\right| + \frac{\left(1-e^{-a^u}\right)}{a^l}\sum_{v=0}^{k-1} e^{-a^l(k-v-1)}\sum_{j=1}^{m}\left\{\begin{array}{c} b_{ij}^u L_j^f\left|\left|y_j(v)-y_j^*\right|(v)\right| \\ +\left(c_{ij}^u+\left|\alpha_{ij}\right|+\left|\beta_{ij}\right|\right)L_j^f\left|y_j\left(v-\tau_{ij}(v)\right)-y_j^*\left(v-\tau_{ij}(v)\right)\right| \\ +d_{ij}^u L_j^\sigma\left|y_j\left(v-\eta_{ij}(v)\right)-y_j^*\left(v-\eta_{ij}(v)\right)\right|\left|\Delta w_j(v)\right| \end{array}\right\},
\end{aligned} \tag{47}$$

$$\begin{aligned}
\left|y_j(k)-y_j^*(k)\right| &\le \prod_{s=0}^{k-1} e^{-\tilde{a}_j(s)}\left|\psi_j(0)-\psi_j^*(0)\right| + \frac{\left(1-e^{-\tilde{a}^u}\right)}{\tilde{a}^l}\sum_{v=0}^{k-1}\prod_{s=v+1}^{k-1} e^{-\tilde{a}_j(s)}\sum_{i=1}^{n} \\
&\cdot\left\{\begin{array}{c} \tilde{b}_{ji}^u L_i^{\tilde{f}}\left|x_i(v)-x_i^*(v)\right| \\ +\left(\tilde{c}_{ji}^u+\left|\tilde{\alpha}_{ji}\right|+\left|\tilde{\beta}_{ji}\right|\right)L_i^{\tilde{f}}\left|x_i\left(v-\tilde{\tau}_{ji}(v)\right)-x_i^*\left(v-\tilde{\tau}_{ji}(v)\right)\right| \\ +\tilde{d}_{ji}^u L_i^{\tilde{\sigma}}\left|x_i\left(v-\tilde{\eta}_{ji}(v)\right)-x_i^*\left(v-\tilde{\eta}_{ji}(v)\right)\right|\left|\Delta\tilde{w}_i(v)\right| \end{array}\right\} \\
&\le e^{-\tilde{a}^l k}\left|\psi_j(0)-\psi_j^*(0)\right| + \frac{\left(1-e^{-\tilde{a}^u}\right)}{\tilde{a}^l}\sum_{v=0}^{k-1} e^{-\tilde{a}^l(k-v-1)}\sum_{i=1}^{n}\left\{\begin{array}{c} \tilde{b}_{ji}^u L_i^{\tilde{f}}\left|x_i(v)-x_i^*(v)\right| \\ +\left(\tilde{c}_{ji}^u+\left|\tilde{\alpha}_{ji}\right|+\left|\tilde{\beta}_{ji}\right|\right)L_i^{\tilde{f}}\left|x_i\left(v-\tilde{\tau}_{ji}(v)\right)-x_i^*\left(v-\tilde{\tau}_{ji}(v)\right)\right| \\ +\tilde{d}_{ji}^u L_i^{\tilde{\sigma}}\left|x_i\left(v-\tilde{\eta}_{ji}(v)\right)-x_i^*\left(v-\tilde{\eta}_{ji}(v)\right)\right|\left|\Delta\tilde{w}_i(v)\right| \end{array}\right\},
\end{aligned} \tag{48}$$

where $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$, and $k \in [-\mu_0, +\infty)$.

For convenience, let

$$a_0 = \frac{1 - e^{-a^u}}{a^l}, \qquad \tilde{a}_0 = \frac{1 - e^{-\tilde{a}^u}}{\tilde{a}^l}, \tag{49}$$

$Z = \{z_i\}$, $W = \{w_j\}$, $z_i(k) = x_i(k) - x_i^*(k)$, and $w_j(k) = y_j(k) - y_j^*(k)$, where $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$, and $k \in \mathbb{Z}$.

Similar to the argument as that in (37), it gets from (47) that

$$\begin{aligned}
|Z(k)|_{2p} &\le e^{-a^l k}\gamma_{2p} + \max_{1\le i\le n}\sum_{j=1}^{m} a_0 b_{ij}^u L_j^f \left\{\left[\sum_{s=0}^{k-1} e^{-a^l(k-s-1)}\right]^{2p-1}\sum_{s=0}^{k-1} e^{-a^l(k-s-1)} E\left|y_j(s) - y_j^*(s)\right|^{2p}\right\}^{1/2p} \\
&\quad + \max_{1\le i\le n}\sum_{j=1}^{m} a_0\left(c_{ij}^u + |\alpha_{ij}| + |\beta_{ij}|\right)L_j^f \left\{\begin{array}{c}\left[\sum_{s=0}^{k-1} e^{-a^l(k-s-1)}\right]^{2p-1} \\ \times\sum_{s=0}^{k-1} e^{-a^l(k-s-1)} E\left|y_j(s-\tau_{ij}(s)) - y_j^*(s-\tau_{ij}(s))\right|^{2p}\end{array}\right\}^{1/2p} \\
&\quad + \max_{1\le i\le n}\sum_{j=1}^{m} a_0 d_{ij}^u L_j^\sigma \left\{\begin{array}{c}\left[\sum_{s=0}^{k-1} e^{-a^l(k-s-1)}\right]^{2p-1} \\ \times\sum_{s=0}^{k-1} e^{-a^l(k-s-1)} E\left|\left[y_j(s-\eta_{ij}(s)) - y_j^*(s-\eta_{ij}(s))\right]\Delta w_j(s)\right|^{2p}\end{array}\right\}^{1/2p} \\
&\le e^{-a^l k}\gamma_{2p} + \max_{1\le i\le n}\sum_{j=1}^{m} a_0 b_{ij}^u L_j^f \left\{\left[\sum_{s=0}^{k-1} e^{-a^l(k-s-1)}\right]^{2p-1}\sum_{s=0}^{k-1} e^{-a^l(k-s-1)} |W(s)|_{2p}^{2p}\right\}^{1/2p} \\
&\quad + \max_{1\le i\le n}\sum_{j=1}^{m} a_0\left(c_{ij}^u + |\alpha_{ij}| + |\beta_{ij}|\right)L_j^f \left\{\begin{array}{c}\left[\sum_{s=0}^{k-1} e^{-a^l(k-s-1)}\right]^{2p-1} \\ \times\sum_{s=0}^{k-1} e^{-a^l(k-s-1)} \left|W(s-\tau_{ij}(s))\right|_{2p}^{2p}\end{array}\right\}^{1/2p} \\
&\quad + \max_{1\le i\le n}\sum_{j=1}^{m} a_0 C_{2p}^{1/2p} d_{ij}^u L_j^\sigma \left\{\left[\sum_{s=0}^{k-1} e^{-a^l(k-s-1)}\right]^{2p-1}\sum_{s=0}^{k-1} e^{-a^l(k-s-1)} \left|W(s-\eta_{ij}(s))\right|_{2p}^{2p}\right\}^{1/2p}.
\end{aligned} \tag{50}$$

Similarly, it follows from (48) that

$$\begin{aligned}
|W(k)|_{2p} &\le e^{-\tilde{a}^l k}\gamma_{2p} + \max_{1\le j\le m}\sum_{i=1}^{n} \tilde{a}_0 \tilde{b}_{ji}^u L_i^{\tilde{f}} \left\{\left[\sum_{s=0}^{k-1} e^{-\tilde{a}^l(k-s-1)}\right]^{2p-1}\sum_{s=0}^{k-1} e^{-\tilde{a}^l(k-s-1)} |Z(s)|_{2p}^{2p}\right\}^{1/2p} \\
&\quad + \max_{1\le j\le m}\sum_{i=1}^{n} \tilde{a}_0\left(\tilde{c}_{ji}^u + |\tilde{\alpha}_{ji}| + |\tilde{\beta}_{ji}|\right)L_i^{\tilde{f}} \left\{\begin{array}{c}\left[\sum_{s=0}^{k-1} e^{-\tilde{a}^l(k-s-1)}\right]^{2p-1} \\ \times\sum_{s=0}^{k-1} e^{-\tilde{a}^l(k-s-1)} \left|Z(s-\tilde{\tau}_{ji}(s))\right|_{2p}^{2p}\end{array}\right\}^{1/2p} \\
&\quad + \max_{1\le j\le m}\sum_{i=1}^{n} \tilde{a}_0 C_{2p}^{1/2p} \tilde{d}_{ji}^u L_i^{\tilde{\sigma}} \left\{\left[\sum_{s=0}^{k-1} e^{-\tilde{a}^l(k-s-1)}\right]^{2p-1}\sum_{s=0}^{k-1} e^{-\tilde{a}^l(k-s-1)} \left|Z(s-\tilde{\eta}_{ji}(s))\right|_{2p}^{2p}\right\}^{1/2p}.
\end{aligned} \tag{51}$$

Be aware of (H_3) in Theorem 1, there exists a constant $\lambda > 0$ small enough such that

$$\max_{1\le i\le n}\sum_{j=1}^{m}\frac{e^{\lambda}a_0}{1-e^{-(a^l-2p\lambda)}}\left[b_{ij}^{u}L_j^{f}+e^{\mu_0\lambda}\left(c_{ij}^{u}+\left|\alpha_{ij}\right|+\left|\beta_{ij}\right|\right)L_j^{f}+e^{\mu_0\lambda}C_{2p}^{1/2p}d_{ij}^{u}L_j^{\sigma}\right]\overset{\text{def}}{=}\rho\le 1,$$
$$\max_{1\le j\le m}\sum_{i=1}^{n}\frac{e^{\lambda}\widetilde{a}_0}{1-e^{-(\widetilde{a}^l-2p\lambda)}}\left[\widetilde{b}_{ji}^{u}L_i^{\widetilde{f}}+e^{\mu_0\lambda}\left(\widetilde{c}_{ji}^{u}+\left|\widetilde{\alpha}_{ji}\right|+\left|\widetilde{\beta}_{ji}\right|\right)L_i^{\widetilde{f}}+e^{\mu_0\lambda}C_{2p}^{1/2p}\widetilde{d}_{ji}^{u}L_i^{\widetilde{\sigma}}\right]\overset{\text{def}}{=}\widetilde{\rho}\le 1. \tag{52}$$

Define $V(k)=\max\left\{|Z(k)|_{2p},|W(k)|_{2p}\right\}$, $k\in\mathbb{Z}$.

Next, we claim that there exists a constant $M_0>1$ such that

$$V(k)\le M_0\gamma_{2p}e^{-\lambda k},\quad \forall k\in[-\mu_0,+\infty). \tag{53}$$

If (53) is invalid, then there must exist $k_0\in(0,+\infty)_{\mathbb{Z}}$ such that one of the following two cases holds:

(C_1) $|Z(k_0)|_{2p}>M_0\gamma_{2p}e^{-\lambda k_0}$ and $V(k)\le M_0\gamma_{2p}e^{-\lambda k}$, $\forall k\in[-\mu_0,k_0)$

(C_2) $|W(k_0)|_{2p}>M_0\gamma_{2p}e^{-\lambda k_0}$ and $V(k)\le M_0\gamma_{2p}e^{-\lambda k}$, $\forall k\in[-\mu_0,k_0)$

If (C_1) is valid, it follows from (50) that

$$\begin{aligned}
\left|Z(k_0)\right|_{2p}&\le e^{-a^lk_0}\gamma_{2p}+\max_{1\le i\le n}\sum_{j=1}^{m}a_0b_{ij}^{u}L_j^{f}M_0\gamma_{2p}\left\{\left[\sum_{s=0}^{k-1}e^{-a^l(k-s-1)}\right]^{2p-1}\sum_{s=0}^{k-1}e^{-a^l(k-s-1)}e^{-2p\lambda s}\right\}^{1/2p}\\
&\quad+\max_{1\le i\le n}\sum_{j=1}^{m}a_0M_0\gamma_{2p}\left[\left(c_{ij}^{u}+\left|\alpha_{ij}\right|+\left|\beta_{ij}\right|\right)L_j^{f}+C_{2p}^{1/2p}d_{ij}^{u}L_j^{\sigma}\right]\\
&\quad\times\left\{\left[\sum_{s=0}^{k-1}e^{-a^l(k-s-1)}\right]^{2p-1}\sum_{s=0}^{k-1}e^{-a^l(k-s-1)}e^{-2p\lambda(s-\mu_0)}\right\}^{1/2p}\\
&\le e^{-a^lk_0}\gamma_{2p}+\max_{1\le i\le n}\sum_{j=1}^{m}a_0b_{ij}^{u}L_j^{f}M_0\gamma_{2p}e^{-\lambda k_0}e^{\lambda}\left[\frac{1-e^{-a^lk_0}}{1-e^{-a^l}}\right]^{1-(1/2p)}\left[\sum_{s=0}^{k_0-1}e^{-(a^l-p\lambda)(k_0-s-1)}\right]^{1/2p}\\
&\quad+\max_{1\le i\le n}\sum_{j=1}^{m}a_0M_0\gamma_{2p}\left[\left(c_{ij}^{u}+\left|\alpha_{ij}\right|+\left|\beta_{ij}\right|\right)L_j^{f}+C_{2p}^{1/2p}d_{ij}^{u}L_j^{\sigma}\right]\\
&\quad\times e^{-\lambda k_0}e^{(\mu_0+1)\lambda}\left[\frac{1-e^{-a^lk_0}}{1-e^{-a^l}}\right]^{1-(1/2p)}\left[\sum_{s=0}^{k_0-1}e^{-(a^l-p\lambda)(k_0-s-1)}\right]^{1/2p}\\
&\le e^{-a^lk_0}\gamma_{2p}+\max_{1\le i\le n}\sum_{j=1}^{m}a_0M_0\gamma_{2p}e^{-\lambda k_0}\left[b_{ij}^{u}L_j^{f}+e^{\mu_0\lambda}\left(c_{ij}^{u}+\left|\alpha_{ij}\right|+\left|\beta_{ij}\right|\right)L_j^{f}+e^{\mu_0\lambda}C_{2p}^{1/2p}d_{ij}^{u}L_j^{\sigma}\right]\\
&\quad\times e^{\lambda}\left[\frac{1-e^{-a^lk_0}}{1-e^{-a^l}}\right]^{1-(1/2p)}\left[\frac{1-e^{-(a^l-2p\lambda)k_0}}{1-e^{-(a^l-2p\lambda)}}\right]^{1/2p}\\
&\le M_0\gamma_{2p}e^{-\lambda k_0}\left\{\begin{array}{c}\dfrac{1}{M_0}e^{-(a^l-\lambda)k_0}\\ +\max\limits_{1\le i\le n}\sum\limits_{j=1}^{m}a_0\left[b_{ij}^{u}L_j^{f}+e^{\mu_0\lambda}\left(c_{ij}^{u}+\left|\alpha_{ij}\right|+\left|\beta_{ij}\right|\right)L_j^{f}+e^{\mu_0\lambda}C_{2p}^{1/2p}d_{ij}^{u}L_j^{\sigma}\right]\dfrac{e^{\lambda}\left[1-e^{-(a^l-\lambda)k_0}\right]}{1-e^{-(a^l-2p\lambda)}}\end{array}\right\}\\
&\le M_0\gamma_{2p}e^{-\lambda k_0}\left\{e^{-(a^l-\lambda)k_0}+\rho\left[1-e^{-(a^l-\lambda)k_0}\right]\right\}\\
&\le M_0\gamma_{2p}e^{-\lambda k_0}.
\end{aligned} \tag{54}$$

In the fourth inequality from the bottom of (54), we use the fact $[1-e^{-a^l k_0}]^{1-(1/2p)}[1-e^{-(a^l-p\lambda)k_0}]^{1/2p} \le 1-e^{-(a^l-\lambda)k_0}$ and $[1-e^{-a^l}]^{1/2p} \ge [1-e^{-(a^l-p\lambda)}]^{1/2p}$. (54) contradicts (C_1). So, (C_1) is invalid. Similarly, we can easily conclude (C_2) is invalid by using (51). Therefore, (53) is satisfied. By (53), system (8) is $2p$-th moment globally exponentially stable. This completes the proof. □

According to Theorems 1 and 2, we can easily obtain the following theorem.

Theorem 4. *Assume that all conditions in Theorem 1 hold, and then system (8) admits a $2p$-th mean almost periodic sequence solution, which is $2p$-th moment globally exponentially stable. Further, if all coefficients in system (8) are periodic sequences, then system (8) admits at least one $2p$-th mean periodic sequence solution, which is globally exponentially stable.*

Proof. The result can be easily obtained by Theorem 3, so we omit it. This completes the proof. □

Remark 2. Assume that $X(k) = (x_1(k), x_2(k), \ldots, x_n(k))$ is a solution of (1), and then the length of $X(k)$ is usually measured by $\|X\|_\infty = \sup_{k\in\mathbb{R}}\max_{1\le i\le n}|x_i(k)|$. However, if $X(k)$ is a solution of stochastic system, its length should not be measured by X_∞ because $X(k)$ is a random variable. In this paper, we use norm $\|X\|_{2p} = \max_{1\le i\le n} \sup_{k\in\mathbb{Z}} (E|x_i(k)|^{2p})^{1/2p}$, $2p > 1$ for random variable $X(k)$. Owing to the expectation E and order p in $\|X\|_{2p}$, the computing processes of this paper are more complicated than those in literatures [41–47]. It is worth mentioning that Minkoswki inequality in Lemma 2 and Hölder inequality in Lemma 3 are crucial to the computing processes. The facts above are obvious from the computations of (32), (42), (50), and (54) in the proofs of Theorems 1 and 3. Further, the stochastic terms $d_{ij}\sigma_j\Delta w_j$ and $\widetilde{d}_{ji}\widetilde{\sigma}_i\Delta\widetilde{w}_i$ $(i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, m)$ in system (8) also increase the complexity of computing. This point is also clear from the computations of (42) and (50).

5. Examples and Computer Simulations

Example 5.1. Consider the following continuous-time BAM neural networks with random perturbation:

$$\begin{cases} dx(t) = [-x(t) + 0.01\sin(\sqrt{5}\,t)\sin(y(t)) \\ \quad + 0.05\cos(\sqrt{7}\,t)\cos(y(t-1))]dt + 0.01\,dw(t), \\ dy(t) = [-0.2y(k) + 0.015\sin(\sqrt{21}\,t)\cos(x(t-1))]dt \\ \quad + 0.03|\cos(\sqrt{23}\,t)|dw(t), \quad \forall t \in \mathbb{R}. \end{cases} \tag{55}$$

5.1. Semidiscrete Model. Based on model (55), we obtain the following semidiscrete model by using the semidiscretization technique:

$$\begin{cases} x(k+1) = e^{-1}x(k) + (1-e^{-1})[0.01\sin(\sqrt{5}\,k)\sin(y(k)) \\ \quad +0.05\cos(\sqrt{7}\,k)\cos(y(k-1)) + 0.01\,\Delta w(k)], \\ y(k+1) = e^{-0.2}y(k) + \dfrac{1-e^{-0.2}}{0.2} \\ \quad [0.15\sin(\sqrt{21}\,k)\cos(x(k-1)) \\ \quad +0.03|\cos(\sqrt{23}\,k)\Delta|w(k)], \end{cases} \tag{56}$$

where $k \in \mathbb{Z}$.

Remark 3. In literature [44], Huang et al. studied model (1) and obtained some sufficient conditions for the existence of a unique almost periodic sequence solution which is globally attractive. In [45], they considered system (2) and studied the dynamics of 2^N almost periodic sequence solutions. But neither of them considered the random factors. Such as, the obtained results in [44, 45] cannot be applied to the study for stochastic model (56). Therefore, the work in this paper complements the corresponding results in [44, 45].

5.2. Discrete Model Formulated by the Euler Scheme. Based on model (55), we obtain the following discrete-time model by using the Euler method:

$$\begin{cases} x(k+1) = 0.01\sin(\sqrt{5}\,k)\sin(y(k)) \\ \quad + 0.05\cos(\sqrt{7}\,k)\cos(y(k-1)) + 0.01\,\Delta w(k), \\ y(k+1) = 0.8y(k) + 0.15\sin(\sqrt{21}\,k)\cos(x(k-1)) \\ \quad + 0.03|\cos(\sqrt{23}\,k)|\Delta w(k), \end{cases} \tag{57}$$

where $k \in \mathbb{Z}$.

In Figures 1 and 2, we give two plots of numerical solutions which are produced by continuous-time model (55), semidiscrete model (56), and Euler-discretization model (57), respectively. Compared with Euler-discretization model (57), semidiscrete model (56) gives a more accurate characterization for continuous-time model (55).

Remark 4. In literature [51–54], the authors discussed the dynamics of periodic solutions of discrete-time cellular neural networks formulated by the Euler scheme. From the above discussion, semidiscrete stochastic system (8) gives a more accurate and realistic formulation for studying the dynamics of discrete-time cellular neural networks. In a way, the work of this paper complements and improves some corresponding results in [51–54].

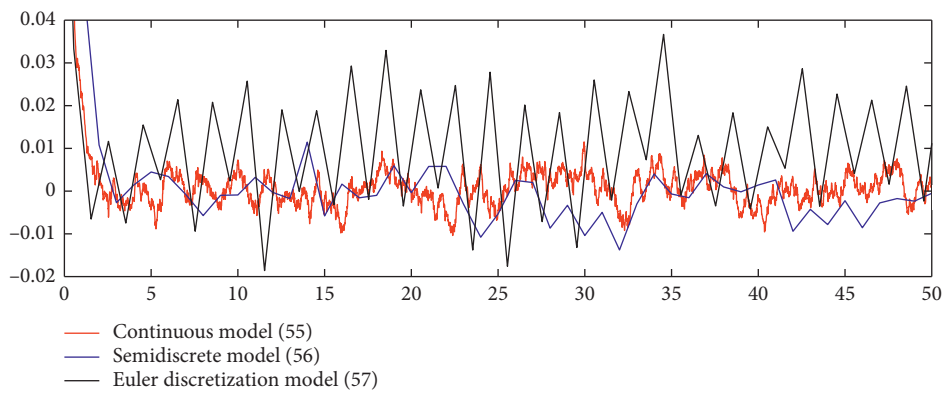

Figure 1: Trajectories of state variable x in models (55), (56), and (57), respectively.

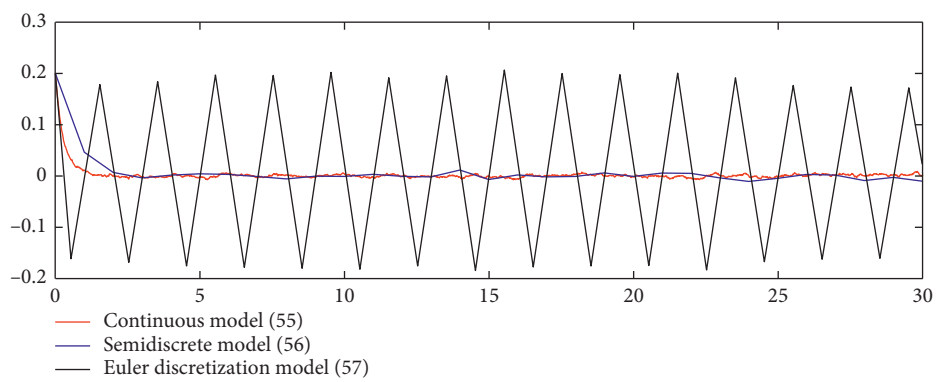

Figure 2: Trajectories of state variable y in models (55), (56), and (57), respectively.

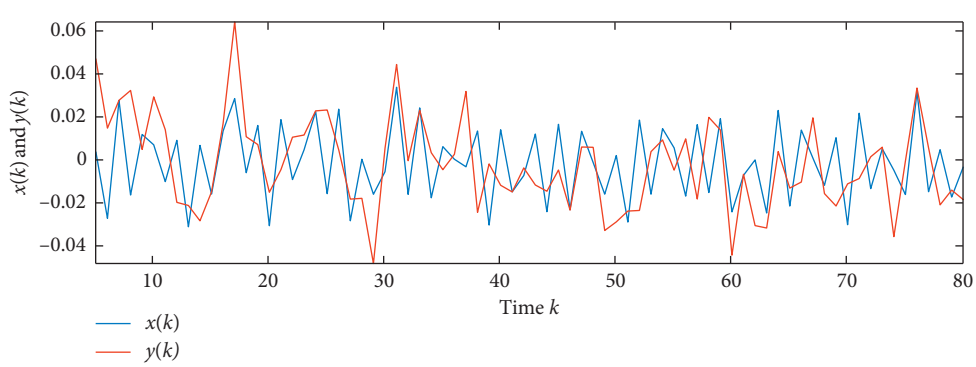

Figure 3: 4-th mean almost periodicity of state variables $(x, y)^T$ in model (56).

Corresponding to system (8), we have $a^l = a^u = 1$, $\tilde{a}^l = \tilde{a}^u = 1$, $L_j^f = L_j^\sigma = \tilde{L}_j^f = \tilde{L}_j^\sigma = 1$, $\bar{b}_{ij} = 0.01$, $\bar{c}_{ij} = 0.05$, $\alpha_{11} = \alpha_{12} = 0.1$, $\beta_{11} = \beta_{12} = 0.02$, $\alpha_{21} = \alpha_{22} = 0.04$, $\beta_{21} = \beta_{22} = 0.2$, $\bar{d}_{ij} = 0.02$, and $i, j = 1, 2$.

Taking $p = 2$, by simple calculation,

$$\begin{aligned} C_4^{1/4} &= 4, \\ D^* &\approx 0.72, \\ K^* &\approx 0.04, \\ r_4 &\approx 0.88 < 1. \end{aligned} \tag{58}$$

According to Theorems 1 and 3, system (56) admits a 4-th mean almost periodic sequence solution, which is 4-th moment globally exponentially stable (see Figures 3–5).

Figure 3 depicts a numerical solution (x, y) of semidiscrete stochastic model (56). Observe that the trajectories of (x, y) demonstrate almost periodic oscillations. Figures 4 and 5 display three numerical solutions of semidiscrete stochastic model (56) at different initial values $(1.5, 2.5)$, $(2.5, 1.5)$, and $(0.2, 0.3)$, respectively. They are shown that semidiscrete stochastic model (56) is 4-th moment globally exponentially stable.

Example 5.2. Consider the corresponding determinant model of system (56) as follows:

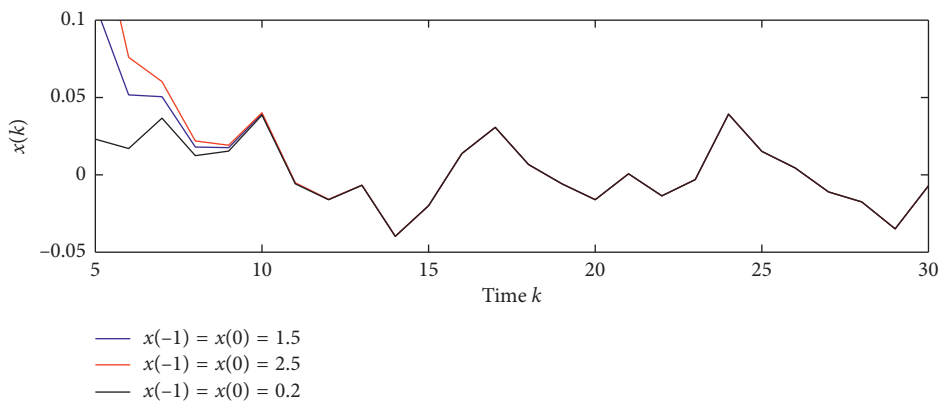

Figure 4: 4-th moment global exponential stability of state variable x of model (56).

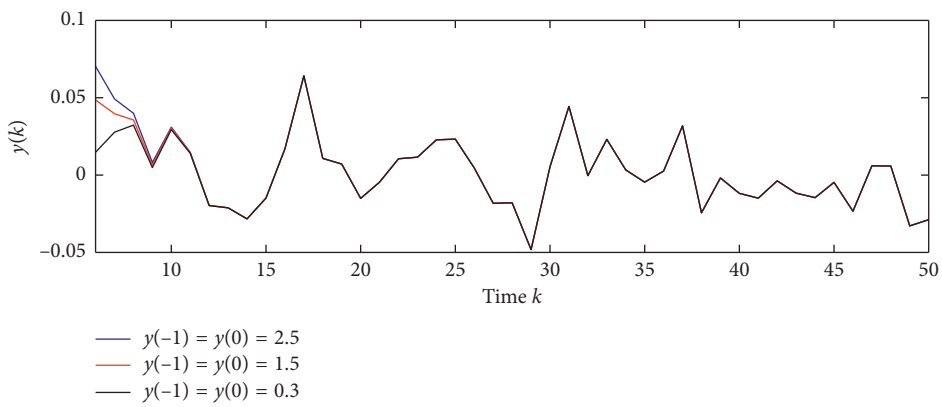

Figure 5: 4-th moment global exponential stability of state variable y of model (56).

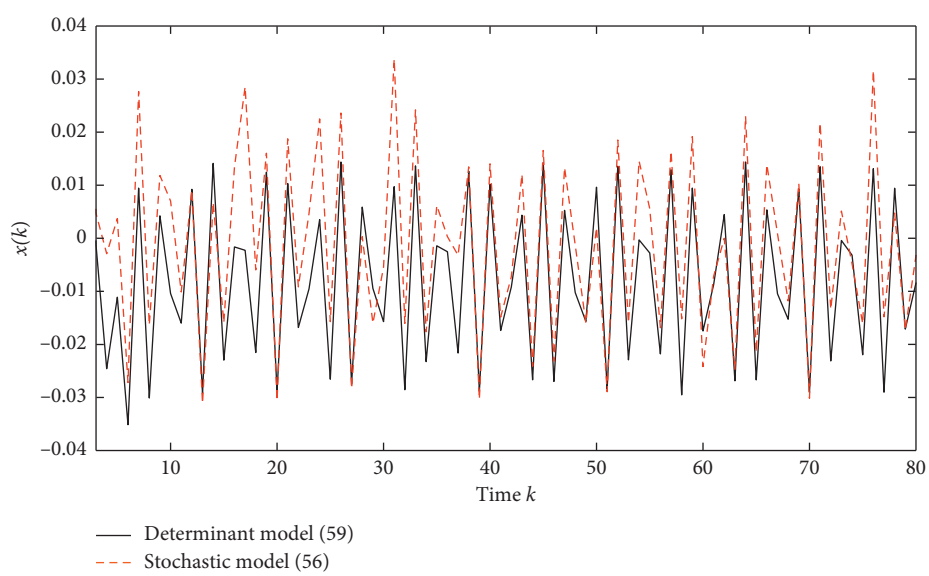

Figure 6: Comparison of state variable x between stochastic model (56) and determinant model (59).

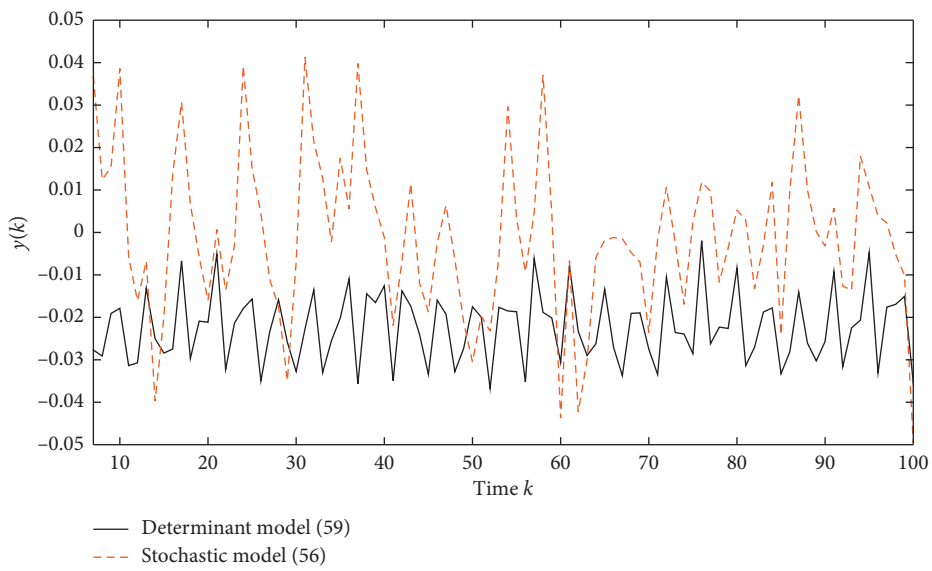

Figure 7: Comparison of state variable y_1 between stochastic model (56) and determinant model (59).

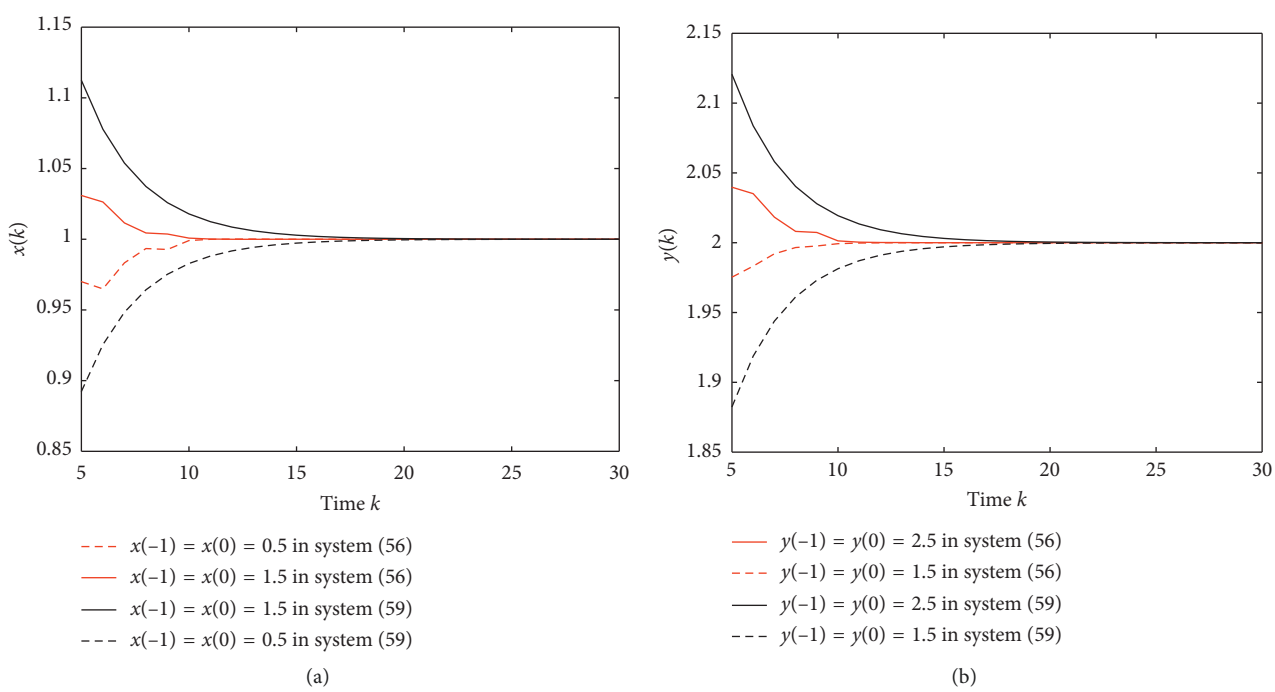

Figure 8: Convergence speed comparison between stochastic model (56) and determinant model (59).

$$\begin{cases} x(k+1) = e^{-1}x(k) + (1-e^{-1})[0.01\sin(\sqrt{5}\,k)\sin(y(k)) \\ \quad + 0.05\cos(\sqrt{7}\,k)\cos(y(k-1)) + 0.01], \\ y(k+1) = e^{-0.2}y(k) + \dfrac{1-e^{-0.2}}{0.2} \\ \quad [0.15\sin(\sqrt{21}\,k)\cos(x(k-1)) + 0.03|\cos(\sqrt{23}\,k)|], \end{cases} \tag{59}$$

where $k \in \mathbb{Z}$. In Figures 6 and 7, we give the results of contrast between stochastic model (56) and determinant model (59). Figures 6 and 7 indicate that the effects of stochastic perturbation on state variables x and y are significant. And the stochastic influence on state variable y is more obvious than that on state variable x. In Figure 8, we give a result of globally exponentially stable contrast between stochastic model (56) and determinant model (59). Figure 8 reveals that the convergence speed of stochastic model (56) is faster than determinant model (59).

Remark 5. From example 5.2, stochastic disturbance brings a positive effect on the global exponential stability of the models.

6. Conclusions and Future Works

In this paper, we formulate a discrete analogue of BAM neural networks with stochastic perturbations and fuzzy operations by using semidiscretization technique. The existence of $2p$-th mean almost periodic sequence solutions and $2p$-th moment global exponential stability for the above models are investigated with the help of Krasnoselskii's fixed point theorem and stochastic theory. The main results obtained in this paper are completely new, and the methods used in this paper provide a possible technique to study $2p$-th mean almost periodic sequence solution and $2p$-th moment global exponential stability of semidiscrete models with stochastic perturbations and fuzzy operations.

With a careful observation of Theorems 1 and 3, it is not difficult to discover that

(1) $2p > 1$ is crucial to the $2p$-th mean almost periodicity and moment global exponential stability of system (8)

(2) The time delays have no effect on the existence of $2p$-th mean almost periodicity and $2p$-th moment global exponential stability of system (8)

(3) Stochastic disturbance may bring a positive effect on the global exponential stability for determinant model

In the future, the following aspects can be explored further:

(1) The methods used in this paper can be applied to study other types of neural networks, such as impulsive neural networks, high-order neural networks, neural networks on time scales, etc.

(2) Other types of fuzzy neural networks could be investigated, such as Takagi-Sugeno fuzzy neural networks

(3) Other dynamic behaviours of system (8) should be further discussed

(4) The case of $2p \in (0, 1]$ could be further explored

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

This work was supported by the National Nature Science Foundation (nos. 11461082, 11601474, and 61472093) and Key Laboratory of Numerical Simulation of Sichuan Province (no. 2017KF002).

References

[1] B. Kosko, "Adaptive bidirectional associative memories," *Applied Optics*, vol. 26, no. 23, pp. 4947–4960, 1987.

[2] B. Kosko, "Bidirectional associative memories," *IEEE Transactions on Systems, Man, and Cybernetics*, vol. 18, no. 1, pp. 49–60, 1988.

[3] C. Wang and S. Rathinasamy, "Double almost periodicity for high-order Hopfield neural networks with slight vibration in time variables," *Neurocomputing*, vol. 282, pp. 1–15, 2018.

[4] C. Xu, P. Li, and Y. Pang, "Exponential stability of almost periodic solutions for memristor-based neural networks with distributed leakage delays," *Neural Computation*, vol. 28, no. 12, pp. 2726–2756, 2016.

[5] S. Han, Y. Li, G. Liu, L. Xiong, and T. Zhang, "Dynamics of two-species delayed competitive stage-structured model described by differential-difference equations," *Open Mathematics*, vol. 17, no. 1, pp. 385–401, 2019.

[6] T. Zhang, L. Yang, and L. Xu, "Stage-structured control on a class of predator-prey system in almost periodic environment," *International Journal of Control*, pp. 1–19, 2018.

[7] T. Zhang and Y. Liao, "Existence and global attractivity of positive almost periodic solutions for a kind of fishing model with pure-delay," *Kybernetika*, vol. 53, pp. 612–629, 2017.

[8] T. Zhang, Y. Li, and Y. Ye, "On the existence and stability of a unique almost periodic solution of Schoener's competition model with pure-delays and impulsive effects," *Communications in Nonlinear Science and Numerical Simulation*, vol. 17, no. 3, pp. 1408–1422, 2012.

[9] T. Zhang and X. Gan, "Almost periodic solutions for a discrete fishing model with feedback control and time delays," *Communications in Nonlinear Science and Numerical Simulation*, vol. 19, no. 1, pp. 150–163, 2014.

[10] C. Xu and P. Li, "Global exponential convergence of neutral-type Hopfield neural networks with multi-proportional delays and leakage delays," *Chaos, Solitons & Fractals*, vol. 96, pp. 139–144, 2017.

[11] C. Xu and P. Li, "On anti-periodic solutions for neutral shunting inhibitory cellular neural networks with time-varying delays and D operator," *Neurocomputing*, vol. 275, pp. 377–382, 2018.

[12] L. Xiong, J. Cheng, J. Cao, and Z. Liu, "Novel inequality with application to improve the stability criterion for dynamical systems with two additive time-varying delays," *Applied Mathematics and Computation*, vol. 321, pp. 672–688, 2018.

[13] H. Zhang, Z. Qiu, L. Xiong, and G. Jiang, "Stochastic stability analysis for neutral-type Markov jump neural networks with additive time-varying delays via a new reciprocally convex combination inequality," *International Journal of Systems Science*, vol. 50, no. 5, pp. 970–988, 2019.

[14] L. L. Xiong, H. Y. Zhang, Y. K. Li, and Z. X. Liu, "Improved stability and H infinity performance for neutral systems with uncertain Markovian jump," *Nonlinear Analysis-Hybrid Systems*, vol. 19, pp. 13–25, 2016.

[15] M. S. Ali, J. Yogambigai, S. Saravanan, and S. Elakkia, "Stochastic stability of neutral-type Markovian-jumping BAM neural networks with time varying delays," *Journal of Computational and Applied Mathematics*, vol. 349, pp. 142–156, 2019.

[16] M. S. Ali, S. Saravanan, M. E. Rani et al., "Asymptotic stability of Cohen-Grossberg BAM neutral type neural networks with distributed time varying delays," *Neural Processing Letters*, vol. 46, no. 3, pp. 991–1007, 2017.

[17] C. Sowmiya, R. Raja, J. D. Cao, G. Rajchakit, and A. Alsaedi, "Enhanced robust finite-time passivity for Markovian jumping discrete-time BAM neural networks with leakage delay," *Advances in Difference Equations*, vol. 318, no. 1, pp. 1–28, 2017.

[18] C. Maharajan, R. Raja, J. Cao, G. Rajchakit, and A. Alsaedi, "Impulsive Cohen-Grossberg BAM neural networks with mixed time-delays: an exponential stability analysis issue," *Neurocomputing*, vol. 275, pp. 2588–2602, 2018.

[19] C. Maharajan, R. Raja, J. Cao, and G. Rajchakit, "Novel global robust exponential stability criterion for uncertain inertial-type BAM neural networks with discrete and distributed time-varying delays via Lagrange sense," *Journal of the Franklin Institute*, vol. 355, no. 11, pp. 4727–4754, 2018.

[20] C. Sowmiya, R. Raja, J. Cao, and G. Rajchakit, "Impulsive discrete-time BAM neural networks with random parameter uncertainties and time-varying leakage delays: an asymptotic stability analysis," *Nonlinear Dynamics*, vol. 91, no. 4, pp. 2571–2592, 2018.

[21] C. Sowmiya, R. Raja, J. Cao et al., "Global exponential stability of antiperiodic solutions for impulsive discrete-time Markovian jumping stochastic BAM neural networks with additive time-varying delays and leakage delay," *International Journal of Adaptive Control and Signal Processing*, vol. 32, no. 6, pp. 908–936, 2018.

[22] P. Vadivel, R. Sakthivel, K. Mathiyalagan, and P. Thangaraj, "New passivity criteria for fuzzy BAM neural networks with Markovian jumping parameters and time-varying delays," *Reports on Mathematical Physics*, vol. 71, no. 1, pp. 69–91, 2013.

[23] C. Xu, M. Liao, P. Li, Y. Guo, Q. Xiao, and S. Yuan, "Influence of multiple time delays on bifurcation of fractional-order neural networks," *Applied Mathematics and Computation*, vol. 361, pp. 565–582, 2019.

[24] C. J. Xu, M. X. Liao, P. L. Li, and Y. Guo, "Bifurcation analysis for simplified five-neuron bidirectional associative memory neural networks with four delays," *Neural Process Letters*, 2019.

[25] K. Ratnavelu, M. Manikandan, and P. Balasubramaniam, "Synchronization of fuzzy bidirectional associative memory neural networks with various time delays," *Applied Mathematics and Computation*, vol. 270, pp. 582–605, 2015.

[26] C. Xu, "Local and global Hopf bifurcation analysis on simplified bidirectional associative memory neural networks with multiple delays," *Mathematics and Computers in Simulation*, vol. 149, pp. 69–90, 2018.

[27] M. Şaylı and E. Yılmaz, "Global robust asymptotic stability of variable-time impulsive BAM neural networks," *Neural Networks*, vol. 60, pp. 67–73, 2014.

[28] C. Xu and P. Li, "Global exponential convergence of fuzzy cellular neural networks with leakage delays, distributed delays and proportional delays," *Circuits, Systems, and Signal Processing*, vol. 37, no. 1, pp. 163–177, 2018.

[29] Y. Liu and W. Tang, "Exponential stability of fuzzy cellular neural networks with constant and time-varying delays," *Physics Letters A*, vol. 323, no. 3-4, pp. 224–233, 2004.

[30] W. Ding and M. Han, "Synchronization of delayed fuzzy cellular neural networks based on adaptive control," *Physics Letters A*, vol. 372, no. 26, pp. 4674–4681, 2008.

[31] P. Balasubramaniam, M. Syed Ali, and S. Arik, "Global asymptotic stability of stochastic fuzzy cellular neural networks with multiple time-varying delays," *Expert Systems With Applications*, vol. 37, no. 12, pp. 7737–7744, 2010.

[32] J. Yu, C. Hu, H. Jiang, and Z. Teng, "Exponential lag synchronization for delayed fuzzy cellular neural networks via periodically intermittent control," *Mathematics and Computers in Simulation*, vol. 82, no. 5, pp. 895–908, 2012.

[33] Y. Li and C. Wang, "Existence and global exponential stability of equilibrium for discrete-time fuzzy BAM neural networks with variable delays and impulses," *Fuzzy Sets and Systems*, vol. 217, pp. 62–79, 2013.

[34] P. H. Bezandry and T. Diagana, *Almost Periodic Stochastic Processes*, Springer, New York, USA, 2011.

[35] S. G. Hu, C. M. Huang, and F. K. Wu, *Stochastic Differential Equations*, Science Press, Beijing, China, 2008.

[36] H. Zhang, Z. Qiu, and L. Xiong, "Stochastic stability criterion of neutral-type neural networks with additive time-varying delay and uncertain semi-Markov jump," *Neurocomputing*, vol. 333, pp. 395–406, 2019.

[37] T. Wu, L. Xiong, J. Cao, and H. Zhang, "Stochastic stability and extended dissipativity analysis for uncertain neutral systems with semi-Markovian jumping parameters via novel free matrix-based integral inequality," *International Journal of Robust and Nonlinear Control*, vol. 29, no. 9, pp. 2525–2545, 2019.

[38] L. Xiong, J. Tian, and X. Liu, "Stability analysis for neutral Markovian jump systems with partially unknown transition probabilities," *Journal of the Franklin Institute-Engineering and Applied Mathematics*, vol. 349, no. 6, pp. 2193–2214, 2012.

[39] S. Mohamad and K. Gopalsamy, "Dynamics of a class of discrete-time neural networks and their continuous-time counterparts," *Mathematics and Computers in Simulation*, vol. 53, no. 1-2, pp. 1–39, 2000.

[40] S. Mohamad and K. Gopalsamy, "Exponential stability of continuous-time and discrete-time cellular neural networks with delays," *Applied Mathematics and Computation*, vol. 135, no. 1, pp. 17–38, 2003.

[41] H. Zhao, L. Li Sun, and G. Wang, "Periodic oscillation of discrete-time bidirectional associative memory neural networks," *Neurocomputing*, vol. 70, no. 16–18, pp. 2924–2930, 2007.

[42] F. Kong and X. Fang, "Pseudo almost periodic solutions of discrete-time neutral-type neural networks with delays," *Applied Intelligence*, vol. 48, no. 10, pp. 3332–3345, 2018.

[43] Z. Zhang and D. Zhou, "Existence and global exponential stability of a periodic solution for a discrete-time interval general BAM neural networks," *Journal of the Franklin Institute*, vol. 347, no. 5, pp. 763–780, 2010.

[44] Z. Huang, X. Wang, and F. Gao, "The existence and global attractivity of almost periodic sequence solution of discrete-time neural networks," *Physics Letters A*, vol. 350, no. 3-4, pp. 182–191, 2006.

[45] Z. Huang, S. Mohamad, and F. Gao, "Multi-almost periodicity in semi-discretizations of a general class of neural networks," *Mathematics and Computers in Simulation*, vol. 101, pp. 43–60, 2014.

[46] Z. Huang, Y. Xia, and X. Wang, "The existence and exponential attractivity of κ-almost periodic sequence solution of discrete time neural networks," *Nonlinear Dynamics*, vol. 50, no. 1-2, pp. 13–26, 2007.

[47] Y. Ji, "Global attractivity of almost periodic sequence solutions of delayed discrete-time neural networks," *Arabian Journal for Science and Engineering*, vol. 36, no. 7, pp. 1447–1459, 2011.

[48] J. C. Kuang, *Applied Inequalities*, Shandong Science and Technology Press, Shandong, China, 2012.

[49] T. Yang and L. B. Yang, "The global stability of fuzzy cellular neural networks," *IEEE Transactions on Circuits and Systems I*

Fundamental Theory and Applications, vol. 43, pp. 880–883, 1996.

[50] D. R. Smart, *Fixed Point Theorems*, Cambridge University Press, Cambridge, UK, 1980.

[51] X. Yang, F. Li, Y. Long, and X. Cui, "Existence of periodic solution for discrete-time cellular neural networks with complex deviating arguments and impulses," *Journal of the Franklin Institute*, vol. 347, no. 2, pp. 559–566, 2010.

[52] S. Gao, R. Shen, and T. Chen, "Periodic solutions for discrete-time Cohen-Grossberg neural networks with delays," *Physics Letters A*, vol. 383, no. 5, pp. 414–420, 2019.

[53] K. Sun, A. Zhang, J. Qiu, X. Chen, C. Yang, and X. Chen, "Dynamic analysis of periodic solution for high-order discrete-time Cohen-Grossberg neural networks with time delays," *Neural Networks*, vol. 61, pp. 68–74, 2015.

[54] J. Wang, H. Jiang, and C. Hu, "Existence and stability of periodic solutions of discrete-time Cohen-Grossberg neural networks with delays and impulses," *Neurocomputing*, vol. 142, pp. 542–550, 2014.