Research Article

Investigation of Interaction Solutions for Modified Korteweg-de Vries Equation by Consistent Riccati Expansion Method

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A consistent Riccati expansion (CRE) method is proposed for obtaining interaction solutions to the modified Korteweg-de Vries (mKdV) equation. Using the CRE method, it is shown that interaction solutions such as the soliton-tangent (or soliton-cotangent) wave cannot be constructed for the mKdV equation. More importantly, exact soliton-cnoidal periodic wave interaction solutions are presented. While soliton-cnoidal interaction solutions were found to degenerate to special resonant soliton solutions for the values of modulus (n) closer to one (upper bound of modulus) in the Jacobi elliptic function, a normal kink-shaped soliton was observed for values of n closer to zero (lower bound).

1. Introduction

The direct study of exact solutions to nonlinear evolution equations (NLEEs) has received much attention from many mathematicians and physicists due to the fact that new strides in nonlinear science, which were made possible by a substantial increase in computational platforms such as Mathematica, Maple, and MATLAB, have enabled improvements in the performance of complicated and tedious numerical computational methods. Indeed, several powerful methods such as the Tanh-function method [1–3], F-expansion method [4], Jacobian elliptic function method [5], and variational approach [6, 7] have been proposed for constructing exact solutions to NLEEs. Despite the successful implementation of such methods, it is still challenging to obtain solutions for interactions among different types of nonlinear excitations such as the soliton-soliton interaction.

Recently, some new soliton structure solutions were obtained for nonlinear systems. Chen *et al.* studied the vortex solitons in Bose-Einstein condensates with spin-orbit coupling and Gaussian optical lattices, based on the analytical and numerical method [8]. Milan *et al.* found exact fundamental soliton solutions in the spiraling guiding structures

by the modified Petviashvili iteration method [9]. Cheng *et al.* investigated the formation and propagation of a multipole soliton in a cold atomic gas with a parity-time symmetric potential using the modified square operator method [10]. Liu *et al.* obtained the three-soliton solutions for high-order nonlinear Schrödinger equation by Hirota's bilinear method [11].

Specially, Lou [12] proposed a consistent Riccati expansion (CRE) method, which is a more generalized yet simpler method to find interaction solutions for various NLEEs [13–17]. The core concept of CRE is the construction of interaction solutions based on the usual Riccati equation method and the consistent equation or the w -equation [12]. The CRE method is critical to finding more new solutions to the w -equation.

In this study, the CRE method is used to construct several types of interaction solutions for the focusing real modified Korteweg-de Vries (mKdV) equation [18] shown in

$$u_t + \alpha u^2 u_x + \beta u_{xxx} = 0, \quad (1)$$

where α and β are arbitrary constants. The mKdV equation plays an important role in describing some physical phenomena, such as optical cycles [19, 20], soliton propagation in

plasma [21] and lattices [22], the Schottky barrier transmission lines [23], and fluid mechanics [24].

To provide better insights into these physical phenomena, finding and analyzing exact solutions to the mKdV equation is important. Previously, many powerful methods have been proposed for constructing exact solutions to the mKdV equation. For instance, in 1972, Hirota obtained an exact solution to the mKdV equation for the case of multiple collisions of solitons with different amplitudes [25]. Subsequently, he also derived the exact envelope soliton solution to the mKdV equation [26]. In 1973, Ablowitz *et al.* obtained exact solutions to the mKdV equation by using the inverse scattering technique [27]. In 1988, Akhmediev *et al.* used the Darboux transformation scheme to obtain second-order periodic solutions to the mKdV equation [28]. In 2004, Kevrekidis *et al.* derived some classes of periodic solutions to the mKdV equation by using direct methods [29]. In 2015, Jiao and Lou constructed a new soliton-cnoidal periodic wave interaction solution by using the CRE method [30]. However, they did not investigate how the soliton-cnoidal interaction solutions may be used to derive soliton-soliton or soliton-periodic wave interaction solutions among other types of solutions. Moreover, new interaction solutions to the mKdV equation involving different types of nonlinear waves must be investigated in depth.

The present article is structured as follows. Section 2 introduces the CRE solvability of the mKdV equation. Section 3 describes new explicit interaction solutions such as soliton-soliton, multiple resonant soliton, soliton-cosine wave, and soliton-cnoidal wave solutions to the mKdV equation obtained using the CRE method. Furthermore, it is demonstrated that interaction solutions such as the soliton-tangent wave solution cannot be constructed for the mKdV equation. The last section presents a summary and discussion.

2. CRE Solvability of the mKdV Equation

Consider the following NLEE, shown in (2), with independent variables $X \equiv (t, x_1, x_2, \dots, x_m)$ and a dependent variable $u \equiv u(X)$

$$P(u, u_t, u_{x_i}, u_{x_i x_i}) = 0, \quad (2)$$

where P is a polynomial function of some arguments with the subscripts denoting partial derivatives. We assume that the solution to (2) is the following possible truncated expansion form

$$u = \sum_{j=0}^n u_j R^j(w), \quad (3)$$

where n is determined from the leading order analysis of (2). All the expansion coefficient functions (u_j) are determined by substituting (3) into (2) and then vanishing all the coefficients for a given power of $R(w)$. Further, u_j and w are functions of (x, y, t) and $R(w)$ satisfies the following simple Riccati equation shown below:

$$R_w = \sigma + R^2, \quad (4)$$

$$R \equiv R(w),$$

which includes the following five special solutions [31]. For $\sigma < 0$,

$$R = -\sqrt{-\sigma} \tanh(\sqrt{-\sigma} w), \quad (5a)$$

$$R = -\sqrt{-\sigma} \coth(\sqrt{-\sigma} w). \quad (5b)$$

For $\sigma > 0$,

$$R = \sqrt{\sigma} \tan(\sqrt{\sigma} w), \quad (6a)$$

$$R = -\sqrt{\sigma} \cot(\sqrt{\sigma} w). \quad (6b)$$

For $\sigma = 0$,

$$R = -\frac{1}{w}. \quad (7)$$

Definition. If the equation for u_j ($j = 0, 1, \dots, n$) and w , obtained by vanishing all the coefficients of each power in $R(w)$ after the substitution of (3) into (2), is either consistent or not overdetermined, then the expansion in (3) is considered a CRE and the nonlinear system in (2) is said to be CRE solvable [8].

According to the CRE method defined above, one can obtain the following form based on the leading order analysis of the mKdV equation in (1)

$$u = u_0 + u_1 R(w), \quad (8)$$

where u_0, u_1 , and w are functions of (x, y, t) and $R(w)$ satisfies the Riccati equation (see (4) above).

Substituting (8) and (4) into (1) and vanishing all the coefficients of different powers of $R(w)$, one obtains

$$u_1 = -\frac{\sqrt{6\beta} w_x}{\sqrt{-\alpha}}, \quad (9)$$

$$\text{or } u_1 = \frac{\sqrt{6\beta} w_x}{\sqrt{-\alpha}},$$

$$u_0 = -\frac{\sqrt{3\beta} w_{xx}}{\sqrt{-2\alpha w_x}}, \quad (10)$$

$$\text{or } u_0 = \frac{\sqrt{3\beta} w_{xx}}{\sqrt{-2\alpha w_x}},$$

$$w_t = 2\sigma\beta w_x^3 - \frac{3\beta w_{xx}^2}{2w_x} + \beta w_{xxx}. \quad (11)$$

Based on the definition above, (11) is the consistent equation of the mKdV equation (or the mKdV w -equation). If w is a solution to the MDWW w -equation in (11), the mKdV equation in (1) is CRE solvable. In this study, we set $\alpha = -6$ and $\beta = 1$. Thus, the solutions to the mKdV equation are expressed as follows.

$$u = -\frac{w_{xx}}{2w_x} - w_x R(w), \quad (12a)$$

$$u = \frac{w_{xx}}{2w_x} + w_x R(w). \quad (12b)$$

3. Interaction Solutions to the mKdV Equation

Upon the determination of solutions to (11) by using (12), the corresponding solutions to the mKdV equation in (1) can be obtained. In this section, we construct interaction solutions to the mKdV equation by using different types of trivial solutions to (11).

3.1. Soliton-Soliton Interaction Solutions to the mKdV Equation. To obtain soliton-soliton interaction solutions to the mKdV equation, we consider the following form in (13) as the trial solution to (11):

$$\begin{aligned} w &= k_1 x + w_1 t + R(\phi), \\ \phi &= k_2 x + w_2 t, \end{aligned} \quad (13)$$

where $R(\phi)$ satisfies the following Riccati equation in

$$R_\phi = r + R(\phi)^2, \quad (14)$$

where r is an arbitrary constant. This equation has special solutions similar to those in (5a), (5b), (6a), (6b), and (7). By vanishing all the coefficients for each power of R_ϕ after the substitution of (13) and (14) into the mKdV w -equation in (11), one can obtain

$$\begin{aligned} \sigma &= -\frac{1}{4}, \\ r &= -1, \\ k_2 &= -\sqrt[3]{\frac{w_2}{2r}}, \\ k_1 &= \frac{k_2 w_1}{w_2}, \\ w_1 &= -w_2. \end{aligned} \quad (15)$$

From (15), it can be seen that both σ and r are less than 0 when $w_1 \neq 0$. Based on (5a), (5b), (6a), and (6b), Eqs. (4) and (14) have only solitary solutions (*viz.*, (5a) and (5b)) but not tangent or cotangent solutions such as the ones described in (6a) and (6b). This shows that interaction solutions such as soliton-tangent (or soliton-cotangent) wave cannot be constructed for the mKdV equation.

Under condition (15), from (5a), (5b), (12a), and (13), the soliton-soliton interaction solutions of the mKdV equation are expressed as

$$\begin{aligned} u &= \frac{2k_2^2 \tanh \phi}{k_1 + 2k_2 + k_1 \cosh(2\phi)} + \frac{1}{2} (k_1 + k_2 \operatorname{sech}^2 \phi) \\ &\quad \cdot \tanh \left(\frac{1}{2} (-k_1 x - w_1 t) - \tanh \phi \right), \end{aligned} \quad (16a)$$

$$u = \frac{k_2^2 \operatorname{csch}^2 \phi \coth \phi}{-k_1 + k_2 \operatorname{csch}^2(\phi)} + \frac{1}{2} (k_1 - k_2 \operatorname{csch}^2 \phi) \quad (16b)$$

$$\cdot \tanh \left(\frac{1}{2} (-k_1 x - w_1 t) - \coth \phi \right),$$

$$u = \frac{k_2^2 \operatorname{sech}^2 \phi \tanh \phi}{k_1 + k_2 \operatorname{sech}^2(\phi)} + \frac{1}{2} (k_1 + k_2 \operatorname{sech}^2 \phi) \quad (16c)$$

$$\cdot \coth \left(\frac{1}{2} (-k_1 x - w_1 t) - \tanh \phi \right),$$

$$u = \frac{-k_2^2 \coth \phi \operatorname{csch}^2 \phi}{k_1 - k_2 \operatorname{csch}^2(\phi)} + \frac{1}{2} (k_1 - k_2 \operatorname{csch}^2 \phi) \quad (16d)$$

$$\cdot \coth \left(\frac{1}{2} (-k_1 x - w_1 t) - \coth \phi \right).$$

3.2. Interaction Solution between and Trigonometric Periodic Wave for the mKdV Equation. To investigate the interaction between a soliton and a periodic wave in the interaction solution to the mKdV equation, we consider a solution of the following form (11):

$$\begin{aligned} w &= k_1 x + w_1 t \\ &\quad + c_0 \ln(\cos(k_2 x + w_2 t) \exp(k_2 x + w_2 t)), \end{aligned} \quad (17)$$

where c_0 is determined constant. Substituting (17) into (11) and then vanishing all coefficients of powers $\tan(k_2 x + w_2 t)$, we can obtain

$$\begin{aligned} \sigma &= -\frac{1}{4}, \\ k_1 &= -\frac{k_2}{\sqrt{-2\sigma}}, \\ c_0 &= \frac{2(2\sigma k_1^3 k_2 - k_1 k_2^3)}{k_2^4 - 8\sigma k_1^2 k_2^2 + 4\sigma^2 k_1^4}, \\ w_1 &= \frac{2(2c_0 k_1 k_2^3 + 3c_0^2 k_2^4 - \sigma k_1^4)}{k_1}, \\ w_2 &= 4(k_2^3 - 3\sigma k_1^2 k_2 - 6\sigma c_0 k_1 k_2^2). \end{aligned} \quad (18)$$

Then, based on (5a), (17), and (12a), one can obtain a soliton-trigonometric periodic wave interaction solution as

$$\begin{aligned} u &= k_2 \frac{(6 \sec^2(M) - \sqrt{2}(1 + 2 \tan^2(M)))}{6 + 12 \tan(M)} \\ &\quad + \frac{\tanh((3k_2(x + k_2^2 t) - 2 \ln(M \cos(M)))/3\sqrt{2})}{6 + 12 \tan(M)}, \end{aligned} \quad (19)$$

where $M = k_2 x + 2k_2^3 t$.

3.3. Interaction Solutions between Soliton and Cnoidal Wave for the mKdV Equation. In [26], Jiao and Lou constructed a solution of the following form for (11)

$$w = k_1 x + w_1 t + AE_\pi [\operatorname{sn}(k_2 x + w_2 t, \mu_1), \nu, \kappa], \quad (20)$$

where $\text{sn}(\zeta, \mu)$ is the usual Jacobi elliptic sine function and $E_\pi(\xi, \nu, \kappa)$ is the third type of incomplete elliptic integral. Jiao and Lou used the following parameters in (21) to obtain a special soliton-cnoidal wave interaction solution to the mKdV equation:

$$\{\mu_1, k_1, k_2\} = \{1.5, 2, 1\}. \quad (21)$$

As seen from (21), Jiao and Lou chose the modulus (μ_1) of Jacobi elliptic function to be 1.5, which is outside the allowed range ($0 < \mu_1 < 1$) [13].

In this study, we will further investigate how soliton-cnoidal interaction solutions can be used to derive soliton-soliton and soliton-periodic wave interaction solutions among other types of solutions. To this end, we performed all the substitutions and evaluations by using the Mathematica software.

Consider a trial solution of the following form for solving (11)

$$w = k_1 x + w_1 t + W(k_2 x + w_2 t), \quad (22)$$

where

$$W(k_2 x + w_2 t) = W(\xi) = W, \quad (23)$$

satisfies the following elliptic equation:

$$\begin{aligned} W_{1\xi}^2 &= C_0 + C_1 W_1 + C_2 W_1^2 + C_3 W_1^3 + C_4 W_1^4, \\ W_1 &\equiv W_\xi. \end{aligned} \quad (24)$$

Substituting (22) and (24) into (11), one obtains

$$\begin{aligned} C_0 &= C_0, \\ C_1 &= \frac{3C_0 k_2^4 - 2k_1 w_1 - 4\sigma k_1^4}{k_1 k_2^3}, \\ C_2 &= \frac{C_1 k_2^4 - k_2 w_1 - k_1 w_2 - 8\sigma k_2 k_1^3}{k_1 k_2^3}, \\ C_3 &= \frac{C_2 k_2^3 - 2w_2 - 24\sigma k_2 k_1^2}{3k_1 k_2^2}, \\ C_4 &= -4\sigma. \end{aligned} \quad (25)$$

From the analysis of (24), we assume the solution of (24) in the following form

$$W_1 = A_0 + A_1 \text{sn}(m(k_2 x + w_2 t), n). \quad (26)$$

Substituting (26) into (24) and setting the coefficient of $\{\text{sn}(m(k_2 x + w_2 t), n), \text{cn}(m(k_2 x + w_2 t), n), \text{dn}(m(k_2 x + w_2 t), n)\}$ equal to zero, one obtains

$$\begin{aligned} A_0 &= A_0, \\ A_1 &= A_0, \\ C_0 &= \frac{-m^2 A_0^2 A_1^2 + m^2 A_1^4 + m^2 n^2 A_0^4 - m^2 n^2 A_0^2 A_1^2}{A_1^2}, \\ C_1 &= \frac{2(2C_0 + m^2 A_0^2 - 2m^2 A_1^2 + m^2 n^2 A_0^2)}{A_0}, \\ C_2 &= \frac{-3C_1 + 4m^2 A_0 + 4m^2 n^2 A_0}{2A_0}, \\ C_3 &= \frac{2(C_2 + m^2 + m^2 n^2)}{3A_0}, \\ C_4 &= \frac{C_3}{4A_0}. \end{aligned} \quad (27)$$

Based on (25) and (27), one can find a group solution

$$\begin{aligned} k_2 &= \sqrt[3]{-\frac{2w_2}{m^2(5-n^2)}}, \\ k_1 &= \frac{600\sigma w_1 w_2 - \sqrt{360000\sigma^2 w_1^2 w_2^2 - 4(N_1)(N_2)}}{2(N_2)}, \\ A_0 &= \frac{-4m^2 k_1 k_2^3 - 4m^2 n^2 k_1 k_2^3 + 3w_1 k_2 + w_2 k_1}{4k_2(m^2 k_2^3 + m^2 n^2 k_2^3 - w_2)}, \\ A_1 &= -\sqrt{\frac{6(m^2 n^2 k_2 k_1^2 + 2m^2 n^2 k_1 k_2^2 A_0 + m^2 n^2 k_2^2 A_0^2)}{m^2 k_2^3 + m^2 n^2 k_2^3 + 2w_2}}, \end{aligned} \quad (28)$$

where $N_1 = -160m^6 k_2^7 + 992m^6 n^2 k_2^7 - 236m^4 w_2 k_2^4 - 616m^4 n^2 w_2 k_2^4 + 20m^4 n^4 w_2 k_2^4 + 168m^2 w_2^2 k_2 + 40m^2 n^2 w_2^2 k_2 + 300\sigma w_1^2 k_2$, and $N_2 = 375\sigma m^4 k_2^5 - 1950\sigma m^4 n^4 k_2^5 + 375\sigma m^4 n^4 k_2^5 + 600\sigma m^2 w_2 k_2^2 + 600\sigma m^2 n^2 w_2 k_2^2$. Under the substitution of (26), (22), and (5a) into (12a), under the conditions imposed by (28), one obtains

$$\begin{aligned} u &= \frac{-mA_1 k_2^2 CD}{2(k_1 + k_2 A_0 + k_2 A_1 S)} \\ &+ \frac{2(k_1 + k_2 A_0 + k_2 A_1 S)^2 \sqrt{-\sigma} \tanh(\sqrt{-\sigma} T)}{2(k_1 + k_2 A_0 + k_2 A_1 S)}, \end{aligned} \quad (29)$$

FIGURE 1: (Color online) The evolution of the soliton-cnoidal wave interaction solution shown in (29) obtained by using the parameters listed in (31). The profiles at $t = 0$ and $x = 0$ are shown in (a) and (b), respectively. Different color traces represent different values for the modulus $n = 0.00001$ (black), $n = 0.5$ (red), and $n = 0.99999$ (blue).

FIGURE 2: (Color online) The evolution of the soliton-cnoidal wave interaction solution shown in (29) obtained by using the parameters listed in (31) with different modulus values as a function of both x and t . The modulus values for different panels are (a) $n = 0.00001$, (b) $n = 0.5$, and (c) $n = 0.99999$.

where

$$\begin{aligned}
 S &= \operatorname{sn}(m(k_2x + w_2t), n), \\
 C &= \operatorname{cn}(m(k_2x + w_2t), n), \\
 D &= \operatorname{dn}(m(k_2x + w_2t), n), \\
 T &= (k_1 + k_2A_0)x + (w_1 + w_2A_0)t \\
 &\quad + \frac{A_1 \ln(D - \sqrt{n}C)}{m\sqrt{n}}.
 \end{aligned} \tag{30}$$

To investigate how the soliton-cnoidal interaction solutions could be used to derive soliton-soliton interaction or other types of solutions, we illustrate the following two cases corresponding to the soliton-cnoidal wave interaction solution described in (29) by selecting different sets of parameters. For the first case, the parameters are chosen as

$$\begin{aligned}
 \sigma &= -1, \\
 w_1 &= 0.5, \\
 w_2 &= 0.8, \\
 m &= 2.
 \end{aligned} \tag{31}$$

While Figure 1 shows two-dimensional views for interaction solution at $t = 0$ and $x = 0$. Figure 2 displays three-dimensional plots for the evolution of soliton-cnoidal wave interaction solution with different values for the modulus in the Jacobian elliptic function, viz., $n = 0.000001$, 0.5 , and 0.99999 . While $n = 0.5$ exhibited a particular periodic-kink soliton wave interaction, the extreme values of $n = 0.00001$ (a value close to the lower modulus limit or 0) showed a normal kink-shaped soliton and $n = 0.99999$ (a value close to the upper modulus limit or 1) displayed an interaction between a periodic wave and another periodic wave.

For the second case, the parameters were altered as shown in (32) by changing the angular frequency w_2

FIGURE 3: (Color online) The evolution of the soliton-cnoidal wave interaction solution shown in (29) obtained by using the parameters in (32). The profiles at $t = 0$ and $x = 0$ are shown in (a) and (b), respectively. Different color traces represent different values for the modulus $n = 0.00001$ (black), $n = 0.5$ (red), and $n = 0.99999$ (blue).

FIGURE 4: (Color online) The evolution of the soliton-cnoidal wave interaction solution shown in (29) obtained by using the parameters listed in (32) with different modulus values as a function of both x and t . The modulus values for different panels are (a) $n = 0.00001$, (b) $n = 0.5$, and (c) $n = 0.99999$.

$$\sigma = -1,$$

$$w_1 = 0.5,$$

$$w_2 = 0.1,$$

$$m = 2.$$

(32)

Similar to the first case, we illustrate the structures of the soliton-cnoidal wave interaction solution for different values of $n = 0.00001$, 0.5 , and 0.99999 . Clearly, as shown in Figures 3 and 4, wavenumbers and the amplitudes in the range of $x(-40,40)$ and $t(-40,40)$ are less than that of the first case (cf. Figures 1 and 2). While there is still a normal kink soliton in the $x-u$ plot for $n = 0.00001$, an incomplete kink soliton is observed in the $t-u$ plot in contrast to the first case shown in Figure 1(b). Building on the above two cases, soliton and soliton-soliton wave interaction solutions are derived from the soliton-cnoidal wave interaction solution by making the limit of the modulus approach either 0 or 1.

4. Summary and Discussion

In this study, we investigated the focusing mKdV equation by using the CRE method. This nonlinear equation was shown to be CRE solvable and interaction solutions; namely, soliton-soliton, soliton-trigonometric periodic waves, and soliton-cnoidal periodic wave for the mKdV equation were explicitly provided by choosing different trial solutions for the mKdV w -equation shown in (11). In addition, analytical solutions for interactions between soliton and cnoidal wave were provided and their properties were discussed graphically. According to the presented analysis, soliton and soliton-soliton wave interaction solutions can be derived from the soliton-cnoidal wave interaction solution by making the limit of the modulus approach either 0 or 1 (i.e., lower or upper bounds for the modulus in the Jacobi elliptical function).

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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