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Research Article

Uncertain Portfolio Selection with Borrowing Constraint and Background Risk

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Due to the complexity of financial markets, there exist situations where security returns and background factor returns are available mainly based on experts' subjective beliefs, such as in the case of lack of historical data. To deal with such indeterminate quantities, uncertain variables are introduced. Based on uncertainty theory, this paper discusses the distribution function of the optimal portfolio return. Two types of new uncertain programming models, namely, the chance-mean model and the measure-mean model, are proposed to make an optimal portfolio selection decision in an uncertain environment. It is proved that there exists an equivalent relation between the chance-mean model and a deterministic linear programming model, which leads to an approach to obtain the optimal solutions of the proposed models. Finally, some numerical examples are illustrated to show the modelling ideas and the effectiveness of the models.

1. Introduction

Portfolio selection involves creating a combination of securities to maximize portfolio return and spread risk. Markowitz [1] first introduced the mean-variance model, which was the beginning of the modern portfolio selection theory. In the mean-variance model, expected return was deemed to be investment return, and the variance was considered as the risk. Though variance has been accepted as a well-known method to measure risk in portfolio selection, application of the mean-variance model was limited to some extent. One limitation is that analysis based on variance suggests high returns are just as undesirable as low returns because high returns will also lead to the extreme of variance (Markowitz [2], Grootveld and Hallerbach [3], and Huang [4]). As a result, the single-factor model was proposed by Sharpe [5]. Moreover, different scholars had various views on risk, and they amplified modern portfolio selection theory by improving the measurement of risk. Some believed that investors paid more attention to excess return exceeding expected return and that a return lower than the expected return should be regarded as risk. Based on this kind of view, Markowitz [2] and Mao [6] proposed the mean-semivariance model, which was more suitable if the distribution of portfolio return is asymmetric. Konno and Yamazaki [7] employed absolute deviation to measure risk and then proposed an expected absolute-deviation portfolio model. Others preferred to use investors' loss to measure risk, and then VaR (Berkowitz et al. [8], Castellacci and Siclari [9]) and CVaR (Lim et al. [10]) were introduced to the portfolio selection problems. In addition, some scholars introduced higher moments to the portfolio selection models. For instance, Samuelson [11] proved that if the first and second moments are the same, investors would tend to make a decision with a larger third-order moment (for good resources on the portfolio selection problem, see Stone [12], Konno and Suzuki [13], Pindoriya et al. [14], and Yu and Lee [15]).

All the studies above assumed that investors only face financial risk; however, in real financial markets, there exist background factors, such as health, labour income, and real estate, that have a considerable effect on the portfolio decision and cannot be hedged. Viceira [16] showed that if labour income risk exists, the optimal allocation to stocks of employed investors is larger than that of retired investors. Rosen and Wu [17] pointed out that health risk and health expenditure both have a considerable effect on investors' optimal portfolio. Baptista [18] made an explanation of optimal delegate portfolio management based on the effect of background risk on the optimal portfolio decision. Jiang et al. [19] studied the influence of background risk on portfolio selection by means of mean-variance and discussed the location of the efficient frontier affected by background risk factors. Huang and Wang [20] further presented the portfolio frontier characteristic with a dependent background risk in the framework of mean-variance.

In the studies mentioned above, returns of securities and background factors are all regarded as random variables, meaning that we have enough historical data to estimate their future values. However, we frequently lack the required observational data, especially for newly listed securities. Even though there exist enough historical data, in many cases, it is impossible to accurately predict security returns since the security market is so complex. In this case, we have no choice but to invite experts to evaluate the values of security returns. Then, security returns may be expressed by belief degrees, which depend heavily on the experience and knowledge of experts. How do we deal with belief degree? Kahneman and Tversky [21] showed that human beings usually give too much weight to unlikely events, which means that belief degree usually contains a much wider range of values than the indeterminate quantity may actually take. In this case, it is not appropriate to employ probability theory to handle belief degrees since it usually deviates far from the actual frequency. Liu [22] introduced a counterexample showing that dealing with the belief degree by probability theory may lead to counterintuitive results. Also, some fuzziologists believe that belief degree can be interpreted as fuzziness and try to employ fuzzy set theory (Zadeh [23]) to manage portfolios. A fuzzy set (Zadeh [23]) is defined by a membership function μ which assigns to each element x a real number $\mu(x)$ ranging between zero and one. The value of $\mu(x)$ denotes the grade of membership of x in the fuzzy set. Within the framework of fuzzy set theory, the portfolio selection decision problem has been widely studied. For instance, Gupta et al. [24] applied fuzzy multicriteria decision-making to develop asset portfolio optimization models. Bhattacharyya et al. [25] proposed a fuzzy mean-variance-skewness model by considering transaction cost. Wang et al. [26] investigated a portfolio selection problem with random fuzzy variables by defining a new equilibrium risk value. Kar et al. [27] considered the Sharpe ratio and the value-at-risk ratio and then proposed a biobjective fuzzy portfolio selection model. In addition, papers by Xia et al. [28], Huang [29], Deng and Li [30], Li et al. [31], Guo et al. [32], Liu and Li [33], and Zhou et al. [34] have also developed the theory of fuzzy portfolio selection. However, further studies show that paradoxes appear when fuzzy set theory is employed to address belief degree (Liu [22]).

To deal with belief degree, an axiomatic uncertainty theory was founded by Liu [35] and then further refined by Liu [36]. Liu [22] also showed that belief degree follows the laws of uncertainty theory, which means that it is suitable to employ uncertainty theory to address belief degree. After that, uncertain programming was proposed by Liu [37] to deal with mathematical programming problems that contain uncertain variables. Uncertainty theory has been widely applied in various decision-making systems, such as transportation problem (Gao et al. [38] and Zhang et al. [39]), optimal assignment problem (Zhang and Peng [40] and Ding and Zeng [41]), and contract design problem (Wu et al. [42], Yang et al. [43]). Specifically, Huang [44] first introduced uncertainty theory to portfolio selection. Since then, many topics in the field of uncertain portfolio selection have attracted the attention of relevant scholars. For example, Huang and Qiao [45] discussed the multiperiod portfolio selection problem and proposed a risk index model. Li et al. [46] further discussed uncertain multiperiod portfolio selection with a bankruptcy constraint. Zhang et al. [47] proposed two innovative uncertain portfolio selection models, the expected-variance-chance model and chance-expected-variance model and designed a hybrid intelligent algorithm to solve the models. Moreover, Li and Qin [48] investigated an interval portfolio selection problem with uncertain information and proposed a mean-semiabsolute deviation model. Huang and Di [49] and Zhai and Bai [50] discussed uncertain portfolio selection problems with background risk. Qin et al. [51] presented mean-semi-absolute deviation adjusting models for an uncertain portfolio optimization problem. Chen et al. [52] proposed two types of mean-semivariance models for optimal portfolio decision-making and designed a hybrid intelligent algorithm to solve the models. Recently, Xue et al. [53] employed mental accounts to reflect different attitudes towards risk and proposed a new uncertain model in which security returns are estimated by experts. Kar et al. [54] introduced a multiobjective uncertain portfolio selection model based on cross-entropy (for more recent research on uncertain portfolio decision-making, see Huang [55]).

The above discussion makes clear that a variety of studies have been proposed to address portfolio selection within the framework of uncertainty theory. They all assume that an enterprise invests with its own capital. However, in reality, enterprises' investment funds are not always owned by the enterprises themselves. Enterprises often borrow money from banks to raise funds for investment. With sufficient investment funds, enterprises may have more choices among more potential projects. In this vein, this paper will consider a portfolio selection problem within the framework of uncertainty theory with a borrowing constraint. In addition, we discuss the effect of background factors on investment decisions.

The paper proceeds as follows. Section 2 introduces some basic concepts and results of uncertainty theory. Section 3 describes the problem of uncertain portfolio selection with the borrowing constraint and background risk and introduces two new types of uncertain portfolio selection models. Section 4 discusses the crisp equivalents of the proposed models and provides the approaches to solve them. Section 5 illustrates the innovations of the paper by comparing it with the existing articles. Section 6 gives some numerical experiments to illustrate our models and shows the impact of borrowing and background risk on the optimal portfolio decision. Finally, Section 7 concludes this paper with a brief summary.

2. Preliminaries

Based on normality, duality, subadditivity, and product axioms, uncertainty theory has now become an axiomatic branch of mathematics. In this section, we will introduce some key concepts of uncertainty theory, such as an uncertain measure, an uncertain variable, and the uncertainty distribution.

Definition 1 (Liu [35]). Suppose $\mathscr{L}$ is a σ-algebra over a nonempty set Γ. Each element $\Lambda \in \mathscr{L}$ is called an event. A real-valued set function $\mathscr{M}\{\Lambda\}$ is said to be an uncertain measure if it satisfies the normality, duality, subadditivity, and product axioms.

Definition 2 (Liu [35]). An uncertain variable ξ is a measurable function from an uncertainty space $(\Gamma, \mathscr{L}, \mathscr{M})$ to the set of real numbers. That is, the set

$$\{\xi \in B\} = \{\gamma \in \Gamma \mid \xi(\gamma) \in B\} \tag{1}$$

is an event for any Borel set B of real numbers.

Definition 3 (Liu [35]). The uncertainty distribution Φ: $\mathscr{R} \longrightarrow [0,1]$ of an uncertain variable ξ is defined as follows:

$$\Phi(x) = \mathscr{M}\{\xi \leq x\}, \tag{2}$$

for any real number x.

Definition 4 (Liu [36]). An uncertainty distribution Φ is said to be regular if it is a continuous and strictly increasing function with respect to x at which $0 < \Phi(x) < 1$ and

$$\begin{aligned} \lim_{x \longrightarrow -\infty} \Phi(x) &= 0, \\ \lim_{x \longrightarrow +\infty} \Phi(x) &= 1. \end{aligned} \tag{3}$$

An uncertain variable ξ is called linear if it has the following linear uncertainty distribution (Liu [35]):

$$\Phi(x) = \begin{cases} 0, & \text{if } x \leq a, \\ \dfrac{x-a}{b-a}, & \text{if } a \leq x \leq b, \\ 1, & \text{if } x \geq b. \end{cases} \tag{4}$$

It is usually denoted by $\xi \sim \mathscr{L}(a, b)$, where a and b are real numbers with $a < b$. In addition, it is easy to verify that the inverse uncertainty distribution of ξ is as follows:

$$\Phi^{-1}(\alpha) = (1-\alpha)a + \alpha b. \tag{5}$$

Theorem 1 (Liu [36]). *Let $\xi_1, \xi_2, \ldots, \xi_n$ be independent uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, \ldots, \Phi_n$, respectively. If $f(x_1, x_2, \ldots, x_n)$ is strictly increasing with respect to $x_1, x_2, \ldots, x_m$ and strictly decreasing with respect to $x_{m+1}, x_{m+2}, \ldots, x_n$, then $\xi = f(\xi_1, \xi_2, \ldots, \xi_n)$ is an uncertain variable with an inverse uncertainty distribution as follows:*

$$\Psi^{-1}(\alpha) = f\left(\Phi_1^{-1}(\alpha), \ldots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \ldots, \Phi_n^{-1}(1-\alpha)\right). \tag{6}$$

Definition 5 (Liu [35]). Let ξ be an uncertain variable. Then, the expected value of ξ is defined as follows:

$$E[\xi] = \int_0^{+\infty} \mathscr{M}\{\xi \geq x\}\mathrm{d}x - \int_{-\infty}^0 \mathscr{M}\{\xi \leq x\}\mathrm{d}x, \tag{7}$$

provided that at least one of the two integrals is finite.

For a linear uncertain variable $\xi \sim \mathscr{L}(a, b)$, it is easy to compute its expected value as follows:

$$E[\xi] = \frac{a+b}{2}. \tag{8}$$

Theorem 2 (Liu [36]). *Let ξ and η be independent uncertain variables with finite expected values. Then, for any real numbers a and b, we have*

$$E[a\xi + b\eta] = aE[\xi] + bE[\eta]. \tag{9}$$

3. Problem Description

As introduced in Section 1, there exist situations in which security returns are indeterminate quantities. In reality, there are background factors, such as health, labour income, and real estate, that influence investors' decisions on portfolio selection and cannot be hedged by diversifying their portfolios. In addition, investors may want to apply for loans and invest them in risky assets such as securities. If the background factor return and securities' return rate are given by experts' belief degree based on their experience rather than historical data, then it is better to consider them as uncertain variables.

Here, let us first introduce the following notations and parameters, which will be used to describe the portfolio selection problem by the mathematical models.

n: number of securities

x_i: the wealth invested in security i, $i = 1, 2, \ldots, n$

r: the borrowing interest rate

w: the wealth of an existing portfolio

v: the upper bound of borrowing

t_i: the lower bound of the wealth invested in security i, $i = 1, 2, \ldots, n$

T_i: the upper bound of the wealth invested in security i, $i = 1, 2, \ldots, n$

ξ_i: return rate of security i, with regular uncertainty distribution Φ_i, $i = 1, 2, \ldots, n$

η: background factor return, with regular uncertainty distribution Ψ

Assume the wealth of an existing portfolio held by an investor is w; in order to pursue more return, the investor plans to reallocate his wealth among n securities by borrowing from bank. Suppose the upper bound of borrowing is v, and the borrowing interest rate is r. Then, the upper bound of possible holding wealth is $w + v$. If the wealth invested in security i is x_i, then the budget constraint can be described as follows:

$$w \le \sum_{i=1}^{n} x_i \le w + v. \tag{10}$$

Then, the portfolio return can be expressed as follows:

$$f(\mathbf{x}; \boldsymbol{\xi}, \eta) = \sum_{i=1}^{n} x_i \xi_i - r\left(\sum_{i=1}^{n} x_i - w\right) + \eta, \tag{11}$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\boldsymbol{\xi} = (\xi_1, \xi_2, \ldots, \xi_n)$.

It is clear that the portfolio return $f(\mathbf{x}; \boldsymbol{\xi}, \eta)$ is an uncertain variable. In uncertain financial markets, investors may require that the expected portfolio return be greater than or equal to a given value u; we then have the following expected return constraint:

$$E[f(\mathbf{x}; \boldsymbol{\xi}, \eta)] \ge u, \tag{12}$$

where u is a predetermined minimal expected return which investors feel satisfactory. The expected return constraint shows that the average value of return is greater than or equal to u. In practical applications, the value of u can be determined by investors or financial experts. It is necessary to point out that the value of this lower bound is dependent on the decision-making conditions and the knowledge of investors or experts.

To control other risk factors, investors can control securities' investment wealth by imposing upper and lower bounds on each security. For each security i, this constraint can be expressed as follows:

$$t_i \le x_i \le T_i. \tag{13}$$

Then, $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ is called a feasible portfolio if it satisfies the following constraints:

$$\begin{cases} E\left[\sum_{i=1}^{n} x_i \xi_i - r\left(\sum_{i=1}^{n} x_i - w\right) + \eta\right] \ge u, \\ w \le \sum_{i=1}^{n} x_i \le w + v, \\ t_i \le x_i \le T_i, \quad i = 1, 2, \ldots, n. \end{cases} \tag{14}$$

The set of feasible portfolios $\mathbf{x}$ is denoted as S.

The goal of this paper is to make a portfolio decision so that the total return is maximized. However, because the portfolio return $f(\mathbf{x}; \boldsymbol{\xi}, \eta)$ is an uncertain variable, we cannot directly obtain the optimal decision by traditional methods. In the following, we will first discuss the uncertainty of the portfolio return and then introduce some optimization decision models based on different points of view.

3.1. Uncertainty of Portfolio Return. Given a feasible portfolio $\mathbf{x}$, the uncertainty distribution of the total portfolio return $f(\mathbf{x}; \boldsymbol{\xi}, \eta)$ is denoted as Υ_x, i.e., for any real number p, we have

$$\Upsilon_x(p) = \mathscr{M}\{f(\mathbf{x}; \boldsymbol{\xi}, \eta) \le p\}. \tag{15}$$

According to Theorem 1, Υ_x is a regular function, since ξ_i and η are regular uncertain variables with uncertainty distributions Φ_i and Ψ, respectively. That is, for any $\alpha \in (0, 1)$, we have the following inverse uncertainty distribution:

$$\Upsilon_x^{-1}(\alpha) = \sum_{i=1}^{n} x_i \Phi_i^{-1}(\alpha) - r\left(\sum_{i=1}^{n} x_i - w\right) + \Psi^{-1}(\alpha). \tag{16}$$

In addition, the following maximum portfolio return:

$$f(\boldsymbol{\xi}, \eta) = \max_{x \in S} f(\mathbf{x}; \boldsymbol{\xi}, \eta), \tag{17}$$

is also an uncertain variable. Denote $\Upsilon_{\max}$ as the uncertainty distribution of $f(\boldsymbol{\xi}, \eta)$. Additionally, $\Upsilon_{\max}$ is a regular function. For any real number p, we have

$$\Upsilon_{\max}(p) = \mathscr{M}\{f(\boldsymbol{\xi}, \eta) \le p\}. \tag{18}$$

According to equations (16)–(18), for any real number p, we have

$$\mathscr{M}\{f(\boldsymbol{\xi}, \eta) \le p\} \le \mathscr{M}\{f(\mathbf{x}; \boldsymbol{\xi}, \eta) \le p\}, \tag{19}$$

or, equivalently,

$$\Upsilon_{\max}(p) \le \Upsilon_x(p). \tag{20}$$

In other words, for any $\alpha \in (0, 1)$, we have

$$\Upsilon_{\max}^{-1}(\alpha) \ge \Upsilon_x^{-1}(\alpha), \tag{21}$$

since $\Upsilon_{\max}$ and Υ_x are both regular distribution functions.

If the returns are deterministic values, i.e., $\boldsymbol{\xi} = (\xi_i)$ and η are replaced with constants $\mathbf{h} = (h_i)$ and t, respectively, then the maximum portfolio return $f(\boldsymbol{\xi}, \eta)$ can be rewritten as $f(\mathbf{h}, t)$. It is easy to verify that $f(\mathbf{h}, t)$ is a strictly increasing function with respect to $h_1, h_2, \ldots, h_n$ and t. That is, for a given $(\mathbf{h}, t)$ and $(\mathbf{h}', t')$, $f(\mathbf{h}, t)$ satisfies the following conditions:

(i) $f(\mathbf{h}, t) \le f(\mathbf{h}', t')$ when $h_i \le h_i'$ for $i = 1, 2, \ldots, n$ and $t \le t'$.

(ii) $f(\mathbf{h}, t) < f(\mathbf{h}', t')$ when $h_i < h_i'$ for $i = 1, 2, \ldots, n$ and $t < t'$.

According to Theorem 1, for any $\alpha \in (0, 1)$, the inverse uncertainty distribution $\Upsilon_{\max}^{-1}(\alpha)$ of $f(\boldsymbol{\xi}, \eta)$ is $f(\boldsymbol{\Phi}^{-1}(\alpha), \Psi^{-1}(\alpha))$, where $\boldsymbol{\Phi}^{-1}(\alpha) = (\Phi_i^{-1}(\alpha))$, $i = 1, 2, \ldots, n$. More precisely, $f(\boldsymbol{\Phi}^{-1}(\alpha), \Psi^{-1}(\alpha))$ is just the optimal objective of the following deterministic model:

$$\begin{cases} \max & \sum_{i=1}^{n} x_i \Phi_i^{-1}(\alpha) - r\left(\sum_{i=1}^{n} x_i - w\right) + \Psi^{-1}(\alpha) \\ \text{s.t.} & \sum_{i=1}^{n} x_i E[\xi_i] - r\left(\sum_{i=1}^{n} x_i - w\right) + E[\eta] \geq u \\ & w \leq \sum_{i=1}^{n} x_i \leq w + v \\ & \qquad t_i \leq x_i \leq T_i, \quad i = 1, 2, \ldots, n. \end{cases} \tag{22}$$

Obviously, model (22) corresponds to the linear programming model. To obtain the value of $\Upsilon_{\max}^{-1}(\alpha)$, we should first solve model (22). Repeating this process, we can get the uncertainty distribution $\Upsilon_{\max}$ by interpolation approximation.

How to get the optimal portfolio plan under an uncertain environment? Some addition criteria should be considered.

3.2. Uncertain Portfolio Models. Investors may first present a satisfying confidence level $\alpha \in (0, 1)$ and hope to obtain a largest value $\overline{f}$ such that the total portfolio return $f(\mathbf{x}; \boldsymbol{\xi}, \eta)$ is greater than or equal to $\overline{f}$ with confidence level α. This leads us to propose the following critical value criterion.

Definition 6. A feasible portfolio $\mathbf{x}^*$ is called α-optimal portfolio if

$$\begin{aligned} &\max\left\{\overline{f} \mid \mathcal{M}\left\{f(\mathbf{x}^*; \boldsymbol{\xi}, \eta) \geq \overline{f}\right\} \geq \alpha\right\} \\ &\quad \geq \max\left\{\overline{f} \mid \mathcal{M}\left\{f(\mathbf{x}; \boldsymbol{\xi}, \eta) \geq \overline{f}\right\} \geq \alpha\right\}, \end{aligned} \tag{23}$$

holds for any feasible portfolio $\mathbf{x}$, where α is a predetermined confidence level.

According to Definition 6, the α-optimal portfolio is essentially the optimal solution of the following chance-mean model:

$$\begin{cases} \max & \overline{f} \\ \text{s.t.} & \mathcal{M}\left\{\sum_{i=1}^{n} x_i \xi_i - r\left(\sum_{i=1}^{n} x_i - w\right) + \eta \geq \overline{f}\right\} \geq \alpha \\ & E\left[\sum_{i=1}^{n} x_i \xi_i - r\left(\sum_{i=1}^{n} x_i - w\right) + \eta\right] \geq u \\ & w \leq \sum_{i=1}^{n} x_i \leq w + v \\ & \qquad t_i \leq x_i \leq T_i, \quad i = 1, 2, \ldots, n. \end{cases} \tag{24}$$

Remark 1. If loans are ignored, then model (24) becomes as follows:

$$\begin{cases} \max & \overline{f} \\ \text{s.t.} & \mathcal{M}\left\{\sum_{i=1}^{n} x_i \xi_i + \eta \geq \overline{f}\right\} \geq \alpha \\ & E\left[\sum_{i=1}^{n} x_i \xi_i + \eta\right] \geq u \\ & \sum_{i=1}^{n} x_i = w \\ & \qquad t_i \leq x_i \leq T_i, \quad i = 1, 2, \ldots, n. \end{cases} \tag{25}$$

Remark 2. If background risk is not taken into account, then model (24) can be rewritten as follows:

$$\begin{cases} \max & \overline{f} \\ \text{s.t.} & \mathcal{M}\left\{\sum_{i=1}^{n} x_i \xi_i - r\left(\sum_{i=1}^{n} x_i - w\right) \geq \overline{f}\right\} \geq \alpha \\ & E\left[\sum_{i=1}^{n} x_i \xi_i - r\left(\sum_{i=1}^{n} x_i - w\right)\right] \geq u \\ & w \leq \sum_{i=1}^{n} x_i \leq w + v \\ & \qquad t_i \leq x_i \leq T_i, \quad i = 1, 2, \ldots, n. \end{cases} \tag{26}$$

Remark 3. If the loans and background risk are not considered, then model (24) degenerates to the following:

$$\begin{cases} \max & \overline{f} \\ \text{s.t.} & \mathcal{M}\left\{\sum_{i=1}^{n} x_i \xi_i \geq \overline{f}\right\} \geq \alpha \\ & E\left[\sum_{i=1}^{n} x_i \xi_i\right] \geq u \\ & \sum_{i=1}^{n} x_i = w \\ & \qquad t_i \leq x_i \leq T_i, \quad i = 1, 2, \ldots, n. \end{cases} \tag{27}$$

From a different perspective, investors may present an appropriate threshold profit $\overline{f}$ and hope to maximize the chance of total return preponderating the given profit level. This modelling idea can be called chance criterion. Based on this criterion, the chance optimal portfolio is defined as follows.

Definition 7. A feasible portfolio $\mathbf{x}^*$ is called the chance optimal portfolio if

$$\mathcal{M}\left\{f(\mathbf{x}^*; \boldsymbol{\xi}, \eta) \geq \overline{f}\right\} \geq \mathcal{M}\left\{f(\mathbf{x}; \boldsymbol{\xi}, \eta) \geq \overline{f}\right\}, \tag{28}$$

holds for any feasible portfolio $\mathbf{x}$, where $\overline{f}$ is a predetermined profit level.

According to Definition 7, the chance optimal portfolio $\mathbf{x}^*$ is simply the optimal solution of the following measure-mean model:

$$\begin{cases} \max & \mathcal{M}\left\{\sum_{i=1}^{n} x_i\xi_i - r\left(\sum_{i=1}^{n} x_i - w\right) + \eta \geq \overline{f}\right\} \\ \text{s.t.} & E\left[\sum_{i=1}^{n} x_i\xi_i - r\left(\sum_{i=1}^{n} x_i - w\right) + \eta\right] \geq u \\ & w \leq \sum_{i=1}^{n} x_i \leq w + v \\ & t_i \leq x_i \leq T_i, \quad i = 1, 2, \ldots, n. \end{cases} \tag{29}$$

Remark 4. If there is no borrowing, then model (29) can be expressed as follows:

$$\begin{cases} \max & \mathcal{M}\left\{\sum_{i=1}^{n} x_i\xi_i + \eta \geq \overline{f}\right\} \\ \text{s.t.} & E\left[\sum_{i=1}^{n} x_i\xi_i + \eta\right] \geq u \\ & \sum_{i=1}^{n} x_i = w \\ & t_i \leq x_i \leq T_i, \quad i = 1, 2, \ldots, n. \end{cases} \tag{30}$$

Remark 5. If there is no background risk, then model (29) can be expressed as follows:

$$\begin{cases} \max & \mathcal{M}\left\{\sum_{i=1}^{n} x_i\xi_i - r\left(\sum_{i=1}^{n} x_i - w\right) \geq \overline{f}\right\} \\ \text{s.t.} & E\left[\sum_{i=1}^{n} x_i\xi_i - r\left(\sum_{i=1}^{n} x_i - w\right)\right] \geq u \\ & w \leq \sum_{i=1}^{n} x_i \leq w + v \\ & t_i \leq x_i \leq T_i, \quad i = 1, 2, \ldots, n. \end{cases} \tag{31}$$

Remark 6. If there are no borrowing and background risk, then model (29) can be rewritten as follows:

$$\begin{cases} \max & \mathcal{M}\left\{\sum_{i=1}^{n} x_i\xi_i \geq \overline{f}\right\} \\ \text{s.t.} & E\left[\sum_{i=1}^{n} x_i\xi_i\right] \geq u \\ & \sum_{i=1}^{n} x_i = w \\ & t_i \leq x_i \leq T_i, \quad i = 1, 2, \ldots, n. \end{cases} \tag{32}$$

4. Equivalents of the Models

Note that there are many uncertain variables in model (24) and model (29). To solve the models, it is necessary for us to discuss the crisp equivalents of them.

Theorem 3. *If ξ_i and η are independent uncertain variables with regular uncertainty distributions Φ_i and Ψ, $i = 1, 2, \ldots, n$, respectively, then the α-optimal portfolio is simply the optimal solution of the following model:*

$$\begin{cases} \max & \sum_{i=1}^{n} x_i\Phi_i^{-1}(1-\alpha) - r\left(\sum_{i=1}^{n} x_i - w\right) + \Psi^{-1}(1-\alpha) \\ \text{s.t.} & \sum_{i=1}^{n} x_i E[\xi_i] - r\left(\sum_{i=1}^{n} x_i - w\right) + E[\eta] \geq u \\ & w \leq \sum_{i=1}^{n} x_i \leq w + v \\ & t_i \leq x_i \leq T_i, i = 1, 2, \ldots, n. \end{cases} \tag{33}$$

Proof. Since $f(\mathbf{x}; \boldsymbol{\xi}, \eta)$ is strictly increasing with respect to $\xi_1, \xi_2, \ldots, \xi_n$ and η, then for any $\alpha \in (0, 1)$, the inverse uncertainty distribution of $f(\mathbf{x}; \boldsymbol{\xi}, \eta)$ is as follows:

$$\Upsilon_x^{-1}(\alpha) = \sum_{i=1}^{n} x_i\Phi_i^{-1}(\alpha) - r\left(\sum_{i=1}^{n} x_i - w\right) + \Psi^{-1}(\alpha). \tag{34}$$

Owing to the fact that ξ_i and η are regular uncertain variables, by using the duality of an uncertain measure, the first constraint of model (24) is equivalent to

$$\begin{aligned} \mathcal{M}\left\{\sum_{i=1}^{n} x_i\xi_i - r\left(\sum_{i=1}^{n} x_i - w\right) + \eta \leq \overline{f}\right\} &= \mathcal{M}\left\{f(\mathbf{x}; \boldsymbol{\xi}, \eta) \leq \overline{f}\right\} \\ &\leq 1 - \alpha. \end{aligned} \tag{35}$$

In detail, we have

$$\Upsilon_x^{-1}(1-\alpha) \geq \overline{f}. \tag{36}$$

It follows from equation (34) that we have

$$\sum_{i=1}^{n} x_i\Phi_i^{-1}(1-\alpha) - r\left(\sum_{i=1}^{n} x_i - w\right) + \Psi^{-1}(1-\alpha) \geq \overline{f}. \tag{37}$$

In addition, by means of the linearity of the expected value operator of the uncertain variable, we have

$$\begin{aligned} E\left[\sum_{i=1}^{n} x_i\xi_i - r\left(\sum_{i=1}^{n} x_i - w\right) + \eta\right] &= \sum_{i=1}^{n} x_i E[\xi_i] - r\left(\sum_{i=1}^{n} x_i - w\right) \\ &\quad + E[\eta]. \end{aligned} \tag{38}$$

Thus, model (24) can be equivalently transformed to the following deterministic model:

$$\begin{cases} \max & \overline{f} \\ \text{s.t.} & \sum_{i=1}^{n} x_i \Phi_i^{-1}(1-\alpha) - r\left(\sum_{i=1}^{n} x_i - w\right) + \Psi^{-1}(1-\alpha) \geq \overline{f} \\ & \sum_{i=1}^{n} x_i E[\xi_i] - r\left(\sum_{i=1}^{n} x_i - w\right) + E[\eta] \geq u \\ & w \leq \sum_{i=1}^{n} x_i \leq w + v \\ & \quad t_i \leq x_i \leq T_i, \quad i = 1, 2, \ldots, n. \end{cases} \tag{39}$$

Obviously, model (39) is equivalent to model (33). As we know, the α-optimal portfolio is simply the optimal solution of model (24). Thus, the theorem is proved.

Comparing model (22) with model (33), the value of the objective function corresponding to the α-optimal portfolio is just the value of $\Upsilon_{\max}^{-1}(1-\alpha)$. □

Theorem 4. *Given a profit level $\overline{f}$, if ξ_i and η are independent uncertain variables with regular uncertainty distributions Φ_i and Ψ, $i = 1, 2, \ldots, n$, respectively, then the chance optimal portfolio is simply the α-optimal portfolio, where $\Upsilon_{\max}(\overline{f}) = 1 - \alpha$.*

Proof. From the above discussion, we can see that the chance optimal portfolio is just the optimal solution of model (29). Apparently, model (29) is equivalent to the following model:

$$\begin{cases} \max & \alpha \\ \text{s.t.} & \mathscr{M}\left\{\sum_{i=1}^{n} x_i \xi_i - r\left(\sum_{i=1}^{n} x_i - w\right) + \eta \geq \overline{f}\right\} \geq \alpha \\ & E\left[\sum_{i=1}^{n} x_i \xi_i - r\left(\sum_{i=1}^{n} x_i - w\right) + \eta\right] \geq u \\ & w \leq \sum_{i=1}^{n} x_i \leq w + v \\ & \quad t_i \leq x_i \leq T_i, \quad i = 1, 2, \ldots, n. \end{cases} \tag{40}$$

As in the proof of Theorem 3, model (40) can be transformed into the following model:

$$\begin{cases} \max & \alpha \\ \text{s.t.} & \sum_{i=1}^{n} x_i \Phi_i^{-1}(1-\alpha) - r\left(\sum_{i=1}^{n} x_i - w\right) + \Psi^{-1}(1-\alpha) \geq \overline{f} \\ & \sum_{i=1}^{n} x_i E[\xi_i] - r\left(\sum_{i=1}^{n} x_i - w\right) + E[\eta] \geq u \\ & w \leq \sum_{i=1}^{n} x_i \leq w + v \\ & \quad t_i \leq x_i \leq T_i, \quad i = 1, 2, \ldots, n. \end{cases} \tag{41}$$

Denote $(\mathbf{x}, \alpha)$ as the feasible solution of model (41). Set $1 - \beta = \Upsilon_{\max}(\overline{f}) = \mathscr{M}\{f(\boldsymbol{\xi}, \eta) \leq \overline{f}\}$. Then, $\Upsilon_{\max}^{-1}(1-\beta) = \overline{f}$ since $\Upsilon_{\max}$ is a regular distribution function. For the given confidence level β, there is a β-optimal portfolio $\mathbf{x}'$, which can be obtained by Theorem 3. In other words, $\mathbf{x}'$ is the optimal solution of model (39), i.e., $\mathbf{x}'$ and β satisfy the following constraints:

$$\begin{cases} \sum_{i=1}^{n} x_i \Phi_i^{-1}(1-\beta) - r\left(\sum_{i=1}^{n} x_i - w\right) + \Psi^{-1}(1-\beta) \geq \overline{f}, \\ \sum_{i=1}^{n} x_i E[\xi_i] - r\left(\sum_{i=1}^{n} x_i - w\right) + E[\eta] \geq u, \\ w \leq \sum_{i=1}^{n} x_i \leq w + v, \\ \quad t_i \leq x_i \leq T_i, \quad i = 1, 2, \ldots, n, \end{cases} \tag{42}$$

which shows that $(\mathbf{x}', \beta)$ is a feasible solution of model (41) and β is the corresponding objective value. In the following, we will prove that β is also the optimal value of the objective function. We account for it as follows.

Set $\gamma > \beta$ and $(\widehat{x}, \gamma)$ is a feasible solution of model (41). We know that $\Upsilon_{\max}$ is a regular distribution function, which means that $\Upsilon_{\max}^{-1}(\alpha)$ is strictly increasing, i.e.,

$$\Upsilon_{\max}^{-1}(1-\gamma) < \Upsilon_{\max}^{-1}(1-\beta) = \overline{f}. \tag{43}$$

According to equation (21), for any $\widehat{x}$, we have

$$\Upsilon_{\widehat{x}}^{-1}(1-\gamma) \leq \Upsilon_{\max}^{-1}(1-\gamma) < \overline{f}. \tag{44}$$

That is,

$$\sum_{i=1}^{n} \widehat{x}_i \Phi_i^{-1}(1-\gamma) - r\left(\sum_{i=1}^{n} \widehat{x}_i - w\right) + \Psi^{-1}(1-\gamma) < \overline{f}, \tag{45}$$

which contradicts the first constraint of model (41). This shows that β is the optimal objective value, and the feasible solution $\mathbf{x}'$, i.e., the β-optimal portfolio, is the optimal solution of model (41). Thus, the theorem is proved.

According to Theorem 4, we can first obtain the uncertainty distribution $\Upsilon_{\max}$. For the given profit level $\overline{f}$, if $\Upsilon_{\max}(\overline{f}) = 1 - \alpha$, then we can obtain the α-optimal portfolio by Theorem 3, which is simply the chance optimal portfolio, and α is the corresponding optimal objective value. □

5. Comparisons and Innovations

To highlight the innovations of the paper, we compare it with the existing articles in two main aspects. On the one hand, we compare the paper with stochastic portfolio selection models. On the other hand, we compare the paper with the existing articles within the framework of uncertainty theory.

Probability theory has been widely employed to deal with indeterminate factors for a long time. As is well known, a fundamental premise of applying probability theory is that we should obtain a probability distribution that is close enough to the real frequency. That is, a large amount of historical data is required. In the stochastic portfolio selection models, the indeterminate factors are regarded as stochastic factors and are described as random variables.

However, without sufficient data, it is impossible for us to accurately predict the values of the indeterminate quantities. In financial markets, we often lack sufficient sample data for the returns of securities, especially for newly listed securities. In this case, we have no choice but to invite financial experts to evaluate the belief degrees about the returns of securities. Uncertainty theory is a powerful technique to deal with belief degrees. Therefore, this paper introduces two novel portfolio selection models based on uncertainty theory. In the proposed uncertain models, the indeterminate quantities come from experts' empirical estimation and are described as uncertain variables.

Thus, compared to stochastic portfolio selection models, the main difference between the existing works and the proposed paper is summarized in Table 1.

As addressed above, some researchers have studied portfolio selection problems within the framework of uncertainty theory. For better readability, we list the characteristics of some closely related works in Table 2, including the main modelling ideas and the main constraints. Clearly, the related studies usually employ variance, semivariance, or semiabsolute deviation to measure risk. Additionally, the considered constraint in the existing literature is usually related to the transaction cost, risk constraint, or return constraint, with the borrowing constraint not taken into account. Meanwhile, many studies ignore the effect of background factors on investment decisions. With this concern in mind, this paper aims to study a portfolio selection with realistic constraints, including the borrowing constraint and background risk, to enhance the practicability of the models.

Stated thus, the main innovations of this paper are summarized as follows. (1) In contrast with the use of probability theory for portfolio selection in extant literature, this paper employs uncertainty theory to handle belief degree. The advantage of uncertainty theory is that belief degree follows the laws of uncertainty theory. (2) In contrast with the existing uncertain portfolio selection models, this paper proposes two novel uncertain portfolio selection models, in which the borrowing constraint and background risk are considered simultaneously.

6. Numerical Experiments

In this section, we will provide some numerical experiments to illustrate the models mentioned above for uncertain portfolio selection problems. We select twenty stocks from the Shanghai Stock Exchange, namely, S_1 (code: 600004), S_2 (code: 600016), S_3 (code: 600018), S_4 (code: 600028), S_5 (code: 600029), S_6 (code: 600048), S_7 (code: 600066), S_8 (code: 600115), S_9 (code: 600177), S_{10} (code: 600340), S_{11} (code: 600377), S_{12} (code: 600383), S_{13} (code: 600398), S_{14} (code: 600463), S_{15} (code: 600496), S_{16} (code: 600499), S_{17} (code: 600519), S_{18} (code: 600522), S_{19} (code: 600601), and S_{20} (code: 600606). According to the experts' estimations, the return rates of the stocks are assumed to be independent linear uncertain variables, which are shown in Table 3. The background factor return $\eta \sim \mathscr{L}(-2, 9)$. The way to obtain the distribution functions of returns based on experts' estimations can be referred to Liu [22].

Suppose the wealth of an existing portfolio held by the investor is $w = 300$ (thousand yuan). We also assume that the upper bound of borrowing $v = 50$ (thousand yuan), the borrowing interest rate $r = 0.01$, the threshold expected return $u = 45$ (thousand yuan), the lower bound $t_i = 0$ (thousand yuan), and the upper bound $T_i = 50$ (thousand yuan).

If the decision-maker wants to maximize the portfolio return at a chance of not less than 0.9 (i.e., $\alpha = 0.9$), according to Theorem 3, he can make an optimum portfolio decision via the following model:

$$\begin{cases} \max & \displaystyle\sum_{i=1}^{20} x_i \Phi_i^{-1}(0.1) - 0.01\left(\sum_{i=1}^{20} x_i - 300\right) + \Psi^{-1}(0.1) \\ \text{s.t.} & \displaystyle\sum_{i=1}^{20} x_i E[\xi_i] - 0.01\left(\sum_{i=1}^{20} x_i - 300\right) + E[\eta] \geq 45 \\ & 300 \leq \displaystyle\sum_{i=1}^{20} x_i \leq 350 \\ & \qquad 0 \leq x_i \leq 50, \quad i = 1, 2, \ldots, 20. \end{cases} \tag{46}$$

First, we can easily obtain that $E[\eta] = 3.5$ and $\Psi^{-1}(0.1) = -0.9$. The expected values and the inverse uncertainty distributions of ξ_i at 0.1 are calculated and listed in Table 3. By using MATLAB 2017, we obtain the optimal portfolio after a computation time of approximately 0.06 s and present it in Table 4. The corresponding objective value is 11.06. Specifically, the decision-maker should borrow 50 thousand yuan from the bank and invest all 350 thousand yuan in the holding wealth in stocks 1, 5, 8, 9, 14, 15, 18, and 20, respectively.

That is, if the decision-maker allocates his wealth according to Table 4, the total return will be greater than or equal to 11.06 thousand yuan with possibility 90%. More precisely, 11.06 thousand yuan is the minimum return that can be obtained under the chance constraint $\alpha = 0.9$.

For different levels of $\alpha \in (0, 1)$, different objective values are shown in Table 5. Table 5 shows that the optimal objective value is nonincreasing with respect to α. As discussed in Section 3.1, the value of the objective function corresponding to the α-optimal portfolio is just the value of $\Upsilon_{\max}^{-1}(1-\alpha)$. Thus, we can obtain the uncertainty distribution $\Upsilon_{\max}$ of $f(\boldsymbol{\xi}, \eta)$ in a numerical sense, which is drawn using MATLAB 2017 in Figure 1. According to Figure 1, we know the distribution of portfolio returns $f(\boldsymbol{\xi}, \eta)$. For example, $\Upsilon_{\max}(115) = \mathscr{M}\{f(\boldsymbol{\xi}, \eta) \leq 115\} = 0.8$, i.e., $\mathscr{M}\{f(\boldsymbol{\xi}, \eta) \geq 115\} = 0.2$. This means that 115 thousand yuan is the minimum return that can be obtained under the chance constraint $\alpha = 0.2$.

Given a threshold return $\overline{f} = 29$ (thousand yuan), if the decision-maker wants to maximize the chance of total return preponderating the given threshold return $\overline{f}$, Theorem 4 shows that the optimal portfolio selection is simply the

Table 1: Comparison with stochastic portfolio selection models.

	Stochastic models	The proposed uncertain models
Sample size	The sample size is large enough	The sample size is too small (or even no sample)
Indeterminate factors	Stochastic factors	Experts' empirical estimation
Theoretical tool	Probability theory	Uncertainty theory
Indeterminate quantities	Random variable	Uncertain variable

Table 2: Comparison with uncertain portfolio selection models.

	The main modelling ideas	The main constraints
Huang and Qiao [45]	An expected model is provided based on an uncertain risk index	Self-financing and risk constraint
Li et al. [46]	A multiperiod portfolio selection is formulated in three steps with two single objective models	Transaction cost and bankruptcy constraint
Zhang et al. [47]	Two uncertain models are provided in the trade-off between risk and return	Return constraint
Qin et al. [51]	A bi-objective mean-semiabsolute deviation linear programming model is provided	Transaction cost
Huang and Di [49]	A mean-chance model is formulated with background risk	Return constraint with background risk
Zhai and Bai [50]	A mean-risk model is provided with background risk and transaction costs	Liquidity constraint
Chen et al. [52]	Two new mean-semivariance models are proposed	Entropy constraint
Kar et al. [54]	Defining risk as variance and divergence among security returns as cross-entropy	Investment proportion constraint
Xue et al. [53]	A mean-chance model is proposed with mental accounts	Return risk and liquidity risk constraint
This paper	A chance-mean model and a measure-mean model are proposed	Borrowing constraint and background risk

Table 3: Uncertain return rate ξ_i of each stock i.

Stock (S_i)	ξ_i	$E[\xi_i]$	$\Phi_i^{-1}(0.1)$
S_1	$\mathscr{L}(0.00, 0.22)$	0.11	0.022
S_2	$\mathscr{L}(-0.05, 0.25)$	0.1	−0.02
S_3	$\mathscr{L}(-0.06, 0.36)$	0.15	−0.018
S_4	$\mathscr{L}(-0.10, 0.30)$	0.1	−0.06
S_5	$\mathscr{L}(0.02, 0.12)$	0.07	0.03
S_6	$\mathscr{L}(-0.02, 0.24)$	0.11	0.006
S_7	$\mathscr{L}(-0.15, 0.40)$	0.125	-0.095
S_8	$\mathscr{L}(0.00, 0.30)$	0.15	0.03
S_9	$\mathscr{L}(0.01, 0.37)$	0.19	0.046
S_{10}	$\mathscr{L}(-0.08, 0.36)$	0.14	−0.036
S_{11}	$\mathscr{L}(-0.02, 0.25)$	0.115	0.007
S_{12}	$\mathscr{L}(-0.20, 0.40)$	0.1	−0.14
S_{13}	$\mathscr{L}(-0.25, 0.55)$	0.15	−0.17
S_{14}	$\mathscr{L}(0.03, 0.19)$	0.11	0.046
S_{15}	$\mathscr{L}(-0.02, 0.32)$	0.15	0.014
S_{16}	$\mathscr{L}(-0.08, 0.40)$	0.16	−0.032
S_{17}	$\mathscr{L}(-0.16, 0.38)$	0.11	−0.106
S_{18}	$\mathscr{L}(0.05, 0.12)$	0.085	0.057
S_{19}	$\mathscr{L}(-0.14, 0.46)$	0.16	−0.08
S_{20}	$\mathscr{L}(0.01, 0.16)$	0.085	0.025

Table 4: Optimal solution of model (46).

Stock (S_i)	Wealth (x_i)
S_1	50
S_2	0
S_3	0
S_4	0
S_5	50
S_6	0
S_7	0
S_8	50
S_9	50
S_{10}	0
S_{11}	0
S_{12}	0
S_{13}	0
S_{14}	50
S_{15}	30.77
S_{16}	0
S_{17}	0
S_{18}	50
S_{19}	0
S_{20}	19.23

α-optimal portfolio such that $\Upsilon_{\max}(29) = 1 - \alpha$. Figure 1 shows that $\Upsilon_{\max}(29) \approx 0.3$, i.e., $\alpha \approx 0.7$. Then, the corresponding objective value is 0.7.

Remark 7. The chance-mean model and measure-mean model are proposed on different modelling ideas. The chance-mean model provides a powerful means of modelling uncertain portfolio decision-making systems under the assumption that the uncertain constraints will hold with expected value and a confidence level, which is predetermined as a safety margin by the decision-maker. The underlying philosophy of the measure-mean model is based on selecting the stocks with the maximal chance to meet the threshold expected return. The chance-mean model can be transformed to a deterministic linear programming model and then it can be easily solved by

Table 5: Objective values for different levels of α.

α	Objective value ($\Upsilon_{\max}^{-1}(1-\alpha)$)
0.10	135.75
0.20	115.2
0.30	95.45
0.40	76.30
0.50	58.50
0.60	42.20
0.70	29.35
0.80	19.70
0.90	11.06
0.95	6.82

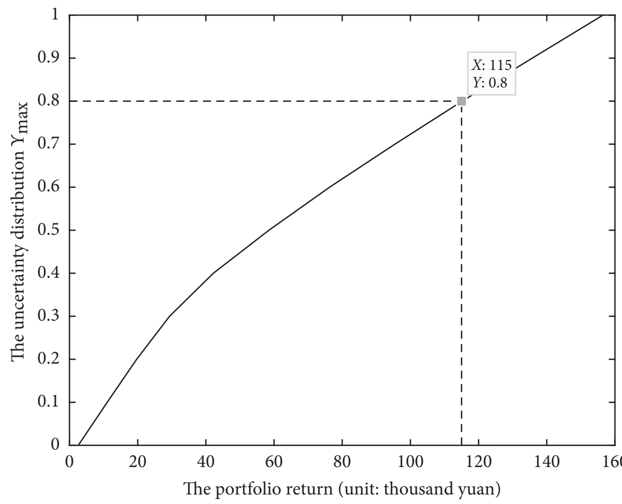

Figure 1: Uncertainty distribution function $\Upsilon_{\max}$ of $f(\boldsymbol{\xi}, \eta)$.

optimization software. However, the measure-mean model cannot be solved directly. Fortunately, we have proved that the optimal solution of the measure-mean model can be obtained by solving the chance-mean model.

6.1. Validity of the Models. To further discuss the validity of the proposed models, we conduct more experiments on the chance-mean model by ignoring background risks and borrowing. The discussion of the measure-mean model can be done in a similar way, so we omit it.

We also set $w = 300$ (thousand yuan), $u = 45$ (thousand yuan), and $\alpha = 0.9$. If the decision-maker does not apply for loans, then the optimal portfolio selection model takes the following form:

$$\begin{cases} \max & \sum_{i=1}^{20} x_i \Phi_i^{-1}(0.1) + \Psi^{-1}(0.1) \\ \text{s.t.} & \sum_{i=1}^{20} x_i E[\xi_i] + E[\eta] \geq 45 \\ & \sum_{i=1}^{20} x_i = 300 \\ & \qquad 0 \leq x_i \leq 50, \quad i = 1, 2, \ldots, 20. \end{cases} \tag{47}$$

Table 6: Objective values for different levels of α without loans.

α	Objective value
0.10	119.95
0.20	101.90
0.30	84.55
0.40	67.60
0.50	51.50
0.60	37.90
0.70	26.55
0.80	17.33
0.90	8.10
0.95	3.49

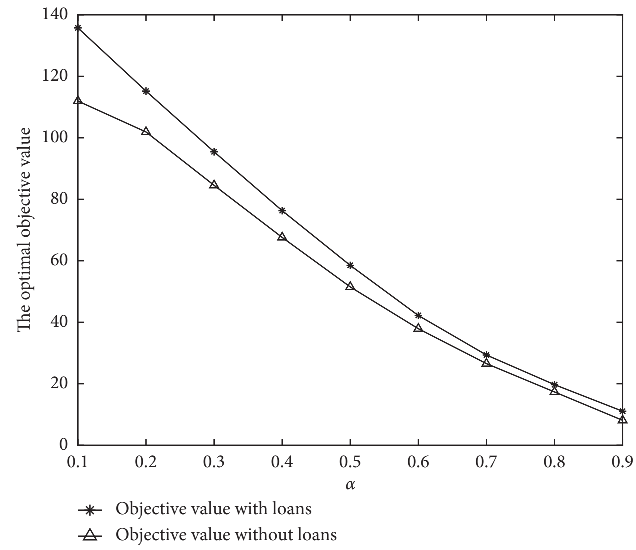

Figure 2: Objective values with loans and without loans.

Table 7: Objective values for different levels of α without background risk.

α	Objective value
0.10	127.85
0.20	108.4
0.30	89.75
0.40	71.7
0.50	55.00
0.60	39.8
0.70	28.05
0.80	19.2
0.90	10.6
0.95	6.3

MATLAB 2017 shows the optimal objective value is 8.10 after a computation time of approximately 0.06 s.

For different confidence levels α, the objective values are listed in Table 6. To further show the influence of loans on portfolio selection, the portfolio returns of the chance-mean model with loans and without loans are presented in Figure 2. According to Figure 2, for the same confidence level α, the portfolio return with loans is greater than without loans.

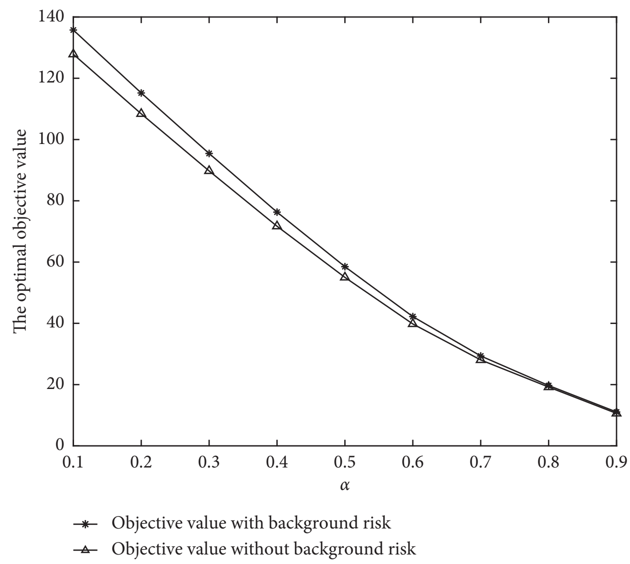

Figure 3: Objective values with background risk and without background risk.

Table 8: Computation results with different n.

n	w	v	u	α	Objective value 1	Objective value 2	Objective value 3	CPU (s)
50	500	100	100	0.2	368.03	324.55	350.78	0.06
50	500	100	100	0.6	287.46	252.51	279.93	0.06
50	500	100	100	0.9	225.48	197.71	224.11	0.06
100	800	300	200	0.2	741.71	577.42	720.36	0.06
100	800	300	200	0.6	534.46	416.44	531.29	0.06
100	800	300	200	0.9	424.74	336.41	425.39	0.06
500	1000	500	300	0.2	1.18×10^3	826.40	1.16×10^3	0.07
500	1000	500	300	0.6	918.77	634.81	914.77	0.07
500	1000	500	300	0.9	752.01	513.38	751.60	0.07

Next, we will illustrate the impact of background risk on investment decision-making. Set $v = 50$ (thousand yuan). If we ignore background risk, then the chance-mean model can be rewritten as follows:

$$\begin{cases} \max & \sum_{i=1}^{20} x_i \Phi_i^{-1}(0.1) - 0.01\left(\sum_{i=1}^{20} x_i - 300\right) \\ \text{s.t.} & \sum_{i=1}^{20} x_i E[\xi_i] - 0.01\left(\sum_{i=1}^{20} x_i - 300\right) \geq 45 \\ & 300 \leq \sum_{i=1}^{20} x_i \leq 350 \\ & \qquad 0 \leq x_i \leq 50, \quad i = 1, 2, \ldots, 20. \end{cases} \tag{48}$$

MATLAB 2017 shows the objective value is 10.6 after a computation time of approximately 0.06 s.

Table 7 illustrates the objective values for different confidence levels α in the chance-mean model without background risk. To compare the optimal portfolio with background risk and without background risk more conveniently, the results are shown in Figure 3, which is drawn by MATLAB 2017. Figure 3 shows that the acceptable threshold portfolio return to an investor without background risk is less than that with background risk.

6.2. Large-Scale Experiments. In the previous section, the experiments were made on problems with a small scale, i.e., $n = 20$. In this section, we do experiments on the chance-mean model with different scales to discuss the impact of borrowing and background risk on investment decisions and results in the general case. In these experiments, the following background factor return:

$$\eta \sim \mathscr{L}(a, b), \tag{49}$$

is assumed to be a linear uncertain variable, where a is generated randomly from the uniform distribution $U(-5, -1)$ and b is generated randomly from the uniform distribution $U(10, 30)$. The following return rate of the stock i:

$$\xi \sim \mathscr{L}(a_i, b_i), \tag{50}$$

is assumed to be a linear uncertain variable, where $b_i = a_i + r_i$, a_i is generated randomly from the uniform distribution $U(-0.5, 0.5)$ and r_i is generated randomly from the uniform distribution $U(0.1, 0.3)$.

According to models (24)–(26) and Theorem 3, we can solve the problems by using MATLAB 2017. The experiment results are listed in Table 8, where "objective value 1" represents the objective value of the initial chance-mean model, "objective value 2" represents the objective value of the chance-mean model without loans, and "objective value 3" represents the objective value of the chance-mean model without background risk.

For each (w, v, u, α), Table 8 shows that the portfolio return with loans is greater than that without loans. Table 8 also shows that the portfolio return with background risk is greater than that without background risk. That is, it is necessary for us to consider portfolio selection problems with loans and background risks.

According to Theorem 3, the proposed chance-mean model can be equivalent transformed to a deterministic linear programming model, which can be efficiently solved by MATLAB. For large-scale experiments, the computation times are listed in the last column of Table 8, which indicates that MATLAB obtains the optimal solutions quickly.

7. Conclusion

Portfolio selection decisions are usually made in the state of indeterminacy since the security market is so complex. This paper deals with portfolio selection problems with a borrowing constraint and background risk, in which security returns and background factor return are indeterminate quantities and are estimated by experts' evaluations rather than historical data. To rationally deal with such indeterminacy, this paper introduces uncertain variables to describe security returns and background factor return in portfolio selection models.

Within the framework of uncertainty theory, the distribution function of the maximum portfolio return is discussed, and an approximate method to calculate the distribution function is obtained. To obtain the optimal portfolio selection by a mathematical model, two decision criteria are proposed: critical value criterion and chance criterion. Based on the critical value criterion, the chance-mean model is proposed. Based on the chance criterion, the measure-mean model is proposed. It is worth pointing out that two new models are proposed from different points of view. We cannot conclude which model is the best for the portfolio selection decision in an uncertain environment. In fact, which model is applied in the real world is heavily dependent on the decision-making conditions and the preferences of investors. Generally, if we want to obtain a largest value $\overline{f}$ such that the portfolio return is greater than or equal to $\overline{f}$ with a predetermined confidence level, then we can use the chance-mean model. On the contrary, if we want to maximize the chance of the return preponderating the given profit level, then we can use the measure-mean model.

We believe that the proposed models can further be able to address the portfolio selection problem in uncertain environments. The main contributions can be summarized as the following three aspects: (1) two novel uncertain portfolio models are proposed based on different modelling methodologies: the chance-mean model and the measure-mean model. (2) The optimal solution to the chance-mean model is proved to be equivalent to a deterministic model, which is essentially a linear programming model. (3) The relationship between the chance-mean model and the measure-mean model is studied, and an approach to solve the measure-mean model is obtained.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

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